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ERROR ANALYSIS OF MIXED-INTERPOLATED ELEMENTS FOR REISSNER–MINDLIN PLATES

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We give an error analysis for the recently introduced mixed-interpolated finite element methods for Reissner–Mindlin plates. Optimal error estimates, which are valid uniformly with respect to the thickness of the plate, are proven for the deflection, rotation and the shear force. In addition, the earlier families are augmented with a new method with linear approximations for the deflection and the rotation. We also introduce a simple postprocessing method by which an improved approximation for the deflection can be obtained.

1. Introduction

The shear locking of finite element discretizations of plates based on the Reissner–Mindlin model has long been a major object of research. An impressive amount of methods and tricks has been introduced for avoiding the problem. These are far too many to review here and instead we refer to any recent engineering text.

However, for the large majority of methods introduced, a mathematical stability and error analysis is missing.

An exception to this are the mixed-interpolated families introduced in Ref. 7. These families generalize some low-order methods earlier introduced.[8,9,5,6] The analysis given in Ref. 7 was, however, not complete. Only the "worst case" of a zero thickness (for the scaled equations), i.e., the classical Kirchhoff model, was analyzed. Furthermore, no estimate for the shear force was given.

The purpose of this paper is to complete the error analysis. We will derive estimates, uniformly valid with respect to the thickness, for all variables involved. In addition, we introduce and analyze a new postprocessing method for obtaining an improved approximation for the deflection. Finally, we complement the earlier families with a new method using linear approximations for the deflection and the rotation. For this method we also perform an error analysis.

The paper is arranged as follows. In the next section we recall some theoretical results on the plate model. In Sec. 3 we first review the procedure of Ref. 7 for designing an element and then give three example families. We then perform a general error analysis for methods designed this way and apply the result for the three concrete families. Our postprocessing method is also introduced and analyzed. The last section is devoted to the new linear method.

If not explicitly defined our notation will be the established, cf. Ref. 15.

2. The Reissner–Mindlin Model

Let Ω be a plane polygonal domain. Assuming, for simplicity, that the plate is clamped along the boundary of Ω, the model is: find $w \in H_0^1(\Omega)$ and $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$ such that

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \lambda t^{-2}(\boldsymbol{\beta} - \nabla w, \boldsymbol{\eta} - \nabla v) = (f, v), \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \qquad v \in H_0^1(\Omega), \tag{2.1}$$

with the bilinear form a defined as

$$a(\boldsymbol{\beta}, \boldsymbol{\eta}) = \frac{E}{12(1 - v^2)} \int_\Omega [(1 - v)\, \boldsymbol{\varepsilon}(\boldsymbol{\beta}) : \boldsymbol{\varepsilon}(\boldsymbol{\eta}) + v \operatorname{div} \boldsymbol{\beta} \operatorname{div} \boldsymbol{\eta}].$$

Here $\boldsymbol{\varepsilon}$ is the small strain operator, t the thickness of the plate and

$$\lambda = \frac{E\kappa}{2(1 + v)},$$

with E and v denoting the Young modulus and Poisson ratio, respectively. κ is the "shear correction factor."

Given the solution $(w, \boldsymbol{\beta})$, the shear force of the plate is obtained from

$$\mathbf{q} = \lambda t^{-2}(\nabla w - \boldsymbol{\beta}).$$

Hence, $\mathbf{q} \in [L^2(\Omega)]^2$ and the problem can equivalently be formulated as

$$\begin{aligned} a(\boldsymbol{\beta}, \boldsymbol{\eta}) + (\mathbf{q}, \nabla v - \boldsymbol{\eta}) &= (f, v), \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \qquad v \in H_0^1(\Omega), \\ \lambda^{-1} t^2(\mathbf{q}, \mathbf{s}) + (\boldsymbol{\beta} - \nabla w, \mathbf{s}) &= 0, \qquad \mathbf{s} \in [L^2(\Omega)]^2. \end{aligned} \tag{2.2}$$

In the limit obtained when letting $t \to 0$ the regularity $\mathbf{q} \in [L^2(\Omega)]^2$ is lost, and the appropriate space for the shear is

$$\mathbf{H}^{-1}(\operatorname{div}; \Omega) = \{\mathbf{q} \in [H^{-1}(\Omega)]^2 \mid \operatorname{div} \mathbf{q} \in H^{-1}(\Omega)\}. \tag{2.3}$$

Therefore the limit problem is to find $w^0 \in H_0^1(\Omega)$, $\boldsymbol{\beta}^0 \in [H_0^1(\Omega)]^2$ and $\mathbf{q}^0 \in \mathbf{H}^{-1}(\mathrm{div}; \Omega)$ such that

$$a(\boldsymbol{\beta}^0, \boldsymbol{\eta}) + (\mathbf{q}^0, \nabla v - \boldsymbol{\eta}) = (f, v), \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \qquad v \in H_0^1(\Omega),$$
$$(\boldsymbol{\beta}^0 - \nabla w^0, \mathbf{s}) = 0, \qquad \mathbf{s} \in \mathbf{H}^{-1}(\mathrm{div}; \Omega). \tag{2.4}$$

This is a typical example of a saddle point problem and the conditions of Ref. 10 are easily seen to be valid.

The first condition

$$a(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq C(\|\boldsymbol{\eta}\|_1^2 + \|v\|_1^2), \qquad (v, \boldsymbol{\eta}) \in \mathbb{Z} \tag{2.5}$$

with

$$\mathbb{Z} = \{ (v, \boldsymbol{\eta}) \in H_0^1(\Omega) \times [H_0^1(\Omega)]^2 \mid (\nabla v - \boldsymbol{\eta}, \mathbf{s}) = 0, \mathbf{s} \in \mathbf{H}^{-1}(\mathrm{div}; \Omega)\},$$

follows from Korn's inequality, while the "inf-sup" condition

$$\sup_{\substack{0 \neq v \in H_0^1(\Omega) \\ \mathbf{0} \neq \boldsymbol{\eta} \in [H_0^1(\Omega)]^2}} \frac{(\nabla v - \boldsymbol{\eta}, \mathbf{s})}{\|v\|_1 + \|\boldsymbol{\eta}\|_1} \geq C\|\mathbf{s}\|_{\mathbf{H}^{-1}(\mathrm{div};\, \Omega)}, \qquad \mathbf{s} \in \mathbf{H}^{-1}(\mathrm{div}; \Omega), \tag{2.6}$$

is a direct consequence of the definition of the norm

$$\|\mathbf{q}\|^2_{\mathbf{H}^{-1}(\mathrm{div};\, \Omega)} = \|\mathbf{q}\|^2_{-1} + \|\mathrm{div}\, \mathbf{q}\|^2_{-1}, \qquad \mathbf{q} \in \mathbf{H}^{-1}(\mathrm{div}; \Omega). \tag{2.7}$$

For the original problem (2.2), the theory of Ref. 10 now gives the following basic existence result.

Proposition 2.1. The system (2.2) has a unique solution satisfying

$$\|w\|_1 + \|\boldsymbol{\beta}\|_1 + \|\mathbf{q}\|_{\mathbf{H}^{-1}(\mathrm{div};\, \Omega)} + t\|\mathbf{q}\|_0 \leq C\|f\|_{-1}. \square \tag{2.8}$$

A more detailed information concerning the regularity of the solution is obtained by splitting the shear using a Helmholtz decomposition theorem proved in Ref. 13. Here and in the sequel we denote

$$\mathrm{rot}\, \mathbf{q} = \mathrm{rot}\,(q_1, q_2) = \partial_1 q_2 - \partial_2 q_1 = \mathrm{div}\, \mathbf{q}^{\perp},$$
$$\mathbf{rot}\, p = (\partial_2 p, -\partial_1 p) = (\nabla p)^{\perp}, \tag{2.9}$$

with the notation $(v_1, v_2)^{\perp} = (v_2, -v_1)$ for the $\pi/2$ clockwise rotation.

Furthermore, we introduce the space

$$\mathbf{H}_0(\mathrm{rot};\,\Omega) = \{\,\mathbf{q}\in[L^2(\Omega)]^2 \mid \mathrm{rot}\,\mathbf{q}\in L^2(\Omega),\, \mathbf{q}\cdot\boldsymbol{\tau} = 0 \text{ on } \partial\Omega\,\}, \tag{2.10}$$

with $\boldsymbol{\tau}$ denoting the tangent to $\partial\Omega$. It can be shown that the dual space of $\mathbf{H}_0(\mathrm{rot};\,\Omega)$ coincides with $\mathbf{H}^{-1}(\mathrm{div};\,\Omega)$.

The following Helmholtz decomposition holds.[13]

Lemma 2.1. Every $\mathbf{q}\in\mathbf{H}^{-1}(\mathrm{div};\,\Omega)$ can uniquely be written as

$$\mathbf{q} = \nabla\psi + \mathbf{rot}\,p. \tag{2.11}$$

with $\psi\in H_0^1(\Omega)$ and $p\in L_0^2(\Omega)$.
Moreover, we have the equivalence of norms

$$\|\mathbf{q}\|_{\mathbf{H}^{-1}(\mathrm{div};\,\Omega)} \equiv \|\nabla\psi\|_0 + \|p\|_0. \ \square \tag{2.12}$$

Here we used the notation

$$L_0^2(\Omega) = \{\,q\in L^2(\Omega) \mid \int_\Omega q = 0\,\}.$$

By using this decomposition it is possible to write (2.3) as a system of two Poisson equations and a singularly perturbated Stokes problem[13]: find $w\in H_0^1(\Omega)$, $\boldsymbol{\beta}\in[H_0^1(\Omega)]^2$, $\psi\in H_0^1(\Omega)$ and $p\in H^1(\Omega)\cap L_0^2(\Omega)$ such that

$$\begin{aligned}
&(\nabla\psi, \nabla v) = (f, v), && v\in H_0^1(\Omega),\\
&a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\mathbf{rot}\,p, \boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = 0, && \boldsymbol{\eta}\in[H_0^1(\Omega)]^2,\\
&\lambda^{-1}t^2(\mathbf{rot}\,p, \mathbf{rot}\,q) + (\boldsymbol{\beta}, \mathbf{rot}\,q) = 0, && q\in H^1(\Omega)\cap L_0^2(\Omega),\\
&(\nabla w, \nabla\xi) - (\boldsymbol{\beta}, \nabla\xi) - \lambda^{-1}t^2(\nabla\psi, \nabla\xi) = 0, && \xi\in H_0^1(\Omega).
\end{aligned} \tag{2.13}$$

Accordingly, the estimate (2.8) becomes

$$\|w\|_1 + \|\boldsymbol{\beta}\|_1 + \|\psi\|_1 + \|p\|_0 + t\|p\|_1 \le C\|f\|_{-1}. \tag{2.14}$$

For deriving L^2-estimates we need a regularity result for the problem with a more general right-hand side.

Proposition 2.2.[13,1] Let Ω be a convex polygonal domain. For $f\in L^2(\Omega)$ and $\mathbf{g}\in[L^2(\Omega)]^2$ the solution to the problem

$$
\begin{aligned}
&(\nabla\psi, \nabla v) = (f, v), \qquad v \in H_0^1(\Omega),\\
&a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (\mathbf{rot}\, p, \boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = (\mathbf{g}, \boldsymbol{\eta}), \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2,\\
&\lambda^{-1}t^2(\mathbf{rot}\, p, \mathbf{rot}\, q) + (\boldsymbol{\beta}, \mathbf{rot}\, q) = 0, \qquad q \in H^1(\Omega) \cap L_0^2(\Omega),\\
&(\nabla w, \nabla \xi) - (\boldsymbol{\beta}, \nabla\xi) - \lambda^{-1}t^2(\nabla\psi, \nabla\xi) = 0, \qquad \xi \in H_0^1(\Omega)
\end{aligned}
\tag{2.15}
$$

satisfies

$$
\|w\|_2 + \|\boldsymbol{\beta}\|_2 + \|\psi\|_2 + \|p\|_1 + t\|p\|_2 \le C(\|f\|_0 + \|\mathbf{g}\|_0). \square
$$

Remark. This regularity estimate cannot in general be much improved. This is due to a boundary layer which exists even for a domain with a smooth boundary; cf. Refs. 2, 3, and 4 for a survey. □

When analyzing the mixed-interpolated finite element methods, it turns out that they can be viewed as discretizations of the system obtained from (2.13) when taking **rot** p as an independent unknown $\boldsymbol{\alpha}$: find $w \in H_0^1(\Omega)$, $\boldsymbol{\beta} \in [H_0^1(\Omega)]^2$, $\psi \in H_0^1(\Omega)$, $p \in H^1(\Omega) \cap L_0^2(\Omega)$ and $\boldsymbol{\alpha} \in \mathbf{H}_0(\mathrm{rot}; \Omega)$ such that

$$
(\nabla\psi, \nabla v) = (f, v), \qquad v \in H_0^1(\Omega), \tag{2.16}
$$

$$
a(\boldsymbol{\beta}, \boldsymbol{\eta}) - (p, \mathrm{rot}\, \boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in [H_0^1(\Omega)]^2, \tag{2.17a}
$$

$$
\lambda^{-1}t^2(\mathrm{rot}\, \boldsymbol{\alpha}, q) + (\mathrm{rot}\, \boldsymbol{\beta}, q) = 0, \qquad q \in L_0^2(\Omega), \tag{2.17b}
$$

$$
(\boldsymbol{\alpha}, \boldsymbol{\delta}) - (p, \mathrm{rot}\, \boldsymbol{\delta}) = 0, \qquad \boldsymbol{\delta} \in \mathbf{H}_0(\mathrm{rot}; \Omega), \tag{2.17c}
$$

$$
(\nabla w, \nabla\xi) - (\boldsymbol{\beta}, \nabla\xi) - \lambda^{-1}t^2(\nabla\psi, \nabla\xi) = 0, \qquad \xi \in H_0^1(\Omega). \tag{2.18}
$$

Remark. Note that since $\boldsymbol{\alpha} = \mathbf{rot}\, p$, we have t^2 rot $\boldsymbol{\alpha} = \lambda$ rot $\boldsymbol{\beta}$. Hence, for $t > 0$ it is correct to seek the solution $\boldsymbol{\alpha}$ in $\mathbf{H}_0(\mathrm{rot}; \Omega)$. Note also that for the limit problem with $t = 0$ we only have $\boldsymbol{\alpha} \in \mathbf{H}^{-1}(\mathrm{div}; \Omega)$. □

3. The Mixed-Interpolated Finite Element Approximation

Most finite element methods for Reissner–Mindlin plates used in practice can be presented as follows. We choose two finite element subspaces $W_h \subset H_0^1(\Omega)$ and $\mathbf{V}_h \subset [H_0^1(\Omega)]^2$ for approximating the deflection and rotation respectively. Then the discretization is defined as: find $w_h \in W_h \subset H_0^1(\Omega)$ and $\boldsymbol{\beta}_h \in \mathbf{V}_h \subset [H_0^1(\Omega)]^2$ such that

$$
a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + \lambda t^{-2}(\mathbf{R}_h(\boldsymbol{\beta}_h - \nabla w_h), \mathbf{R}_h(\boldsymbol{\eta} - \nabla v)) = (f, v), \qquad v \in W_h, \qquad \boldsymbol{\eta} \in \mathbf{V}_h, \tag{3.1}
$$

where $\mathbf{R}_h$ is some suitably chosen "reduction operator" introduced in order to circumvent the shear locking phenomenon. For the class of methods to be analyzed in this paper the reduction operator is defined in the space of piecewise smooth functions of $\mathbf{H}_0(\mathrm{rot};\,\Omega)$ and takes its values in a subspace $\mathbf{\Gamma}_h$ of $\mathbf{H}_0(\mathrm{rot};\,\Omega)$. Furthermore, the subspaces W_h and $\mathbf{\Gamma}_h$, and the operator $\mathbf{R}_h$ are chosen so that

$$\mathbf{R}_h \nabla v = \nabla v, \qquad v \in W_h.$$

Hence, the discrete problem is of the form

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + \lambda t^{-2}(\mathbf{R}_h\boldsymbol{\beta}_h - \nabla w_h, \mathbf{R}_h \boldsymbol{\eta} - \nabla v) = (f, v), \qquad v \in W_h, \qquad \boldsymbol{\eta} \in \mathbf{V}_h. \tag{3.2}$$

The approximation for the shear $\mathbf{q}_h \in \mathbf{\Gamma}_h$ is then obtained from

$$\mathbf{q}_h = \lambda t^{-2}(\nabla w_h - \mathbf{R}_h \boldsymbol{\beta}_h).$$

Hence the equivalent mixed form corresponding to (2.2) is

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + (\mathbf{q}_h, \nabla v - \mathbf{R}_h \boldsymbol{\eta}) = (f, v), \qquad \boldsymbol{\eta} \in \mathbf{V}_h, \qquad v \in W_h, \tag{3.3a}$$

$$\lambda^{-1}t^2(\mathbf{q}_h, \mathbf{s}) + (\mathbf{R}_h\boldsymbol{\beta}_h - \nabla w_h, \mathbf{s}) = 0, \qquad \mathbf{s} \in \mathbf{\Gamma}_h. \tag{3.3b}$$

In Refs. 6 and 7 an analysis of this class of methods was performed for the limiting case $t = 0$. It was shown that the deflection and rotation converge optimally if there exist an auxiliary space $Q_h \subset L_0^2(\Omega)$ such that it, together with the spaces W_h, $\mathbf{V}_h$, and $\mathbf{\Gamma}_h$, satisfies the following conditions:

P1. $\nabla W_h \subset \mathbf{\Gamma}_h$.
P2. $\mathrm{rot}\,\mathbf{\Gamma}_h \subset Q_h$.
P3. $\mathrm{rot}\,\mathbf{R}_h\boldsymbol{\eta} = P_h \mathrm{rot}\boldsymbol{\eta}$, $\boldsymbol{\eta} \in [H_0^1(\Omega)]^2$, with $P_h : L_0^2(\Omega) \to Q_h$ denoting the L^2-projection.
P4. If $\mathbf{s} \in \mathbf{\Gamma}_h$ satisfies $\mathrm{rot}\,\mathbf{s} = 0$, then $\mathbf{s} = \nabla v$ for some $v \in W_h$.
P5. $(\mathbf{V}_h^\perp, Q_h)$ is a stable pair for the Stokes problem, i.e., we have

$$\sup_{\mathbf{0} \neq \boldsymbol{\eta} \in \mathbf{V}_h} \frac{(\mathrm{rot}\,\boldsymbol{\eta}, q)}{\|\boldsymbol{\eta}\|_1} \geq C\|q\|_0, \qquad q \in Q_h.$$

These properties give the following recipe for building a method:

1. We pick a pair $(\mathbf{V}_h, \mathbf{Q}_h) \subset [H_0^1(\Omega)]^2 \times L_0^2(\Omega)$ which is known to be stable for the Stokes problem.
2. Q_h which is now fixed, we have to find a space $\mathbf{\Gamma}_h \subset \mathbf{H}_0(\mathrm{rot};\,\Omega)$ and an operator $\mathbf{R}_h : [H_0^1(\Omega)]^2 \to \mathbf{\Gamma}_h$ which together with Q_h and P_h satisfies the commuting diagram property (CDP)[18]:

$$
\begin{array}{ccc}
[H_0^1(\Omega)]^2 & \xrightarrow{\text{rot}} & L_0^2(\Omega) \\
\mathbf{R}_h \downarrow & & \downarrow P_h \\
\mathbf{\Gamma}_h & \xrightarrow{\text{rot}} & Q_h
\end{array} .
$$

3. The final task is to find a space $W_h \subset H_0^1(\Omega)$ such that

$$\nabla W_h = \{ \mathbf{s} \in \mathbf{\Gamma}_h \mid \text{rot}\, \mathbf{s} = 0 \} .$$

By the construction, the method now satisfies the properties P1–P5.

The theory for mixed methods for the Stokes problem is by now rather complete, and there exists a wide choice of stable combinations; cf. Refs. 23 and 14. The same applies for mixed finite element methods for second order elliptic problem; several complete families satisfying CDP are now known; cf. Refs. 27, 11, 12, and 14 for a unified presentation. As a consequence, the above procedure gives rise to several families of Reissner–Mindlin elements, some of which can be found in Ref. 7.

When we follow the above procedure for triangular elements, we naturally arrive at three different families of methods, one based on the classical Raviart–Thomas mixed methods[27] and the others on the more recent Brezzi–Douglas–Marini[11] and Brezzi–Douglas–Fortin–Marini[12,14] families. In all three families C_h denotes a triangulation of $\bar{\Omega}$ satisfying the usual regularity and compatibility conditions.

We let $k \geq 2$ and choose the spaces $\mathbf{V}_h$ and Q_h as the same for all families:

$$\mathbf{V}_h = \{ \boldsymbol{\eta} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|K} \in \mathbf{V}_k(K),\, K \in C_h \}, \tag{3.4}$$

where

$$\mathbf{V}_k(K) = \begin{cases} [P_k(K)]^2, & \text{for } k \geq 4, \\ [S_k(K)]^2, & \text{for } k = 2, 3, \end{cases}$$

with

$$S_k(K) = \{ v \in P_{k+1}(K) \mid v_{|e} \in P_k(e) \text{ for every edge } e \text{ of } K \} \tag{3.5}$$

and

$$Q_h = \{ p \in L_0^2(\Omega) \mid p_{|K} \in P_{k-1}(K),\, K \in C_h \}. \tag{3.6}$$

It is well-known that $(\mathbf{V}_h, Q_h)$ satisfies P5[17,29,23] with only minor assumptions on the mesh for $k \geq 4$.

Remark. For $k = 3$ the space for the rotation above could be chosen slightly smaller by defining

$$\mathbf{V}_k(K) = [P_k(K)]^2 \oplus \{ \boldsymbol{\eta} \mid \boldsymbol{\eta} = b_K \,\mathbf{rot}\, v, v \in \tilde{P}_{k-1}(K) \},$$

where $b_K = \lambda_1\lambda_2\lambda_3$ is the "bubble function" on K and $\tilde{P}_{k-1}(K)$ denotes the homogeneous polynomials of degree $k-1$. But this choice gives degrees of freedom in which both components of the rotation are represented and hence the first choice has some practical advantages. This is especially the case when these elements are used for shells. □

Let us now complete the definition of the three families.

Family I.

We define

$$W_h = \{ v \in H_0^1(\Omega) \mid v_{|K} \in P_k(K), K \in C_h \} . \tag{3.7}$$

For $\boldsymbol{\Gamma}_h$ we take the $\pi/2$-rotation of the Raviart–Thomas space

$$\boldsymbol{\Gamma}_h = \{ \mathbf{s} \in \mathbf{H}_0(\mathrm{rot};\Omega) \mid \mathbf{s}_{|K} \in [P_{k-1}(K)]^2 + (x_2, -x_1)P_{k-1}(K), K \in C_h \}, \tag{3.8}$$

which together with Q_h satisfies CDP with $\mathbf{R}_h$ defined through

$$\int_e [(\mathbf{R}_h\mathbf{s} - \mathbf{s}) \cdot \boldsymbol{\tau}]\, v = 0, \qquad v \in P_{k-1}(e) \text{ for every edge } e \text{ of } K, \tag{3.9a}$$

$$\int_K (\mathbf{R}_h\, \mathbf{s} - \mathbf{s}) \cdot \mathbf{r} = 0, \qquad \mathbf{r} \in [P_{k-2}(K)]^2, \tag{3.9b}$$

for every $K \in C_h$. □

Family II.

We let $S_k(K)$ be as in (3.5) and define

$$W_h = \{ v \in H_0^1(\Omega) \mid v_{|K} \in S_k(K), K \in C_h \}, \tag{3.10}$$

$$\boldsymbol{\Gamma}_h = \{ \mathbf{s} \in \mathbf{H}_0(\mathrm{rot};\Omega) \mid \mathbf{s}_{|K} \in [P_k(K)]^2, \mathbf{s} \cdot \boldsymbol{\tau}_{|e} \in P_{k-1}(e) \text{ for every edge } e \text{ of } K, K \in C_h \}. \tag{3.11}$$

This space combined with Q_h is the triangular BDFM pair (cf. Refs. 14 and 12 where the corresponding space for rectangles was first introduced). The reduction operator we now defined, differently from Refs. 12 and 14, by

$$\int_e [(\mathbf{R}_h\, \mathbf{s} - \mathbf{s}) \cdot \boldsymbol{\tau}]\, v = 0, \qquad v \in P_{k-1}(e) \text{ for every edge } e \text{ of } K, \tag{3.12a}$$

$$\int_K (\mathbf{R}_h\,\mathbf{s} - \mathbf{s}) \cdot \mathbf{r} = 0, \qquad \mathbf{r} \in [P_{k-2}(K)]^2, \tag{3.12b}$$

$$\int_K (\mathbf{R}_h\,\mathbf{s} - \mathbf{s}) \cdot \nabla\varphi_j = 0, \qquad j = 0, \ldots, k-2, \tag{3.12c}$$

where $\varphi_0,\ldots,\varphi_{k-2}$ are (arbitrary) polynomials in $P_k(K)$, chosen once and for all, with $\Delta\varphi_j = x_1^j x_2^{k-j-2}, j=0,\ldots,k-2.$ □

Remark. The advantage of this choice compared with the previous family is that for $k = 2,3$ the same basis functions are used for the deflection and the two components of the rotation. □

Family III.

In this choice we have

$$W_h = \{ v \in H_0^1(\Omega) \mid v_{|K} \in P_{k+1}(K), K \in C_h \}. \tag{3.13}$$

For $\mathbf{\Gamma}_h$ we now use the space

$$\mathbf{\Gamma}_h = \{ \mathbf{s} \in \mathbf{H}_0(\mathrm{rot}; \Omega) \mid \mathbf{s}_{|K} \in [P_k(K)]^2, K \in C_h \}, \tag{3.14}$$

which together with Q_h gives the rotated Brezzi–Douglas–Marini family. The reduction operator we now define through

$$\int_e [(\mathbf{R}_h\,\mathbf{s} - \mathbf{s}) \cdot \boldsymbol{\tau}]\, v = 0 \qquad v \in P_k(e) \text{ for every edge } e \text{ of } K, \tag{3.15a}$$

$$\int_K (\mathbf{R}_h\,\mathbf{s} - \mathbf{s}) \cdot \mathbf{r} = 0, \qquad \mathbf{r} \in [P_{k-2}(K)]^2, \tag{3.15b}$$

$$\int_K (\mathbf{R}_h\,\mathbf{s} - \mathbf{s}) \cdot \nabla\varphi_j = 0, \qquad j = 0, \ldots, k-2, \tag{3.15c}$$

where $\varphi_0,\ldots,\varphi_{k-2}$ are as in (3.12c).

Remark. Note that for $k \geq 4$ we have for family III that $\mathbf{V}_h \subset \mathbf{\Gamma}_h$ and hence $\mathbf{R}_h\boldsymbol{\eta} = \boldsymbol{\eta}$ for all $\boldsymbol{\eta} \in \mathbf{V}_h$. This means that the solution is obtained by a direct minimization of the energy in the finite element subspaces (i.e., "full integration" is used). □

Following Refs. 14 and 27 it is easily proved that the reduction operators for the families II and III are well-defined and that they satisfy the commuting diagram property P3. Properties P1 and P4 are evident. Hence, we have three families satisfying P1 to P5. Let us now give an error analysis of the methods in which we obtain more refined error estimates, uniform in t, for all variables involved. The key for the analysis of the methods is a discrete analog of the Helmholtz decomposition in Lemma 2.1.

Lemma 3.1. Suppose that P1 to P4 are valid. Then for every $\mathbf{q} \in \mathbf{\Gamma}_h$ there exist unique $\psi \in W_h$, $p \in Q_h$ and $\boldsymbol{\alpha} \in \mathbf{\Gamma}_h$ such that

$$\mathbf{q} = \nabla\psi + \boldsymbol{\alpha} \tag{3.16}$$

and

$$(\boldsymbol{\alpha}, \mathbf{s}) = (\operatorname{rot} \mathbf{s}, p), \qquad \mathbf{s} \in \mathbf{\Gamma}_h. \tag{3.17}$$

Proof: Due to P3 the following mixed finite element discretization has a unique solution $(\boldsymbol{\alpha}, p) \in \mathbf{\Gamma}_h \times Q_h$

$$(\boldsymbol{\alpha}, \mathbf{s}) - (\operatorname{rot} \mathbf{s}, p) = 0, \qquad \mathbf{s} \in \mathbf{\Gamma}_h,$$

$$(\operatorname{rot} \boldsymbol{\alpha}, q) = (\operatorname{rot} \mathbf{q}, q), \qquad q \in Q_h.$$

From P2 we now have

$$\operatorname{rot} (\mathbf{q} - \boldsymbol{\alpha}) = 0,$$

and P4 shows that there is $\psi \in W_h$ such that

$$\mathbf{q} - \boldsymbol{\alpha} = \nabla\psi.$$

The uniqueness of ψ follows from the uniqueness of $\boldsymbol{\alpha}$. □

Remark. Note that, due to P1, (3.17) implies the following orthogonality result

$$(\boldsymbol{\alpha}, \nabla z) = 0, \qquad z \in W_h. \; \square \tag{3.18}$$

Using this decomposition we get the following splitting.

Theorem 3.1. Suppose that P1 to P4 are valid. Then the solution $(w_h, \boldsymbol{\beta}_h, \mathbf{q}_h) \in W_h \times \mathbf{V}_h \times \mathbf{\Gamma}_h$ of (3.3) can be found by solving the following problem: find $(w_h, \boldsymbol{\beta}_h, \psi_h, p_h, \boldsymbol{\alpha}_h) \in W_h \times \mathbf{V}_h \times W_h \times Q_h \times \mathbf{\Gamma}_h$ such that:

$$(\nabla\psi_h, \nabla v) = (f, v), \qquad v \in W_h, \tag{3.19}$$

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - (p_h, \operatorname{rot} \boldsymbol{\eta}) - (\nabla\psi_h, \mathbf{R}_h \boldsymbol{\eta}) = 0, \qquad \boldsymbol{\eta} \in \mathbf{V}_h, \tag{3.20a}$$

$$\lambda^{-1}t^2(\text{rot}\,\boldsymbol{\alpha}_h, q) + (\text{rot}\,\boldsymbol{\beta}_h, q) = 0, \qquad q \in Q_h, \tag{3.20b}$$

$$(\boldsymbol{\alpha}_h, \boldsymbol{\delta}) - (p_h, \text{rot}\,\boldsymbol{\delta}) = 0, \qquad \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h, \tag{3.20c}$$

$$(\nabla w_h, \nabla \xi) - (\mathbf{R}_h\,\boldsymbol{\beta}_h, \nabla \xi) - \lambda^{-1}t^2(\nabla \psi_h, \nabla \xi) = 0, \qquad \xi \in W_h, \tag{3.21}$$

and setting $\mathbf{q}_h = \nabla \psi_h + \boldsymbol{\alpha}_h$.

Proof: Let $(w_h, \boldsymbol{\beta}_h, \mathbf{q}_h)$ be the solution to (3.3) and use Lemma 3.1 to write

$$\mathbf{q}_h = \nabla \psi_h + \boldsymbol{\alpha}_h \tag{3.22}$$

and

$$(\boldsymbol{\alpha}_h, \mathbf{s}) = (\text{rot}\,\mathbf{s}, p_h), \qquad \mathbf{s} \in \boldsymbol{\Gamma}_h \tag{3.23}$$

with unique $\psi_h \in W_h$, $\boldsymbol{\alpha}_h \in \boldsymbol{\Gamma}_h$ and $p_h \in Q_h$. We will now show that $(w_h, \boldsymbol{\beta}_h, \psi_h, p_h, \boldsymbol{\alpha}_h)$ satisfy (3.19)–(3.21). Testing by v in (3.3a) and using the orthogonality (3.18) we get (3.19). From (3.3a) we also get that

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) - (\boldsymbol{\alpha}_h, \mathbf{R}_h\boldsymbol{\eta}) - (\nabla \psi_h, \mathbf{R}_h\,\boldsymbol{\eta}) = 0.$$

From (3.23) and P3 we now have

$$(\boldsymbol{\alpha}_h, \mathbf{R}_h\,\boldsymbol{\eta}) = (p_h, \text{rot}\,\mathbf{R}_h\boldsymbol{\eta}) = (p_h, P_h\,\text{rot}\,\boldsymbol{\eta}) = (P_h\,p_h, \text{rot}\,\boldsymbol{\eta}) = (p_h, \text{rot}\,\boldsymbol{\eta}).$$

Hence, (3.20a) is satisfied. Next, for $q \in Q_h$ let $\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_h$ be defined through

$$(\boldsymbol{\gamma}, \boldsymbol{\delta}) = (\text{rot}\,\boldsymbol{\delta}, q), \qquad \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h.$$

Testing by this $\boldsymbol{\gamma}$ in (3.3b), we get using P3

$$\begin{aligned} 0 &= \lambda^{-1}t^2(\nabla \psi_h + \boldsymbol{\alpha}_h, \boldsymbol{\gamma}) + (\mathbf{R}_h\,\boldsymbol{\beta}_h - \nabla w_h, \boldsymbol{\gamma}) \\ &= \lambda^{-1}t^2(\text{rot}\,\boldsymbol{\alpha}_h, q) + (\text{rot}\,\mathbf{R}_h\,\boldsymbol{\beta}_h, q) = \lambda^{-1}t^2(\text{rot}\,\boldsymbol{\alpha}_h, q) + (\text{rot}\,\boldsymbol{\beta}_h, q). \end{aligned}$$

Finally, (3.21) is obtained by choosing $\mathbf{s} = \nabla \xi$ in (3.3b) and using the orthogonality (3.18). The asserted equivalence is now proved if the solution to (3.19)–(3.21) is unique. To this end let $f = 0$. Then (3.19) shows that $\psi_h = 0$. Next, we choose $\boldsymbol{\eta} = \boldsymbol{\beta}_h$, $q = p_h$, $\boldsymbol{\delta} = \boldsymbol{\alpha}_h$, multiply (3.20c) by $\lambda^{-1}t^2$ and add all the equations of (3.20). This gives

$$a(\boldsymbol{\beta}_h, \boldsymbol{\beta}_h) + \lambda^{-1}t^2\,\|\boldsymbol{\alpha}_h\|_0^2 = 0.$$

Hence, $\boldsymbol{\alpha}_h = \mathbf{0}$ and $\boldsymbol{\beta}_h = \mathbf{0}$. (3.20c) now reduces to

$$(\text{rot}\,\boldsymbol{\delta}, p_h) = 0, \qquad \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h.$$

Using P3 a standard argument used in connection with mixed methods shows that $p_h = 0$. The last equation (3.21) now shows that $w_h = 0$. □

From above we see that the problem decouples into a Poisson problem (3.19), a Stokes-type problem (3.20) and another Poisson problem (3.21). Hence, the error analysis is naturally performed in three steps. First, we immediately have

Lemma 3.2. For the solutions ψ and ψ_h to (2.16) and (3.19), respectively, we have

$$\|\psi - \psi_h\|_1 \leq C \inf_{\phi \in W_h} \|\psi - \phi\|_1. \;\square \tag{3.24}$$

Next, let us compare the solution $(\boldsymbol{\beta}, p, \boldsymbol{\alpha})$ of (2.17) with the solution $(\boldsymbol{\beta}_h, p_h, \boldsymbol{\alpha}_h)$ of (3.20). For this we introduce the following notation

$$A(\boldsymbol{\beta}, \boldsymbol{\alpha}; \boldsymbol{\eta}, \boldsymbol{\delta}) = a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \lambda^{-1} t^2 (\boldsymbol{\alpha}, \boldsymbol{\delta}),$$

$$B(\boldsymbol{\beta}, \boldsymbol{\alpha}; p) = (\text{rot}\,\boldsymbol{\beta}, p) + \lambda^{-1} t^2 (\text{rot}\,\boldsymbol{\alpha}, p),$$

and note that the problems (2.17) and (3.20) can be written as

$$A(\boldsymbol{\beta}, \boldsymbol{\alpha}; \boldsymbol{\eta}, \boldsymbol{\delta}) - B(\boldsymbol{\eta}, \boldsymbol{\delta}; p) = (\nabla \psi, \boldsymbol{\eta}), \qquad (\boldsymbol{\eta}, \boldsymbol{\delta}) \in [H_0^1(\Omega)]^2 \times \mathbf{H}_0(\text{rot}; \Omega),$$

$$B(\boldsymbol{\beta}, \boldsymbol{\alpha}; q) = 0, \qquad q \in L_0^2(\Omega),$$

and

$$A(\boldsymbol{\beta}_h, \boldsymbol{\alpha}_h; \boldsymbol{\eta}, \boldsymbol{\delta}) - B(\boldsymbol{\eta}, \boldsymbol{\delta}; p_h) = (\nabla \psi_h, \mathbf{R}_h \boldsymbol{\eta}) \qquad (\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbf{V}_h \times \boldsymbol{\Gamma}_h,$$

$$B(\boldsymbol{\beta}_h, \boldsymbol{\alpha}_h; q) = 0, \qquad q \in Q_h,$$

respectively. Furthermore, we note that the bilinear forms A and B are continuous on $([H_0^1(\Omega)]^2 \times \mathbf{H}_0(\text{rot}; \Omega)) \times ([H_0^1(\Omega)]^2 \times \mathbf{H}_0(\text{rot}; \Omega))$ and $([H_0^1(\Omega)]^2 \times \mathbf{H}_0(\text{rot}; \Omega)) \times L_0^2(\Omega)$ if in $[H_0^1(\Omega)]^2 \times \mathbf{H}_0(\text{rot}; \Omega)$ we make use of the norm

$$|\,\|(\boldsymbol{\beta}, \boldsymbol{\alpha})\|\,|^2 = \|\boldsymbol{\beta}\|_1^2 + t^2 \|\boldsymbol{\alpha}\|_0^2 + t^4 \|\text{rot}\,\boldsymbol{\alpha}\|_0^2$$

(and the usual L^2-norm in $L_0^2(\Omega)$).

Remark. Recall that

$$t\boldsymbol{\alpha} = t\,\mathbf{rot}\,p \qquad \text{and} \qquad t^2 \operatorname{rot} \boldsymbol{\alpha} = \lambda \operatorname{rot} \boldsymbol{\beta}$$

so that this norm is natural. □

Now we easily get the following result.

Lemma 3.3. Suppose that P1 to P5 are valid. For the solutions $(\boldsymbol{\beta}, p, \boldsymbol{\alpha})$ and $(\boldsymbol{\beta}_h, p_h, \boldsymbol{\alpha}_h)$ to (2.17) and (3.20), we then have

$$\begin{aligned} &|\,\|(\boldsymbol{\beta}, \boldsymbol{\alpha}) - (\boldsymbol{\beta}_h, \boldsymbol{\alpha}_h)\|\,| + \|p - p_h\|_0 \\ &\le C\Big\{ \inf_{(\boldsymbol{\eta}, \boldsymbol{\delta})\in \mathbf{V}_h \times \boldsymbol{\Gamma}_h} |\,\|(\boldsymbol{\beta}, \boldsymbol{\alpha}) - (\boldsymbol{\eta}, \boldsymbol{\delta})\|\,| + \inf_{q\in Q_h} \|p - q\|_0 \\ &\quad + \|\psi - \psi_h\|_1 + \sup_{\mathbf{0}\neq \boldsymbol{\eta}\in \mathbf{V}_h} \frac{(\nabla\psi, \mathbf{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_1}\Big\}. \end{aligned}$$

Proof: We note that the problem naturally fits into the abstract theory of Ref. 10. Hence we have to prove that

$$A(\boldsymbol{\beta}, \boldsymbol{\alpha}; \boldsymbol{\beta}, \boldsymbol{\alpha}) \ge C\,|\,\|(\boldsymbol{\beta}, \boldsymbol{\alpha})\|\,|^2, \qquad (\boldsymbol{\beta}, \boldsymbol{\alpha})\in \mathbf{K}_h,$$

with

$$\mathbf{K}_h = \{\, (\boldsymbol{\beta}, \boldsymbol{\alpha})\in \mathbf{V}_h \times \boldsymbol{\Gamma}_h \mid B(\boldsymbol{\beta}, \boldsymbol{\alpha}; q) = 0, \qquad q\in Q_h \,\},$$

and

$$\sup_{\substack{(\boldsymbol{\eta}, \boldsymbol{\alpha})\in \mathbf{V}_h \times \boldsymbol{\Gamma}_h \\ (\boldsymbol{\eta}, \boldsymbol{\alpha})\neq (\mathbf{0}, \mathbf{0})}} \frac{B(\boldsymbol{\eta}, \boldsymbol{\alpha}; q)}{|\,\|(\boldsymbol{\eta}, \boldsymbol{\alpha})\|\,|} \ge C\,\|q\|_0, \qquad q\in Q_h.$$

Of these the first condition follows directly from P2 (and Korn's inequality). The second condition follows trivially from P5. The saddle point theory then gives

$$\begin{aligned} &|\,\|(\boldsymbol{\beta}, \boldsymbol{\alpha}) - (\boldsymbol{\beta}_h, \boldsymbol{\alpha}_h)\|\,| + \|p - p_h\|_0 \\ &\le C\Big\{ \inf_{(\boldsymbol{\eta}, \boldsymbol{\delta})\in \mathbf{V}_h \times \boldsymbol{\Gamma}_h} |\,\|(\boldsymbol{\beta}, \boldsymbol{\alpha}) - (\boldsymbol{\eta}, \boldsymbol{\delta})\|\,| + \inf_{q\in Q_h} \|p - q\|_0 \\ &\quad + \sup_{\mathbf{0}\neq \boldsymbol{\eta}\in \mathbf{V}_h} \frac{(\nabla\psi_h, \mathbf{R}_h\boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_1}\Big\}. \end{aligned}$$

Furthermore we have

$$(\nabla\psi_h, \mathbf{R}_h\,\boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = (\nabla(\psi_h - \psi), \mathbf{R}_h\,\boldsymbol{\eta}) + (\nabla\psi, \mathbf{R}_h\,\boldsymbol{\eta} - \boldsymbol{\eta})$$

$$\leq ||\psi_h - \psi||_1\,||\mathbf{R}_h\boldsymbol{\eta}||_0 + (\nabla\psi, \mathbf{R}_h\,\boldsymbol{\eta} - \boldsymbol{\eta})$$

$$\leq C\,||\psi_h - \psi||_1\,||\boldsymbol{\eta}||_1 + (\nabla\psi, \mathbf{R}_h\,\boldsymbol{\eta} - \boldsymbol{\eta}),$$

which proves the asserted estimate. □

The third step in the analysis is given by

Lemma 3.4. For the solutions w and w_h to (2.18) and (3.21) we have

$$||w - w_h||_1 \leq ||\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}||_0 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0 + Ch\,||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_1 + \inf_{v\in W_h} |w - v|_1.$$

Proof: Let $w^I \in W_h$ be the solution of

$$(\nabla w^I, \nabla\xi) = (\nabla w, \nabla\xi), \qquad \xi\in W_h.$$

From (2.16) and (3.19) we have $(\nabla\psi_h, \nabla v) = (\nabla\psi, \nabla v)$. From (2.18) and (3.21) we have

$$||\nabla(w^I - w_h)||_0^2 = (\nabla(w^I - w_h), \nabla(w^I - w_h)) = (\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h, \nabla(w^I - w_h))$$

$$\leq ||\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h||_0\,||\nabla(w^I - w_h)||_0$$

which gives

$$||\nabla(w^I - w_h)||_0 \leq ||\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h||_0. \tag{3.25}$$

Next, we note that any useful reduction operator satisfies the basic interpolation estimate

$$||\,(\mathbf{I} - \mathbf{R}_h)\boldsymbol{\eta}||_0 \leq Ch\,||\boldsymbol{\eta}||_1, \qquad \boldsymbol{\eta}\in[H_0^1(\Omega)]^2,$$

with $\mathbf{I}$ denoting the identity.

Hence, we can conclude the estimation as follows:

$$||\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h||_0 = ||(\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}) + (\boldsymbol{\beta} - \boldsymbol{\beta}_h) - (\mathbf{I} - \mathbf{R}_h)\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h)||_0$$

$$\leq ||\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}||_0 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0 + ||(\mathbf{I} - \mathbf{R}_h)\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h)||_0 \tag{3.26}$$

$$\leq ||\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}||_0 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0 + Ch\,||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_1\,. \square$$

Next we give estimates on $\boldsymbol{\alpha} - \boldsymbol{\alpha}_h$ without a t factor.

Lemma 3.5. For the solutions $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}_h$ of (2.17c) and (3.20c) we have

$$||\boldsymbol{\alpha} - \boldsymbol{\alpha}_h||_{-1} \leq C \{ ||p - p_h||_0 + h \inf_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} ||\boldsymbol{\alpha} - \boldsymbol{\delta}||_0 \}.$$

For a quasi-uniform mesh we additionally have

$$h||\boldsymbol{\alpha} - \boldsymbol{\alpha}_h||_0 \leq C ||p - p_h||_0 + h \inf_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} ||\boldsymbol{\alpha} - \boldsymbol{\delta}||_0 .$$

Proof: We recall first that

$$(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h, \boldsymbol{\delta}) = (p - p_h, \operatorname{rot} \boldsymbol{\delta}), \qquad \boldsymbol{\delta} \in \boldsymbol{\Gamma}_h. \tag{3.27}$$

Letting $\boldsymbol{\alpha}^I$ be the L^2-projection of $\boldsymbol{\alpha}$ onto $\boldsymbol{\Gamma}_h$ we then have

$$\sup_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} \frac{(\boldsymbol{\alpha}^I - \boldsymbol{\alpha}_h, \boldsymbol{\delta})}{||\boldsymbol{\delta}||_{\mathbf{H}_0(\mathrm{rot};\, \Omega)}} = \sup_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} \frac{(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h, \boldsymbol{\delta})}{||\boldsymbol{\delta}||_{\mathbf{H}_0(\mathrm{rot};\, \Omega)}} \leq ||p - p_h||_0 ,$$

which, using local scaling arguments, gives

$$\left(\sum_{K \in C_h} h_K^2 \, ||\boldsymbol{\alpha}^I - \boldsymbol{\alpha}_h||_0^2 \right)^{1/2} \leq C \, ||p - p_h||_0 .$$

Hence we have

$$\left(\sum_{K \in C_h} h_K^2 \, ||\boldsymbol{\alpha} - \boldsymbol{\alpha}_h||_0^2 \right)^{1/2} \leq C \, ||p - p_h||_0 + h \inf_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} ||\boldsymbol{\alpha} - \boldsymbol{\delta}||_0 , \tag{3.28}$$

which for a quasi-uniform triangulation reduces to

$$h \, ||\boldsymbol{\alpha} - \boldsymbol{\alpha}_h||_0 \leq C \, ||p - p_h||_0 + h \inf_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} ||\boldsymbol{\alpha} - \boldsymbol{\delta}||_0.$$

Moreover, using (3.27) and (3.28) we get

$$\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_{-1} = \sup_{\boldsymbol{\delta} \in [H_0^1(\Omega)]^2} \frac{(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h, \boldsymbol{\delta})}{\|\boldsymbol{\delta}\|_1}$$

$$= \sup_{\boldsymbol{\delta} \in [H_0^1(\Omega)]^2} \left\{ \frac{(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h, \boldsymbol{\delta} - \mathbf{R}_h\boldsymbol{\delta})}{\|\boldsymbol{\delta}\|_1} + \frac{(p - p_h, \operatorname{rot} \mathbf{R}_h\boldsymbol{\delta})}{\|\boldsymbol{\delta}\|_1} \right\}$$

$$\leq C \left\{ \left(\sum_{K \in C_h} h_K^2 \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0^2 \right)^{1/2} + \|p - p_h\|_0 \right\}$$

$$\leq C \left\{ \|p - p_h\|_0 + h \inf_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} \|\boldsymbol{\alpha} - \boldsymbol{\delta}\|_0 \right\}. \square$$

We can collect all the above error estimates in the following theorem.

Theorem 3.2. Suppose that the properties P1 to P5 are valid. For the solutions to (2.16)–(2.18) and (3.19)–(3.21) we then have

$$\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|p - p_h\|_0 + t\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0$$
$$+ t^2\|\operatorname{rot}(\boldsymbol{\alpha} - \boldsymbol{\alpha}_h)\|_0 + \|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_{-1} + \|\psi - \psi_h\|_1 \leq CE(h),$$

with

$$E(h) = \left\{ \inf_{v \in W_h} \|w - v\|_1 + \inf_{\boldsymbol{\theta} \in \mathbf{V}_h} \|\boldsymbol{\beta} - \boldsymbol{\theta}\|_1 + \inf_{\phi \in W_h} \|\psi - \phi\|_1 + \inf_{q \in Q_h} \|p - q\|_0 \right.$$
$$+ \|\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}\|_0 + h \inf_{\boldsymbol{\delta} \in \boldsymbol{\Gamma}_h} \|\boldsymbol{\alpha} - \boldsymbol{\delta}\|_0 + \inf_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_h} [t\|\boldsymbol{\alpha} - \boldsymbol{\gamma}\|_0$$
$$\left. + t^2\|\operatorname{rot}(\boldsymbol{\alpha} - \boldsymbol{\gamma})\|_0] + \sup_{\mathbf{0} \neq \boldsymbol{\eta} \in \mathbf{V}_h} \frac{(\nabla\psi, \mathbf{R}_h \boldsymbol{\eta} - \boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_1} \right\}.$$

Moreover, for a quasi-uniform mesh we have

$$h\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_0 \leq CE(h). \square$$

For the example families we hence get

Corollary 3.1. Suppose that for the solution of (2.13) we have $w \in H^{k+1}(\Omega)$, $\boldsymbol{\beta} \in [H^{k+1}(\Omega)]^2$, $\psi \in H^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$. For the approximation with the families I, II and III we then have the estimate

$$\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + \|\mathbf{q} - \mathbf{q}_h\|_{-1}$$

$$\leq Ch^k(|w|_{k+1} + |\boldsymbol{\beta}|_{k+1} + |\psi|_{k+1} + |p|_k + t|p|_{k+1}).$$

For a convex domain we have in addition

$$\|w - w_h\|_0 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_0 \leq Ch^{k+1}(|w|_{k+1} + |\boldsymbol{\beta}|_{k+1} + |\psi|_{k+1} + |p|_k + t|p|_{k+1}).$$

Proof: We note that

$$\|\mathbf{q} - \mathbf{q}_h\|_{-1} \leq C(\|\boldsymbol{\alpha} - \boldsymbol{\alpha}_h\|_{-1} + \|\psi - \psi_h\|_1).$$

Hence, we have to evaluate the error expression $E(h)$. Recalling that $t^2 \text{rot}\, \boldsymbol{\alpha} = \lambda \text{ rot } \boldsymbol{\beta}$, and using P3 we get the estimate

$$\inf_{\boldsymbol{\gamma} \in \boldsymbol{\Gamma}_h} [t\|\boldsymbol{\alpha} - \boldsymbol{\gamma}\|_0 + t^2\|\text{rot}\,(\boldsymbol{\alpha} - \boldsymbol{\gamma})\|_0]$$

$$\leq t\|\boldsymbol{\alpha} - \mathbf{R}_h\boldsymbol{\alpha}\|_0 + t^2\|\text{rot}\,(\boldsymbol{\alpha} - \mathbf{R}_h\boldsymbol{\alpha})\|_0 = t\|\boldsymbol{\alpha} - \mathbf{R}_h\boldsymbol{\alpha}\|_0 + t^2\|\text{rot}\,\boldsymbol{\alpha} - P_h \text{ rot } \boldsymbol{\alpha}\|_0$$

$$= t\|\boldsymbol{\alpha} - \mathbf{R}_h\boldsymbol{\alpha}\|_0 + \lambda\,\|\text{rot}\,\boldsymbol{\beta} - P_h \text{ rot } \boldsymbol{\beta}\|_0 \leq Ch^k(t|\boldsymbol{\alpha}|_k + |\boldsymbol{\beta}|_{k+1}).$$

From the definitions of the reduction operators, we see that all three families satisfy

$$\int_K (\mathbf{R}_h\boldsymbol{\eta} - \boldsymbol{\eta}) \cdot \mathbf{r} = 0, \qquad \mathbf{r} \in [P_{k-2}(K)]^2, \qquad K \in C_h.$$

Letting $\boldsymbol{\Pi}_h$ be the orthogonal projection onto the space

$$\{\mathbf{r} \in [L^2(\Omega)]^2 \mid \mathbf{r} \in [P_{k-2}(K)]^2, \qquad K \in C_h\}$$

we therefore get

$$(\nabla\psi, \boldsymbol{\eta} - \mathbf{R}_h\boldsymbol{\eta}) = (\nabla\psi - \boldsymbol{\Pi}_h\nabla\psi, \boldsymbol{\eta} - \mathbf{R}_h\boldsymbol{\eta})$$

$$\leq \|\nabla\psi - \boldsymbol{\Pi}_h\nabla\psi\|_0\,\|\boldsymbol{\eta} - \mathbf{R}_h\boldsymbol{\eta}\|_0 \leq Ch^k|\psi|_{k+1}|\boldsymbol{\eta}|_1.$$

This gives the estimate

$$\sup_{\mathbf{0} \neq \boldsymbol{\eta} \in \mathbf{V}_h} \frac{(\nabla\psi, \mathbf{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_1} \leq Ch^k|\psi|_{k+1}.$$

The rest of the interpolation estimates in $E(h)$ are standard, and hence we have proved the first of the asserted estimates.

Let us turn to the L^2-estimates. We proceed as usual and consider the solution $\boldsymbol{\theta}\in[H_0^1(\Omega)]^2$, $\boldsymbol{\gamma}\in\mathbf{H}_0(\mathrm{rot};\Omega)$, $r\in L_0^2(\Omega)$ to

$$A(\boldsymbol{\theta},\boldsymbol{\gamma};\boldsymbol{\eta},\boldsymbol{\delta}) - B(\boldsymbol{\eta},\boldsymbol{\delta};r) = (\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\eta}), \qquad (\boldsymbol{\eta},\boldsymbol{\delta})\in[H_0^1(\Omega)]^2\times H_0(\mathrm{rot};\Omega),$$

$$B(\boldsymbol{\theta},\boldsymbol{\gamma};q) = 0, \qquad q\in L_0^2(\Omega).$$

The standard manipulations then give

$$\begin{aligned}||\boldsymbol{\beta}-\boldsymbol{\beta}_h||_0^2 &= A(\boldsymbol{\theta}-\boldsymbol{\theta}^I,\boldsymbol{\gamma}-\mathbf{R}_h\boldsymbol{\gamma};\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)\\ &\quad - B(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h;r-P_hr)\\ &\quad - B(\boldsymbol{\theta}-\boldsymbol{\theta}^I,\boldsymbol{\gamma}-\mathbf{R}_h\boldsymbol{\gamma};p-p_h) + (\nabla\psi_h,\mathbf{R}_h\boldsymbol{\theta}^I) - (\nabla\psi,\boldsymbol{\theta}^I),\end{aligned}$$

for any $\boldsymbol{\theta}^I\in\mathbf{V}_h$. Let us now fix $\boldsymbol{\theta}^I$ to be the Clemént interpolant (cf. Ref. 16 and Ref. 23, pp. 109–111) of $\boldsymbol{\theta}$ in the following subspace of W_h

$$\{\boldsymbol{\eta}\in[H_0^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|K}\in[P_1(K)]^2,\ K\in C_h\}.$$

With this we have

$$||\boldsymbol{\theta}^I||_1 \le C||\boldsymbol{\theta}||_1 \qquad \text{and} \qquad ||\boldsymbol{\theta}-\boldsymbol{\theta}^I||_1 \le Ch|\boldsymbol{\theta}|_2,$$

and in particular $\mathbf{R}_h\boldsymbol{\theta}^I = \boldsymbol{\theta}^I$, which gives

$$\begin{aligned}(\nabla\psi_h,\mathbf{R}_h\boldsymbol{\theta}^I) - (\nabla\psi,\boldsymbol{\theta}^I) &= (\nabla\psi_h - \nabla\psi,\boldsymbol{\theta}^I)\\ &= (\psi-\psi_h,\mathrm{div}\boldsymbol{\theta}^I) \le ||\psi-\psi_h||_0\,||\boldsymbol{\theta}^I||_1 \le C||\psi-\psi_h||_0\,||\boldsymbol{\theta}||_1.\end{aligned}$$

Now we get

$$\begin{aligned}||\boldsymbol{\beta}-\boldsymbol{\beta}_h||_0^2 &\le (|\,||(\boldsymbol{\theta}-\boldsymbol{\theta}^I,\boldsymbol{\gamma}-\mathbf{R}_h\boldsymbol{\gamma})||\,| + ||r-r^I||_0)(|\,||(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)||\,|\\ &\quad + ||p-p_h||_0) + C||\psi-\psi_h||_0\,||\boldsymbol{\theta}||_1\\ &\le Ch\,(|\boldsymbol{\theta}|_2 + |r|_1 + t|\boldsymbol{\gamma}|_1)\,|\,||(\boldsymbol{\beta}-\boldsymbol{\beta}_h,\boldsymbol{\alpha}-\boldsymbol{\alpha}_h)||\,| + ||p-p_h||_0)\\ &\quad + C||\psi-\psi_h||_0\,||\boldsymbol{\theta}||_1.\end{aligned}$$

(Note that here we have to use the fact that $t^2\,\mathrm{rot}\boldsymbol{\gamma} = \lambda\,\mathrm{rot}\,\boldsymbol{\theta}$ for estimating

$$t^2||\mathrm{rot}\,(\boldsymbol{\gamma} - \mathbf{R}_h\boldsymbol{\gamma})||_0 = \lambda||\mathrm{rot}\,\boldsymbol{\theta} - P_h\,\mathrm{rot}\,\boldsymbol{\theta}||_0 \le Ch|\boldsymbol{\theta}|_2.)$$

From Proposition 2.2 we have

$$(|\boldsymbol{\theta}|_2 + |r|_1 + t|\boldsymbol{\gamma}|_1) \le C\,||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0,$$

which gives

$$||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0 \le C[h(|\,||(\boldsymbol{\beta} - \boldsymbol{\beta}_h, \boldsymbol{\alpha} - \boldsymbol{\alpha}_h)||\,| + ||p - p_h||_0) + ||\psi - \psi_h||_0].$$

The estimate

$$||\psi - \psi_h||_0 \le Ch^{k+1}|\psi|_{k+1}$$

is standard and hence we have derived the L^2-estimate for the rotation. To derive the L^2-estimate for the deflection we use the usual duality argument. We let $z \in H_0^1(\Omega)$ be the solution to

$$(\nabla z, \nabla v) = (w - w_h, v), \qquad v \in H_0^1(\Omega),$$

and let z^I be its interpolant in the space of continuous piecewise linear functions. Using (2.16), (2.18), (3.19), and (3.21) we then have

$$\begin{aligned} ||w - w_h||_0^2 &= (\nabla(w - w_h), \nabla z) = (\nabla(w - w_h), \nabla(z - z^I)) + (\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h, \nabla z^I) \\ &\le Ch|z|_2||\nabla(w - w_h)||_0 + (\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h, \nabla z^I)\,. \end{aligned}$$

Next, we note that due to (3.9b), (3.12b), and (3.15b) $(\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}, \nabla z^I) = 0$ holds, and hence we have

$$\begin{aligned} &(\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h, \nabla z^I) = (\mathbf{R}_h\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h, \nabla z^I) \le ||\mathbf{R}_h\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h||_0\,||\nabla z^I||_0 \\ &\le C||\mathbf{R}_h\boldsymbol{\beta} - \mathbf{R}_h\boldsymbol{\beta}_h||_0\,||z||_2 = C||(\mathbf{R}_h - \mathbf{I})\,(\boldsymbol{\beta} - \boldsymbol{\beta}_h) + (\boldsymbol{\beta} - \boldsymbol{\beta}_h)||_0\,||z||_2 \\ &\le (Ch|\boldsymbol{\beta} - \boldsymbol{\beta}_h|_1 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0)\,||z||_2\,. \end{aligned}$$

The assertion now follows from the regularity estimate $||z||_2 \le C||w - w_h||_0$. □

Remark. Due to the boundary layer the assumptions above are unrealistic. However, if f is smooth, we know that the solution is smooth in any compact subset Ω^i of Ω. For a convex domain we also know by Proposition 2.2 that $w \in H^2(\Omega)$, $\boldsymbol{\beta} \in [H^2(\Omega)]^2$, $\psi \in H^2(\Omega)$ and $p \in H^2(\Omega)$. Denoting $\Omega^b = \Omega \backslash \Omega^i$, we then get the following rigorous estimate for Ω convex:

$$||w - w_h||_0 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0 \le Ch^2(|w|_{2,\Omega^b} + |\boldsymbol{\beta}|_{2,\Omega^b} + |\psi|_{2,\Omega^b} + |p|_{1,\Omega^b} + t|p|_{2,\Omega^b})$$

$$+ Ch^{k+1}(|w|_{k+1,\Omega^i} + |\boldsymbol{\beta}|_{k+1,\Omega^i} + |\psi|_{k+1,\Omega^i} + |p|_{k,\Omega^i} + t|p|_{k+1,\Omega^i}).$$

This is clearly a quite crude estimate. It would be interesting to know if, e.g., some optimal interior estimates could be derived for this problem. □
Next we note that for the third family Lemma 3.4 gives a better estimate for the deflection.

***Corollary** 3.2.* Suppose that Ω is convex and that we have $w \in H^{k+2}(\Omega)$, $\boldsymbol{\beta} \in [H^{k+1}(\Omega)]^2$, $\psi \in H^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$. For the solution of (3.3) with the family III, we then have

$$||w - w_h||_1 \le Ch^{k+1}(|w|_{k+2} + |\boldsymbol{\beta}|_{k+1} + |\psi|_{k+1} + |p|_k + t|p|_{k+1}). \square$$

On the other hand, the same result can also be obtained by "postprocessing" the solution with the families I and II. Given the approximate solution $(w_h, \boldsymbol{\beta}_h, \mathbf{q}_h) \in W_h \times \mathbf{V}_h \times \boldsymbol{\Gamma}_h$ obtained by the original method, we calculate a new approximation for the deflection

$$w_h^* \in W_h^* = \{ v \in H_0^1(\Omega) \mid v_{|K} \in P_{k+1}(K), \qquad K \in C_h \},$$

by solving the system

$$(\nabla w_h^*, \nabla v) = (\boldsymbol{\beta}_h, \nabla v) + \lambda^{-1} t^2 (f, v), \qquad v \in W_h^*.$$

For this new approximation for the deflection we get the following error estimate.

***Theorem** 3.3.* Suppose that Ω is convex and that we have $w \in H^{k+2}(\Omega)$, $\boldsymbol{\beta} \in [H^{k+1}(\Omega)]^2$, $\psi \in H^{k+1}(\Omega)$ and $p \in H^{k+1}(\Omega)$. For the deflection w_h^* obtained by postprocessing the solution with the families I and II we then have

$$||w - w_h^*||_1 \le Ch^{k+1}(|w|_{k+2} + |\boldsymbol{\beta}|_{k+1} + |\psi|_{k+1} + |p|_k + t|p|_{k+1}).$$

Proof: First, we note that the exact solution $(w, \boldsymbol{\beta}, \mathbf{q})$ satisfies

$$(\nabla w, \nabla v) = (\boldsymbol{\beta}, \nabla v) + \lambda^{-1} t^2 (\mathbf{q}, \nabla v) = (\boldsymbol{\beta}, \nabla v) + \lambda^{-1} t^2 (f, v), \qquad v \in W_h^*.$$

Let $\tilde{w} \in W_h^*$ be the interpolant to w. We then have

$$||\nabla(w_h^* - \tilde{w})||_0^2 = (\nabla(w_h^* - \tilde{w}), \nabla(w_h^* - \tilde{w}))$$

$$= (\nabla(w - \tilde{w}), \nabla(w_h^* - \tilde{w})) + (\boldsymbol{\beta}_h - \boldsymbol{\beta}, \nabla(w_h^* - \tilde{w})),$$

which gives

$$||\nabla(w - w_h^*)||_0 \leq 2||\nabla(w - \tilde{w})||_0 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_0.$$

The assertion now follows from a standard interpolation estimate and Corollary 3.1. □

4. A New Linear Element

From the preceding section the reason why linear elements in general cannot be used is clearly seen; it would require the linear-constant velocity-pressure combination to be stable. It is a classical result that this is not the case. For quadrilateral elements it is also well known that the bilinear-constant is not a uniformly stable Stokes element. Nevertheless, this method can, under some restrictions on the mesh, be proven to give an optimally convergent solution provided that the pressure has been filtered in an appropriate way (cf. Ref. 26). These results can be carried over to give optimal error estimates for the deflection and the rotation obtained with the MITC4 element.[5,6]

The lowest order method obtained with the theory presented above, is a method in which linear deflections are combined with a rotation space consisting of linear functions augmented with piecewise quadratic functions with the values of the tangential component of the rotation at mid-edge nodes as degrees of freedom.[19] Another variant of this idea would be to use linear approximations for one component of the rotation and quadratic for the other, cf. Ref. 28.

However, there is an alternative which is both simpler and cheaper. In this a recent stabilization technique developed for mixed finite element methods (cf. e.g. Refs. 20 and 24) is used. The idea is to augment the normal "Galerkin" formulation with appropriately scaled least-squares forms of the differential equations. In the context of Reissner–Mindlin plates this approach was first used in Refs. 21 and 25. For the lowest order methods for Reissner–Mindlin plates this technique becomes particularly simple; the stability is achieved by simply replacing t^2 with $t^2 + \gamma h^2$ (γ being a positive constant) in the shear term of the energy expression.

Let us now describe and analyze the method so obtained. Let C_h be a triangular decomposition and define the spaces as

$$W_h = \{ v \in H_0^1(\Omega) \mid v_{|K} \in P_1(K), K \in C_h \}, \tag{4.1}$$

$$\mathbf{V}_h = \{ \boldsymbol{\eta} \in [H_0^1(\Omega)]^2 \mid \boldsymbol{\eta}_{|K} \in [P_1(K)]^2, K \in C_h \}. \tag{4.2}$$

The method is then defined as : find $w_h \in W_h$ and $\boldsymbol{\beta}_h \in \mathbf{V}_h$ such that

$$\begin{gathered} a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + \lambda(t^2 + \gamma h^2)^{-1}(\mathbf{R}_h\boldsymbol{\beta}_h - \nabla w_h, \mathbf{R}_h\boldsymbol{\eta} - \nabla v) = (f, v), \\ v \in W_h, \qquad \boldsymbol{\eta} \in \mathbf{V}_h, \end{gathered} \tag{4.3}$$

with $\mathbf{R}_h$ as the reduction operator to the lowest order rotated Raviart–Thomas space

$$\mathbf{\Gamma}_h = \{ \mathbf{s} \in \mathbf{H}_0(\mathrm{rot};\Omega) \mid \mathbf{s}_{|K} \in [P_0(K)]^2 + (x_2, -x_1) P_0(K), K \in C_h \}, \tag{4.4}$$

i.e., $\mathbf{R}_h$ is defined through

$$\int_e [(\mathbf{R}_h \mathbf{s} - \mathbf{s}) \cdot \boldsymbol{\tau}] = 0 \text{ for every edge } e \text{ of } K, \tag{4.5}$$

for every $K \in C_h$.

The positive parameter γ is supposed to lie in a fixed range: $C_1 \le \gamma \le C_2$. We recall that h denotes the global mesh parameter.

The approximation $\mathbf{q}_h \in \mathbf{\Gamma}_h$ for the shear is then calculated through

$$\mathbf{q}_h = \lambda (t^2 + \gamma h^2)^{-1} (\nabla w_h - \mathbf{R}_h \boldsymbol{\beta}_h).$$

Hence, $(w_h, \boldsymbol{\beta}_h, \mathbf{q})$ solves the problem

$$\begin{aligned} a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + (\mathbf{q}_h, \nabla v - \mathbf{R}_h \boldsymbol{\eta}) &= (f, v), \qquad \boldsymbol{\eta} \in \mathbf{V}_h, \quad v \in W_h, \\ \lambda^{-1}(t^2 + \gamma h^2)(\mathbf{q}_h, \mathbf{s}) + (\mathbf{R}_h \boldsymbol{\beta}_h - \nabla w_h, \mathbf{s}) &= 0, \qquad \mathbf{s} \in \mathbf{\Gamma}_h. \end{aligned} \tag{4.6}$$

For this method we are able to prove the following estimate

Theorem 4.1. There is a positive constant C such that

$$\begin{aligned} &\|w - w_h\|_1 + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_1 + (t + h)\, \|\mathbf{q} - \mathbf{q}_h\|_0 \\ &\qquad \le Ch(|w|_2 + |\boldsymbol{\beta}|_2 + |\psi|_2 + |p|_1 + t|p|_2). \ \square \end{aligned}$$

For the analysis we first note that with Q_h defined as

$$Q_h = \{ p \in L_0^2(\Omega) \mid p_{|K} \in P_0(K), K \in C_h \}$$

the spaces $\mathbf{\Gamma}_h$ and W_h satisfy the properties P1 to P4.
Hence, Lemma 3.1 is valid and the discrete problem can equivalently be writen as

$$(\nabla \psi_h, \nabla v) = (f, v), \qquad v \in W_h, \tag{4.7}$$

$$A_h(\boldsymbol{\beta}_h, \boldsymbol{\alpha}_h; \boldsymbol{\eta}, \boldsymbol{\delta}) - B_h(\boldsymbol{\eta}, \boldsymbol{\delta}; p_h) - (\nabla \psi_h, \mathbf{R}_h \boldsymbol{\eta}) = 0, \qquad (\boldsymbol{\eta}, \boldsymbol{\delta}) \in \mathbf{V}_h \times \mathbf{\Gamma}_h, \tag{4.8a}$$

$$B_h(\boldsymbol{\beta}_h, \boldsymbol{\alpha}_h; q) = 0, \qquad q \in Q_h, \tag{4.8b}$$

$$(\nabla w_h, \nabla \xi) - (\mathbf{R}_h \boldsymbol{\beta}_h, \nabla \xi) - \lambda^{-1}(t^2 + \gamma h^2)(\nabla \psi_h, \nabla \xi) = 0, \qquad \xi \in W_h, \tag{4.9}$$

with

$$A_h(\boldsymbol{\beta}, \boldsymbol{\alpha}; \boldsymbol{\eta}, \boldsymbol{\delta}) = a(\boldsymbol{\beta}, \boldsymbol{\eta}) + \lambda^{-1}(t^2 + \gamma h^2)(\boldsymbol{\alpha}, \boldsymbol{\delta}),$$

$$B_h(\boldsymbol{\beta}, \boldsymbol{\alpha}; p) = (\mathrm{rot}\, \boldsymbol{\beta}, p) + \lambda^{-1}(t^2 + \gamma h^2)(\mathrm{rot}\, \boldsymbol{\alpha}, p),$$

and $\mathbf{q}_h = \nabla \psi_h + \boldsymbol{\alpha}_h$.

The analysis again divides into three steps of which essentially only the Stokes part differs from that of the preceding section.

We now define the norms

$$|\,||(\boldsymbol{\beta}, \boldsymbol{\alpha})||\,|_h^2 = ||\boldsymbol{\beta}||_1^2 + (t^2 + h^2)||\boldsymbol{\alpha}||_0^2,$$

$$||p||_h^2 = (t^2 + h^2)\,(\sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e |\,[\![p]\!]\,|^2),$$

where $\mathcal{E}_h$ stands for the collection of element edges in the interior of Ω, $([\![p]\!])_{|e}$ denotes the jump in p along e and h_e is the length of e.

With these norms the first of the stability conditions is always satisfied:

$$A_h(\boldsymbol{\beta}, \boldsymbol{\alpha}; \boldsymbol{\beta}, \boldsymbol{\alpha}) \geq C|\,||(\boldsymbol{\beta}, \boldsymbol{\alpha})||\,|_h^2, \qquad (\boldsymbol{\beta}, \boldsymbol{\alpha}) \in \mathbf{V}_h \times \boldsymbol{\Gamma}_h. \tag{4.10}$$

The second one is then the following.

Lemma 4.1. There is a positive constant C such that

$$\sup_{\substack{(\boldsymbol{\eta}, \boldsymbol{\alpha}) \in \mathbf{V}_h \times \boldsymbol{\Gamma}_h \\ (\boldsymbol{\eta}, \boldsymbol{\alpha}) \neq (\mathbf{0}, \mathbf{0})}} \frac{B_h(\boldsymbol{\eta}, \boldsymbol{\alpha}; q)}{|\,||(\boldsymbol{\eta}, \boldsymbol{\alpha})||\,|_h} \geq C||q||_h, \qquad q \in Q_h. \tag{4.11}$$

Proof: The degrees of freedom for $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}_h$ are now the constant values of $(\boldsymbol{\alpha} \cdot \boldsymbol{\tau})_{|e}$ for every $e \in \mathcal{E}_h$.

Hence, for a given $q \in Q_h$ we can define $\boldsymbol{\alpha} \in \boldsymbol{\Gamma}_h$ through

$$(\boldsymbol{\alpha} \cdot \boldsymbol{\tau})_{|e} = h_e^{-1}([\![p]\!])_{|e}, \qquad e \in \mathcal{E}_h.$$

This gives

$$B_h(\mathbf{0}, \boldsymbol{\alpha}; q) \geq C_1||q||_h^2.$$

Local scaling arguments give

$$|\,\|(\mathbf{0}, \boldsymbol{\alpha})\|\,|_h \leq C_2 \|q\|_h,$$

which proves the claim. □

Let us now complete the

Proof of Theorem 4.1. We directly get

$$\|\psi - \psi_h\|_1 \leq Ch|\psi|_2. \tag{4.12}$$

Next, the stability estimates (4.10) and (4.11) imply the stability of the discrete Stokes system, i.e., there exists $(\boldsymbol{\eta}, \boldsymbol{\delta}, q) \in \mathbf{V}_h \times \boldsymbol{\Gamma}_h \times Q_h$ such that

$$|\,\|(\boldsymbol{\eta}, \boldsymbol{\delta})\|\,|_h + \|q\|_0 \leq C$$

and

$$\begin{aligned} &|\,\|(\boldsymbol{\beta}_h - \boldsymbol{\beta}^I, \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^I)\|\,|_h + \|p_h - P_h p\|_0 \\ &\leq A_h(\boldsymbol{\beta}_h - \boldsymbol{\beta}^I, \boldsymbol{\alpha}_h - \mathbf{R}_h\boldsymbol{\alpha}; \boldsymbol{\eta}, \boldsymbol{\delta}) - B_h(\boldsymbol{\eta}, \boldsymbol{\delta}; p_h - P_h p) \\ &\quad + B_h(\boldsymbol{\beta}_h - \boldsymbol{\beta}^I, \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^I; q), \end{aligned} \tag{4.13}$$

where $\boldsymbol{\beta}^I \in \mathbf{V}_h$ is the interpolant to $\boldsymbol{\beta}$ and $\boldsymbol{\alpha}^I$ is the L^2-projection of $\boldsymbol{\alpha}$.
Using (2.17) and (4.8) we now get

$$\begin{aligned} &A_h(\boldsymbol{\beta}_h - \boldsymbol{\beta}^I, \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^I; \boldsymbol{\eta}; \boldsymbol{\delta}) - B_h(\boldsymbol{\eta}, \boldsymbol{\delta}; p_h - P_h p) + B_h(\boldsymbol{\beta}_h - \boldsymbol{\beta}^I, \boldsymbol{\alpha}_h - \boldsymbol{\alpha}^I; q) \\ &= a(\boldsymbol{\beta} - \boldsymbol{\beta}^I, \boldsymbol{\eta}) - (\mathrm{rot}\,\boldsymbol{\eta}, p - P_h p) + \lambda^{-1} t^2 (\mathrm{rot}(\boldsymbol{\alpha} - \boldsymbol{\alpha}^I), q) \\ &\quad + (\mathrm{rot}\,(\boldsymbol{\beta} - \boldsymbol{\beta}^I), q) + \lambda^{-1}(t^2 + \gamma h^2)(\boldsymbol{\alpha} - \boldsymbol{\alpha}^I, \boldsymbol{\delta}) \\ &\quad - \lambda^{-1}(t^2 + \gamma h^2)(\mathrm{rot}\,\boldsymbol{\delta}, p - P_h p) - \lambda^{-1}\gamma h^2 (\mathrm{rot}\,\boldsymbol{\alpha}^I, q) \\ &\quad + (\nabla\psi_h, \mathbf{R}_h\boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) \\ &= a(\boldsymbol{\beta} - \boldsymbol{\beta}^I, \boldsymbol{\eta}) + \lambda^{-1} t^2 (\mathrm{rot}\,(\boldsymbol{\alpha} - \boldsymbol{\alpha}^I), q) + (\mathrm{rot}\,(\boldsymbol{\beta} - \boldsymbol{\beta}^I), q) \\ &\quad - \lambda^{-1}\gamma h^2 (\mathrm{rot}\,\boldsymbol{\alpha}^I, q) + (\nabla\psi_h, \mathbf{R}_h\boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}). \end{aligned} \tag{4.14}$$

The second and the third term on the right-hand side above can be written as

$$(\mathrm{rot}\,(\boldsymbol{\beta} - \boldsymbol{\beta}^I), q) = \sum_{e \in \mathcal{E}_h} \int_e ((\boldsymbol{\beta} - \boldsymbol{\beta}^I) \cdot \boldsymbol{\tau})([\![q]\!])$$
$$\leq h^{-1} \Big(\sum_{e \in \mathcal{E}_h} h_e \int_e |\boldsymbol{\beta} - \boldsymbol{\beta}^I|^2 \Big)^{1/2} ||q||_h \leq Ch|\boldsymbol{\beta}|_2 \tag{4.15}$$

and

$$h^2(\mathrm{rot}\,\boldsymbol{\alpha}^I, q) = h^2 \sum_{e \in \mathcal{E}_h} \int_e (\boldsymbol{\alpha}^I \cdot \boldsymbol{\tau})([\![q]\!])$$
$$\leq h \Big(\sum_{e \in \mathcal{E}_h} h_e \int_e |\boldsymbol{\alpha}^I|^2 \Big)^{1/2} ||q||_h \leq Ch||\boldsymbol{\alpha}^I||_0 \leq Ch||\boldsymbol{\alpha}||_0. \tag{4.16}$$

Here we used an interpolation estimate and an equivalence of norms, both easily proven by scaling arguments.

Furthermore, we have

$$(\nabla\psi_h, \mathbf{R}_h\boldsymbol{\eta}) - (\nabla\psi, \boldsymbol{\eta}) = (\nabla(\psi_h - \psi), \mathbf{R}_h\boldsymbol{\eta}) + (\nabla\psi, \mathbf{R}_h\boldsymbol{\eta} - \boldsymbol{\eta})$$
$$\leq ||\psi_h - \psi||_1||\mathbf{R}_h\boldsymbol{\eta}||_0 + ||\psi||_1||\boldsymbol{\eta} - \mathbf{R}_h\boldsymbol{\eta}||_0 \leq C(||\psi_h - \psi||_1||\boldsymbol{\eta}||_1 + h||\psi||_1||\boldsymbol{\eta}||_1). \tag{4.17}$$

The rest of the interpolation estimates needed are standard and hence a combination of (4.12)–(4.17) gives

$$||\psi - \psi_h||_1 + ||\boldsymbol{\beta} - \boldsymbol{\beta}_h||_1 + (t + h)||\boldsymbol{\alpha} - \boldsymbol{\alpha}_h||_0 \leq Ch(|\boldsymbol{\beta}|_2 + |\psi|_2 + |\boldsymbol{\alpha}|_0 + t|\boldsymbol{\alpha}|_1).$$

The estimates for $||w - w_h||_1$ is now obtained as in Lemma 3.4. □

Remark. The same extension can, of course, also be made to the family III. This leads to a method with $\mathbf{V}_h$ as in (4.2) and

$$W_h = \{ v \in H_0^1(\Omega) \mid v_{|K} \in P_2(K), K \in C_h \}. \tag{4.18}$$

This discretization is then defined as

$$a(\boldsymbol{\beta}_h, \boldsymbol{\eta}) + \lambda(t^2 + \gamma h^2)^{-1}(\boldsymbol{\beta}_h - \nabla w_h, \boldsymbol{\eta} - \nabla v) = (f, v),$$
$$v \in W_h, \qquad \boldsymbol{\eta} \in \mathbf{V}_h. \tag{4.19}$$

Note that since $\mathbf{V}_h$ is a subspace of the lowest order Brezzi–Douglas–Marini space, the "reduction operator" reduces to the identity, i.e., "full integration" is used.

This method has earlier been proposed by Pitkäranta (Ref. 25, Remark 4.3.). □

Let us finally mention that in Ref. 22 we proposed a similar modification for a recent method by Arnold and Falk.[1]

References

1. D. N. Arnold and R. S. Falk, *A uniformly accurate finite element method for the Reissner–Mindlin plate*, *SIAM J. Numer. Anal.* **26** (1989) 1276–1290.
2. D. N. Arnold and R. S. Falk, *The boundary layer for the Reissner–Mindlin plate model*, *SIAM J. Math. Anal.* **21** (1990) 10–40.
3. D. N. Arnold and R. S. Falk, *The boundary layer for the Reissner–Mindlin plate model: soft simply supported, soft clamped and free plates*, to appear.
4. D. N. Arnold and R. S. Falk, *Edge effects in the Reissner–Mindlin plate theory. Analytical and Computation Models of Shells*; eds. A. K. Noor, T. Belytschko, and J. C. Simo, (ASME, 1989), pp. 71–89.
5. K. J. Bathe and F. Brezzi, *On the convergence of a four node plate bending element based on Reissner–Mindlin theory*, *The Mathematics of Finite Elements and Applications*, ed. J. R. Whiteman (Academic Press, 1985), pp. 491–503.
6. K. J. Bathe and F. Brezzi, *A simplified analysis of two plate bending elements — The MITC4 and MITC9 elements*, in *NUMETA 87 Numerical Techniques for Engineering Analysis and Design*, eds. G. N. Pande and J. Middleton (Martinus Nijhoff, 1987), Vol. 1, D46/1.
7. K. J. Bathe, F. Brezzi, and M. Fortin, *Mixed-interpolated elements for Reissner–Mindlin plates*, *Internat. J. Numer. Methods. Engrg.* **28** (1989) 1787–1801.
8. K. J. Bathe and E. Dvorkin, *A four node plate bending element based on Mindlin–Reissner plate theory and mixed interpolation*, *Internat. J. Numer. Methods. Engrg.* **21** (1985) 367–383.
9. K. J. Bathe and E. Dvorkin, *A formulation of general shell elements — The use of mixed interpolation of tensorial components*, *Internat. J. Numer. Methods. Engrg.* **22** (1986) 697–722.
10. F. Brezzi, *On the existence, uniqueness and approximation of saddle point problems arising from Lagrangian multipliers*, *RAIRO Modél Math. Anal. Numér.* **2** (1974) 129–151.
11. F. Brezzi, J. Douglas, Jr., and L. D. Marini, *Two families of mixed finite elements for second order elliptic problems*, *Numer. Math.* **47** (1985) 217–235.
12. F. Brezzi, J. Douglas, Jr., M. Fortin, and L. D. Marini, *Efficient rectangular mixed finite elements in two and three space variables*, *RAIRO Modél Math. Anal. Numér.* **21** (1987) 237–250.
13. F. Brezzi and M. Fortin, *Analysis of some low-order finite element schemes for Mindlin–Reissner plates*, *Math. Comp.* **47** (1986) 151–158.
14. F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, to appear.
15. P. G. Giarlet, *The Finite Element Method for Elliptic Problems* (North-Holland, 1978).
16. P. Clemént, *Approximation of finite element functions using local regularization*, *RAIRO Modél Math. Anal. Numér.* **9** (1975) 77–84.
17. M. Crouzeix and P. A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations*, *RAIRO Modél Math. Anal. Math. Numér.* **3** (1973) 33–76.
18. J. Douglas, Jr. and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, *Math. Comp.* **44** (1985) 39–52.

19. M. Fortin, *Old and new finite elements for incompressible flows*, *Internat. J. Numer. Methods. in Fluids* **1** (1981) 347–364.
20. T. J. R. Hughes and L. P. Franca, *A new finite element formulation for computational fluid dynamics: VII. The Stokes problem with various well-posed boundary conditions: Symmetric formulations that converge for all velocity/pressure spaces*, *Comp. Meths. Appl. Mech. Engrg.* **65** (1987) 85–96.
21. T. J. R. Hughes and L. P. Franca, *A mixed finite element formulation for Reissner– Mindlin plate theory: Uniform convergence of all higher-order spaces*, *Comput. Methods Appl. Mech. Engrg.* **67** (1988) 223–240.
22. L. P. Franca and R. Stenberg, *A modification of a low-order Reissner–Mindlin plate bending element*, to appear in *The Mathematics of Finite Elements and Applications VII*, ed. J. R. Whiteman (Academic Press, 1990).
23. V. Girault and P. A. Raviart, *Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms* (Springer, 1986).
24. A. F. D. Loula, T. J. R. Hughes, L. P. Franca, and I. Miranda, *Mixed Petrov-Galerkin methods for the Timoshenko beam*, *Comput. Methods Appl. Mech. Engrg.* **63** (1987) 133–154.
25. J. Pitkäranta, *Analysis of some low-order finite element schemes for Mindlin–Reissner and Kirchhoff plates*, *Numer. Math.* **53** (1988) 237–254.
26. J. Pitkäranta and R. Stenberg, *Error bounds for the approximation of Stokes problem with bilinear/constant elements on irregular quadrilateral meshes. The Mathematics of Finite Elements and Applications V*, ed. J. R. Whiteman (Academic Press, 1985), pp. 325–334.
27. P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems*, *Mathematical Aspects of the Finite Element Method*, Lecture Notes in Math. (Springer, 1977), vol. 606 pp. 292–315.
28. R. Stenberg, *Error analysis of some finite element methods for the Stokes problem*, *Math. Comp.* **54** (1990) 495–508.
29. L. R. Scott and M. Vogelius, *Norm estimates for a maximal right inverse of the divergence operator in spaces of piecewise polynomials. RAIRO Modél Math. Anal. Numér.* **19** (1985) 111–143.