TWO LOW-ORDER MIXED METHODS FOR THE ELASTICITY PROBLEM

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1. INTRODUCTION

We will consider the finite element approximation of the equations of linear elasticity:

$$\begin{aligned} A\sigma - \varepsilon(u) &= 0 && \text{in } \Omega, \\ \operatorname{div} \sigma + f &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma_1, \\ \sigma \cdot n &= g && \text{on } \Gamma_2, \end{aligned} \tag{1.1}$$

where the bounded domain $\Omega \subset \mathbf{R}^N$, $N = 2,3$, is assumed to have a polygonal or polyhedral boundary $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$, and $\Gamma_i \neq \emptyset$, $i = 1,2$. The unknowns of the problem are the displacement $u : \Omega \to \mathbf{R}^N$ and the stress tensor $\sigma : \Omega \to \mathbf{R}^N \times \mathbf{R}^N$, and the data is given by the body force f and the boundary traction g. $\varepsilon(u)$ denotes the linear strain tensor and $\operatorname{div} \sigma$ stands for the divergence of σ. A is the fourth order tensor expressing the constitutive law for a homogeneous, isotropic and linearly elastic material, i.e.

$$A\sigma = \alpha\sigma + \beta \operatorname{tr}(\sigma)\delta,$$

where δ denotes the unit tensor, $\operatorname{tr}(\sigma)$ is the trace of σ,

$$\alpha = \frac{(1+\nu)}{E}, \qquad \beta = -\frac{\nu(1+\nu)}{E}$$

for the plane strain problem and

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for the plane stress and the three-dimensional problem. Here E denotes the Young modulus and ν the Poisson ratio. Our notation is standard and the same as used in our paper [10].

If one assumes for example that $f \in [L^2(\Omega)]^N$ and $g \in [L^2(\Gamma_2)]^N$ then (1.1) has a unique solution satisfying

$$\|u\|_1 + \|\sigma\|_0 \le C(\|f\|_0 + \|g\|_{0,\Gamma_2}).$$

In particular, one can show [1] that the positive constant C can be chosen to be independent of the Poisson ratio. This will be the case below; the generic positive constant C will be independent of the Poisson ratio. Hence, all the estimates are valid also for incompressible and nearly incompressible elasticity.

In order to obtain optimal L^2-estimates for the displacement variable we will use standard duality arguments, and for these we have to assume that

$$\|u\|_2 + \|\sigma\|_1 \le C\|f\|_0, \tag{1.2}$$

for the solution of (1.1) with $g = 0$.

In the mixed approximation of the elasticity problem one directly approximates the system (1.1), instead of first eliminating the stress tensor which would lead to the classical displacement method. For the motivations given for using a mixed method, and for discussions of the problems connected with this approach, we refer directly to papers cited in the list of references. In general, however, one can claim that most traditional mixed methods are rather cumbersome to implement, and as a consequence they have rarely been used in practice.

Recently [1,10] it has been shown that most of the problems connected with mixed methods can be overcomed by using an idea originally proposed by Fraijs de Veubeke [4], and it now seems to be possible to design mixed methods which are competitive with displacement methods.

In the approach of Fraijs de Veubeke the assumption of a symmetric stress tensor is dropped, and instead a new unknown skew symmetric second order tensor γ, with the physical meaning of the rotation, is introduced. The equations to be approximated are then

$$\begin{aligned} A\sigma - \nabla u + \gamma &= 0 && \text{in } \Omega,\\ \sigma - \sigma^T &= 0 && \text{in } \Omega,\\ \operatorname{div} \sigma + f &= 0 && \text{in } \Omega,\\ u &= 0 && \text{on } \Gamma_1,\\ \sigma \cdot n &= g && \text{on } \Gamma_2, \end{aligned} \tag{1.3}$$

where σ^T is the transpose of σ and

$$(\nabla u)_{ij} = \partial_j u_i \ , \quad i,j = 1,2,...,N,$$

$(\partial_j = \partial/\partial x_j)$. Now, denoting by $\omega(u)$ the rotation of u :

$$\omega(u)_{ij} = \frac{1}{2}(\partial_j u_i - \partial_i u_j), \quad i,j = 1,2,...,N,$$

we note that the solution of (1.3) coincides with the solution of (1.1) where $\gamma = \omega(u)$.

In [10] we introduced and analyzed a family of mixed methods based on the above formulation. We will not give the details of this family here. Instead, we will review our method of analysis and apply it on two new methods. Our methods very much resemble some classical methods first introduced by Fraijs de Veubeke [3] and Watwood and Hartz [11], and later analyzed by Johnson and Mercier [6] and Hlaváček [5]. The difference is, however, that the methods of the present paper are considerably simpler to implement.

The methods to be introduced can without loss of generality be analyzed assuming homogeneous boundary conditions. Hence, we will assume that $g = 0$ and then the basis for the mixed method is the following variational form of (1.3): Find $(\sigma, u, \gamma) \in H \times V \times W$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau; u, \tau) &= 0, \quad \tau \in H, \\ b(\sigma; v, \eta) + (f, v) &= 0, \quad (v,\eta) \in V \times W, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} b(\sigma; u, \gamma) &= (\text{div}\ \sigma, u) + (\sigma, \gamma), \\ a(\sigma,\tau) &= (A\sigma, \tau), \end{aligned}$$

and

$$\begin{aligned} V &= [L^2(\Omega)]^N, \\ H &= \{\ \sigma \in [L^2(\Omega)]^{N\times N} \mid \text{div}\ \sigma \in V,\ \sigma\cdot n = 0 \quad \text{on} \quad \Gamma_2\ \}, \\ W &= \{\ \gamma \in [L^2(\Omega)]^{N\times N} \mid \ \gamma + \gamma^T = 0\ \}. \end{aligned}$$

Above $(\cdot,\cdot)$ stands for the inner product in $[L^2(\Omega)]^N$ or $[L^2(\Omega)]^{N\times N}$.

As usual, the finite element approximation is sought in a finite dimensional subspace: Find $(\sigma_h, u_h, \gamma_h) \in H_h \times V_h \times W_h \subset H \times V \times W$ such that

$$\begin{aligned} a(\sigma_h,\tau) + b(\tau; u_h, \tau_h) &= 0, \quad \tau \in H_h, \\ b(\sigma_h; v, \eta) + (f, v) &= 0, \quad (v,\eta) \in V_h \times W_h. \end{aligned} \tag{1.5}$$

2. CONVERGENCE ANALYSIS

The formulation (1.5) is an example of a saddle point problem and hence it can be analyzed by the general theory of Brezzi [2]. In [10] we showed that the analysis becomes particularly simple if the following mesh dependent norms are employed:

$$\|\sigma\|_{0,h}^2 = \|\sigma\|_0^2 + \sum_{T\in\Gamma_h,\ T\subset\Gamma} h_T \int_T |\sigma\cdot n|^2 ds, \quad \sigma \in \mathcal{H}_h,$$

where

$$\mathcal{H}_h = \{\ \sigma \in [L^2(\Omega)]^{N\times N} \mid \ \sigma\cdot n_{|T} \in [L^2(T)]^N,\ \ T\in\Gamma_h,\ \ T\subset\Gamma\ \},$$

$$\|u\|_{1,h}^2 = \sum_{K\in C_h} \|\varepsilon(u)\|_{0,K}^2 + \sum_{T\in\Gamma_h} h_T^{-1} \int_T [u]^2 ds + \sum_{T\subset\Gamma_1} h_T^{-1}\int_T u^2 ds, \quad u\in V_h,$$

and

$$\|(u,\gamma)\|_h^2 = \|u\|_{1,h}^2 + \sum_{K\in C_h} \|\gamma-\omega(u)\|_{0,K}^2\ , \quad u\in V_h,\ \ \gamma\in W_h.$$

Here C_h denotes the regular finite element partitioning of $\bar{\Omega}$. T denotes an edge of an element of C_h and h_T stands for the length of T. Γ_h denotes the collection of edges in the interior of Ω and $[u]$ denotes the value of the jump in u at the inter-element boundaries. Due to the assumption $\Gamma_1 \neq \emptyset$, $\|\cdot\|_{1,h}$ and $\|\cdot\|_h$ are norms in V_h and $V_h\times W_h$, respectively.

The conditions to be verified when using the above norms are a coercivity condition for the bilinear form a :

$$a(\sigma,\sigma) \geq C\|\sigma\|_{0,h}^2, \qquad \sigma\in Z_h, \tag{2.1}$$

where

$$Z_h = \{\ \sigma\in H_h \mid \ b(\sigma;u,\gamma)=0,\ \ u\in V_h,\ \ \gamma\in W_h\ \ \},$$

and the Babuška-Brezzi condition for b :

$$\sup_{0\neq\sigma\in H_h} \frac{b(\sigma;u,\gamma)}{\|\sigma\|_{0,h}} \geq C\|(u,\gamma)\|_h, \quad (u,\gamma)\in V_h\times W_h. \tag{2.2}$$

By integrating by parts on each element and using the Schwarz inequality, one easily shows that

$$b(\sigma;u,\gamma) \leq \|\sigma\|_{0,h}\|(u,\gamma)\|_h, \ \ \sigma\in H_h, \ \ (u,\gamma)\in V_h\times W_h,$$

Hence the stability inequality (2.2) is the best possible with our choice of norms. In our papers [8,9,10] we have developed a "macroelement technique", the use of which considerably facilitates the verification of the Babuška-Brezzi inequality. Using this method of analysis the stability inequality (2.2) can be verified by a simple "patch test"; cf. the next section.

Using standard scaling arguments one can show that

$$\|\sigma\|_0 \leq \|\sigma\|_{0,h} \leq C\|\sigma\|_0, \quad \sigma \in H_h.$$

Thus the condition (2.1) can equally well be stated with $\|\cdot\|_{0,h}$ replaced with the L^2-norm $\|\cdot\|_0$, and from the definition of A it then follows that the condition is valid for a constant C depending on the Poisson ratio ν (with $C = 0$ when $\nu = 1/2$ for the plain strain and the three-dimensional problem). In [1] it was shown that the inequality with a constant independent of the Poisson ratio follows from the following equilibrium condition:

$$\text{If } \sigma \in H_h \text{ satisfies } (\operatorname{div} \sigma, u) = 0 \text{ for all } u \in V_h, \text{ then } \operatorname{div} \sigma = 0. \tag{2.3}$$

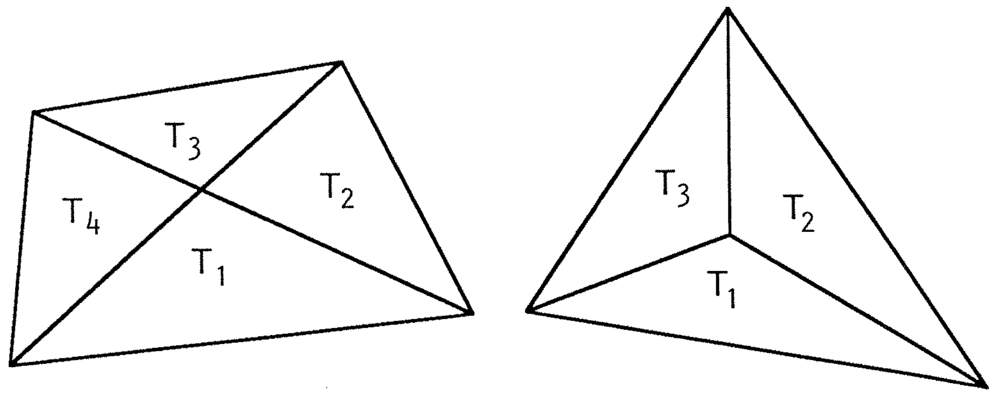
This condition is also needed in the analysis in order that the error estimates obtained may be optimal. We also note that the equilibrium condition implies the existence of an operator $P_h : V \to V_h$ such that

$$(\operatorname{div} \sigma, u - P_h u) = 0, \quad \sigma \in H_h, \quad u \in V. \tag{2.4}$$

Summarizing, we can state that optimal error estimates for both the stress tensor, the rotation and the displacement follow if the method can be proved to satisfy the conditions (2.2) and (2.3).

3. THE FINITE ELEMENT METHODS

Let $\Omega \subset \mathbf{R}^2$ and introduce a partitioning C_h of $\bar{\Omega}$ into regular triangles or quadrilaterals (a mixing of triangles and quadrilaterals is not excluded). The quadrilaterals are further subdivided into four triangles by drawing the two diagonals, and the triangles are divided into three smaller triangles by connecting the center of gravity to the vertices; see the figure below.



Let $T_1, T_2, ..., T_\kappa$ be the triangles of K, with $\kappa = 3$ when K is a triangle and $\kappa = 4$ for a quadrilateral.

Denote by b_K the function that is continuous on K, linear on each $T_i \subset K$, $i = 1, 2, ..., \kappa$, equal to unity at the interior node and vanishes on ∂K. On $K \in \mathcal{C}_h$ we define

$$B(K) = \{ \tau \in [L^2(K)]^{2\times 2} \mid (\tau_{i1}, \tau_{i2}) = c_i(\partial_2 b_K, -\partial_1 b_K),\ c_i \in \mathbf{R},\ i = 1,2 \},$$
$$D(K) = \{ \tau \in [C(K)]^{2\times 2} \mid \tau_{ij|T_k} \in P_1(T_k),\ i,j = 1,2,\quad k = 1,2,...,\kappa \},$$

and the finite element spaces are then defined through

$$\begin{aligned} H_h &= \{ \tau \in H \mid \tau_{|K} \in B(K) \oplus D(K),\ K \in \mathcal{C}_h \}, \\ V_h &= \{ v \in V \mid v_{|K} \in [P_1(K)]^2,\ K \in \mathcal{C}_h \}, \\ W_h &= \{ \gamma \in W \mid \gamma_{|K} \in [P_1(K)]^{2\times 2},\ K \in \mathcal{C}_h \}. \end{aligned} \tag{3.1}$$

We note that as the degrees of freedom for $\tau \in D(K)$ one can take

$$\text{the values } \tau \cdot n \text{ at two distinct points of each side of } K, \tag{3.2a}$$

and

$$\int_K \tau_{ij}\, dx, \quad \text{for } i,j = 1,2. \tag{3.2b}$$

Since for $\tau \in H_h$ only the continuity of $\tau \cdot n$ across inter-element boundaries is required, we also note that as the degrees of freedom for $\tau \in H_h$ one can for each $K \in \mathcal{C}_h$ take the values (3.2) together with the degrees of freedom for $B(K)$.

Now, let us carry out the error analysis of the above methods.

The equilibrium condition (2.3) follows from

Lemma 1. *If* $\tau \in H(K) = B(K) \oplus D(K)$ *satisfies*

$$\int_K v \cdot \operatorname{div} \tau \, dx = 0 \quad \text{for} \quad v \in [P_1(K)]^2, \tag{3.3}$$

$$\text{then} \quad \operatorname{div} \tau = 0 \quad \text{on } K.$$

Proof. First we note that $\operatorname{div} \tau = 0$ for each $\tau \in B(K)$, and hence we have to prove the condition for $\tau \in D(K)$.

Let us prove the condition for the more difficult quadrilateral case. In this case the proof follows from the fact that the dimension of the space $\operatorname{div} D(K)$ is only six; cf. [7]. This is easily seen upon noting that each $\tau \in D(K)$ can be uniquely written as $\tau^1 + \tau^2 + \tau^3$ where

$$\tau^i{}_{|L_i} \in [P_1(L_i)]^{2\times 2}, \quad \tau^i = 0 \quad \text{on} \quad K \setminus L_i,$$

with $L_1 = K$, $L_2 = T_1 \cup T_2$ and $L_3 = T_2 \cup T_3$. Denoting by $(\operatorname{div} \tau)^i$ the constant value of $\operatorname{div} \tau^i$ on L_i, we can write

$$\int_K v \cdot \operatorname{div} \tau \, dx = \sum_{i=1}^{3} (\operatorname{div} \tau)^i \cdot \int_{L_i} v \, dx = \sum_{i=1}^{3} v(G_i) \cdot (\operatorname{div} \tau)^i \operatorname{area}(L_i),$$

where G_i denotes the center of gravity of L_i. Now, K is convex and nondegenerated and hence G_1, G_2, G_3 are not located on a straight line. We can thus choose v such that $v = (1,0)$ (and $v = (0,1)$, respectively) on G_i and $v = 0$ on G_j, $j \neq i$, for $i = 1,2,3$. This shows that the condition (3.3) gives $(\operatorname{div} \tau)^i = 0$, $i = 1,2,3$, which proves the assertion for quadrilaterals.

The proof for triangular elements is analogous to that given above.

□

Lemma 2. *The stability inequality* (2.2) *is valid.*

Proof. For the methods under consideration it suffices to apply the macroelement technique of [8,9,10] with macroelements consisting of two elements. Hence we define a macroelement to be the union of two elements with one common side. For a macroelement we define

$$\begin{aligned}
H_{M,L} &= \{ \sigma \in [L^2(M)]^{2\times 2} \mid \operatorname{div} \sigma \in [L^2(M)]^2,\ \sigma_{|K} \in H(K) \quad K \subset M, \\
&\qquad \sigma \cdot n = 0 \quad \text{on} \quad \partial M \setminus L,\ \ L \subset \partial M \quad \}, \\
V_M &= \{ v \in [L^2(M)]^2 \mid v_{|K} \in [P_1(K)]^2, \quad K \subset M \quad \}, \\
W_M &= \{ \gamma \in [L^2(M)]^{2\times 2} \mid \gamma + \gamma^T = 0,\ \gamma_{|K} \in [P_1(K)]^{2\times 2},\ K \subset M \}, \\
N_M &= \{ (u,\gamma) \in V_M \times W_M \mid b_M(\sigma; u, \gamma) = 0, \quad \sigma \in H_{M,L} \quad \},
\end{aligned}$$

where L is either empty or the union of one or more of the edges of the elements of M and

$$b_M(\sigma; u, \gamma) = (\text{div } \sigma, u)_M + (\sigma, \gamma)_M.$$

Now, in [8,9,10] it is shown that, in order to verify the stability condition (2.2), it suffices to check that we have

$$N_M = \begin{cases} \{ (0,0) \}, & \text{if } L \neq \emptyset, \\ \{ (r, \omega(r)) \mid r \in R_M \}, & \text{if } L = \emptyset, \end{cases} \tag{3.4}$$

where R_M denotes the rigid body motions on M :

$$R_M = \{ v \in [L^2(M)]^2 \mid v = (a, b) + c(-x_2,\ x_1),\quad a, b, c \in \mathbf{R} \ \}.$$

Now, for the methods in question the verification of the above condition is identical to the corresponding proofs for the methods of [10]. For completeness let us repeat these arguments.

Let $(u, \gamma) \in V_M \times W_M$ be arbitrary and denote $z = \gamma_{12} = -\gamma_{21}$. Define $\sigma \in H_{M,L}$ through

$$(\sigma_{i1}, \sigma_{i2})_{|K} = (\partial_2(b_K \partial_i z), -\partial_1(b_K \partial_i z)) \quad, i = 1, 2,$$

on each $K \subset M$. Now, div $\sigma = 0$ on M, and hence an integration by parts yields

$$\begin{aligned} b_M(\sigma; u, \gamma) = (\sigma, \gamma)_M &= \int_M (\sigma_{12} - \sigma_{21})\ \gamma_{12}\ dx \\ = \sum_{K \subset M} \int_K (-\partial_1(b_K \partial_1 z) - \partial_2(b_K \partial_2 z))\ z\ dx &= \sum_{K \subset M} \int_K b_K |\nabla z|^2 dx, \end{aligned}$$

since $\sigma \cdot n = 0$ on ∂K, $K \subset M$. This shows that if $(v, \gamma) \in N_M$ then $\gamma_{12} = -\gamma_{21}$ is a constant on each $K \subset M$. Hence, on each $K \subset M$ γ can be represented through

$$\gamma_{|K} = \omega(w_K) = \nabla w_K,$$

for some $w_K \in R_K$. A substitution and an integration by parts then gives

$$\begin{aligned} b_M(\sigma; u, \gamma) &= (\text{div } \sigma, u)_M + (\sigma, \gamma)_M \\ &= \sum_{K \subset M} (\sigma, \nabla(w_K - u))_K + \int_S \sigma \cdot n\ [u]\ ds + \int_L \sigma \cdot n\ u\ ds, \end{aligned}$$

where S denotes the common edge of the two elements of M. Now, $\nabla(w_K - u)$ is a constant tensor on each $K \subset M$, whereas $[u]$ and u are linear on S and L, respectively. Hence (3.2) shows that we can choose $\sigma \in H_{M,L}$ such that $b_M(\sigma; u, \gamma) = 0$ forces $\nabla(w_K - u)$ to vanish on each $K \subset M$, u to be continuous along S and u to vanish on L when $L \neq \emptyset$. Hence we conclude that u is a rigid body mode on M with $u = 0$ for the case $L \neq \emptyset$. In addition we have $\gamma = \omega(u)$.

The stability of the methods is thus proved.

□

Lemmas 1 and 2 now imply (cf. [10]) the following

Theorem. *For the method* (3.1) *we have the following error estimates*

$$\|\sigma - \sigma_h\|_0 + \|\gamma - \gamma_h\|_0 \leq Ch^2(|\sigma|_2 + |\gamma|_2)$$

and

$$\|u - u_h\|_0 \leq Ch^2(|u|_2 + |\sigma|_2 + |\gamma|_2).$$

Moreover, if the regularity estimate (1.2) *is valid, we have the optimal estimate*

$$\|u - u_h\|_0 \leq Ch^2(|u|_2 + |\sigma|_1 + |\gamma|_1).$$

□

Remark. For the method we also obtain (cf. [10]) the estimate

$$\|P_h u - u_h\|_0 \leq Ch^3(|\sigma|_2 + |\gamma|_2),$$

where P_h is the projection operator defined through (2.4). This estimate can be applied for the analysis of some postprocessing schemes developed for the improvement of the displacement approximation; cf. [1,10].

□

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