A MODIFICATION OF A LOW-ORDER REISSNER-MINDLIN PLATE BENDING ELEMENT

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1. INTRODUCTION

During the last decade there has been a continuous search for simple "locking free" plate bending elements based on the Reissner-Mindlin theory. A great number of methods have been proposed and many of these have been shown to perform rather well in practical computations, cf. e.g. [2, 13] and the references therein.

However, relatively few methods have allowed a rigorous mathematical stability and error analysis [1,3,4,14,15,16], a fact which suggest that the task of finding a good Reissner-Mindlin element is a non trivial problem. From a practical side, this conclusion is supported by the fact that so many methods have been, and are being, proposed.

To our knowledge the simplest method proposed for which the optimal order of convergence has rigorously been proved, is a recent method by Arnold and Falk [1].

The purpose of this note is to point out a modification of Arnold and Falk's element, which is considerably simpler to implement. Furthermore, for the modification it is possible to prove error estimates which are identical to those of the original method.

The same idea has independently been introduced bu Durán, Ghioldi and Wolanski [9].

In the next section we recall some theoretical results on the Reissner-Mindlin model and the method by Arnold and Falk. In section 3 we give our modification, discuss its advantage, and give the error analysis.

Our notation is standard (cf. [7]) and consistent, though not completely equivalent, with that of [1].

2. THE REISSNER-MINDLIN MODEL AND THE ARNOLD-FALK METHOD

Let Ω be the region occupied by the plate, the thickness of which is denoted by t. Denote by w and $\boldsymbol{\phi} = (\phi_1, \phi_2)$ the transverse deflection of Ω, and the

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rotation of the normals to Ω, respectively. Assuming a clamped boundary, the model is: Find $w \in H_0^1(\Omega)$ and $\boldsymbol{\phi} \in [H_0^1(\Omega)]^2$ such that

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) + \lambda t^{-2}(\boldsymbol{\phi} - \mathbf{grad}\, w, \boldsymbol{\psi} - \mathbf{grad}\, v) = (g, v), \qquad v \in H_0^1(\Omega), \quad \boldsymbol{\psi} \in [H_0^1(\Omega)]^2. \tag{2.1}$$

Here g is the (appropriately scaled, cf. [4]) load and

$$\lambda = \frac{E\kappa}{2(1+\nu)}$$

is the shear modulus multiplied with the shear correction factor κ. As usual, E and ν denote Young's modulus and Poisson's ratio, respectively. The bilinear form a is defined through

$$a(\boldsymbol{\phi},\boldsymbol{\psi}) = \frac{E}{12(1-\nu^2)} \int_\Omega [(1-\nu)\, \boldsymbol{\varepsilon}(\boldsymbol{\phi}) : \boldsymbol{\varepsilon}(\boldsymbol{\psi}) + \nu \operatorname{div} \boldsymbol{\phi} \operatorname{div} \boldsymbol{\psi}],$$

where $\boldsymbol{\varepsilon}$ is the linear strain operator.

We recall that Korn's inequality implies

$$a(\boldsymbol{\phi},\boldsymbol{\phi}) \geq C\|\boldsymbol{\phi}\|_1^2, \qquad \boldsymbol{\phi} \in [H_0^1(\Omega)]^2,$$

for $-1 < \nu \leq 1/2$.

Above and below C, C_1, C_2, ... denote positive constants independent of t, g and the mesh parameter h.

Introducing the shear

$$\mathbf{q} = \lambda t^{-2}(\mathbf{grad}\, w - \boldsymbol{\phi})$$

as an independent variable, (2.1) can equivalently be written as:
Find $w \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in [H_0^1(\Omega)]^2$ and $\mathbf{q} \in [L^2(\Omega)]^2$, such that

$$\begin{aligned} a(\boldsymbol{\phi},\boldsymbol{\psi}) - (\mathbf{q},\boldsymbol{\psi}) &= 0, && \boldsymbol{\psi} \in [H_0^1(\Omega)]^2, \\ (\mathbf{q}, \mathbf{grad}\, v) &= (g, v), && v \in H_0^1(\Omega), \\ \lambda^{-1}t^2(\mathbf{q},\mathbf{s}) + (\boldsymbol{\phi} - \mathbf{grad}\, w, \mathbf{s}) &= 0, && \mathbf{s} \in [L^2(\Omega)]^2. \end{aligned} \tag{2.2}$$

For the analysis the following Helmholtz decomposition proved in [4] is useful.

LEMMA 1. *Every* $\mathbf{q} \in [L^2(\Omega)]^2$ *can be uniquely written as*

$$\mathbf{q} = \mathbf{grad}\, r + \mathbf{curl}\, p, \quad r \in H_0^1(\Omega), \quad p \in H^1(\Omega) \cap L_0^2(\Omega). \quad \blacksquare$$

Here and in the sequel we denote

$$\mathbf{curl}\, v = \left(-\frac{\partial v}{\partial x_2}, \frac{\partial v}{\partial x_1}\right) \quad \text{for } v \in H^1(\Omega),$$

and

$$\operatorname{curl} \boldsymbol{\psi} = \frac{\partial \psi_1}{\partial x_2} - \frac{\partial \psi_2}{\partial x_1} \quad \text{for } \boldsymbol{\psi} = (\psi_1, \psi_2) \in [H^1(\Omega)]^2.$$

In [1,4] the following result for a generalization of (2.2) is proved.

PROPOSITION 1. *Let Ω be a convex polygonal or smoothly bounded domain in the plane. For any t, $0 < t \le C$, $g \in H^{-1}(\Omega)$, and $\mathbf{f} \in [H^{-1}(\Omega)]^2$, there is a unique solution $w \in H_0^1(\Omega)$, $\boldsymbol{\phi} \in [H_0^1(\Omega)]^2$ and $\mathbf{q} \in [L^2(\Omega)]^2$ to*

$$\begin{aligned} a(\boldsymbol{\phi}, \boldsymbol{\psi}) - (\mathbf{q}, \boldsymbol{\psi}) &= (\mathbf{f}, \boldsymbol{\psi}), && \boldsymbol{\psi} \in [H_0^1(\Omega)]^2, \\ (\mathbf{q}, \mathbf{grad}\, v) &= (g, v), && v \in H_0^1(\Omega), \\ \lambda^{-1} t^2 (\mathbf{q}, \mathbf{s}) + (\boldsymbol{\phi} - \mathbf{grad}\, w, \mathbf{s}) &= 0, && \mathbf{s} \in [L^2(\Omega)]^2. \end{aligned}$$

Moreover, if $\mathbf{f} \in [L^2(\Omega)]^2$, then $\boldsymbol{\phi} \in [H^2(\Omega)]^2$, and we have

$$\|r\|_1 + \|\boldsymbol{\phi}\|_2 + \|p\|_1 + t\|p\|_2 + \|w\|_1 \le C(\|g\|_{-1} + \|\mathbf{f}\|_0),$$

with $\mathbf{q} = \mathbf{grad}\, r + \mathbf{curl}\, p$.

If additionally $g \in L^2(\Omega)$, then $r, w \in H^2(\Omega)$, and we have

$$\|r\|_2 + \|w\|_2 \le C(\|g\|_0 + \|\mathbf{f}\|_0). \quad \blacksquare$$

Next, let us recall the method of [1]. We introduce a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$, which henceforth is assumed to be polygonal. As usual the mesh parameter is defined through

$$h = \max_{T \in \mathcal{T}_h} h_T,$$

where h_T denotes the diameter of T. The triangulation is not assumed to be quasiuniform.

For approximating the deflection the space of piecewise linear nonconforming elements is used:

$$W_h = \{ v \in L^2(\Omega) \mid v_{|T} \in P_1(T),\ T \in \mathcal{T}_h \text{ and } v \text{ is continuous at midpoints of element edges and vanishes at midpoints of boundary edges} \}. \tag{2.3}$$

The space for the rotation is

$$\mathbf{V}_h^B = \{ \boldsymbol{\phi} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\phi}_{|T} \in [P_1(T) \oplus B(T)]^2,\ T \in \mathcal{T}_h \}, \tag{2.4}$$

where $B(T)$ denotes the spaces of cubic "bubbles" on T:

$$B(T) = \{ v \in P_3(T) \mid v_{|\partial T} = 0 \}.$$

Futhermore, denote

$$\mathbf{Q}_h = \{ \mathbf{q} \in [L^2(\Omega)]^2 \mid \mathbf{q}_{|T} \in [P_0(T)]^2,\ T \in \mathcal{T}_h \} \tag{2.5}$$

and let $\mathbf{P}_0 : [L^2(\Omega)]^2 \to \mathbf{Q}_h$ be the orthogonal projection. For $v \in W_h + H^1(\Omega)$ we define $\mathbf{grad}_h v$ to be the $[L^2(\Omega)]^2$ function whose restriction to each $T \in \mathcal{T}_h$ is given by $\mathbf{grad}\, v_{|T}$.

The method is then defined as: Find $w_h \in W_h$ and $\boldsymbol{\phi}_h \in \mathbf{V}_h^B$ such that

$$\begin{aligned} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda t^{-2}(\mathbf{P}_0\boldsymbol{\phi}_h - \mathbf{grad}_h w_h, \mathbf{P}_0\boldsymbol{\psi} - \mathbf{grad}_h v) &= (g, v), \\ v \in W_h,\ \ \boldsymbol{\psi} &\in \mathbf{V}_h^B. \end{aligned} \tag{2.6}$$

The error estimate proved in [1] is the following.

PROPOSITION 2. *Suppose that* Ω *is convex,* $g \in L^2(\Omega)$, *and that* $0 < t \leq C$. *For the unique solutions* $(w, \boldsymbol{\phi})$ *and* $(w_h, \boldsymbol{\phi}_h)$ *to (2.1) and (2.6), respectively, we have*

$$||w - w_h||_0 + ||\boldsymbol{\phi} - \boldsymbol{\phi}_h||_0 \leq Ch^2 ||g||_0. \quad \blacksquare$$

3. THE MODIFIED METHOD

The modification we are proposing is the following: The space for the deflection is kept as defined in (2.3). For the rotations we use the standard space of continuous piecewise linear functions

$$\mathbf{V}_h = \{ \boldsymbol{\phi} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\phi}_{|T} \in [P_1(T)]^2,\ T \in \mathcal{T}_h \}, \tag{3.1}$$

i.e. the bubble degrees of freedom in (2.4) are dropped. We again denote by $\mathbf{P}_0$ the orthogonal projection onto the space $\mathbf{Q}_h$ as defined in (2.5). The method is then defined as: Find $w_h \in W_h$ and $\boldsymbol{\phi}_h \in \mathbf{V}_h$ such that

$$\begin{aligned} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) + \lambda \sum_{T \in \mathcal{T}_h} (t^2 + \alpha_T h_T^2)^{-1} (\mathbf{P}_0\boldsymbol{\phi}_h - \mathbf{grad}_h w_h, \mathbf{P}_0\boldsymbol{\psi} - \mathbf{grad}_h v)_T \\ = (g, v), \qquad v \in W_h,\ \boldsymbol{\psi} \in \mathbf{V}_h, \end{aligned} \tag{3.2}$$

were α_T are positive parameters restricted to lie in a fixed range, $C_1 \leq \alpha_T \leq C_2$.

For the modification we can prove error estimates analogous to those of the original method.

THEOREM. *Suppose that* Ω *is convex,* $g \in L^2(\Omega)$, *and that* $0 < t \leq C$. *For the unique solutions* $(w, \boldsymbol{\phi})$ *and* $(w_h, \boldsymbol{\phi}_h)$ *to (2.1) and (3.2), respectively, we have*

$$||w - w_h||_0 + ||\boldsymbol{\phi} - \boldsymbol{\phi}_h||_0 \leq Ch^2 ||g||_0 . \quad \blacksquare$$

Before turning to the error analysis of the method, let us discuss the difference in implementing the two methods.

First, considering the original method (2.6), we see that when calculating the contribution to the stiffness matrix from the bilinear form a an integration formula exact for fourth degree polynomials has to be used due to the presence of the bubble functions. In addition, when calculating the contribution from the shear energy, the local projections of the rotations have to be calculated. Furthermore, when for an element the local stiffness matrix has been calculated, it is preferable to eliminate the bubble degrees of freedom by condensation. Taken together, all this leads to rather cumbersome calculations.

Looking at the modification (3.2), we first note that since the rotations are piecewise linear, the constant value of the projection of a function in $\mathbf{V}_h$ is merely the value of the function at the midpoint (i.e. center of gravity) of the element. Since the functions of W_h and $\mathbf{V}_h$ are piecewise linear, this means that *the local stiffness matrix in (3.2) is obtained by the midpoint rule.* Hence, it is evident that our modification implies a considerably cheaper and simpler calculation of the stiffness matrix.

Let us remark that (3.2) can equivalently be written as:
Find $w_h \in W_h$, $\boldsymbol{\phi}_h \in \mathbf{V}_h$ and $\mathbf{q}_h \in \mathbf{Q}_h$, such that

$$\begin{aligned} a(\boldsymbol{\phi}_h, \boldsymbol{\psi}) - (\mathbf{q}_h, \boldsymbol{\psi}) &= 0, & \boldsymbol{\psi} &\in \mathbf{V}_h, \\ (\mathbf{q}_h, \mathbf{grad}_h v) &= (g, v), & v &\in W_h, \\ \lambda^{-1} \sum_{T \in \mathcal{T}_h} (t^2 + \alpha_T h_T^2)(\mathbf{q}_h, \mathbf{s})_T + (\boldsymbol{\phi}_h - \mathbf{grad}_h w_h, \mathbf{s}) &= 0, & \mathbf{s} &\in \mathbf{Q}_h. \end{aligned} \tag{3.3}$$

Comparing with (2.2), we see that (3.3) is a "Galerkin-least-squares", or "stabilized", mixed method. Recently these techniques have been applied to a number of different problems; cf. [5,6,10,11] and the references therein. For Reissner-Mindlin plates methods of this kind have been proposed in [14,15,16].

For the analysis of the method we need the discrete Helmholtz decomposition theorem of Arnold and Falk.

LEMMA 2 (Arnold and Falk [1]).

$$\mathbf{Q}_h = \mathbf{grad}_h W_h \oplus \mathbf{curl}\, \hat{S}_h,$$

with

$$\hat{S}_h = \{\, v \in H^1(\Omega) \cap L_0^2(\Omega) \mid v_{|T} \in P_1(T),\ T \in \mathcal{T}_h \,\}.$$

This is an orthogonal decomposition in $[L^2(\Omega)]^2$. ■

Furthermore a classical estimate for nonconforming methods will be needed [1,8].

LEMMA 3. *There is a positive constant* C *such that*

$$|\sum_{T\in\mathcal{T}_h}\int_{\partial T} v\,\boldsymbol{\psi}\cdot\mathbf{n}_T| \le Ch\|\boldsymbol{\psi}\|_1 \inf_{q\in H_0^1(\Omega)}\|\mathbf{grad}_h(v-q)\|_0,$$

$$\boldsymbol{\psi}\in[H^1(\Omega)]^2,\quad v\in W_h+H_0^1(\Omega).\quad ■$$

Let us introduce some additional notation.

By $\boldsymbol{\Pi}_h:\mathbf{Q}_h\to\mathbf{grad}_h W_h$ we denote the orthogonal projection, and we define a norm in $\mathbf{Q}_h$ through

$$\|\mathbf{s}\|_h^2 = \|\boldsymbol{\Pi}_h\mathbf{s}\|_0^2 + \sum_{T\in\mathcal{T}_h}(t^2+h_T^2)\|\mathbf{s}\|_{0,T}^2.$$

REMARK. It is also possible to perform the error analysis using the same norm as in [1] for the shear, i.e.

$$\|\mathbf{s}\|^2 = \|\mathbf{grad}_h k\|_0^2 + \|l\|_0^2 + t^2\|l\|_1^2,$$

where the decomposition $\mathbf{s}=\mathbf{grad}_h k+\mathbf{curl}\,l$, $k\in W_h$, $l\in\hat{S}_h$, is used for $\mathbf{s}\in\mathbf{Q}_h$ (note that $\boldsymbol{\Pi}_h\mathbf{s}=\mathbf{grad}_h k$). For this some extra technical details are needed (cf. Lemmas 3.2 and 3.3 of [11]) and it gives the optimal estimate

$$\|\mathbf{grad}\,r-\mathbf{grad}_h r_h\|_0 + \|p-p_h\|_0 + t\|p-p_h\|_1 \le Ch\|g\|_0,$$

with $\mathbf{q}=\mathbf{grad}\,r+\mathbf{curl}\,p$ and $\mathbf{q}_h=\mathbf{grad}_h r_h+\mathbf{curl}\,p_h$. However, this result does not seem to be very useful, and hence we prefer to present a more straightforward error analysis. ■

For this we write the method (3.3) with a more compact notation as

$$\mathcal{B}_h(w_h,\boldsymbol{\phi}_h,\mathbf{q}_h;v,\boldsymbol{\psi},\mathbf{s}) = (g,v),\quad v\in W_h,\ \boldsymbol{\psi}\in\mathbf{V}_h,\ \mathbf{s}\in\mathbf{Q}_h,$$

with

$$\mathcal{B}_h(w,\boldsymbol{\phi},\mathbf{q};v,\boldsymbol{\psi},\mathbf{s}) = a(\boldsymbol{\phi},\boldsymbol{\psi}) + (\mathbf{q},\mathbf{grad}_h v-\boldsymbol{\psi}) + (\mathbf{s},\mathbf{grad}_h w-\boldsymbol{\phi}) - \lambda^{-1}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)(\mathbf{q},\mathbf{s})_T.$$

Introducing the notation

$$|||(w,\boldsymbol{\phi},\mathbf{q})|||_h^2 = \|\mathbf{grad}_h w\|_0^2 + \|\boldsymbol{\phi}\|_1^2 + \|\mathbf{q}\|_h^2,$$

our stability estimate is the following.

LEMMA 4.

$$\sup_{\substack{(v,\psi,s)\in W_h\times V_h\times Q_h\\ (v,\psi,s)\neq(0,0,0)}} \frac{\mathcal{B}_h(w,\boldsymbol{\phi},\mathbf{q};v,\boldsymbol{\psi},\mathbf{s})}{|||(v,\boldsymbol{\psi},\mathbf{s})|||_h} \geq C|||(w,\boldsymbol{\phi},\mathbf{q})|||_h,$$

$$w\in W_h,\quad \boldsymbol{\phi}\in \mathbf{V}_h,\quad \mathbf{q}\in\mathbf{Q}_h.$$

Proof: Let $w\in W_h$, $\boldsymbol{\phi}\in\mathbf{V}_h$ and $\mathbf{q}\in\mathbf{Q}_h$ be given. Further, let $z\in W_h$ be such that $\mathbf{grad}_h z = \boldsymbol{\Pi}_h\mathbf{q}$.

Choosing $v = w+z$, $\boldsymbol{\psi}=\boldsymbol{\phi}$, $\mathbf{s} = -\mathbf{q}+\delta\,\mathbf{grad}_h w$, and letting $\delta>0$, $\varepsilon>0$, we get

$$\begin{aligned}
&\mathcal{B}_h(w,\boldsymbol{\phi},\mathbf{q};v,\boldsymbol{\psi},\mathbf{s})\\
&= \mathcal{B}_h(w,\boldsymbol{\phi},\mathbf{q};w+z,\boldsymbol{\phi},-\mathbf{q}) + \delta\,\mathcal{B}_h(w,\boldsymbol{\phi},\mathbf{q};0,\mathbf{0},\mathbf{grad}_h w)\\
&= a(\boldsymbol{\phi},\boldsymbol{\phi}) + \lambda^{-1}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)\|\mathbf{q}\|_{0,T}^2 + \|\boldsymbol{\Pi}_h\mathbf{q}\|_0^2 - \delta\,(\mathbf{grad}_h w,\boldsymbol{\phi})\\
&\quad + \delta\,\|\mathbf{grad}_h w\|_0^2 - \delta\,\lambda^{-1}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)(\mathbf{grad}_h w,\mathbf{q})_T\\
&\geq (C_1-\frac{\delta}{2})\|\boldsymbol{\phi}\|_1^2 + \lambda^{-1}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)\|\mathbf{q}\|_{0,T}^2\\
&\quad + \|\boldsymbol{\Pi}_h\mathbf{q}\|_0^2 + \frac{\delta}{2}\|\mathbf{grad}_h w\|_0^2 - \frac{\delta\varepsilon}{2\lambda}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)\|\mathbf{grad}_h w\|_{0,T}^2\\
&\quad - \frac{\delta}{2\varepsilon\lambda}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)\|\mathbf{q}\|_{0,T}^2\\
&\geq (C_1-\frac{\delta}{2})\|\boldsymbol{\phi}\|_1^2 + (1-\frac{\delta}{2\varepsilon})\lambda^{-1}\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)\|\mathbf{q}\|_{0,T}^2\\
&\quad + \|\boldsymbol{\Pi}_h\mathbf{q}\|_0^2 + \frac{\delta}{2}[1-\frac{\varepsilon}{\lambda}(t^2+C_2h^2)]\|\mathbf{grad}_h w\|_0^2\\
&\geq C(\|\boldsymbol{\phi}\|_1^2 + \|\mathbf{q}\|_h^2 + \|\mathbf{grad}_h w\|_0^2),
\end{aligned}$$

if ε is small enough and $\delta<\min\{2C_1,\ 2\varepsilon\}$.

Since we also have

$$|||(v,\boldsymbol{\psi},\mathbf{s})|||_h \leq C|||(w,\boldsymbol{\phi},\mathbf{q})|||_h,$$

the assertion is proven. ■

We will now close the paper by giving the

Proof of the Theorem : We first use Lemmas 1 and 2 to decompose the exact and approximate shear:

$$\mathbf{q} = \mathbf{grad}\, r + \mathbf{curl}\, p, \quad r \in H_0^1(\Omega),\ p \in H^1(\Omega) \cap L_0^2(\Omega),$$
$$\mathbf{q}_h = \mathbf{grad}_h r_h + \mathbf{curl}\, p_h, \quad r_h \in W_h,\ p_h \in \hat{S}_h,$$

and to construct an interpolant $\tilde{\mathbf{q}}$ to $\mathbf{q}$ as

$$\tilde{\mathbf{q}} = \mathbf{grad}\, \tilde{r} + \mathbf{curl}\, \tilde{p},$$

where $\tilde{r} \in W_h \cap H_0^1(\Omega)$ and $\tilde{p} \in \hat{S}_h$ are the Clemént interpolants (cf. [12, pp. 109-111]) to r and p, respectively.

We remark that $\mathbf{\Pi}_h \mathbf{q}_h = \mathbf{grad}_h r_h$ and $\mathbf{\Pi}_h \tilde{\mathbf{q}} = \mathbf{grad}\, \tilde{r}$.

Further, let $\boldsymbol{\phi}$ be interpolated by $\tilde{\boldsymbol{\phi}} \in \mathbf{V}_h$, and w by $\tilde{w} \in W_h \cap H_0^1(\Omega)$.

Our stability estimate now supplies us with a triple $(v, \boldsymbol{\psi}, \mathbf{s}) \in W_h \times \mathbf{V}_h \times \mathbf{Q}_h$, such that

$$|||(v, \boldsymbol{\psi}, \mathbf{s})|||_h \leq C \tag{3.4}$$

and

$$|||(w_h - \tilde{w}, \boldsymbol{\phi}_h - \tilde{\boldsymbol{\phi}}, \mathbf{q}_h - \tilde{\mathbf{q}})|||_h \leq \mathcal{B}_h(w_h - \tilde{w}, \boldsymbol{\phi}_h - \tilde{\boldsymbol{\phi}}, \mathbf{q}_h - \tilde{\mathbf{q}}; v, \boldsymbol{\psi}, \mathbf{s}). \tag{3.5}$$

Using (2.2), (3.3) and noting that

$$(\mathbf{q}_h, \mathbf{grad}_h v) = (\mathbf{grad}_h r_h, \mathbf{grad}_h v) = (g, v), \quad v \in W_h,$$
$$(\mathbf{grad}\, r, \mathbf{grad}_h v) = \sum_{T \in \mathcal{T}_h} \int_{\partial T} v\, \mathbf{grad}\, r \cdot \mathbf{n}_T + (g, v), \quad v \in W_h,$$

the normal technique gives

$$\begin{aligned}
&\mathcal{B}_h(w_h - \tilde{w}, \boldsymbol{\phi}_h - \tilde{\boldsymbol{\phi}}, \mathbf{q}_h - \tilde{\mathbf{q}}; v, \boldsymbol{\psi}, \mathbf{s}) \\
&= a(\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, \boldsymbol{\psi}) - (\mathbf{grad}\,(r - \tilde{r}), \boldsymbol{\psi}) - (\mathbf{curl}\,(p - \tilde{p}), \boldsymbol{\psi}) \\
&+ (\mathbf{grad}\,(r - \tilde{r}), \mathbf{grad}_h v) + (\mathbf{grad}\,(w - \tilde{w}), \mathbf{s}) - (\boldsymbol{\phi} - \tilde{\boldsymbol{\phi}}, \mathbf{s}) \\
&- \lambda^{-1} \sum_{T \in \mathcal{T}_h} (t^2 + \alpha_T h_T^2)(\mathbf{q} - \tilde{\mathbf{q}}, \mathbf{s})_T \\
&+ \lambda^{-1} \sum_{T \in \mathcal{T}_h} \alpha_T h_T^2 (\mathbf{q}, \mathbf{s})_T - \sum_{T \in \mathcal{T}_h} \int_{\partial T} v\, \mathbf{grad}\, r \cdot \mathbf{n}_T.
\end{aligned} \tag{3.6}$$

Let us estimate the different terms above.

Integrating by parts gives

$$|(\mathbf{curl}\,(p - \tilde{p}), \boldsymbol{\psi})| = |(p - \tilde{p}, \mathrm{curl}\, \boldsymbol{\psi})| \leq Ch|p|_1 |\boldsymbol{\psi}|_1. \tag{3.7}$$

Next, we have

$$|(\boldsymbol{\phi}-\tilde{\boldsymbol{\phi}},\mathbf{s})| \le C\sum_{T\in\mathcal{T}_h} h_T^2|\boldsymbol{\phi}|_{2,T}||\mathbf{s}||_{0,T} \le Ch|\boldsymbol{\phi}|_2(\sum_{T\in\mathcal{T}_h} h_T^2||\mathbf{s}||_{0,T}^2)^{1/2}, \quad (3.8)$$

and using Lemma 2

$$|(\mathbf{grad}\,(w-\tilde{w}),\mathbf{s})| = |(\mathbf{grad}\,(w-\tilde{w}),\mathbf{\Pi}_h\mathbf{s})| \le Ch|w|_2||\mathbf{\Pi}_h\mathbf{s}||_0. \quad (3.9)$$

Lemma 3 gives

$$|\sum_{T\in\mathcal{T}_h}\int_{\partial T} v\,\mathbf{grad}\,r\cdot\mathbf{n}_T| \le Ch||r||_2||\mathbf{grad}_h v||_0. \quad (3.10)$$

From the definition of $\tilde{\mathbf{q}}$ we get

$$\begin{aligned}&|\sum_{T\in\mathcal{T}_h}(t^2+\alpha_T h_T^2)(\mathbf{q}-\tilde{\mathbf{q}},\mathbf{s})_T|\\ &\le Ch[t(|r|_2+|p|_2)+|r|_1+|p|_1](\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)||\mathbf{s}||_{0,T}^2)^{1/2}.\end{aligned} \quad (3.11)$$

The estimation of the rest of the terms in the right hand side of (3.6) is straightforward, and combining (3.4) -(3.11) we obtain

$$\begin{aligned}&|||(w_h-\tilde{w},\boldsymbol{\phi}_h-\tilde{\boldsymbol{\phi}},\mathbf{q}_h-\tilde{\mathbf{q}})|||_h\\ &\le Ch\{|\boldsymbol{\phi}|_2+|w|_2+||r||_2+|p|_1+t(|r|_2+|p|_2)\}|||(v,\boldsymbol{\psi},\mathbf{s})|||_h\\ &\le Ch\{|\boldsymbol{\phi}|_2+|w|_2+||r||_2+|p|_1+t(|r|_2+|p|_2)\}.\end{aligned}$$

Hence, the use of the triangle inequality and Proposition 1 gives

$$\begin{aligned}&||\mathbf{grad}_h(w-w_h)||_0+||\boldsymbol{\phi}-\boldsymbol{\phi}_h||_1+||\mathbf{grad}_h(r-r_h)||_0\\ &\qquad+(\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)||\mathbf{q}-\mathbf{q}_h||_{0,T}^2)^{1/2} \le Ch||g||_0.\end{aligned} \quad (3.12)$$

To proceed, we let $z\in H_0^1(\Omega)$, $\boldsymbol{\theta}\in[H_0^1(\Omega)]^2$ and $\mathbf{r}\in[L^2(\Omega)]^2$ solve

$$\begin{aligned} a(\boldsymbol{\theta},\boldsymbol{\psi})-(\mathbf{r},\boldsymbol{\psi}) &= (\boldsymbol{\phi}-\boldsymbol{\phi}_h,\boldsymbol{\psi}), && \boldsymbol{\psi}\in[H_0^1(\Omega)]^2,\\ (\mathbf{r},\mathbf{grad}\,v) &= (w-w_h,v), && v\in H_0^1(\Omega),\\ \lambda^{-1}t^2(\mathbf{r},\mathbf{s})+(\boldsymbol{\theta}-\mathbf{grad}\,z,\mathbf{s}) &= 0, && \mathbf{s}\in[L^2(\Omega)]^2.\end{aligned} \quad (3.13)$$

Using Lemma 1 to write $\mathbf{r} = \mathbf{grad}\, k + \mathbf{curl}\, l$, Proposition 1 yields

$$||\boldsymbol{\theta}||_2 + ||z||_2 + ||k||_2 + ||l||_1 + t||l||_2 \leq C(||\boldsymbol{\phi} - \boldsymbol{\phi}_h||_0 + ||w - w_h||_0). \quad (3.14)$$

Integrating by parts in the second equation of (3.13) gives

$$-\text{div}\, \mathbf{r} = -\Delta\, k = w - w_h,$$

and thus we get

$$||w-w_h||_0^2 = (\mathbf{grad}\, k, \mathbf{grad}_h(w-w_h)) - \sum_{T\in\mathcal{T}_h} \int_{\partial T} (w-w_h)\, \mathbf{grad}\, k\cdot\mathbf{n}_T. \quad (3.15)$$

Let now $\tilde{z} \in W_h \cap H_0^1(\Omega)$ and $\tilde{\boldsymbol{\theta}} \in \mathbf{V}_h$ be the Lagrange interpolants to z and $\boldsymbol{\theta}$, respectively. We again use the Clemént construction to define $\tilde{k} \in W_h \cap H_0^1(\Omega)$ interpolating k, and $\tilde{l} \in \hat{S}_h$ interpolating l. The interpolant $\tilde{\mathbf{r}}$ to $\mathbf{r}$ is then defined through $\tilde{\mathbf{r}} = \mathbf{grad}\, \tilde{k} + \mathbf{curl}\, \tilde{l}$.

Using (2.2), (3.3), (3.13) , (3.15) and Lemma 2, we now get

$$\begin{aligned}
&||\boldsymbol{\phi} - \boldsymbol{\phi}_h||_0^2 + ||w - w_h||_0^2 \\
&= a(\boldsymbol{\phi} - \boldsymbol{\phi}_h, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}) - (\mathbf{q} - \mathbf{q}_h, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}}) + (\mathbf{q} - \mathbf{q}_h, \mathbf{grad}\,(z - \tilde{z})) \\
&\quad - (\mathbf{r} - \tilde{\mathbf{r}}, \boldsymbol{\phi} - \boldsymbol{\phi}_h) + (\mathbf{grad}\,(k - \tilde{k}), \mathbf{grad}_h(w - w_h)) \\
&\quad - \lambda^{-1} t^2 (\mathbf{q} - \mathbf{q}_h, \mathbf{r} - \tilde{\mathbf{r}}) - \sum_{T\in\mathcal{T}_h} \int_{\partial T} (w - w_h)\, \mathbf{grad}\, k \cdot \mathbf{n}_T \\
&\quad - \lambda^{-1} \sum_{T\in\mathcal{T}_h} \alpha_T h_T^2 (\mathbf{q}_h, \tilde{\mathbf{r}}).
\end{aligned} \quad (3.16)$$

Standard interpolation estimates give

$$\begin{aligned}
|(\mathbf{q} - \mathbf{q}_h, \boldsymbol{\theta} - \tilde{\boldsymbol{\theta}})| &\leq C \sum_{T\in\mathcal{T}_h} h_T^2 ||\mathbf{q} - \mathbf{q}_h||_{0,T} |\boldsymbol{\theta}|_{2,T} \\
&\leq Ch (\sum_{T\in\mathcal{T}_h} h_T^2 ||\mathbf{q} - \mathbf{q}_h||_{0,T}^2)^{1/2} |\boldsymbol{\theta}|_2 .
\end{aligned} \quad (3.17)$$

Lemmas 1 and 2 give

$$\begin{aligned}
&|(\mathbf{q} - \mathbf{q}_h, \mathbf{grad}\,(z - \tilde{z}))| = |(\mathbf{grad}_h(r - r_h), \mathbf{grad}\,(z - \tilde{z}))| \\
&\leq Ch ||\mathbf{grad}_h(r - r_h)||_0 |z|_2 .
\end{aligned} \quad (3.18)$$

An integration by parts yields

$$
\begin{aligned}
|(\mathbf{r}-\tilde{\mathbf{r}},\boldsymbol{\phi}-\boldsymbol{\phi}_h)| &\le |(\mathbf{grad}\,(k-\tilde{k}),\boldsymbol{\phi}-\boldsymbol{\phi}_h)| + |(\mathbf{curl}\,(l-\tilde{l}),\boldsymbol{\phi}-\boldsymbol{\phi}_h)| \\
&= |(\mathbf{grad}\,(k-\tilde{k}),\boldsymbol{\phi}-\boldsymbol{\phi}_h)| + |(l-\tilde{l},\mathrm{curl}\,(\boldsymbol{\phi}-\boldsymbol{\phi}_h))| \\
&\le Ch(|k|_2+|l|_1)||\boldsymbol{\phi}-\boldsymbol{\phi}_h||_1 .
\end{aligned}
\tag{3.19}
$$

Since $||\tilde{\mathbf{r}}||_0 \le C||\mathbf{r}||_0$, we get

$$
\begin{aligned}
|\sum_{T\in\mathcal{T}_h}\alpha_T h_T^2(\mathbf{q}_h,\tilde{\mathbf{r}})| &\le |\sum_{T\in\mathcal{T}_h}\alpha_T h_T^2(\mathbf{q}-\mathbf{q}_h,\tilde{\mathbf{r}})| + |\sum_{T\in\mathcal{T}_h}\alpha_T h_T^2(\mathbf{q},\tilde{\mathbf{r}})| \\
&\le Ch(\sum_{T\in\mathcal{T}_h}\alpha_T h_T^2||\mathbf{q}-\mathbf{q}_h||_{0,T}^2)^{1/2}||\mathbf{r}||_0 + Ch^2||\mathbf{q}||_0||\mathbf{r}||_0 .
\end{aligned}
\tag{3.20}
$$

Further, Lemma 3 implies

$$
|\sum_{T\in\mathcal{T}_h}\int_{\partial T}(w-w_h)\,\mathbf{grad}\,k\cdot\mathbf{n}_T| \le Ch||k||_2||\mathbf{grad}_h(w-w_h)||_0 . \tag{3.21}
$$

Collecting (3.16) through (3.21) and estimating the rest of the terms in the standard manner we obtain

$$
\begin{aligned}
&||\boldsymbol{\phi}-\boldsymbol{\phi}_h||_0^2 + ||w-w_h||_0^2 \\
&\le Ch\{||\mathbf{grad}_h(w-w_h)||_0 + ||\boldsymbol{\phi}-\boldsymbol{\phi}_h||_1 + ||\mathbf{grad}_h(r-r_h)||_0 \\
&+ (\sum_{T\in\mathcal{T}_h}(t^2+h_T^2)||\mathbf{q}-\mathbf{q}_h||_{0,T}^2)^{1/2} + h||\mathbf{q}||_0\} \cdot \\
&\{|\boldsymbol{\theta}|_2 + |z|_2 + ||k||_2 + ||l||_1 + t|l|_2\}.
\end{aligned}
\tag{3.22}
$$

Since $||\mathbf{q}||_0 \le C||g||_0$, the final estimate now follows from (3.22), (3.14) and (3.12). ■

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