ERROR BOUNDS FOR THE APPROXIMATION OF THE STOKES PROBLEM USING BILINEAR/CONSTANT ELEMENTS ON IRREGULAR QUADRILATERAL MESHES

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1. INTRODUCTION

One of the most popular methods to numerically solve the Stokes equations in fluid mechanics is to use a mixed finite element method where the velocities are approximated with continuous isoparametric bilinear elements on quadrilateral meshes whereas a piecewise constant approximation is used for the pressure. After eliminating the pressure by simple perturbation techniques one obtains a positive definite system for the velocities alone. Another way to obtain the same method is to apply penalty techniques with reduced/selective integration (cf. [6], [8]). In numerical computations this method has been shown to give excellent results for the computed velocities and also for the pressure provided that the latter has been "smoothed" in an appropriate way (cf. e.g. [6]). From a theoretical point of view this success has been considered somewhat surprising since it is well known that the method is not uniformly stable in the sense of Babuška [1] and Brezzi [3]. For rectangular meshes, however, it has been possible to analyze the method, cf. [7]. The analysis of [7] relies on a weak Babuška-Brezzi-type stability condition together with a careful consistency estimate and shows that the method converges with the optimal rate provided the exact solution is smooth enough. In this note we will extend and improve the analysis of [7]. We will show that the method in fact converges (after a pressure smoothing) for a very general class of meshes and that this happens without any extra smoothness assumptions on the exact solution. The fact that the extra smoothness assumption of [7] is not required was also observed by Boland and Nicolaides [2] in the case of a rectangular grid.

2. NOTATION AND PRELIMINARIES

Let Ω be a polygonal domain in $\mathbb{R}^2$ with boundary Γ. The

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problem under consideration consists of the stationary Stokes equations for an incompressible viscous fluid:

$$
\begin{aligned}
-\nu\Delta u + \nabla p &= f \quad \text{in } \Omega,\\
\operatorname{div} u &= 0 \quad \text{in } \Omega,\\
u &= 0 \quad \text{on } \Gamma,
\end{aligned}
\tag{2.1}
$$

where u is the fluid velocity, p is the pressure, f is the body force and $\nu > 0$ is the kinematic viscosity.

We denote by $|\cdot|_{s,T}$ and $\|\cdot\|_{s,T}$, respectively, the semi-norm and norm of the Sobolev space $[H^s(T)]^\alpha$ where s and α are integers. As usual $H_0^1(T)$ denotes the subspace of $H^1(T)$ consisting of functions with vanishing trace on ∂T. We will also introduce the space

$$L_0^2(T) = \{p \in L^2(T) \mid \int_T p\, dx = 0\}.$$

The inner product in $[L^2(T)]^\alpha$, for integral α, is denoted by $(\cdot,\cdot)_T$. The subscript T will be dropped if $T = \Omega$. As usual we will denote by C and C_j positive constants, possibly different at different occurences, which are independent of the mesh parameter h.

In variational form (2.1) reads: Find $u \in [H_0^1(\Omega)]^2$ and $p \in L_0^2(\Omega)$ such that

$$
\begin{aligned}
&\nu(\nabla u,\nabla v) - (\operatorname{div} v,p) = (f,v) \quad \forall v \in [H_0^1(\Omega)]^2,\\
&(\operatorname{div} u,\mu) = 0 \quad \forall \mu \in L_0^2(\Omega).
\end{aligned}
\tag{2.2}
$$

In the finite approximation of (2.2) the spaces $[H_0^1(\Omega)]^2$ and $L_0^2(\Omega)$ are replaced by the finite dimensional subspaces V_h and P_h, respectively. Below we define the subspaces as

$$V_h = \{v \in [H_0^1(\Omega)]^2 \mid v|_K \in [Q_1(K)]^2 \quad \forall K \in C_h\}$$

and

$$P_h = \{p \in L_0^2(\Omega) \mid p|_K \text{ is constant} \quad \forall K \in C_h\},$$

where C_h stands for a partitioning of Ω into convex quadrilaterals and $Q_1(K)$ is the space of (isoparametrically) transformed bilinear functions [4]. As usual, the mesh parameter h is defined as $h = \max_{K \in C_h} h_K$, where h_K denotes the diameter of K.

We now specify our assumptions on the partitioning C_h. First, we assume that C_h is a refinement of a coarser partitioning C_{2h}, obtained by subdividing each $\tilde{K} \in C_{2h}$ into four quadrilaterals by joining the midpoints of the opposite sides of $\tilde{K}$ by straight lines. Second, we assume that C_{2h} is also a similar refinement of a still coarser partitioning C_{4h}. Third, regarding C_{4h}, we merely assume that C_{4h} is regular. By this we mean that there are the constants $\sigma > 1$ and $0 < \gamma < 1$ independent of h such that

$$h_K \leq \sigma \rho_K, \quad |\cos \theta_{iK}| \leq \gamma; \; i = 1,2,3,4, \quad \forall K \in C_{4h},$$

where h_K, ρ_K and θ_{iK} are respectively the diameter of K, the diameter of the largest circle contained in K, and the angles of K.

Below we refer to the quadrilaterals of C_{2h} or C_{4h} as "macroelements" and denote them by M. We also introduce the subspace.

$$V_{2h} = \{v \in [H_0^1(\Omega)]^2 \mid v_{|M} \in [Q_1(M)]^2 \quad \forall M \in C_{2h}\}$$

where $Q_1(M)$ is as above. The space P_h will be written as the sum of three subspaces. The unit square $\hat{K}$ is partitioned into subdomains $\hat{K}_{ij} = \{(x_1,x_2) \in \hat{K} \mid \frac{(i-1)}{2} \leq x_1 \leq \frac{i}{2}, \frac{(j-1)}{2} \leq x_2 \leq \frac{j}{2}\}$, $i,j = 1,2$ and on $\hat{K}$ we define the function η through

$$\eta_{|\hat{K}_{ij}} = (-1)^{i+j}, \quad i,j = 1,2.$$

We then define the subspaces

$$P_{h1} = \{p \in P_h \mid p_{|M} \text{ is constant } \forall M \in C_{2h}\},$$

$$P_{h3} = \{p \in P_h \mid p_{|M} = c_M \eta \circ F_M^{-1}, \; c_M \in \mathbb{R}, \forall M \in C_{2h}\}$$

where F_M is the bilinear mapping of $\hat{K}$ onto M. The orthogonal complement of P_h with respect to $P_{h1} \oplus P_{h3}$ is denoted by P_{h2}. Finally we introduce a "pressure smoothing operator" $\pi: P_h \to P_{h1} \oplus P_{h2}$. Every $p \in P_h$ can be written uniquely as $p = \sum_{i=1}^{3} p_i$, $p_i \in P_{hi}$. The filtered pressure πp is then defined as $\pi p = p_1 + p_2$.

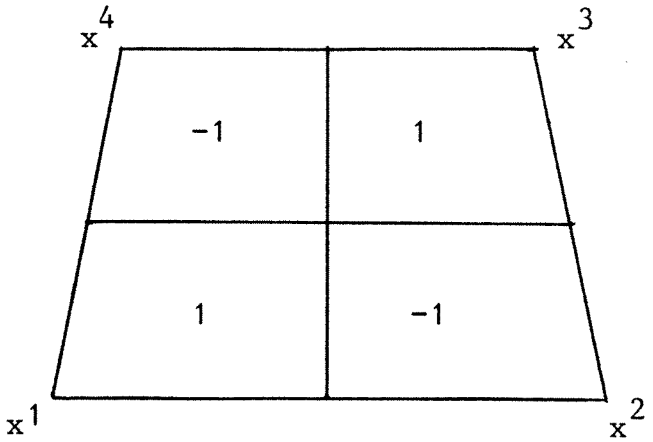
3. ERROR ANALYSIS

Let us start with a consistency estimate which is crucial for the analysis in this paper.

Lemma 3.1. For each $v \in V_{2h}$ and $p \in P_{h3}$ we have

$$(\operatorname{div} v, p) = 0.$$

Proof. Consider a macroelement $M \in C_{2h}$ with nodes x^i, $i = 1,2,3,4$, as in the figure below and suppose $p_M = \eta \circ F_M^{-1}$ takes the values ± 1 as in the figure.



Denote by $v^i = v(x^i)$, $i = 1,2,3,4$, the degrees of freedom of $v \in V_{2h|M}$ and write

$$x^i x^j = x^i - x^j, \quad i,j = 1,2,3,4.$$

It will further be convenient to use the (scalar valued) vector product in $\mathbb{R}^2$, i.e. if $a = (a_1,a_2)$ and $b = (b_1,b_2)$ we define

$$a \wedge b = a_1 b_2 - a_2 b_1.$$

Using Green's formula and integrating over the sides in M one obtains

$$(\text{div } v, p_M)_M = \frac{1}{4}\Bigg[(v^1-v^4) \wedge x^4x^1$$

$$+ 2\left(-\left(\frac{v^1+v^2}{2}\right) + \left(\frac{v^3+v^4}{2}\right)\right) \wedge \left(\frac{x^4x^1+x^3x^2}{2}\right)$$

$$+ (v^2-v^3) \wedge x^3x^2 + (-v^1+v^2) \wedge x^2x^1$$

$$+ \left(\left(\frac{v^1+v^4}{2}\right) - \left(\frac{v^2+v^3}{2}\right)\right) \wedge \left(\frac{x^2x^1+x^3x^4}{2}\right) + (-v^4+v^3) \wedge x^3x^4\Bigg]$$

$$= \frac{1}{8}\,[(v^1-v^4-v^2+v^3) \wedge (x^4x^1-x^3x^2)$$

$$+ (-v^1+v^2+v^4-v^3) \wedge (x^2x^1-x^3x^4)] = 0,$$

since by the definition of x^ix^j, $i,j = 1,2,3,4$ we have $x^4x^1-x^3x^2 = x^2x^1-x^3x^4$.

Since $p_{|M} = c_M p_M$, $c_M \in \mathbb{R}$, for every $p \in P_{h3}$, the assertion is proved. □

Next we will turn to the stability estimate, the proof of which will only be sketched since the arguments are very similar to those given in [9].

Lemma 3.2. There is a constant $C > 0$ such that

$$\sup_{\substack{u \in V_h \\ u \neq 0}} \frac{(\text{div } u, p)}{|u|_1} \geq C\|\pi p\|_0 \qquad \forall p \in P_h.$$

Proof. Consider a macroelement $M \in C_{4h}$ and define

$$V_{0,M} = \{v \in [H_0^1(M)]^2 \mid v_{|K} \in [Q_1(K)]^2 \quad \forall K \subset M,\ K \in C_h\}$$

and

$$N_M = \{p \in P_{h|M} \mid (\text{div } v, p)_M = 0 \quad \forall v \in V_{0,M}\}.$$

A straightforward calculation shows that $N_M = \{c_1\psi_1^M + c_2\psi_3^M,\ c_1,c_2 \in \mathbb{R}\}$, where ψ_1^M is constant on M and ψ_3^M takes the values ± 1 in a chessboard - like manner on the subrectangles of M. Let $\tilde{P}_{hi} = \{p \in Q_h \mid p_{|M} = c_M\psi_i^M,\ c_M \in \mathbb{R}\}$, $i = 1,3$, and let $\tilde{P}_{h2}$ be the orthogonal complement of P_h to $\tilde{P}_{h1} \oplus \tilde{P}_{h3}$. By the same arguments as those leading to the macroelement principle introduced in [9] (cf. Lemma 3.1 and Lemma 3.2 of

[9]) one now concludes that for every $p \in P_h$, $p = \sum_{i=1}^{3} \tilde{p}_i$, $\tilde{p}_i \in \tilde{P}_{hi}$, there is a $v \in V_h$ such that $v_{|M} \in V_{0,M}$ $\forall M \in C_{4h}$ and

$$(\text{div } v, p) \geq C_1 \|\tilde{p}_2\|_0^2 \tag{3.1}$$

and

$$|v|_1 \leq \|\tilde{p}_2\|_0 . \tag{3.2}$$

By the same reasoning as in Lemma 3.3 of [9] one can also show that for every $\tilde{p}_1 \in \tilde{P}_{h1}$ there is a $g \in V_{2h}$ such that

$$(\text{div } g, \tilde{p}_1) = \|\tilde{p}_1\|_0^2 \tag{3.3}$$

and

$$|g|_1 \leq C_2 \|\tilde{p}_1\| . \tag{3.4}$$

Since $g \in V_{2h}$ we have by Lemma 3.1

$$(\text{div } g, \tilde{p}_3) = 0. \tag{3.5}$$

Let now $p \in P_h$ be arbitrary and write $p = \sum_{i=1}^{3} \tilde{p}_i$. Define $z = v + \frac{2C_1}{(1+C_2^2)} g$, where v, g, C_1 and C_2 are as above. A straightforward calculation, using the relations (3.1) to (3.5), then gives (cf. the proof of Theorem 3.1 in [9])

$$\frac{(\text{div } z, p)}{|z|_1} \geq C(\|\tilde{p}_1\|_0 + \|\tilde{p}_2\|_0) \geq C\|\pi p\|_0 . \qquad \square$$

As a final preparation for our error estimate we will introduce a seminorm on P_h defined through

$$|p|_h = \sup_{\substack{v \in V_h \\ v \neq 0}} \frac{(\text{div } v, p)}{|v|_1} \qquad \forall p \in P_h .$$

The following estimate is an immediate consequence of Lemma 3.2 and the definition of the seminorm $|\cdot|_h$

$$\|p\|_0 \geq |p|_h \geq C\|\pi p\|_0 \qquad \forall p \in P_h . \tag{3.6}$$

We are now ready to prove

Theorem 3.1. Let (u,p) be the solution to (2.1) and let $(u_h,p_h) \in V_h \times P_h$ be its finite element approximation defined as above. Then we have the error estimate

$$|u-u_h|_1 + \|p-\pi p_h\|_0 \le Ch(|u|_2 + |p|_1),$$

provided $u \in [H^2(\Omega)]^2$ and $p \in H^1(\Omega)$.

Moreover, if Ω is a convex region, we have the additional estimate

$$\|u-u_h\|_0 \le Ch^2(|u|_2 + |p|_1).$$

Proof. Let $\tilde{u} \in V_h$ be the interpolant to u and let $\tilde{p}$ be the L^2-projection of p onto P_h. By the general theory of Babuška [1] and Brezzi [3] (cf. also [7]) one concludes that there exists $v \in V_h$ and $\mu \in P_h$ such that

$$|v|_1 + |\mu|_h \le C,$$

and

$$|u_h-\tilde{u}|_1 + |p_h-\tilde{p}|_h \le C\{|(\nabla u-\tilde{u}),\nabla v)| + |(\text{div } v,p-\tilde{p})| + \tag{3.7}$$
$$+ \ (\text{div}(u-\tilde{u}),\mu)|\}.$$

By standard interpolation theory [4] the first two terms on the right hand side of (3.7) can be estimated as

$$|(\nabla(u-\tilde{u}),\nabla v)| \le |u-\tilde{u}|_1 |v|_1 \le Ch|u|_2, \tag{3.8}$$

$$|(\text{div } v,p-\tilde{p})| \le |v|_1 \|p-\tilde{p}\|_0 \le Ch|p|_1. \tag{3.9}$$

To estimate the third term we write $\mu = \pi\mu + (I-\pi)\mu$ so as to obtain

$$|(\text{div}(u-\tilde{u}),\mu)| \le |(\text{div}(u-\tilde{u}),\pi\mu)| + \tag{3.10}$$
$$+ \ |(\text{div}(u-\tilde{u}),(I-\pi)\mu)|$$

For the first term on the right hand side of (3.10) we obtain, using the estimate (3.6),

$$|(\text{div}(u-\tilde{u}),\pi\mu)| \le |u-\tilde{u}|_1 \|\pi\mu\|_0 \le Ch|u|_2. \tag{3.11}$$

To estimate the second term on the right hand side of (3.10) we introduce the interpolant $\approx{u} \in V_{2h}$ to u. Since $\text{div } u = 0$,

$(I-\pi)\mu \in P_{h3}$ and $\tilde{\tilde{u}} \in V_{2h}$ we have by Lemma 3.1

$$|(\mathrm{div}(u-\tilde{u}),(I-\pi\mu))| = |(\mathrm{div}(\tilde{u}-\tilde{\tilde{u}}),(I-\pi)\mu)| \leq \tag{3.12}$$

$$\leq |\tilde{u}-\tilde{\tilde{u}}|_1(|\mu|_h + \|\pi\mu\|_0) \leq C|\tilde{u}-\tilde{\tilde{u}}|_1|\mu|_h \leq C|\tilde{u}-\tilde{\tilde{u}}|_1 \leq Ch|u|_2.$$

Here the last inequality is a consequence of standard interpolation error estimates. Upon collecting the estimates (3.7) through (3.12) we obtain

$$|u_h-\tilde{u}|_1 + |p_h-\tilde{p}|_h \leq Ch(|u|_2 + |p|_1). \tag{3.13}$$

The asserted estimate for $|u-u_h|_1$ now follows using the triangle inequality. To obtain the estimate for $\|p-\pi p_h\|_0$ we use (3.6) and get

$$\|\pi p_h - \pi\tilde{p}\|_0 \leq Ch(|u|_2 + |p|_1). \tag{3.14}$$

The asserted estimate is now obtained upon applying the triangle inequality together with the estimate

$$\|p-\pi\tilde{p}\|_0 \leq Ch|p|_1. \tag{3.15}$$

In order to obtain the L^2-estimate for the velocity we first note that if we replace $\tilde{p}$ by $\pi\tilde{p}$ in (3.7) through (3.12) we obtain

$$|p_h-\pi\tilde{p}|_h \leq Ch(|u|_2 + |p|_1). \tag{3.16}$$

From (3.16), (3.6) and (3.14) we then obtain

$$|(I-\pi)p_h|_h \leq |p_h-\pi\tilde{p}|_h + |\pi\tilde{p}-\pi p_h|_h \tag{3.17}$$

$$\leq |p_h-\pi\tilde{p}|_h + \|\pi\tilde{p}-\pi p_h\|_0 \leq Ch(|u|_2 + |p|_1).$$

We now proceed using the Aubin-Nitsche trick. Let $(z,\lambda) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ be the solution to the problem

$$\nu(\nabla z,\nabla v) - (\mathrm{div}\ \lambda,v) = (u-u_h,v) \quad \forall v \in [H_0^1(\Omega)]^2,$$

$$(\mathrm{div}\ z,\mu) = 0 \qquad \forall\mu \in L_0^2(\Omega).$$

In the usual way (cf. e.g. [5]) we then obtain

$$\|u-u_h\|_0^2 \leq Ch\|u-u_h\|_1(|z|_2 + |\lambda|_1) + |(\mathrm{div}(z-\tilde{z}),p-p_h)|, \tag{3.18}$$

where $\tilde{z} \in V_h$ is the interpolant to z. To estimate the second term on the right hand side of (3.18) we repeat the arguments used in proving the estimates (3.10) through (3.12). Let $\tilde{\tilde{z}}$ be V_{2h}-interpolant to z. Since $\operatorname{div} z = 0$ and $\tilde{\tilde{z}} \in V_{2h}$ we obtain

$$|(\operatorname{div}(z-\tilde{z}), p-p_h)| \leq |(\operatorname{div}(z-\tilde{z}), p-\pi p_h)| + \tag{3.19}$$
$$+ |(\operatorname{div}(\tilde{z}-\tilde{\tilde{z}}), (I-\pi)p_h)| \leq Ch|z|_2(\|p-\pi p_h\|_0 +$$
$$+ |(I-\pi)p_h|_h).$$

The asserted estimate now follows upon combining (3.18), (3.19) and (3.17) and using the regularity estimate

$$|z|_2 + |\lambda|_1 \leq C\|u-u_h\|_0 . \qquad \square$$

Remark. We could have simplified the above analysis by choosing the interpolants $\tilde{u}$ and $\tilde{z}$ in V_{2h}, i.e. setting $\tilde{u} = \tilde{\tilde{u}}$, $\tilde{z} = \tilde{\tilde{z}}$. The reason for not doing this was to show that the error constants in the final estimates are not substantially larger than what they would be if the method were uniformly stable in the classical sense - a fact that has also been confirmed by numerous numerical calculations.

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