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ON WEAKLY IMPOSED BOUNDARY CONDITIONS FOR SECOND ORDER PROBLEMS

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Abstract

We consider the finite element method in which essential boundary conditions are imposed in a weak sense using a technique introduced by Nitsche. This technique is applied in a convection-diffusion-reaction problem and for slip boundary conditions in Stokes flow. We give the error estimates for the methods and also some numerical results.

Introduction

Traditionally, one of the main advantages of the standard finite element method has been its efficiency in treating complicated geometries and the accompanying boundary conditions. There exists, however, problems for which is seems to be worthwhile to explore the use of non-standard techniques for treating boundary conditions. Two of these problems will be considered in this paper; a convection-diffusion-reaction problem and slip boundary conditions for viscous incompressible flow. For these we will apply a technique introduced by Nitsche [6].

The plan of the paper is the following. In the next section we recall the method of Nitsche and introduce a new variant of it. Then we apply the techniques for the two problems mentioned above. We give the theoretical error estimates and discuss the advantages with the technique. In the last section we give the results of some numerical computations.

The Nitsche method

We let C_h denote the regular finite element division of Ω into triangles or tetrahedrons. For simplicity we assume that $\overline{\Omega} = \cup_{K \in C_h} \overline{K}$. We use the notation $E = \partial K \cap \Gamma$ for an edge or side lying on the boundary Γ of the domain Ω. We also define

$$(v, w)_{s,h} = \sum_{K \in C_h} h_K^s (v, w)_K$$

for $s = 1$ or $s = 2$, and

$$\langle v, w \rangle_{-1,h,G} = \sum_{E \subset G} h_E^{-1} \langle v, w \rangle_E,$$

for a subset $G \subset \Gamma$. The notation $(\cdot,\cdot)_K$ and $\langle \cdot,\cdot \rangle_E$ is used for the L_2-inner products on K and E, respectively. The mesh parameters are defined by

$$h_K = \operatorname{diam} K, \quad h_E = \operatorname{diam} E \quad \text{and} \quad h = \max_{K \in C_h} h_K.$$

Let us next consider Nitsche's method [6] for the model Dirichlet problem

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega, \\ u - u_0 &= 0 \quad \text{on } \Gamma. \end{aligned} \tag{1}$$

In the method no boundary conditions are imposed on the finite element subspace defined as

$$V_h = \left\{ v \in H^1(\Omega) \mid v|_K \in P_k(K) \ \forall K \in C_h \right\}.$$

The method is defined as: find $u_h \in V_h$ such that

$$\mathcal{B}(u_h, v) = \mathcal{F}(v) \qquad \forall v \in V_h, \tag{2}$$

with

$$\begin{aligned} \mathcal{B}(u,v) &= (\nabla u, \nabla v) - \langle \nabla u \cdot \vec{n}, v \rangle - \langle u, \nabla v \cdot \vec{n} \rangle + \beta \langle u, v \rangle_{-1,h,\Gamma}, \\ \mathcal{F}(v) &= (f, v) - \langle u_0, \nabla v \cdot \vec{n} \rangle + \beta \langle u_0, v \rangle_{-1,h,\Gamma}, \end{aligned} \tag{3}$$

and

where $\beta > 0$ is constant and $\vec{n}$ is the outward unit normal vector to the boundary. The method is consistent and one has an optimal error estimate in the norm

$$\|v\|_{1,h}^2 = \|v\|_1^2 + \langle v, v \rangle_{-1,h,\Gamma}.$$

Theorem 1. *There are positive constants* C_1 *and* C *so that if* $\beta > C_1$ *it holds*

$$\|u - u_h\|_{1,h} \le Ch^k \|u\|_{k+1} \cdot$$

The proof is given in [6] (see also [7]). Let us here simply indicate the main steps of it. First one has

$$\mathcal{B}(u,u) = (\nabla u, \nabla u) - 2\langle \nabla u \cdot \vec{n}, u\rangle + \beta\langle u,u\rangle_{-1,h,\Gamma}.$$

Next, using the following inverse inequality

$$\sum_{E\subset\Gamma} h_E \|\nabla u \cdot \vec{n}\|_{0,E}^2 \le C_1 \|\nabla u\|_0^2 \tag{4}$$

and the arithmetic-geometric-mean inequality, it follows that if $\beta > C_1$ then the following stability estimate holds

$$B(u,u) \ge C_2 \|u\|_{1,h}^2 \quad \forall u \in V_h.$$

The error estimate then follows from the interpolation estimate in the norm $\|\cdot\|_{1,h}$.

The biggest drawback with the above method is the condition $\beta > C_1$, where C_1 is the constant in the inverse estimate of Eq. 4. This condition is avoided if the forms of Eq. 3 are modified to be

$$\mathcal{B}(u,v) = (\nabla u, \nabla v) - \langle \nabla u \cdot \vec{n}, v\rangle + \langle u, \nabla v \cdot \vec{n}\rangle + \beta\langle u,v\rangle_{-1,h,\Gamma}, \tag{5}$$

and

$$\mathcal{F}(v) = (f,v) + \langle u_0, \nabla v \cdot \vec{n}\rangle + \beta\langle u_0, v\rangle_{-1,h,\Gamma}.$$

With this the method is still consistent and, in particular, stable for all positive values of β:

$$\mathcal{B}(u,u) = (\nabla u, \nabla u) + \beta\langle u,u\rangle_{-1,h,\Gamma} \ge C\|u\|_{1,h}^2.$$

Therefore one obtains:

Theorem 2. *Suppose that* $\beta > 0$. *Then the problem defined by* Eq. 2 *and* Eq. 5 *has a unique solution satisfying*

$$\|u - u_h\|_{1,h} \le Ch^k \|u\|_{k+1} \,.$$

Compared to the original Nitsche method this formulation has the drawback that the symmetry of the discretization is lost. Hence, it does not seem to be too attractive. In the diffusion-convection-reaction problem considered below the situation is different.

Let us here remark that there is a close connection between Nitsche's approach and recent stabilizing techniques for saddle point problems. This is discussed in [4]. In connection with the Stokes problem Douglas and Wang [2] have proposed non-symmetric formulation similar to our modification of Nitsche's method. Many other variants of Nitsche's method exist and some of them are discussed in [7].

The diffusion-convection equation

We consider the steady scalar diffusion-convection-reaction problem

$$\begin{aligned} \mathcal{L}u - f \equiv -d\Delta u + \vec{a}\cdot\nabla u + cu - f = 0 \quad &\text{in } \Omega, \\ u = u_0 \quad &\text{on } \Gamma, \end{aligned} \tag{6}$$

where the diffusion coefficient $d \ge 0$, the velocity vector $\vec{a}$ and the sink factor $c \ge 0$ are constants. This problem is numerically difficult both in the convection- and reaction-dominated cases.

The method we propose combines the SUPG [1] formulation and the modified Nitsche method: find $u_h \in V_h$ such that

$$\mathcal{B}_\Omega(u_h, v) + \mathcal{B}_\Gamma(u_h, v) = \mathcal{F}(v) \quad \forall v \in V_h \,, \tag{7}$$

where

$$\begin{aligned} \mathcal{B}_\Omega(u,v) &= (d\nabla u, \nabla v) + (\vec{a}\cdot\nabla u, v) + (cu, v) + (\mathcal{L}u, \vec{\tau}\cdot\nabla v)_{1,h} \\ \mathcal{B}_\Gamma(u,v) &= -\langle \vec{a}\cdot\vec{n}\, u, v\rangle_{\Gamma_-} - \langle d\nabla u\cdot\vec{n}, v\rangle_\Gamma + \langle u, d\nabla v\cdot\vec{n}\rangle_\Gamma + \beta\langle u, dv\rangle_{-1,h,\Gamma}, \\ \mathcal{F}(v) &= (f, v) + (f, \vec{\tau}\cdot\nabla v)_{1,h} - \langle \vec{a}\cdot\vec{n}\, u_0, v\rangle_{\Gamma_-} + \langle u_0, d\nabla v\cdot\vec{n}\rangle_\Gamma + \beta\langle u_0, dv\rangle_{-1,h,\Gamma}, \end{aligned} \tag{8}$$

and V_h is defined as for the previous problem. The stability vector expression is $\vec{\tau} = \alpha\vec{a}/|\vec{a}|$, where α is positive.

As the problem is not symmetric it is natural to use the modified form of Nitsche's method. By this one has more freedom in choosing the positive parameter β. We would like to point out that in the limiting case $d=0$ the boundary condition $u=u_0$ is only left on the inflow part $\Gamma_-=\{\vec{x}\in\Gamma \mid \vec{a}\cdot\vec{n}\leq 0\}$ of the boundary. An advantage of our formulation is that this is correctly taken into account by the method. Similarly, in the reaction limit when $d=0$ and $\vec{a}=\vec{0}$ no boundary condition is left in the problem and the same happens in the numerical scheme.

For brevity we will give the error estimate for the case of a small diffusion. It will be given in the norm

$$\|v\|_h^2 = d\|v\|_1^2 + c\|v\|_0^2 + |\vec{a}|^{-1}(\vec{a}\cdot\nabla v, \vec{a}\cdot\nabla v)_{1,h} + \langle v, v|\vec{a}\cdot\vec{n}|\rangle_\Gamma + d\langle v, v\rangle_{-1,h,\Gamma}.$$

Theorem 3. *Suppose that* $d\leq C|\vec{a}|h_K \ \forall K\in C_h$, $\beta>0$ *and that* $\alpha>0$ *is sufficiently small. Then the solution* $u_h\in V_h$ *to the problem of* Eq. 7 *and* Eq. 8 *is unique and satisfies*

$$\|u-u_h\|_h \leq Ch^{k+1/2}\|u\|_{k+1}\ .$$

The proof makes use of standard techniques explained for example in [3].

Let us also remark that for the case when the diffusion is greater , i.e. for elements where $d>C|\vec{a}|h_K$, one chooses $\alpha=0$.

The Stokes equations

We consider the problem scaled so that the viscosity equals unity: find the velocity $\vec{u}$ and the pressure p such that

$$\begin{aligned}
\mathcal{L}(\vec{u},p) \equiv -\Delta\vec{u}+\nabla p = \vec{f} \quad &\text{in } \Omega,\\
\nabla\cdot\vec{u}=0 \quad &\text{in } \Omega,\\
\vec{u}=\vec{u}_0 \quad &\text{on } \Gamma_1,\\
\vec{t}=\vec{0} \quad &\text{on } \Gamma_2,\\
\vec{u}\cdot\vec{n}=g,\ \vec{t}-(\vec{t}\cdot\vec{n})\,\vec{n}=\vec{0} \quad &\text{on } \Gamma_3,
\end{aligned} \tag{9}$$

where $\vec{t} = \sigma(\vec{u}, p) \cdot \vec{n}$, with $\sigma(\vec{u}, p) = 2\varepsilon(\vec{u}) - pI$, and $\varepsilon(\vec{u})$ is the symmetric part of $\nabla\vec{u}$. We assume that the velocity is given on Γ_1 and that the traction vanish on Γ_2. On Γ_3 we impose a slip type of condition. We will also assume that the disjoint boundary parts Γ_i, $i = 1,2,3$, all have a positive measure and that $\partial\Omega = \overline{\Gamma_1} \cup \overline{\Gamma_2} \cup \overline{\Gamma_3}$.

We will use a well known stabilizing technique [5] to handle the divergence constraint. The slip condition $\vec{u} \cdot \vec{n} = 0$ on Γ_3 we treat by the original symmetric Nitsche technique. This seems to be more practical than imposing it in strong form as that would involve both local co-ordinate transformations and the approximation of the normal at corners (either appearing in the domain or arising in its finite element modelling). We remark that the essential condition $\vec{u} = \vec{u}_0$ on Γ_1 could of course also be treated by Nitsche's method.

The finite element method is thus defined as: find $(\vec{u}_h - \vec{u}_0, p_h) \in V_h \times P_h$ such that

$$\mathcal{B}_\Omega(\vec{u}_h, p_h; \vec{v}, q) + \mathcal{B}_\Gamma(\vec{u}_h, p_h; \vec{v}, q) = \mathcal{F}(\vec{v}, q) \qquad \forall(\vec{v}, p) \in V_h \times P_h\ , \tag{10}$$

where

$$\mathcal{B}_\Omega(\vec{u}, p; \vec{v}, q) = 2(\varepsilon(\vec{u}), \varepsilon(\vec{v})) - (\nabla \cdot \vec{u}, q) - \alpha(\mathcal{L}(\vec{u}, p), \mathcal{L}(\vec{v}, q))_{2,h},$$

$$\mathcal{B}_\Gamma(\vec{u}, p; \vec{v}, q) = -\langle \vec{n} \cdot \vec{t}, \vec{v} \cdot \vec{n} \rangle_{\Gamma_3} - \langle \vec{u} \cdot \vec{n}, \vec{s} \cdot \vec{n} \rangle_{\Gamma_3} + \beta \langle \vec{u} \cdot \vec{n}, \vec{v} \cdot \vec{n} \rangle_{-1,h,\Gamma_3},$$

$$\mathcal{F}(\vec{v}, q) = (\vec{f}, \vec{v}) + \alpha(\vec{f}, \mathcal{L}(\vec{v}, q))_{2,h} - \langle g, \vec{s} \cdot \vec{n} \rangle_{\Gamma_3} + \beta \langle g, \vec{v} \cdot \vec{n} \rangle_{-1,h,\Gamma_3} \tag{11}$$

and $\vec{s} = \sigma(\vec{v}, q) \cdot \vec{n}$. It is readily verified that the formulation is consistent, i.e. the exact solution $(\vec{u}, p)$ satisfies Eq. 10. (Here one has to use the fact that since $\nabla \cdot \vec{u} = 0$ it holds $\Delta\vec{u} = 2\nabla \cdot \varepsilon(\vec{u})$).

The finite element space for the velocity is defined by

$$V_h = \left\{ \vec{v} \in [H^1(\Omega)]^N \mid \vec{v}|_K \in [P_k(K)]^N \, \forall K \in C_h \mid \vec{v}|_{\Gamma_1} = 0 \right\}.$$

The pressure is approximated either discontinuously

$$P_h = \left\{ p \in L_2(\Omega) \mid p|_K \in P_l(K) \, \forall K \in C_h \right\},$$

with $l \geq 0$ or continuously $(l \geq 1)$

$$P_h = \{p \in C(\Omega) \mid p|_K \in P_l(K)\ \forall K \in C_h\}.$$

For the error estimate we use the following norm

$$\|\vec{v}\|_{1,h}^2 = \|\vec{v}\|_1^2 + \langle \vec{v}, \vec{v} \rangle_{-1,h,\Gamma_3}^2 .$$

With this we get a measure of how well the essential boundary condition $\vec{u} \cdot \vec{n} = g$ on Γ_3 is approximated.

Theorem 4. *Suppose that either* $k \geq N$ *or* $P_h \subset C(\Omega)$. *There are positive constants* C_1 *and* C_2 *such that if* $0 < \alpha < C_1$ *and* $\beta > C_2$ *then the solution* $(\vec{u}_h - \vec{u}_0, p) \in V_h \times P_h$ *to the problem of* Eq. 10 *and* Eq. 11 *is unique and satisfies*

$$\|\vec{u} - \vec{u}_h\|_{1,h} + \|p - p_h\|_0 \leq Ch^s \left(\|\vec{u}\|_{s+1} + \|p\|_s \right),$$

where $s = \min\{k, l+1\}$.

The proof of this result is obtained by combining techniques given in [5] and [6].

Examples

Although the bilinear and linear forms of the variational formulations are somewhat more lengthy than when enforcing the Dirichlet type of conditions strongly, no particular problems arise in practice. The examples are solved using linear triangular elements.

Example 1

The diffusion-convection-reaction problem can be quite difficult from the numerical point of view when the convection term dominates, i.e., when $\hat{a} = h|\vec{a}|/d$ is large and the Dirichlet data is non-smooth. In that case a boundary layer much thinner than the element size appear at the outflow boundaries and also inside the domain. Similar behavior is possible also in the reaction dominated case when $\hat{c} = h^2 c / d$ is large.

Fig.1 shows the numerical results when the solution domain is the rectangle $\Omega = \{(x, y) \mid 0 < x < 1 \text{ and } 0 < y < 1\}$ and the Dirichlet data is zero except on $\Gamma = \{(x, y) \mid x = 0 \text{ and } 0.3 < y < 1\}$ where the value is one. The stability parameters used were $\alpha = \min\{6, \hat{a}\}/9$ and $\beta = 10$ and the problem parameters were varied to end up with representative combinations of $\hat{a}$ and $\hat{c}$. It is noteworthy that the oscillations of the SUPG-formulation, when the boundary layers are skewed to the flow, are reduced substantially.

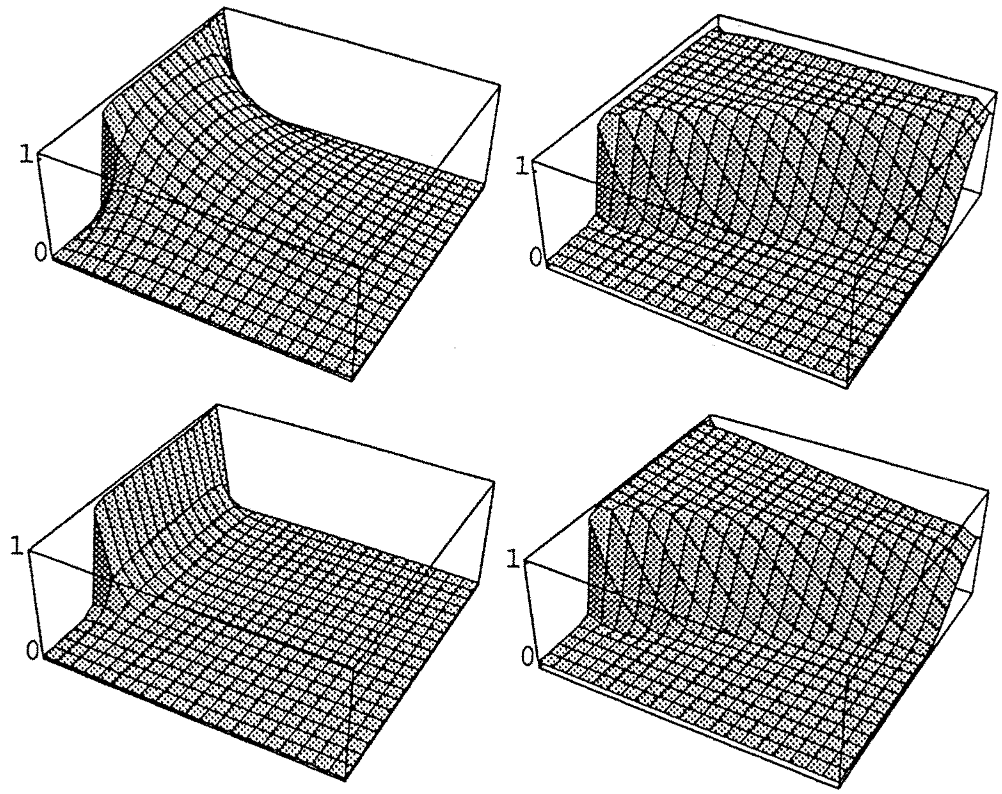

Fig.1. Numerical results on a regular 800 triangular element mesh where each small rectangle of the figures is divided into two triangles. The columns correspond to the dimensionless parameter values $\hat{a} = 0$ (first) and $\hat{a} = 50$ (second) and the rows $\hat{c}=0$ (first) and $\hat{c} = 1$ (second).

Example 2

The second example problem describes flow past a frictionless cylinder positioned at the center of a rectangular domain. Due to the symmetry only the upper half

$$\Omega = \{(x,y) \mid -1 < x < 1,\ 0 < y < 1 \text{ and } \sqrt{x^2 + y^2} > 3/10\}$$

of the original domain was used in the calculations. The boundary conditions describe the channel flow, where the right hand side edge is of the type Γ_2

(outflow), the upper and left hand side edges are of the type Γ_1 (wall and inflow). The remaining part, consisting of the lower edge with a cylindrical section, is the type Γ_3. The velocity components given on the inflow part represented fully developed channel flow.

In this example we applied piecewise linear approximations for the pressure and velocity components on a mesh consisting of triangles. The stability parameters were $\alpha = 1/10$, $\beta = 10$. Numerical results of Fig.2. show that the frictionless wall conditions are satisfied correctly.

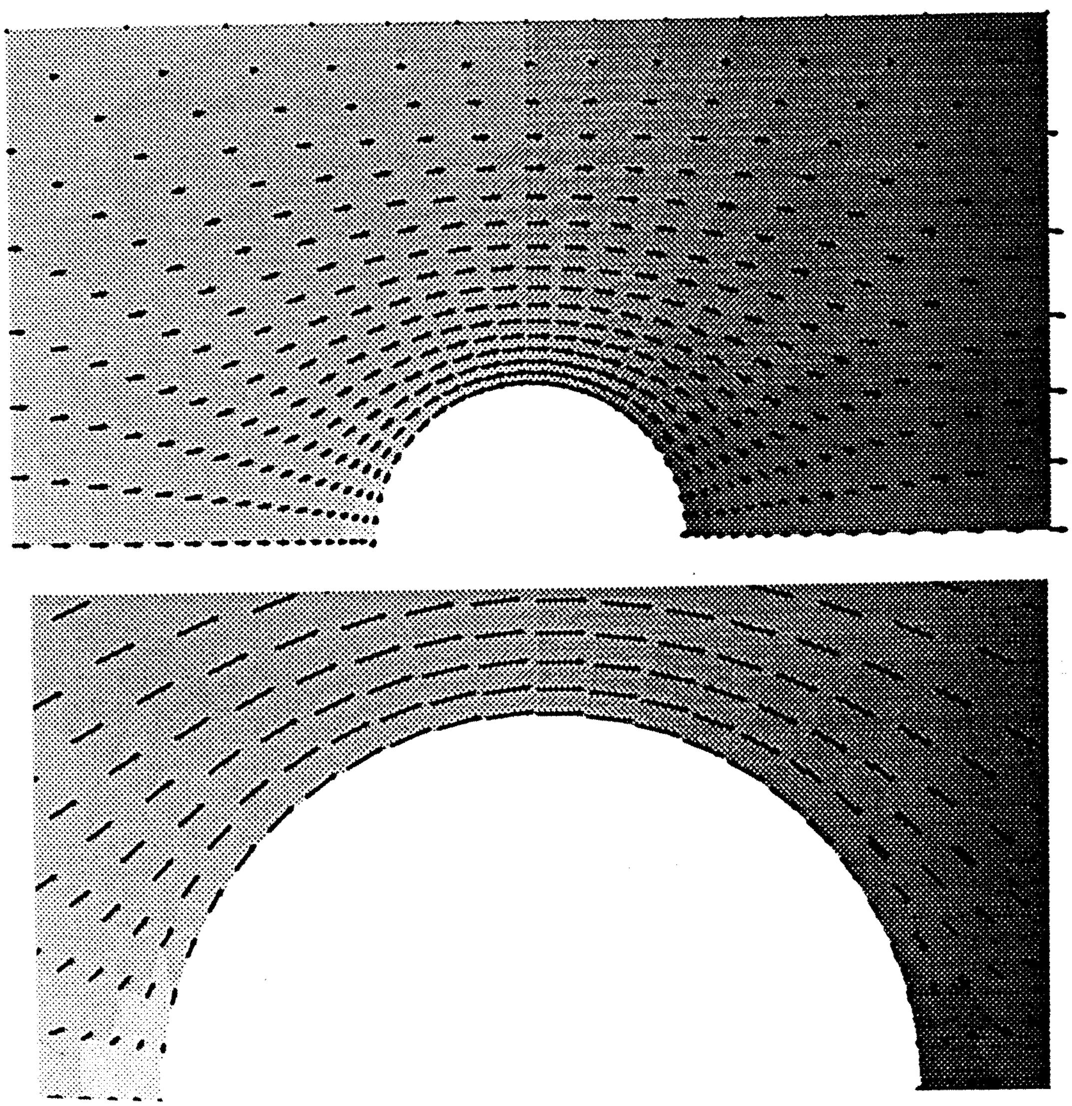

Fig.2. Channel flow past a frictionless cylinder on a mesh consisting of 850 triangles. Dark background coloring represents small pressure values and light larger ones. The highest velocity component in the vertical direction at the inlet is one. The velocity vectors shown

by dots have zero magnitude. The lower figure is a magnification of the neighbourhood of the cylinder.

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