MATHEMATICAL MODELLING AND NUMERICAL ANALYSIS
MODÉLISATION MATHÉMATIQUE ET ANALYSE NUMÉRIQUE

(Vol. 25, n° 1, 1991, p.151 à 168)

POSTPROCESSING SCHEMES FOR SOME MIXED FINITE ELEMENTS (*)

Rolf STENBERG ([1])

Communicated by J. DOUGLAS

Abstract. — *We consider some mixed finite element methods for scalar second and fourth order elliptic equations. For these methods we introduce and analyze some new postprocessing schemes. It is shown that by a simple postprocessing, performed separately on each element, one can obtain a considerably better approximation for the scalar variable than the original one.*

Résumé. — *Nous considérons quelques méthodes d'éléments finis mixtes pour des équations aux dérivées partielles scalaires, elliptiques, du second ou du quatrième ordre. Pour ces méthodes, nous introduisons et analysons quelques techniques nouvelles de postraitement. On montre qu'un postraitement simple, effectué séparément sur chaque élément, permet d'obtenir une approximation bien meilleure sur la variable scalaire.*

1. INTRODUCTION

The purpose of this note is to discuss some mixed finite element approximations of two model problems ; the Poisson equation and the biharmonic equation. For some problems of these types, mixed methods have been applied with considerably success.

Equations for which the Poisson equation can be taken as a prototype arise in some geophysical problems (*cf.* e.g. [7, 18] and the references therein) and problems in semiconductor physics [13], and for these two applications very good results have been obtained with the mixed methods of the Raviart-Thomas-Nedelec (RTN) [14, 15] and Brezzi-Douglas-Marini (BDM) [2, 4] families.

The standard model problem for fourth order elliptic equations is the biharmonic equation which arise as the equation for the deflection of a thin

(*) Received April 1988.

([1]) INRIA, B.P. 105, 78153 Le Chesnay, France and (present address) Faculty of Mechanical Engineering, Helsinki University of Technology, 02150 Espoo, Finland.

elastic plate. The other main application of the biharmonic problem is the stream function formulation of Stokes and Navier-Stokes equations. For the approximate solution of the biharmonic equation some mixed methods were among the very first successful finite element methods introduced [10, 11].

In some recent papers F. Brezzi and co-workers [1, 4] discussed some mixed methods for the afore-mentioned problems. They considered a technique of implementing the methods where Lagrange multipliers are utilized in order to impose interelement continuity of some of the variables. The advantage of this technique is that by using local condensation techniques the final linear system to be solved is positive definite. In addition, they showed that this new Lagrange multiplier can be exploited in some postprocessing methods for producing better approximations for some of the original variables.

In [6] a similar postprocessing method for the Hellan-Herrmann-Johnson (HHJ) family [10, 11, 12] for approximating the biharmonic equation was developed.

In this paper we will first introduce an alternative to the postprocessing methods of [1, 4] for the BDM family. Then we will develop an analog postprocessing procedure for the HHJ methods. Our postprocessing approach is rather general (and natural) ; it can be used for all methods in the RTN, BDM and HHJ families. In addition, it does not require that the methods have been implemented by the Lagrange multiplier technique of [1]. In [17] we introduced the corresponding postprocessing scheme for some mixed methods for the linear elasticity problem.

Our exposition will be rather brief, since most of the estimates we will need for our analysis are found in [2, 3, 4, 8]. Our notation will be the established one, *cf.* [5]. For the specific mixed methods we will mainly use the same notation as in [2, 3, 4, 8].

2. SECOND ORDER ELLIPTIC PROBLEMS

Consider as the model problem the Poisson equation with non-homogeneous Dirichlet boundary conditions :

$$\begin{aligned} -\Delta u &= f \quad \text{in } \Omega\,, \\ u &= u_0 \quad \text{on } \Gamma\,, \end{aligned} \tag{2.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$, $N = 2, 3$, which for simplicity is assumed to have a polygonal or polyhedral boundary Γ.

For the mixed approximation the equation is first written as an elliptic system :

$$\begin{aligned} \mathbf{q} + \mathbf{grad}\, u &= 0 \quad \text{in } \Omega\,, \\ \operatorname{div} \mathbf{q} &= f \quad \text{in } \Omega\,, \\ u &= u_0 \quad \text{on } \Gamma\,. \end{aligned} \tag{2.2}$$

Next, one introduces the variational formulation

$$\begin{aligned} (\mathbf{q}, \mathbf{p}) - (\operatorname{div} \mathbf{p}, u) &= - \langle u_0, \mathbf{p} \cdot \mathbf{n} \rangle, \quad \mathbf{p} \in \mathbf{H}, \\ (\operatorname{div} \mathbf{q}, v) &= (f, v), \quad v \in V, \end{aligned} \tag{2.3}$$

and then the finite element method

$$\begin{aligned} (\mathbf{q}_h, \mathbf{p}) - (\operatorname{div} \mathbf{p}, u_h) &= - \langle u_0, \mathbf{p} \cdot \mathbf{n} \rangle, \quad \mathbf{p} \in \mathbf{H}_h \subset \mathbf{H}, \\ (\operatorname{div} \mathbf{q}_h, v) &= (f, v), \quad v \in V_h \subset V. \end{aligned} \tag{2.4}$$

Above we have used the notation

$$\mathbf{H} = \mathbf{H}(\operatorname{div}; \Omega) = \{ \mathbf{p} \in [L^2(\Omega)]^N \mid \operatorname{div} \mathbf{p} \in L^2(\Omega) \},$$

$$V = L^2(\Omega), \quad (u, v) = \int_\Omega uv \, dx,$$

$$(\mathbf{p}, \mathbf{q}) = \int_\Omega \mathbf{p} \cdot \mathbf{q} \, dx, \quad \langle u, v \rangle = \int_\Gamma uv \, ds.$$

$\mathbf{n}$ stands for the unit outward normal to Γ.

For clarity of exposition we will perform our analysis for the triangular or tetrahedral BDM family. The extension to the other mixed methods of [2, 4, 14, 15] is trivial. Hence, we let $\mathcal{T}_h$ be a regular partitioning of $\bar{\Omega}$ into closed triangles or tetrahedrons and define the finite element spaces as [2, 4]

$$\mathbf{H}_h = \{ \mathbf{p} \in \mathbf{H} \mid \mathbf{p}|_T \in [P_k(T)]^N, \ T \in \mathcal{T}_h \}, \tag{2.5a}$$

$$V_h = \{ u \in V \mid u|_T \in P_{k-1}(T), \ T \in \mathcal{T}_h \}, \tag{2.5b}$$

where $P_l(T)$, $l = k, k-1$, $l \geq 0$, denotes the polynomials of degree l on T.

In [2, 4] quasioptimal error estimates have been derived for the above method. The analysis of [2, 4] relies on the existence of two special interpolation operators $\Pi_h : \mathbf{H} \to \mathbf{H}_h$ and $P_h : V \to V_h$. Here we only recall the properties of P_h:

$$(\operatorname{div} \mathbf{p}, u - P_h u) = 0, \quad \mathbf{p} \in \mathbf{H}_h, \quad u \in V, \tag{2.6}$$

and

$$\| u - P_h u \|_0 \leq C h^r |u|_r \quad \text{if} \quad u \in H^r(\Omega) \quad \text{for} \quad 0 \leq r \leq k. \tag{2.7}$$

For the finite element spaces at hand the operator P_h clearly coincides with the L^2-projection from V onto V_h.

Let us also remark that the analysis can be performed without the explicit construction of the operator Π_h. This is easily seen from the following line of arguments : consider, for a given index k, the pair $(\mathbf{H}_h, V_h)$ as defined in (2.5). Then there is a corresponding method $(\tilde{\mathbf{H}}_h, \tilde{V}_h)$ in the RTN-family such that $\tilde{V}_h = V_h$ and $\tilde{\mathbf{H}}_h \subset \mathbf{H}_h$ [14, 15]. Now, it is well known that the pair $(\tilde{\mathbf{H}}_h, \tilde{V}_h)$ is stable, i.e. it satisfies the Babuška-Brezzi condition with an apropriate choice of norms, e.g. the mesh dependent ones introduced in [16]. Hence, the pair $(\mathbf{H}_h, V_h)$ is also stable with respect to the same norms and as a consequence one can perform an error analysis as in [16, Theorem 3.1]. Recalling the mesh dependent norm $\|\cdot\|_{0,h}$ as defined in [16]

$$\|\mathbf{q}\|_{0,h}^2 = \|\mathbf{q}\|_0^2 + \sum_{T \in \mathcal{T}_h} h_T \int_{\partial T} |\mathbf{q} \cdot \mathbf{n}|^2 \, ds$$

for

$$\mathbf{q} \in \{\mathbf{p} \in \mathbf{H} \mid \mathbf{p} \cdot \mathbf{n} \in L^2(\partial T), T \in \mathcal{T}_h\}$$

the error estimates obtained are the following.

THEOREM 2.1 : *Suppose that the solution of* (2.1) *satisfies* $u \in H^r(\Omega)$ *with* $r > 3/2$. *Then we have*

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,h} \leq Ch^s |\mathbf{q}|_s, \quad s = \min\{r-1, k+1\}, \tag{2.8}$$

and

$$\|u - u_h\|_0 \leq Ch^l(|\mathbf{q}|_l + |u|_l), \quad l = \min\{r-1, k\}. \tag{2.9}$$

For a convex region Ω *we have*

$$\|u - u_h\|_0 \leq Ch^l(|\mathbf{q}|_{l-1} + |u|_l), \quad l = \min\{r, k\}, \tag{2.10}$$

and

$$\|u_h - P_h u\|_0 \leq \begin{cases} Ch^{s+1}|\mathbf{q}|_s, \quad s = \min\{r-1, k+1\} \quad \textit{for} \quad k \geq 2, & (2.11) \\ Ch^2 |\mathbf{q}|_2 \quad \textit{for} \quad k = 1. & (2.12) \end{cases}$$

If we in addition have $f \in V_h$, *then the estimate* (2.11) *also holds for* $k = 1$.

Proof: All the above estimates except the last result are essentially derived in [2, 4].

Hence, let us prove that (2.11) is also valid for $k = 1$ when $f \in V_h$. To this end, let $(\mathbf{z}, w) \in \mathbf{H} \times V$ be the solution to

$$\begin{aligned} (\mathbf{z}, \mathbf{p}) - (\operatorname{div} \mathbf{p}, w) &= 0, & \mathbf{p} \in \mathbf{H}, \\ (\operatorname{div} \mathbf{z}, v) &= (u_h - P_h u, v), & v \in V. \end{aligned} \tag{2.13}$$

Due to the convexity of Ω we have

$$\|\mathbf{z}\|_1 + \|w\|_2 \leq C \|u_h - P_h u\|_0 . \tag{2.14}$$

Now, let $(\mathbf{z}_h, w_h)$ be the mixed finite element approximation to (2.13). By choosing $v = u_h - P_h u$, $\mathbf{p} = \mathbf{q} - \mathbf{q}_h$ in (2.13) we obtain in the usual manner

$$\begin{aligned}
\|u_h - P_h u\|_0^2 &= (\operatorname{div} \mathbf{z}, u_h - P_h u) + (\mathbf{z}, \mathbf{q} - \mathbf{q}_h) \\
&\quad - (\operatorname{div} (\mathbf{q} - \mathbf{q}_h), w) - (\mathbf{q} - \mathbf{q}_h, \mathbf{z}_h) + (\operatorname{div} \mathbf{z}_h, u - u_h) \\
&\quad + (\operatorname{div} (\mathbf{q} - \mathbf{q}_h), P_h w) \\
&= (\mathbf{z} - \mathbf{z}_h, \mathbf{q} - \mathbf{q}_h) - (\operatorname{div} (\mathbf{q} - \mathbf{q}_h), w - P_h w) \\
&\quad + (\operatorname{div} (\mathbf{z} - \mathbf{z}_h), u_h - P_h u) + (\operatorname{div} \mathbf{z}_h, u - P_h u) .
\end{aligned}$$

Now, the last two terms above vanish by virtue of (2.6) and the definition of $\mathbf{z}_h$. Next, consider the term $(\operatorname{div} (\mathbf{q} - \mathbf{q}_h), w - P_h w)$. Since we assume that $\operatorname{div} \mathbf{q} = f \in V_h$, we note that also this term vanishes. Using (2.14) and (2.8) we thus obtain

$$\begin{aligned}
\|u_h - P_h u\|_0^2 &= (\mathbf{z} - \mathbf{z}_h, \mathbf{q} - \mathbf{q}_h) \leq \|\mathbf{z} - \mathbf{z}_h\|_0 \|\mathbf{q} - \mathbf{q}_h\|_0 \\
&\leq Ch |\mathbf{z}|_1 \|\mathbf{q} - \mathbf{q}_h\|_0 \leq Ch \|u_h - P_h u\|_0 \|\mathbf{q} - \mathbf{q}_h\|_0
\end{aligned}$$

which together with (2.8) proves the assertion. ■

Remark 2.1 : For the lowest order method in the RTN family the assumption $f \in V_h$ yields the estimate

$$\|u_h - P_h u\|_0 \leq Ch^2 |\mathbf{q}|_1 .$$

The estimate one gets without this assumption is [1]

$$\|u_h - P_h u\|_0 \leq Ch^2 \|u\|_3 .$$

Hence, by assuming $f \in V_h$ the maximal convergence rate is not improved, but the regularity requirement on the exact solution is relaxed. ■

Remark 2.2 : The assumption $f \in V_h$ does not seem to be a severe restriction since in practice we often have $f = 0$. Also, if $f \notin V_h$ it is often possible to find a vector field $\mathbf{q}_0$ such that $\operatorname{div} \mathbf{q}_0 = f$. Then one can use the mixed method to approximate $\mathbf{q} - \mathbf{q}_0$. ■

Remark 2.3 : In the case when one can neither assume that $f \in V_h$ nor find a field $\mathbf{q}_0$ with $\operatorname{div} \mathbf{q}_0 = f$, the lowest order method can be modified with the technique elaborated in [16] : each $T \in \mathcal{C}_h$ is subdivided into N subtriangles or subtetrahedrons by adjoining the center of gravity of T with the vertices. Let $\mathcal{T}_{h/2}$ be the finer triangulation so obtained. The modified method is then defined as

$$\mathbf{H}_h = \left\{\mathbf{p} \in \mathbf{H} \mid \mathbf{p}\,|_T \in [C(T)]^N, T \in \mathcal{T}_h, \mathbf{p}\,|_K \in [P_1(K)]^N, K \in \mathcal{T}_{h/2}\right\}$$

$$V_h = \left\{u \in V \mid u\,|_T \in P_1(T), T \in \mathcal{T}_h\right\}.$$

This method is easily proved to be stable and to satisfy the « equilibrium condition » which implies the existence of a projection operator P_h with the properties (2.6) and (2.7). Hence one obtains the error estimates

$$\|\mathbf{q} - \mathbf{q}_h\|_{0,h} \le C h^s |\mathbf{q}|_s, \quad s = \min\{r-1, 2\},$$

$$\|u - u_h\|_0 \le C h^l (|\mathbf{q}|_l + |u|_l), \quad l = \min\{r-1, 2\}.$$

For a convex region Ω we get

$$\|u - u_h\|_0 \le C h^l (|\mathbf{q}|_{l-1} + |u|_l), \quad l = \min\{r, 2\}$$

and, in particular,

$$\|u_h - P_h u\|_0 \le C h^{s+1} |\mathbf{q}|_s, \quad s = \min\{r-1, 2\}.$$

This modified method does not seem to be substantially more costly to implement than the original lowest order BDM method, since when implemented e.g. as suggested in [1] the size of the linear system to solve is not increased. ■

Let us now define the

POSTPROCESSING METHOD

Let

$$V_h^* = \left\{v \in L^2(\Omega) \mid v\,|_T \in P_{k+1}(T), T \in \mathcal{T}_h\right\}$$

and define the approximation $u_h^* \in V_h^*$ to u separately on each $T \in \mathcal{T}_h$ as the solution to the system

$$\int_T \mathbf{grad}\, u_h^* \cdot \mathbf{grad}\, v \, dx = \int_T f v \, dx + \int_{\partial T} \mathbf{q}_h \cdot \mathbf{n} v \, ds \qquad \forall v \in (I - Q_T)\, V_h^*|_T, \quad (2.16a)$$

$$Q_T u_h^* = Q_T u_h, \quad (2.16b)$$

where either $Q_T = P_h|_T$ or Q_T is the L^2-projection from $L^2(T)$ onto $P_0(T)$. ■

For this new approximation we obtain the following error estimate.

THEOREM 2.2 : *If $u \in H^r(\Omega)$, $r > 3/2$, and Ω is convex, then we have*

$$\|u - u_h^*\| \leqslant \begin{cases} C h^{s+1}(|\mathbf{q}|_s + |u|_{s+1}), & s = \min\{r-1, k+1\}, \quad for \quad k \geqslant 2, \quad (2.17) \\ C h^2(|\mathbf{q}|_2 + |u|_2), & for \quad k = 1. \quad (2.18) \end{cases}$$

If we in addition have $f \in V_h$, then (2.17) is also valid for $k = 1$.

Proof : Let $\tilde{u} \in V_h^*$ be the L^2-projection of u and define $v \in V_h^*$ through $v|_T = (I - Q_T)(\tilde{u} - u_h^*)$ for each $T \in \mathcal{T}_h$.

We now write

$$\begin{aligned} |v|_{1,T}^2 &= \int_T \mathbf{grad}\ ((I - Q_T)(\tilde{u} - u_h^*)) \cdot \mathbf{grad}\ v\ dx \\ &= \int_T \mathbf{grad}\ (\tilde{u} - u_h^*) \cdot \mathbf{grad}\ v\ dx - \int_T \mathbf{grad}\ (Q_T(\tilde{u} - u_h^*)) \cdot \mathbf{grad}\ v\ dx . \end{aligned} \tag{2.19}$$

Next, using (2.16*a*) we obtain

$$\begin{aligned} \int_T \mathbf{grad}\ (\tilde{u} - u_h^*) \cdot \mathbf{grad}\ v\ dx &= \int_T \mathbf{grad}\ (\tilde{u} - u) \cdot \mathbf{grad}\ v\ dx + \\ &\quad + \int_{\partial T} (\mathbf{q} \cdot \mathbf{n} - \mathbf{q}_h \cdot \mathbf{n})\ v\ ds \qquad (2.20) \\ &\leqslant |u - \tilde{u}|_{1,T} |v|_{1,T} + h_T^{1/2} \|\mathbf{q} \cdot \mathbf{n} - \mathbf{q}_h \cdot \mathbf{n}\|_{0,\partial T} \cdot h_T^{-1/2} \|v\|_{0,\partial T} . \end{aligned}$$

By scaling and the fact that $(I - Q_T) w = 0$ if $w \in P_0(T)$, we get

$$h_T^{-1/2} \|v\|_{0,\partial T} \leqslant C |v|_{1,T} \tag{2.21}$$

and

$$\|v\|_{0,T} \leqslant C h_T |v|_{1,T} . \tag{2.22}$$

Combining (2.19)-(2.21) gives

$$\begin{aligned} |v|_{1,T} \leqslant |u - \tilde{u}|_{1,T} &+ h_T^{1/2} \|\mathbf{q} \cdot \mathbf{n} - \mathbf{q}_h \cdot \mathbf{n}\|_{0,\partial T} \\ &+ |Q_T(\tilde{u} - u_h^*)|_{1,T} . \end{aligned} \tag{2.23}$$

Hence, (2.22) and the inverse estimate

$$|Q_T(\tilde{u} - u_h^*)|_{1,T} \leqslant C h_T^{-1} \|Q_T(\tilde{u} - u_h^*)\|_{0,T}$$

give

$$\|(I - Q_T)(\tilde{u} - u_h^*)\|_{0,T} = \|v\|_{0,T} \leq$$

$$C h_T(|u - \tilde{u}|_{1,T} + h_T^{1/2}\|\mathbf{q}\cdot\mathbf{n} - \mathbf{q}_h\cdot\mathbf{n}\|_{0,\partial T}) + \|Q_T(\tilde{u} - u_h^*)\|_{0,T}. \quad (2.24)$$

A squaring and summation over all $T \in \mathcal{C}_h$ yields

$$\|(I - Q_T)(\tilde{u} - u_h^*)\|_0 \leq$$

$$C h_T\left\{\left(\sum_{T\in\mathcal{C}_h} |u - \tilde{u}|_{1,T}^2\right)^{1/2} + \|\mathbf{q}\cdot\mathbf{n} - \mathbf{q}_h\cdot\mathbf{n}\|_{0,h}\right\} + \|Q_T(\tilde{u} - u_h^*)\|_0. \quad (2.25)$$

By the definition(s) of Q_T and (2.16*b*) we have

$$\|Q_T(\tilde{u} - u_h^*)\|_0 = \|Q_T(P_h u - u_h)\|_0 \leq \|P_h u - u_h\|_0. \quad (2.26)$$

Hence, the final estimates follow from (2.25), (2.26) and the estimates of Theorem 2.1. ■

Remark 2.4 : The estimate one gets for a nonconvex domain Ω, is

$$\|u - u_h^*\|_0 \leq C h^s(|u|_s + |q|_s), \quad s = \min\{r - 1, k + 1\},$$

and this estimate is also valid for $k = 1$. ■

3. THE BIHARMONIC EQUATION

In this section we will introduce and analyze a postprocessing scheme for the HHJ family for approximating the biharmonic equation.

In the presentation we will have the application to the plate bending problem in mind (for an account of the application of the method for the Stokes and Navier-Stokes equations we refer to [9]). Hence we consider the problem

$$\begin{aligned} D\,\Delta^2\psi &= g \quad \text{in} \quad \Omega \subset \mathbf{R}^2, \\ \psi = \frac{\partial\psi}{\partial\nu} &= 0 \quad \text{on} \quad \Gamma. \end{aligned} \quad (3.1)$$

Here ψ denotes the deflection of a thin plate due to the transverse loading g. D denotes the bending stiffness of the plate :

$$D = \frac{Ed^3}{12(1 - \sigma^2)},$$

where d, E, σ are the thickness of the plate, the Young modulus and the Poisson ratio, respectively. The unit outward normal to Γ is in this section denoted by $\nu = (\nu_1, \nu_2)$.

For simplicity we will assume that the boundary Γ is polygonal and that the plate is clamped along Γ.

If $g \in H^{-2}(\Omega)$ then there is a unique solution $\psi \in H_0^2(\Omega)$ to (3.1). It is also well known that the regularity of the solution ψ depends on the singularities arising at the corners of Ω. For instance, if all interior angles of Ω are less or equal to π, i.e. if Ω is convex, then we have

$$\|\psi\|_3 \leq C \|g\|_{-1}, \tag{3.2}$$

provided that $g \in H^{-1}(\Omega)$. In the sequel we will assume that Ω is convex so that this estimate is valid. For the estimates for the lowest order method we in addition have to assume that $g \in L^2(\Omega)$.

In the HHJ method one does not directly approximate (3.1). Instead (3.1) is written as the system

$$\begin{gathered}
\frac{1}{D(1-\sigma^2)} \{(1+\sigma) u_{ij} - \sigma\delta_{ij}(u_{11} + u_{22})\} + \frac{\partial^2 \psi}{\partial x_i\, \partial x_j} = 0\,, \qquad i,j = 1,2\,, \quad \text{in } \Omega\,, \\
\sum_{i,j=1}^{2} \frac{\partial^2 u_{ij}}{\partial x_i\, \partial x_j} + g = 0 \quad \text{in } \Omega\,, \\
\psi = \frac{\partial \psi}{\partial \nu} = 0 \quad \text{on } \Gamma\,.
\end{gathered} \tag{3.3}$$

Here the symmetric tensor $\mathbf{u} = \{u_{ij}\}$, $i,j = 1, 2$, has the physical meaning of bending moments.

The variational formulation of (3.3), upon which the finite element method is based, can be stated in different ways ; *cf.* [3, 8]. They all, however, lead to the same discretization and hence we will turn directly to that. For the index k, $k \geq 1$, and for a regular triangular paritioning $\mathcal{T}_h$, the finite element spaces are defined through

$$W_h = \left\{\psi \in H_0^1(\Omega) \mid \psi|_T \in P_k(T), T \in \mathcal{T}_h\right\}, \tag{3.4a}$$

and

$$V_h = \{\mathbf{u} \in \mathcal{V}_h \mid u_{ij}|_T \in P_{k-1}(T), i,j = 1, 2, T \in \mathcal{T}_h\}, \tag{3.4b}$$

where

$$\mathcal{V}_h = \left\{\mathbf{u} \in [L^2(\Omega)]^{2\times 2} \mid u_{12} = u_{21}, u_{ij}|_T \in H^1(T), i,j = 1, 2,\right.$$

$$\left. T \in \mathcal{T}_h,\ M_\nu(\mathbf{u}) \text{ is continuous across interelement boundaries}\right\}.$$

Above and in the sequel we denote

$$M_\nu(\mathbf{u}) = \sum_{i,j=1}^{2} u_{ij}\, \nu_i\, \nu_j$$

and

$$M_{\nu\tau}(\mathbf{u}) = \sum_{i,j=1}^{2} u_{ij}\, \nu_i\, \tau_j\,,$$

where $\tau = (\tau_1, \tau_2) = (\nu_2, -\nu_1)$ is the unit tangent to ∂T for $T \in \mathcal{C}_h$. The approximate method is now defined as follows : find $(\psi_h, \mathbf{u}_h) \in W_h \times V_h$ such that

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, \psi_h) &= 0\,, \quad \mathbf{v} \in V_h\,, \\ b(\mathbf{u}_h, \varphi) + (g, \varphi) &= 0\,, \quad \varphi \in W_h\,, \end{aligned} \tag{3.5}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \frac{1}{D(1-\sigma^2)} \sum_{i,j=1}^{2} \int_\Omega \{(1+\sigma)\, u_{ij} - \sigma\delta_{ij}(u_{11} + u_{22})\}\, v_{ij}\, dx\,,$$

$$b(\mathbf{u}, \varphi) = \sum_{T \in \mathcal{C}_h} \left\{ - \sum_{i,j=1}^{2} \int_T \frac{\partial u_{ij}}{\partial x_j} \frac{\partial \varphi}{\partial x_i}\, dx + \int_{\partial T} M_{\nu\tau}(\mathbf{u}) \frac{\partial \varphi}{\partial \tau}\, ds \right\}$$

and

$$(g, \varphi) = \int_\Omega g\, \varphi\, dx\,.$$

For the error analysis of the method we refer directly to the papers [3] and [8].

The analysis of [3, 8] relies on two special interpolation operators $\Sigma_h : H^2(\Omega) \to W_h$ and $\Pi_h : \mathcal{V}_h \to V_h$. For the analysis of our postprocessing scheme we will need the properties of Σ_h and therefore we recall its definition. For $\psi \in H^2(\Omega)$ given, $\Sigma_h\, \psi$ is defined through

$$\int_T (\psi - \Sigma_h\, \psi)\, q\, dx = 0\,, \quad \forall q \in P_{k-3}(T) \quad \text{and} \quad \forall T \in \mathcal{C}_h\,, \tag{3.6a}$$

$$\int_{T'} (\psi - \Sigma_h\, \psi)\, q\, ds = 0\,, \quad \forall q \in P_{k-2}(T') \quad \text{and} \quad \forall T' \in I_h\,, \tag{3.6b}$$

$$(\psi - \Sigma_h\, \psi)(a) = 0 \quad \forall a \in J_h\,, \tag{3.6c}$$

where I_h and J_h are the sets of all sides and vertices of $\mathcal{C}_h$, respectively.

Σ_h has the following properties for $\psi \in H^r(\Omega)$, $r \geq 2$,

$$b(\mathbf{v}, \psi - \Sigma_h\, \psi) = 0\,, \quad \mathbf{v} \in V_h\,, \tag{3.7}$$

and

$$\|\psi - \Sigma_h \psi\|_j \leqslant C h^{l-j} \|\psi\|_l \quad \text{for} \quad j = 0, 1 \quad \text{and} \quad l = \min\{r, k+1\}. \tag{3.8}$$

In this section the mesh dependent norm $\|\cdot\|_{0,h}$ is defined through

$$\|\mathbf{v}\|_{0,h}^2 = \sum_{i,j=1}^{2} \left\{ \|v_{ij}\|_0^2 + \sum_{T \in \mathcal{T}_h} h_T \int_{\partial T} |v_{ij}|^2 \, ds \right\}. \tag{3.9}$$

Since only the component $M_\nu(\mathbf{v})$ is assumed to be continuous along interelement boundaries, $v_{ij}|_{\partial T}$ is here defined as the limit of v_{ij} when approaching ∂T from the interior of T.

Note that the definition of the norm $\|\cdot\|_{0,h}$ is now slightly different from that given in [3]. However, one easily checks that the following estimates still hold. For some of the estimates for the lowest order method we now need the assumption $g \in L^2(\Omega)$.

THEOREM 3.1 : *Suppose that the solution of* (3.1) *satisfies* $\psi \in H^r(\Omega)$ *with* $r \geqslant 3$. *Then we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,h} \leqslant C h^{\delta} \|\psi\|_{\delta+2} \quad \textit{where} \quad \delta = \min\{r-2, k\}, \tag{3.10}$$

$$\|\psi - \psi_h\|_1 \leqslant \begin{cases} C h^{s-1} \|\psi\|_s & \textit{for} \quad k \geqslant 2 \quad \textit{where} \quad s = \min\{r, k+1\}, \\ C h \|\psi\|_3 & \textit{for} \quad k = 1, \end{cases} \tag{3.11}$$

and

$$\|\psi - \psi_h\|_0 \leqslant \begin{cases} C h^{\bar{s}} \|\psi\|_{\bar{s}+1} & \textit{for} \quad k \geqslant 2 \quad \textit{where} \quad \bar{s} = \min\{r-1, k+1\} \\ C h^2 (\|\psi\|_3 + \|g\|_0) & \textit{for} \quad k = 1. \quad \blacksquare \end{cases} \tag{3.12}$$

For the analysis of our postprocessing method we will need an additional estimate which can be derived by adapting the arguments given in [3] and using the property (3.7) of Σ_h, cf. [6, THEOREM 4.2].

LEMMA 3.1 : *For* $\psi \in H^r(\Omega)$, $r \geqslant 3$, *we have*

$$\|\psi_h - \Sigma_h \psi\|_1 \leqslant \begin{cases} C h^{s} \|\psi\|_{s+1} & \textit{for} \quad k \geqslant 2, \quad \textit{where} \quad s = \min\{r-1, k+1\}, \\ C h^2 (\|\psi\|_3 + \|g\|_0) & \textit{for} \quad k = 1. \quad \blacksquare \end{cases} \tag{3.13}$$

Before introducing our postprocessing scheme we recall the that the normal shear force along an edge $T' \in I_h$ is given by

$$Q_\nu = -D\,\frac{\partial\,\Delta\psi}{\partial\nu} = \frac{1}{(1+\sigma)}\,\frac{\partial}{\partial\nu}\,(u_{11}+u_{22}) = Q_\nu(\mathbf{u})\,. \tag{3.14}$$

Hence, from the finite element solution $(\mathbf{u}_h, \psi_h)$ we can calculate an approximation to the shear force

$$Q_\nu(\mathbf{u}_h) = \frac{1}{(1+\sigma)}\,\frac{\partial}{\partial\nu}\,(u_{11,h}+u_{22,h})\,. \tag{3.15}$$

(Note that for the lowest order method this « approximation » vanishes.)
Now, let us define our

POSTPROCESSING PROCEDURE

Let

$$W_h^* = \{\varphi \in L^2(\Omega) \mid \varphi|_T \in P_{k+1}(T),\, T \in \mathcal{C}_h\}$$

and

$$A_T(\psi,\varphi) = $$
$$= D\int_T \left\{\Delta\psi\,\Delta\varphi - (1-\sigma)\left(\frac{\partial^2\psi}{\partial x_1^2}\frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial x_2^2}\frac{\partial^2\varphi}{\partial x_1^2} - 2\,\frac{\partial^2\psi}{\partial x_1\,\partial x_2}\,\frac{\partial^2\varphi}{\partial x_1\,\partial x_2}\right)\right\}dx\,.$$

The improved approximation $\psi_h^* \in W_h^*$ to ψ is now calculated separately on each $T \in \mathcal{C}_h$ through the conditions

$$\psi_h^*(a) = \psi_h(a)\,, \quad \forall a \in J_h \cap T\,, \tag{3.16a}$$

$$A_T(\psi_h^*,\varphi) = \int_T g\varphi\,dx + $$
$$+ \int_{\partial T}\left\{M_\nu(\mathbf{u}_h)\,\frac{\partial\varphi}{\partial\nu} + M_{\nu\tau}(\mathbf{u}_h)\,\frac{\partial\varphi}{\partial\tau} - Q_\nu(\mathbf{u}_h)\,\varphi\right\}ds\,, \tag{3.16b}$$

$$\forall\varphi \in W_h^*|_T \quad \textit{with} \quad \varphi(a) = 0 \quad \textit{for} \quad a \in J_h \cap T\,. \quad \blacksquare$$

Remark 3.1 : Since φ in (3.16*b*) vanishes at the vertices of $T \in \mathcal{C}_h$, the concentrated forces at the corners do not have to be calculated and the condition is equivalent to

$$A_T(\psi_h^*,\varphi) = \int_T g\varphi\,dx + \int_{\partial T}\left\{M_\nu(\mathbf{u}_h)\,\frac{\partial\varphi}{\partial\nu} - V_\nu(\mathbf{u}_h)\,\varphi\right\}ds\,,$$
$$\forall\varphi \in W_h^*|_T \quad \text{with} \quad \varphi(a) = 0 \quad \text{for} \quad a \in J_h \cap T\,,$$

where

$$V_\nu(\mathbf{u}_h) = Q_\nu(\mathbf{u}_h) + \frac{\partial M_{\nu\tau}(\mathbf{u}_h)}{\partial \tau}$$

is the approximation to the « effective (or Kirchhoff) shear force »

$$V_\nu(\mathbf{u}) = Q_\nu(\mathbf{u}) + \frac{\partial M_{\nu\tau}(\mathbf{u})}{\partial \tau}. \quad \blacksquare$$

The error estimate for the new approximation ψ_h^* will be given in the following norm

$$\|\psi - \psi_h^*\|_{1,h} = \Big(\sum_{T \in \mathcal{T}_h} \|\psi - \psi_h^*\|_{1,T}^2 \Big)^{1/2}.$$

For the error analysis of the higher order methods we assume that $\psi \in H^r(\Omega)$ with $r > 7/2$, which implies that the shear force Q_ν is in $L^2(\partial T)$ for $T \in \mathcal{T}_h$. (When this assumption is not valid, one can apply the estimate for the lowest order method.)

THEOREM 3.2: *For the postprocessing scheme* (3.16) *we have the following error estimates.*
For $k = 1$ *and* $g \in L^2(\Omega)$:

$$\|\psi - \psi_h^*\|_{1,h} \leq C h^2 (\|\psi\|_3 + \|g\|_0). \tag{3.17}$$

For $k \geq 2$ *and* $\psi \in H^r(\Omega)$ *with* $r > 7/2$:

$$\|\psi - \psi_h^*\|_{1,h} \leq C h^s \|\psi\|_{s+1} \quad \textit{where} \quad s = \min\{r-1, k+1\}. \tag{3.18}$$

Proof: Let Q_h be the Lagrange interpolation operator onto the space of continuous piecewise linear functions:

$$\{f \in C(\bar{\Omega}) \mid f|_T \in P_1(T), T \in \mathcal{T}_h\}.$$

Further, we denote by $\tilde{\psi} \in W_h^* \cap C(\bar{\Omega})$ the Lagrange interpolate to ψ.

First, using (3.6*c*) and (3.16*a*) we obtain

$$\|Q_h(\tilde{\psi} - \psi_h^*)\|_{1,h} = \|Q_h(\Sigma_h \psi - \psi_h)\|_{1,h} =$$

$$\|Q_h(\Sigma_h \psi - \psi_h)\|_1 \leq C \|\Sigma_h \psi - \psi_h\|_1. \tag{3.19}$$

Next, let us estimate $\|(I - Q_h)(\tilde{\psi} - \psi_h^*)\|_{1,h}$. For convenience let us denote $z = (I - Q_h)(\tilde{\psi} - \psi_h^*)$. Since $Q_h z = 0$, standard interpolation theory gives

$$\|z\|_{1,T} = \|z - Q_h z\|_{1,T} \leq C h_T |z|_{2,T}. \tag{3.20}$$

Now, the exact solution $(\psi, \mathbf{u})$ of (3.3) satisfies

$$A_T(\psi, \varphi) =$$

$$\int_T g\varphi\, dx + \int_{\partial T} \left\{ M_\nu(\mathbf{u}) \frac{\partial \varphi}{\partial \nu} + M_{\nu\tau}(\mathbf{u}) \frac{\partial \varphi}{\partial \tau} - Q_\nu(\mathbf{u})\, \varphi \right\} ds\,, \quad \varphi \in H^2(T)\,.$$

Using this, (3.19*b*) and recalling the definition of z we get

$$\begin{aligned} A_T(z, z) &= A_T(\tilde{\psi} - \psi_h^*, z) = A_T(\tilde{\psi} - \psi, z) \\ &+ \int_{\partial T} \left\{ M_\nu(\mathbf{u} - \mathbf{u}_h) \frac{\partial z}{\partial \nu} + M_{\nu\tau}(\mathbf{u} - \mathbf{u}_h) \frac{\partial z}{\partial \tau} - Q_\nu(\mathbf{u} - \mathbf{u}_h)\, z \right\} ds\,. \end{aligned} \tag{3.21}$$

Let us estimate the terms in (3.21). Since $0 \leqslant \sigma < 1/2$ we have

$$C\, |z|_{2,T}^2 \leqslant A_T(z, z) \tag{3.22}$$

and

$$A_T(\tilde{\psi} - \psi, z) \leqslant C\, |\tilde{\psi} - \psi|_{2,T} |z|_{2,T}\,. \tag{3.23}$$

Further, Schwarz inequality and a scaling argument yield

$$\begin{aligned} &\int_{\partial T} \left\{ M_\nu(\mathbf{u} - \mathbf{u}_h) \frac{\partial z}{\partial \nu} + M_{\nu\tau}(\mathbf{u} - \mathbf{u}_h) \frac{\partial z}{\partial \tau} \right\} ds \leqslant \\ &\quad C \left(h_T \int_{\partial T} (|M_\nu(\mathbf{u} - \mathbf{u}_h)|^2 + |M_{\nu\tau}(\mathbf{u} - \mathbf{u}_h)|^2)\, ds \right)^{1/2} \\ &\quad \cdot \left(h_T^{-1} \int_{\partial T} \left(\left| \frac{\partial z}{\partial \nu} \right|^2 + \left| \frac{\partial z}{\partial \tau} \right|^2 \right) ds \right)^{1/2} \\ &\leqslant C \left(h_T \int_{\partial T} (|M_\nu(\mathbf{u} - \mathbf{u}_h)|^2 + |M_{\nu\tau}(\mathbf{u} - \mathbf{u}_h)|^2)\, ds \right)^{1/2} \cdot |z|_{2,T}\,. \end{aligned} \tag{3.24}$$

To estimate the last term in the right hand side of (3.21) we treat separately the cases $k = 1$ and $k \geqslant 2$.

For $k = 1$ we have $Q_\nu(\mathbf{u}_h) = 0$ and since we assume that $D\,\Delta^2\psi = g \in L^2(\Omega)$ we can use a trace theorem [9, Theorem 2.5, p. 27] to estimate as follows

$$\begin{aligned} &- \int_{\partial T} Q_\nu(\mathbf{u} - \mathbf{u}_h)\, z\, ds = - \int_{\partial T} Q_\nu(\mathbf{u})\, z\, ds = D \int_{\partial T} \frac{\partial\, \Delta\psi}{\partial \nu}\, z\, ds \leqslant \\ &D \left\| \frac{\partial \Delta\psi}{\partial \nu} \right\|_{-1/2, \partial T} \|z\|_{1/2, \partial T} \leqslant C(\|\psi\|_{3,T} + \|g\|_{0,T})\, \|z\|_{1,T} \\ &\qquad \leqslant C h_T(\|\psi\|_{3,T} + \|g\|_{0,T}) |z|_{2,T}\,, \end{aligned} \tag{3.25}$$

where we in the last step used (3.20)

For $k \geqslant 2$ we assume that $r > 7/2$, and hence we get

$$- \int_{\partial T} Q_\nu(\mathbf{u} - \mathbf{u}_h)\, z\, ds \leqslant \left(h_T^3 \int_{\partial T} |Q_\nu(\mathbf{u} - \mathbf{u}_h)|^2\, ds \right)^{1/2} \left(h_T^{-3} \int_{\partial T} z^2\, ds \right)^{1/2}$$

$$\leqslant C \left(h_T^3 \int_{\partial T} |Q_\nu(\mathbf{u} - \mathbf{u}_h)|^2\, ds \right)^{1/2} |z|_{2, T}, \qquad (3.26)$$

where we again used a scaling argument in the last step. Combining (3.20) through (3.26) now gives

$$\|z\|_{1, T} \leqslant C h_T \left\{ |\psi - \tilde{\psi}|_{2, T} + \left(h_T \int_{\partial T} (|M_\nu(\mathbf{u} - \mathbf{u}_h)|^2 + \right.\right.$$
$$\left.\left. + |M_{\nu\tau}(\mathbf{u} - \mathbf{u}_h)|^2\, ds \right)^{1/2} + E_T \right\} \qquad (3.27)$$

with

$$E_T = h_T(\|\psi\|_{3, T} + \|g\|_{0, T}) \quad \text{for} \quad k = 1$$

and

$$E_T = \left(h_T^3 \int_{\partial T} |Q_\nu(\mathbf{u} - \mathbf{u}_h)|^2\, ds \right)^{1/2} \quad \text{for} \quad k \geqslant 2\,.$$

Recalling the definitions of z, M_ν, $M_{\nu\tau}$ and $\|\cdot\|_{0,h}$, (3.27) now gives

$$\left\|(I - Q_h)(\tilde{\psi} - \psi_h^*)\right\|_{1, h} \leqslant C h_T \left\{ \left(\sum_{T \in \mathcal{C}_h} |\psi - \tilde{\psi}|_{2, T}^2 \right)^{1/2} + \|\mathbf{u} - \mathbf{u}_h\|_{0, h} + E \right\} \qquad (3.28)$$

with

$$E = h(\|\psi\|_3 + \|g\|_0) \quad \text{for} \quad k = 1$$

and

$$E = \left(\sum_{T \in \mathcal{C}_h} h_T^3 \int_{\partial T} |Q_\nu(\mathbf{u} - \mathbf{u}_h)|^2\, ds \right)^{1/2} \quad \text{for} \quad k \geqslant 2\,.$$

Now, by local scaling arguments (*cf.* [4]) one can show that the following estimate

$$\left(\sum_{T \in \mathcal{C}_h} h_T^3 \int_{\partial T} |Q_\nu(\mathbf{u} - \mathbf{u}_h)|^2\, ds \right)^{1/2} \leqslant$$
$$C h^\delta \|\psi\|_{\delta + 2} \quad \text{with} \quad \delta = \min\{r - 2, k\},$$

follows from (3.10). Hence, the asserted estimates follow from (3.19), (3.13), (3.28) and standard interpolation estimates. ■

Remark 3.1 : Note that when the method is used for the approximation of Stokes and Navier-Stokes equations (*cf.* [9]), then the estimate above contains a quasioptimal L^2-estimate for the postimproved approximation of the velocity. ■

Remark 3.2 : If the stronger regularity estimate

$$\|\psi\|_4 \leq C \|g\|_0$$

is valid, then one obtains the following error estimate for the higher order methods with $k \geq 3$

$$\|\psi - \psi_h^*\|_0 \leq Ch^s \|\psi\|_s \quad \text{with} \quad s = \min\{r, k+2\}, \text{when } \psi \in H^r(\Omega)\,. \quad ■$$

Remark 3.3 : In [1] it is shown that the lowest order method in the HHJ family can be implemented as a slight modification of Morleys nonconforming method. It was also shown that the approximation for the deflection so obtain converges with the same order as our postprocessed approximation. Hence, at least in applications to the plate bending problem, the lowest order HHJ method is most efficiently implemented as suggested in [1]. ■

REFERENCES

[1] D. N. ARNOLD and F. BREZZI, *Mixed and Nonconforming Finite Element Methods : Implementation, Postprocessing and Error Estimates,* RAIRO, M²AN, Vol. 19, 1985, pp. 7-32.

[2] F. BREZZI, J. DOUGLAS, R. DURÁN and M. FORTIN, *Mixed Finite Elements for Second Order Elliptic Problems in Three Variables,* Numer. Math., Vol. 51, 1987, pp. 237-250.

[3] I. BABUŠKA, J. E. OSBORN and J. PITKÄRANTA, *Analysis of Mixed Methods using Mesh Dependent Norms,* Math. Comp., Vol. 35, 1980, pp. 1039-1062.

[4] F. BREZZI, J. DOUGLAS and L. D. MARINI, *Two Families of Mixed Finite Elements for Second Order Elliptic Equations,* Numer. Math., Vol. 47, 1985, pp. 19-34.

[5] P. G. CIARLET, *The Finite Element Method for Elliptic Problems,* North-Holland, 1978.

[6] M. I. COMODI, *The Hellan-Herrmann-Johnson Method : Estimates for the Lagrange Multiplier and Postprocessing,* Math. Comp. Vol. 52, 1989, pp. 17-29.

[7] J. DOUGLAS and J. E. SANTOS, *Approximation of Waves in Composite Media,* The Mathematics of Finite Elements and Applications VI. MAFELAP 1987, J. R. Whiteman (Ed.), Academic Press, 55-74.

[8] R. S. FALK and J. E. OSBORN, *Error Estimates for Mixed Methods,* RAIRO, Anal. Numer., Vol. 14, 1980, pp. 249-278.

[9] V. GIRAULT and P. A. RAVIART, *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms,* Springer 1986.

[10] K. HELLAN, *Analysis of Elastic Plates in Flexure by a Simplified Finite Element Method,* Acta Polytechnica Scandinavica, Ci 46, Trondheim, 1967.

[11] L. HERRMANN, *Finite Element Bending Analysis for Plates,* J. Eng. Div. ASCE, a3, EM5, 1967, pp. 49-83.

[12] C. JOHNSON, *On the Convergence of a Mixed Finite Method for Plate Bending Problems,* Numer. Math. Vol. 21, 1973, pp. 43-62.

[13] L. D. MARINI and A. SAVINI, *Accurate Computation of Electric Field in Reverse Biased Semiconductor Devices. A Mixed Finite Element Approach,* Compel, Vol. 3, 1984, pp. 123-135.

[14] J. C. NEDELEC, *Mixed Finite Elements in* $\mathbf{R}^3$, Numer. Math., Vol. 35, 1980, pp. 315-341.

[15] P. A. RAVIART and J. M. THOMAS, *A Mixed Finite Element Method for 2nd Order Elliptic Problems,* Proceedings of the Symposium on the Mathematical Aspects of the Finite Element Method. Lecture Notes in Mathematics 606, Springer 1977, pp. 292-315.

[16] R. STENBERG, *On the Construction of Optimal Mixed Finite Element Methods for the Linear Elasticity Problem,* Numer. Math., Vol. 48, 1986, pp. 447-462.

[17] R. STENBERG, *On the Postprocessing of Mixed Equilibrium Finite Element Methods,* Numerical Techniques in Continuum Mechanics. Proceedings of the Second GAMM-Seminar, Kiel, January 17 to 19, 1986. W. Hackbusch, K. Witsch (Eds.), Vieweg, Braunschweig 1987, pp. 102-109.

[18] M. WHEELER and R. GONZALES, *Mixed Finite Element Methods for Petroleum Reservoir Simulation,* Computing Methods in Applied Sciences and Engineering, VI, R. Glowinski, J. L. Lions (Eds.), North-Holland 1984, pp. 639-658.