ELSEVIER

Comput. Methods Appl. Mech. Engrg. 190 (2001) 3893–3914

Computer methods in applied mechanics and engineering

www.elsevier.com/locate/cma

GLS and EVSS methods for a three-field Stokes problem arising from viscoelastic flows

J. Bonvin [a,1], M. Picasso [a,*], R. Stenberg [b]

[a] Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland
[b] Department of Mathematics, Tampere University of Technology, Korkeakoulunkatu 1, P.O. Box 692, 33101 Tampere, Finland

Received 2 February 2000

Abstract

In this paper, order one finite elements together with Galerkin least-squares (GLS) methods are used for solving a three-field Stokes problem arising from the numerical study of viscoelastic flows. Stability and convergence results are established, even when the solvent viscosity is small compared to the viscosity due to the polymer chains. An iterative algorithm decoupling velocity–pressure and stress calculations is proposed. The link with the modified elastic viscous split stress (EVSS) method studied in (M. Fortin, R. Guénette, R. Pierre, Comput. Methods Appl. Mech. Engrg. 143 (1997) 79–95; R. Guénette, M. Fortin, J. Non-Newtonian Fluid Mech. 60 (1995) 27–52) is presented. Numerical results are in agreement with theoretical predictions, and with those presented in (M. Fortin, R. Guénette, R. Pierre, Comput. Methods Appl. Mech. Engrg. 143 (1997) 79–95). © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Numerical simulation of viscoelastic flows is of interest for many industrial applications such as extrusion or injection processes. Macroscopic modelling of viscoelastic flows usually consists in supplementing the mass and momentum equations with a differential constitutive equation involving the stress and the velocity field. When considering this set of partial differential equations in the frame of a finite element method, three problems may arise.

The first one is related to the "Babuška–Brezzi" or "inf–sup" condition which restricts the choice of the finite element spaces used to discretize the velocity, pressure and stress fields.

The second problem is due to the presence of convective terms in the differential constitutive equation. It can be solved by using appropriate upwinding procedures, such as streamline upwind Petrov Galerkin (SUPG) [3], Galerkin least-squares (GLS) [4,5], discontinuous finite elements [6–8] or characteristic-Galerkin methods [9].

The third problem arises form the strong non-linear nature of the model. Existence and uniqueness results for viscoelastic models are established only at low Deborah numbers (see for instance [10] for a review), this being consistent with the fact that numerical procedures fail to converge at high Deborah numbers.

The goal of this paper is to focus on the first of these three problems, namely the inf–sup condition and, for this purpose, a three-field Stokes problem is introduced. Various approaches have been considered to

* Corresponding author. Tel.: +41-216-9342-97; fax: +41-216-9343-03.
E-mail address: marco.picasso@epfl.ch (M. Picasso).
[1] Supported by the Swiss National Fund for Scientific Research.

PII: S0045-7825(00)00307-8

obtain stable and convergent methods. The first of these approaches considers the usual Galerkin method and makes use of finite element spaces (for the velocity, pressure and stress) satisfying the inf–sup condition. For instance, continuous approximations were proposed in [3] and justified in [11], discontinuous approximations in [6,8]. An alternative is to use equal-order approximations for the velocity, pressure and stress, while adding stabilization terms to the usual weak formulation in a GLS fashion, see for instance [5] for computations and [12] for mathematical studies. The third possible manner to cope with the inf–sup condition (half way between the two methods hereabove) consists in splitting the stress into two parts, an "elastic" part and a "viscous" part. The velocity and pressure spaces still have to satisfy the inf–sup condition but no additional compatibility condition apply on the two split stresses. This so-called elastic viscous split stress (EVSS) method was introduced in [2,13,14] and a numerical analysis was proposed in [1] for a three-field Stokes problem arising from viscoelastic models. Recently, many extensions have been proposed and we refer to [15] for a review. For instance, an adaptive splitting of the stress is proposed in [16], together with SUPG techniques to account for convection, whereas discontinuous stresses were considered in [7].

In this paper, GLS methods are proposed and the connection with EVSS methods is proposed for a three-field Stokes problem arising from viscoelastic models. Continuous, piecewise linear velocity, pressure and stress are considered on triangles. Stabilization terms are added and, as for the modified EVSS method of [1], it is proved that the method remains stable even when the solvent viscosity is small. Two GLS schemes are presented.

The first GLS method corresponds to the case when all the possible stabilization terms are added in a least-squares fashion, following the ideas of [17]. Stability and convergence results are established using techniques borrowed from [12,17,18]. The stability parameters are chosen in order to ensure convergence, even when the solvent viscosity is much smaller than the polymer viscosity.

In the second GLS method, only a selected number of these terms are added. Stability and convergence results are also given. The reasons for introducing this second, "reduced" GLS method are the following. Firstly, the number of stabilizing terms is reduced, which simplifies the implementation. It must be kept in mind that our goal is to apply this GLS method to "real-life" constitutive relationships, the simplest being the Oldroyd-B equation. Secondly, this reduced GLS method is closely linked to the modified EVSS method of [1,2] which is now very popular in the field of non-newtonian fluid mechanics. Finally, as for the modified EVSS method, the velocity–pressure and stress computations can be decoupled, which reduces the memory requirements.

The outline of the paper is the following. The three-field Stokes model is presented in Section 2. The first GLS method is presented and studied in Section 3, the second GLS method in Section 4. The link with the modified EVSS method of [1,2] is proposed in Section 5. An iterative procedure to decouple the velocity–pressure and stress computations is presented in Section 6. In Section 7, numerical results are reported, matching those of [1], and confirming our theoretical predictions.

2. A three-field Stokes problem

Let Ω be a bounded polygonal domain of $\mathbb{R}^2$ with boundary $\partial\Omega$. We consider the following formulation of the Stokes problem. Given force terms $\mathbf{f}_1$, f_2, $\mathbf{f}_3$, constant solvent and polymer viscosities $\eta_s \geqslant 0$ and $\eta_p > 0$, find the velocity $\mathbf{u}$, pressure p and extra-stress $\boldsymbol{\sigma}$ such that

$$-2\eta_s \operatorname{div} \epsilon(\mathbf{u}) + \nabla p - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}_1, \quad \operatorname{div} \mathbf{u} = f_2, \quad \frac{1}{2\eta_p}\boldsymbol{\sigma} - \epsilon(\mathbf{u}) = \mathbf{f}_3 \tag{1}$$

in Ω, where the velocity $\mathbf{u}$ vanishes on $\partial\Omega$ and $\epsilon(\mathbf{u}) = (1/2)(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ is the rate of deformation tensor. The aim of the paper is to propose a numerical procedure for solving (1) with continuous, piecewise linear finite elements, which is stable and convergent for any values of $\eta_s \geqslant 0$ and $\eta_p > 0$. The reason for choosing the above formulation of Stokes problem is due to the fact that, in number of viscoelastic codes, the velocity–pressure computations are decoupled from the extra-stress computations at each iteration, thus reducing memory requirements and possibly CPU time. This decoupling is based on the usual splitting of the total stress for a polymeric liquid into three contributions: the pressure $-p\mathbf{I}$, the stress due to the (newtonian) solvent $2\eta_s\epsilon(\mathbf{u})$, and the extra-stress due to the polymer chains $\boldsymbol{\sigma}$, which we shall simply refer to as the "stress" in this paper. For instance, when considering the steady Oldroyd-B model, the convective term $\rho(\mathbf{u}\cdot\nabla)\mathbf{u}$ should be added in the first equation of (1), and the last equation of (1) should be replaced by the extra-stress constitutive equation

$$\boldsymbol{\sigma} + \lambda\big((\mathbf{u}\cdot\nabla)\boldsymbol{\sigma} - \nabla\mathbf{u}\boldsymbol{\sigma} - \boldsymbol{\sigma}\nabla\mathbf{u}^{\mathrm{T}}\big) = 2\eta_{\mathrm{p}}\epsilon(\mathbf{u}), \tag{2}$$

where the relaxation time λ characterizes the elastic properties of the fluid. Note that when $\eta_s = 0$, Maxwell's model is obtained, and we want to design methods which are still stable and convergent.

The variational formulation for problem (1) consists in seeking $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W$ such that

$$B(\mathbf{u}, p, \boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau}) = F(\mathbf{v}, q, \boldsymbol{\tau}) \tag{3}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W$. Here B is the symmetric, bilinear form defined by

$$B(\mathbf{u}, p, \boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau}) = 2\eta_s(\epsilon(\mathbf{u}), \epsilon(\mathbf{v})) - (p, \operatorname{div}\mathbf{v}) + (\boldsymbol{\sigma}, \epsilon(\mathbf{v})) - (\operatorname{div}\mathbf{u}, q) - \frac{1}{2\eta_{\mathrm{p}}}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\epsilon(\mathbf{u}), \boldsymbol{\tau}) \tag{4}$$

and F is the form defined by

$$F(\mathbf{v}, q, \boldsymbol{\tau}) = (\mathbf{f}_1, \mathbf{v}) - (f_2, q) - (\mathbf{f}_3, \boldsymbol{\tau}),$$

where $(\cdot,\cdot)$ stands for the usual $L^2(\Omega)$ inner product for scalars, vectors or tensors. The appropriate functional space W is defined by

$$W = H_0^1(\Omega)^2 \times L_0^2(\Omega) \times L^2(\Omega)_s^4.$$

Here, $H_0^1(\Omega)^2$ stands for the space of square integrable velocities defined on Ω, vanishing on the boundary and with first derivatives also square integrable, $L_0^2(\Omega)$ stands for the space of square integrable functions having zero average, and $L^2(\Omega)_s^4$ denotes the space of 2×2 symmetric tensors whose components are square integrable.

Well posedness of the continuous problem. Let $\|\cdot\|_W$ be the norm on W defined by

$$\|\mathbf{u}, p, \boldsymbol{\sigma}\|_W^2 = 2\eta\|\epsilon(\mathbf{u})\|^2 + \frac{1}{2\eta}\|p\|^2 + \frac{1}{2\eta_{\mathrm{p}}}\|\boldsymbol{\sigma}\|^2, \tag{5}$$

where $\eta = \eta_s + \eta_p$ is the total viscosity of the fluid. Assume that $(\mathbf{f}_1, f_2, \mathbf{f}_3) \in L^2(\Omega)^2 \times L^2(\Omega) \times L^2(\Omega)_s^4$ and let $\|\cdot\|_*$ be the norm on this space defined by

$$\|\mathbf{f_1}, f_2, \mathbf{f}_3\|_*^2 = \frac{1}{2\eta}\left(\sup_{0\neq\mathbf{v}\in H_0^1(\Omega)^2} \frac{(\mathbf{f}_1, \mathbf{v})}{\|\epsilon(\mathbf{v})\|}\right)^2 + 2\eta\|f_2\|^2 + 2\eta_{\mathrm{p}}\|\mathbf{f}_3\|^2. \tag{6}$$

We have the following existence and unicity result for the exact solution $(\mathbf{u}, p, \boldsymbol{\sigma})$ of (3).

Theorem 1 (Existence and uniqueness of the exact solution). *Let $\eta_s \geqslant 0$ and $\eta_p > 0$. Problem* (3) *admits a unique solution $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W$. Moreover, the solution depends continuously on the data and there exists a positive constant C, independent of η_s and η_p, such that*

$$\|\mathbf{u}, p, \boldsymbol{\sigma}\|_W \leqslant C\|\mathbf{f}_1, f_2, \mathbf{f}_3\|_*. \tag{7}$$

This theorem results from the Babuška–Ladyzhenskaya theorem (see for instance [19]) and is true since the inf–sup condition holds for B, this being stated in the following lemma.

Lemma 2 (Stability of the continuous problem). *Let $\eta_s \geqslant 0$ and $\eta_p > 0$. There exist two positive constants C_1 and C_2, independent of η_s and η_p, such that*

$$\sup_{0\neq \mathbf{v}\in H_0^1(\Omega)^2} \frac{(\operatorname{div}\mathbf{v},p)}{\|\boldsymbol{\epsilon}(\mathbf{v})\|} \geqslant C_1\|p\| \quad \forall p\in L_0^2(\Omega), \tag{8}$$

$$\sup_{0\neq \boldsymbol{\tau}\in L^2(\Omega)_s^4} \frac{(\boldsymbol{\tau},\boldsymbol{\epsilon}(\mathbf{u}))}{\|\boldsymbol{\tau}\|} \geqslant C_2\|\boldsymbol{\epsilon}(\mathbf{u})\| \quad \forall \mathbf{u}\in H_0^1(\Omega)^2. \tag{9}$$

Therefore, there exists a positive constant C_3, independent of η_s and η_p, such that

$$\sup_{0\neq (\mathbf{v},q,\boldsymbol{\tau})\in W} \frac{B(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{v},q,\boldsymbol{\tau})}{\|\mathbf{v},q,\boldsymbol{\tau}\|_W} \geqslant C_3\|\mathbf{u},p,\boldsymbol{\sigma}\| \quad \forall(\mathbf{u},p,\boldsymbol{\sigma})\in W. \tag{10}$$

Before proving this lemma, let us note that condition (8) is nothing but the standard inf–sup condition for the usual two-field Stokes problem (see for instance [19,20]), whereas (9) is trivial (given $\mathbf{u}$, choose $\boldsymbol{\tau}=\boldsymbol{\epsilon}(\mathbf{u})$).

From the numerical point of view, conditions (8) and (9) must also be satisfied in the finite element subspaces. Choosing equal order approximations for $\mathbf{u}$, p, and $\boldsymbol{\sigma}$ (for instance, continuous, piecewise linears), does not lead to a stable scheme, as in the case of the classical (two fields) Stokes problem.

A possible remedy to this situation is to enrich the finite element subspaces for the velocity and stress in order to satisfy both stability conditions (8) and (9), see for instance [3,8,9,11].

The GLS method is designed to avoid both compatibility conditions (8) and (9) by adding extra terms to the bilinear form B and to the linear form F. More precisely, the extra terms are least-square-like expressions of the equation residual, weighted on each triangle of the finite element mesh. The two methods that are proposed in this paper consist in adding such stabilization terms to (3), while maintaining order one approximations for the velocity, the pressure and the stress.

The EVSS method of [1,2] is such that (8) is satisfied while (9) is avoided by a technique very close to GLS. In other words, the EVSS formulation of [1,2] offers an alternative which consists in using compatible velocity–pressure finite elements and low order stresses, but with the extra cost of having to add an extra variable to stabilize the discretization of (1), as will be pointed out in Section 5.

Proof of Lemma 2. First, note that for any $(\mathbf{u},p,\boldsymbol{\sigma})\in W$ we have

$$B(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{u},-p,-\boldsymbol{\sigma}) = 2\eta_s\|\boldsymbol{\epsilon}(\mathbf{u})\|^2 + \frac{1}{2\eta_p}\|\boldsymbol{\sigma}\|^2. \tag{11}$$

Condition (8) implies that for any $p\in L_0^2(\Omega)$ there exists $\tilde{\mathbf{v}}\in H_0^1(\Omega)^2$, such that

$$(\operatorname{div}\tilde{\mathbf{v}},p) \geqslant C_1\|p\|\|\boldsymbol{\epsilon}(\tilde{\mathbf{v}})\|.$$

Choosing $\tilde{\mathbf{v}}$ such that $\|\boldsymbol{\epsilon}(\tilde{\mathbf{v}})\| = 1/2\eta\|p\|$ yields

$$(\operatorname{div}\tilde{\mathbf{v}},p) \geqslant \frac{C_1}{2\eta}\|p\|^2.$$

Similarly, condition (9) implies that for any $\mathbf{u}\in H_0^1(\Omega)^2$ there exists $\tilde{\boldsymbol{\tau}}\in L^2(\Omega)_s^4$ such that $\|\tilde{\boldsymbol{\tau}}\| = 2\eta_p\|\boldsymbol{\epsilon}(\mathbf{u})\|$ and

$$\left(\tilde{\boldsymbol{\tau}},\boldsymbol{\epsilon}(\mathbf{u})\right) \geqslant 2\eta_p C_2\|\boldsymbol{\epsilon}(\mathbf{u})\|^2.$$

We want to prove that there exists a constant C_3 such that, for all $(\mathbf{u},p,\boldsymbol{\sigma})\in W$, we have

$$B(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{u}-\delta\tilde{\mathbf{v}},-p,-\boldsymbol{\sigma}+\delta\tilde{\boldsymbol{\tau}}) \geqslant C_3\|\mathbf{u},p,\boldsymbol{\sigma}\|_W\|\mathbf{u}-\delta\tilde{\mathbf{v}},-p,-\boldsymbol{\sigma}+\delta\tilde{\boldsymbol{\tau}}\|_W$$

for some value δ. Using the Cauchy–Schwarz inequality we have

$$B(\mathbf{u},p,\boldsymbol{\sigma};-\tilde{\mathbf{v}},0,\tilde{\boldsymbol{\tau}}) = -2\eta_{\mathrm{s}}\Big(\epsilon(\mathbf{u},\epsilon(\tilde{\mathbf{v}})\Big) + (p,\operatorname{div}\tilde{\mathbf{v}}) - \Big(\boldsymbol{\sigma},\epsilon(\tilde{\mathbf{v}})\Big) - \frac{1}{2\eta_{\mathrm{p}}}(\boldsymbol{\sigma},\tilde{\boldsymbol{\tau}}) + \Big(\epsilon(\mathbf{u},\tilde{\boldsymbol{\tau}}\Big)$$
$$\geqslant -2\eta_{\mathrm{s}}\|\epsilon(\mathbf{u})\|\|\epsilon(\tilde{\mathbf{v}})\| + \frac{C_1}{2\eta}\|p\|^2 - \|\boldsymbol{\sigma}\|\|\epsilon(\tilde{\mathbf{v}})\| - \frac{1}{2\eta_{\mathrm{p}}}\|\boldsymbol{\sigma}\|\|\tilde{\boldsymbol{\tau}}\| + C_2 2\eta_{\mathrm{p}}\|\epsilon(\mathbf{u})\|^2.$$

Consider Young's inequality, which writes $2ab \leqslant (1/\gamma)a^2 + \gamma b^2$, for all real numbers a,b and for any $\gamma > 0$. Using this inequality and then the definitions of $\tilde{\mathbf{v}}$ and $\tilde{\boldsymbol{\tau}}$, we get

$$B(\mathbf{u},p,\boldsymbol{\sigma};-\tilde{\mathbf{v}},0,\tilde{\boldsymbol{\tau}}) \geqslant -\frac{2\eta_{\mathrm{s}}}{2\gamma_1}\|\epsilon(\mathbf{u})\|^2 - 2\eta_{\mathrm{s}}\frac{\gamma_1}{2}\|\epsilon(\tilde{\mathbf{v}})\|^2 + \frac{C_1}{2\eta}\|p\|^2 - \frac{1}{2\gamma_2}\|\boldsymbol{\sigma}\|^2$$
$$-\frac{\gamma_2}{2}\|\epsilon(\tilde{\mathbf{v}})\|^2 - \frac{1}{2\eta_{\mathrm{p}}}\frac{1}{2\gamma_3}\|\boldsymbol{\sigma}\|^2 - \frac{1}{2\eta_{\mathrm{p}}}\frac{\gamma_3}{2}\|\tilde{\boldsymbol{\tau}}\|^2 + C_2 2\eta_{\mathrm{p}}\|\epsilon(\mathbf{u})\|^2$$
$$= \left(2\eta_{\mathrm{p}}\Big(C_2 - \frac{\gamma_3}{2}\Big) - \frac{\eta_{\mathrm{s}}}{\gamma_1}\right)\|\epsilon(\mathbf{u})\|^2 + \frac{1}{2\eta}\left(C_1 - \frac{\gamma_1\eta_{\mathrm{s}}}{2\eta} - \frac{\gamma_2}{4\eta}\right)\|p\|^2 - \frac{1}{2\eta_{\mathrm{p}}}\left(\frac{1}{2\gamma_3} + \frac{\eta_{\mathrm{p}}}{\gamma_2}\right)\|\boldsymbol{\sigma}\|^2.$$

From (11) and the previous inequality we get the following lower bound:

$$B(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{u}-\delta\tilde{\mathbf{v}},-p,-\boldsymbol{\sigma},+\delta\tilde{\boldsymbol{\tau}}) \geqslant \left(2\eta_{\mathrm{s}}\Big(1 - \frac{\delta}{2\gamma_1}\Big) + 2\eta_{\mathrm{p}}\delta\Big(C_2 - \frac{\gamma_3}{2}\Big)\right)\|\epsilon(\mathbf{u})\|^2$$
$$+\frac{\delta}{2\eta}\left(C_1 - \frac{\gamma_1\eta_{\mathrm{s}}}{2\eta} - \frac{\gamma_2}{4\eta}\right)\|p\|^2 + \frac{1}{2\eta_{\mathrm{p}}}\left(1 - \delta\Big(\frac{1}{2\gamma_3} + \frac{\eta_{\mathrm{p}}}{\gamma_2}\Big)\right)\|\boldsymbol{\sigma}\|^2.$$

Let us choose $\gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0$ and $\delta > 0$ such that

$$1 - \frac{\delta}{2\gamma_1} > 0, \quad C_2 - \frac{\gamma_3}{2} > 0, \quad C_1 - \frac{\gamma_1\eta_{\mathrm{s}}}{2\eta} - \frac{\gamma_2}{4\eta} > 0, \quad 1 - \delta\left(\frac{1}{2\gamma_3} + \frac{\eta_{\mathrm{p}}}{\gamma_2}\right) > 0.$$

If we choose for instance γ_1, γ_2 and γ_3 such that

$$\gamma_1 = \frac{C_1}{2}, \quad \gamma_2 = C_1\eta \quad \text{and} \quad \gamma_3 = \frac{C_2}{2}$$

and using the fact that $\eta_{\mathrm{s}}/\eta \leqslant 1$ and $\eta_{\mathrm{p}}/\eta \leqslant 1$, we obtain

$$B(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{u}-\delta\tilde{\mathbf{v}},-p,-\boldsymbol{\sigma}+\delta\tilde{\boldsymbol{\tau}}) \geqslant \left(2\eta_{\mathrm{s}}\Big(1 - \frac{\delta}{C_1}\Big) + 2\eta_{\mathrm{p}}\frac{3\delta C_2}{4}\right)\|\epsilon(\mathbf{u})\|^2 + \frac{1}{2\eta}\frac{\delta C_1}{2}\|p\|^2$$
$$+\frac{1}{2\eta_{\mathrm{p}}}\left(1 - \delta\Big(\frac{1}{C_2} + \frac{1}{C_1}\Big)\right)\|\boldsymbol{\sigma}\|^2.$$

In the inequality hereabove, the coefficients are all positive if δ is chosen sufficiently small, i.e., such that

$$0 < \delta < \min\left(C_1, \Big(\frac{1}{C_2} + \frac{1}{C_1}\Big)^{-1}\right).$$

Thus, for any $(\mathbf{u},p,\boldsymbol{\sigma}) \in W$ we have found $(\mathbf{v},q,\boldsymbol{\tau}) = (\mathbf{u}-\delta\tilde{\mathbf{v}},-p,-\boldsymbol{\sigma}+\delta\tilde{\boldsymbol{\tau}}) \in W$ such that

$$B(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{v},q,\boldsymbol{\tau}) \geqslant C_4\|\mathbf{u},p,\boldsymbol{\sigma}\|_W^2,$$

where

$$C_4 = \min\left(1 - \frac{\delta}{C_1}, \frac{3\delta C_2}{4}, \frac{\delta C_1}{2}, 1 - \delta\Big(\frac{1}{C_2} + \frac{1}{C_1}\Big)\right)$$

is clearly independent of η_{s} and η_{p}. The previous inequality together with the fact that

$$\|\mathbf{u}-\delta\tilde{\mathbf{v}},-p,-\boldsymbol{\sigma}+\delta\tilde{\boldsymbol{\tau}}\|_W \leqslant (1+\delta)\|\mathbf{u},p,\boldsymbol{\sigma}\|_W$$

yield inequality (10) with $C_3 = C_4/(1+\delta)$.

3. A first GLS method

The GLS method was originally developed for the Stokes problem in [18]. It was then extended to diffusion–convection problems in [21]. In [17], the GLS method was presented in a general framework and applied to Navier–Stokes equations. In [4,12,22], the GLS method was applied to the three-field Stokes problem (1) with $\eta_s = 0$. However, it seems that nobody has studied in detail the GLS method for solving formulation (1) of the Stokes problem, for any values $\eta_s \geqslant 0$ and $\eta_p > 0$.

In this paper, only continuous, piecewise linears will be considered. For any $h > 0$, let $\mathscr{T}_h$ be a mesh of $\overline{\Omega}$ into triangles K with diameters h_K less than h. As usual, the mesh is assumed to be regular in the sense of [23], which means that the smallest angle is bounded below by some positive constant independent of the mesh size h. We consider W_h the finite dimensional subspace of W consisting in the continuous, piecewise linear velocities, pressures, and stresses on the mesh $\mathscr{T}_h$. More precisely $W_h \subset W$ is defined by

$$W_h = V_h \times Q_h \times T_h,$$

where

$$\begin{aligned} V_h &= \left\{ \mathbf{v} \in \mathscr{C}^0(\overline{\Omega})^2 \,|\, \mathbf{v}_{|K} \in (\mathbb{P}_1)^2,\ \forall K \in \mathscr{T}_h \right\} \cap H_0^1(\Omega)^2, \\ Q_h &= \left\{ q \in \mathscr{C}^0(\overline{\Omega}) \,|\, q_{|K} \in \mathbb{P}_1,\ \forall K \in \mathscr{T}_h \right\} \cap L_0^2(\Omega), \\ T_h &= \left\{ \boldsymbol{\tau} \in \mathscr{C}^0(\overline{\Omega})^4 \,|\, \boldsymbol{\tau}_{|K} \in (\mathbb{P}_1)^4,\ \forall K \in \mathscr{T}_h \right\} \cap L^2(\Omega)_s^4. \end{aligned} \tag{12}$$

We now use the general framework of [17] in order to present a GLS formulation corresponding to (3). First, we rewrite (1) as

$$\mathscr{L}_1(\mathbf{u}, p, \boldsymbol{\sigma}) = \mathbf{f}_1, \quad \mathscr{L}_2(\mathbf{u}, p, \boldsymbol{\sigma}) = f_2, \quad \mathscr{L}_3(\mathbf{u}, p, \boldsymbol{\sigma}) = \mathbf{f}_3 \tag{13}$$

with

$$\mathscr{L}_1(\mathbf{u}, p, \boldsymbol{\sigma}) = -2\eta_s \operatorname{div} \boldsymbol{\epsilon}(\mathbf{u}) + \nabla p - \operatorname{div} \boldsymbol{\sigma}, \quad \mathscr{L}_2(\mathbf{u}, p, \boldsymbol{\sigma}) = \operatorname{div} \mathbf{u}, \quad \mathscr{L}_3(\mathbf{u}, p, \boldsymbol{\sigma}) = \frac{1}{2\eta_p} \boldsymbol{\sigma} - \boldsymbol{\epsilon}(\mathbf{u}).$$

Following [17], the GLS method corresponding to the weak formulation (3) may be stated as follows:

Find $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ such that

$$B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = F_h(\mathbf{v}, q, \boldsymbol{\tau}) \tag{14}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$. Here the symmetric, bilinear form B_h and the form F_h are defined by

$$B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = B(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) + \sum_{i=1}^{3} \sum_{K \in \mathscr{T}_h} \kappa_i (\mathscr{L}_i(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h), \mathscr{L}_i(\mathbf{v}, q, \boldsymbol{\tau}))_K,$$

$$F_h(\mathbf{v}, q, \boldsymbol{\tau}) = F(\mathbf{v}, q, \boldsymbol{\tau}) + \sum_{i=1}^{3} \sum_{K \in \mathscr{T}_h} \kappa_i (\mathbf{f}_i, \mathscr{L}_i(\mathbf{v}, q, \boldsymbol{\tau}))_K,$$

where $(\cdot,\cdot)_K$ denotes the $L^2(K)$ inner product. The stabilization coefficients $\kappa_i, i = 1, 2, 3$ are designed to obtain optimal convergence. Following [4,12], we choose:

$$\kappa_1 = -\frac{\alpha h_K^2}{2\eta_p}, \quad \kappa_2 = 0, \quad \kappa_3 = 2\eta_p \beta,$$

where $\alpha > 0$ and $0 < \beta < 1$ are dimensionless parameters. Using (13), the bilinear form B_h is then defined by

$$\begin{aligned} B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = {} & B(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) - \sum_{K \in \mathscr{T}_h} \frac{\alpha h_K^2}{2\eta_p} (- 2\eta_s \operatorname{div} \boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \operatorname{div} \boldsymbol{\sigma}_h \\ & - 2\eta_s \operatorname{div} \boldsymbol{\epsilon}(\mathbf{v}) + \nabla q - \operatorname{div} \boldsymbol{\tau})_K + 2\eta_p \beta \left(\frac{1}{2\eta_p} \boldsymbol{\sigma}_h - \boldsymbol{\epsilon}(\mathbf{u}_h), \frac{1}{2\eta_p} \boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v}) \right) \end{aligned} \tag{15}$$

and the form F_h by

$$F_h(\mathbf{v},q,\boldsymbol{\tau}) = F(\mathbf{v},q,\boldsymbol{\tau}) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_{\rm p}}(\mathbf{f}_1, -2\eta_{\rm s}\operatorname{div}\boldsymbol{\epsilon}(\mathbf{v}) + \nabla q - \operatorname{div}\boldsymbol{\tau})_K + 2\eta_{\rm p}\beta\left(\mathbf{f}_3, \frac{1}{2\eta_{\rm p}}\boldsymbol{\tau} - \boldsymbol{\epsilon}(\mathbf{v})\right).$$

Note that, since we have chosen $\kappa_2 = 0$, the GLS term corresponding to the second equation of (1), namely $(\operatorname{div}\mathbf{u};\operatorname{div}\mathbf{v})$, is not included. This is due to the fact that this term essentially produces terms which are already present in $(\boldsymbol{\epsilon}(\mathbf{u}),\boldsymbol{\epsilon}(\mathbf{v}))$.

The GLS scheme (14) with B_h given by (15) is said to be *consistent* in the following sense. If the solution $(\mathbf{u},p,\boldsymbol{\sigma}) \in W$ of the weak formulation (3) is regular enough, that is, if $(\mathbf{u},p,\boldsymbol{\sigma}) \in H^2(\Omega)^2 \times H^1(\Omega) \times H^1(\Omega)^4$, then $(\mathbf{u},p,\boldsymbol{\sigma})$ satisfies (1) in the L^2 sense in each triangle, and we have

$$B_h(\mathbf{u},p,\boldsymbol{\sigma};\mathbf{v},q,\boldsymbol{\tau}) = F_h(\mathbf{v},q,\boldsymbol{\tau}) \quad \forall(\mathbf{v},q,\boldsymbol{\tau}) \in W_h.$$

Consequently, if $(\mathbf{u}_h,p_h,\boldsymbol{\sigma}_h) \in W_h$ is the solution of (14), we obtain

$$B_h(\mathbf{u}-\mathbf{u}_h,p-p_h,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h;\mathbf{v},q,\boldsymbol{\tau}) = 0 \quad \forall(\mathbf{v},q,\boldsymbol{\tau}) \in W_h. \tag{16}$$

We are now in a position to present some theoretical results concerning stability and convergence of the method. In Lemma 3, stability is first established in the discrete norm $\|\cdot\|_h$ defined, for all $(\mathbf{v},q,\boldsymbol{\tau}) \in W_h$ by

$$\|\mathbf{v},q,\boldsymbol{\tau}\|_h^2 = 2\eta\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \frac{1}{2\eta}\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla q\|_K^2 + \frac{1}{2\eta_{\rm p}}\|\boldsymbol{\tau}\|^2. \tag{17}$$

Then stability and convergence are established in the (natural) $\|\cdot\|_W$ norm, following [12,24]. Our goal is to prove convergence of the method, keeping as much information as possible on the influence of physical parameters $\eta_{\rm s},\eta_{\rm p}$. Again, we want to prove that the method is reliable for any values $\eta_{\rm s} \geqslant 0$ and $\eta_{\rm p} > 0$, even when $\eta_{\rm s}$ is much smaller than $\eta_{\rm p}$.

The following inverse inequality for the stress will be needed. Since the mesh is regular in the sense of [23], there exists a positive constant C_I, independent of the mesh size, such that

$$C_I \sum_{K\in\mathcal{T}_h} h_K^2\|\operatorname{div}\boldsymbol{\tau}\|_K^2 \leqslant \|\boldsymbol{\tau}\|^2 \quad \forall\boldsymbol{\tau} \in T_h. \tag{18}$$

In the sequel, it will always be assumed that C_I represents the largest positive constant satisfying inequality (18). We then have the following stability result.

Lemma 3 (Stability in norm $\|\cdot\|_h$). *Let $\eta_{\rm s} \geqslant 0$ and $\eta_{\rm p} > 0$. Let B_h be defined by* (15) *with $\alpha > 0$ and $0 < \beta < 1$. There exists a constant $C > 0$, independent of h, $\eta_{\rm s}$ and $\eta_{\rm p}$, but depending on α,β and on the shape of the mesh triangles, such that for all $(\mathbf{v},q,\boldsymbol{\tau}) \in W_h$*

$$B_h(\mathbf{v},q,\boldsymbol{\tau};\mathbf{v},-q,-\boldsymbol{\tau},) \geqslant C\|\mathbf{v},q,\boldsymbol{\tau}\|_h^2. \tag{19}$$

Therefore (14) *has a unique solution.*

Proof of Lemma 3. Let $(\mathbf{v},q,\boldsymbol{\tau})$ be an element of W_h. Since $\mathbf{v}$ is piecewise linear, we have $\operatorname{div}\boldsymbol{\epsilon}(\mathbf{v}) = 0$ in each triangle K of the mesh. Using (15) we then obtain

$$B_h(\mathbf{v},q,\boldsymbol{\tau};\mathbf{v},-q,-\boldsymbol{\tau}) = (2\eta_{\rm s} + 2\eta_{\rm p}\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \frac{1-\beta}{2\eta_{\rm p}}\|\boldsymbol{\tau}\|^2 + \sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_{\rm p}}\|\nabla q - \operatorname{div}\boldsymbol{\tau}\|_K^2.$$

By Young's inequality, we obtain

$$B_h(\mathbf{v},q,\boldsymbol{\tau};\mathbf{v},-q,-\boldsymbol{\tau}) \geqslant (2\eta_s+2\eta_p\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2+\frac{1-\beta}{2\eta_p}\|\boldsymbol{\tau}\|^2 + \frac{\alpha}{2\eta_p}\sum_{K\in\mathcal{T}_h} h_K^2\left((1-\gamma)\|\nabla q\|_K^2+\left(1-\frac{1}{\gamma}\right)\|\operatorname{div}\boldsymbol{\tau}\|_K^2\right)$$

for any $\gamma>0$. To retrieve control on the pressure we have to choose $\gamma<1$. Using (18) we have

$$-\left(\frac{1}{\gamma}-1\right)C_I\sum_{K\in\mathcal{T}_h} h_K^2\|\operatorname{div}\boldsymbol{\tau}\|_k^2 \geqslant -\left(\frac{1}{\gamma}-1\right)\|\boldsymbol{\tau}\|^2.$$

We then obtain the following inequality:

$$B_h(\mathbf{v},q,\boldsymbol{\tau};\mathbf{v},-q,-\boldsymbol{\tau}) \geqslant (2\eta_s+2\eta_p\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2+\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}(1-\gamma)\|\nabla_q\|_K^2+\frac{1}{2\eta_p}\left(1-\beta+\tilde{\alpha}\left(1-\frac{1}{\gamma}\right)\right)\|\boldsymbol{\tau}\|^2, \tag{20}$$

where we have set $\tilde{\alpha}=\alpha/C_I$. It then remain to choose γ such that

$$0<\gamma<1 \quad\text{and}\quad 1-\beta+\tilde{\alpha}\left(1-\frac{1}{\gamma}\right)>0,\quad\text{i.e.,}\quad 1<\frac{1}{\gamma}<1+\frac{1-\beta}{\tilde{\alpha}}.$$

If we choose for instance γ such that

$$\frac{1}{\gamma}=\frac{1}{2}\left(1+1+\frac{1-\beta}{\tilde{\alpha}}\right)=1+\frac{1-\beta}{2\tilde{\alpha}},$$

a simple calculation shows that

$$1-\beta+\tilde{\alpha}\left(1-\frac{1}{\gamma}\right)=\frac{1-\beta}{2}\quad\text{and}\quad 1-\gamma=\left(1+\frac{2\tilde{\alpha}}{1-\beta}\right)^{-1}. \tag{21}$$

Introducing (21) into (20) yields

$$B_h(\mathbf{v},q,\boldsymbol{\tau};\mathbf{v},-q,-\boldsymbol{\tau}) \geqslant (2\eta_s+2\eta_p\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2+\left(1+\frac{2\alpha}{C_I(1-\beta)}\right)^{-1}\sum_{K\in\mathcal{T}_h}\frac{\alpha h_K^2}{2\eta_p}\|\nabla_q\|_K^2+\frac{1-\beta}{4\eta_p}\|\boldsymbol{\tau}\|^2.$$

Since $0<\beta<1$ and $\eta_p\leqslant\eta$ we get

$$B_h(\mathbf{v},q,\boldsymbol{\tau};\mathbf{v},-q,-\boldsymbol{\tau}) \geqslant \beta 2\eta\|\boldsymbol{\epsilon}(\mathbf{v})\|^2+\left(\frac{1}{\alpha}+\frac{2}{C_I(1-\beta)}\right)^{-1}\frac{1}{2\eta}\sum_{K\in\mathcal{T}_h}h_K^2\|\nabla_q\|_K^2+\frac{1-\beta}{2}\frac{1}{2\eta_p}\|\boldsymbol{\tau}\|^2 \tag{22}$$

and thus (19) holds with

$$C=\min\left(\beta,\frac{1-\beta}{2},\left(\frac{1}{\alpha}+\frac{2}{C_I(1-\beta)}\right)^{-1}\right).$$

Finally, existence and uniqueness of the solution of (14) is a direct consequence of (19). Indeed, (19) implies that the kernel of B_h is zero. Therefore the corresponding mapping is injective, and thus bijective since W_h is a finite dimensional subspace of W.

Remark 4. Our numerical interpretation of Lemma 3 is the following. When $\alpha > 0$ and $0 < \beta < 1$, the velocity is controlled for any solvent viscosity η_s, even when $\eta_s = 0$. However, a careful look at the constant C of (19) shows that when η_s and β approach zero simultaneously, the method fails. Similarly, when α approaches zero, or when β approaches one, the method also fails. These predictions are compared to numerical experiments in Section 7.

By Lemma 3, stability can be proved in the (natural) $\|\cdot\|_W$ norm defined in (5), using arguments similar to those introduced in [12].

Lemma 5 (Stability in norm $\|\cdot\|_W$). *Let $\eta_s \geqslant 0$ and $\eta_p > 0$. Let B_h be defined by* (15) *with $\alpha > 0$ and $0 < \beta < 1$. There exists a positive constant C, independent of h, η_s* and *η_p, but depending on Ω, α, β and the shape of the triangles, such that, for all $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W_h$, we have*

$$\sup_{\mathbf{0} \neq (\mathbf{v}, q, \boldsymbol{\tau}) \in W_h} \frac{B_h(\mathbf{u}, p, \boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau})}{\|\mathbf{v}, q, \boldsymbol{\tau}\|_W} \geqslant C \|\mathbf{u}, p, \boldsymbol{\sigma}\|_W. \tag{23}$$

This stability result is a consequence of Lemma 3 together with Lemma 3.3 in [12]. The proof of Lemma 5 is not given here since it is very similar to that of Lemma 2, as well as to that of Lemma 3.2 in [12]. Using Lemma 5, convergence in norm $\|\cdot\|_W$ can be proved.

Theorem 6 (Convergence in norm $\|\cdot\|_W$). *Let $\eta_s \geqslant 0$ and $\eta_p > 0$. Let $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W$ be the solution of* (3), *let $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ be the solution of* (14), *where B_h is defined by* (15) *with $\alpha > 0$ and $0 < \beta < 1$. Moreover, assume that $(\mathbf{u}, p, \boldsymbol{\sigma}) \in H^2(\Omega)^2 \times H^1(\Omega) \times H^1(\Omega)^4$. Then, there exists a positive constant C, independent of h, η_s and η_p, but depending on Ω, α, β and the shape of the triangles, such that*

$$\|\mathbf{u} - \mathbf{u}_h, p - p_h, \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_W^2 \leqslant C\left(1 + \frac{\eta_s}{\eta_p}\right)^2 h^2 \left(\eta \|\mathbf{u}\|^2_{H^2(\Omega)^2} + \frac{1}{\eta}\|p\|^2_{H^1(\Omega)} + \frac{1}{\eta_p}\|\boldsymbol{\sigma}\|^2_{H^1(\Omega)^4}\right). \tag{24}$$

In order to prove Theorem 6, we need as in [12] a technical lemma.

Lemma 7. *Let $\eta_s \geqslant 0$ and $\eta_p > 0$. Consider the bilinear form B_h defined by* (15) *with $\alpha > 0$ and $0 < \beta < 1$. Then, there exists a positive constant C, independent of h, η_s and η_p, but depending on Ω, α, β and the shape of the triangles, such that, for all*

$$(\mathbf{u}, p, \boldsymbol{\sigma}) \in W \cap \prod_{K \in \mathscr{T}_h} \left(H^2(K)^2 \times H^1(K) \times H^1(K)^4\right)$$

and for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$ the following inequality holds:

$$|B_h(\mathbf{u}, p, \boldsymbol{\sigma}; \mathbf{v}, q, \boldsymbol{\tau})| \leqslant C\left(1 + \frac{\eta_s}{\eta_p}\right) |G_h(\mathbf{u}, p, \boldsymbol{\sigma})| \|\mathbf{v}, q, \boldsymbol{\tau}\|_W, \tag{25}$$

where G_h is defined by

$$|G_h(\mathbf{u}, p, \boldsymbol{\sigma})|^2 = \eta \|\boldsymbol{\epsilon}(\mathbf{u})\|^2 + \frac{1}{\eta}\|p\|^2 + \frac{1}{\eta_p}\|\boldsymbol{\sigma}\|^2 + \frac{1}{\eta}\sum_{K \in \mathscr{T}_h} h_K^2 \|\mathscr{L}_1(\mathbf{u}, p, \boldsymbol{\sigma})\|_K^2. \tag{26}$$

Proof of Lemma 7. Let us define $\mathbf{w}_1 = (\mathbf{u}, p, \boldsymbol{\sigma})$ and $\mathbf{w}_0 = (\mathbf{v}, q, \boldsymbol{\tau})$. Using (15) and since $\operatorname{div} \boldsymbol{\epsilon}(\mathbf{v}) = 0$ on each triangle, we have

$$B_h(\mathbf{w}_1,\mathbf{w}_0) = (2\eta_s + 2\eta_p\beta)(\boldsymbol{\epsilon}(\mathbf{u}),\boldsymbol{\epsilon}(\mathbf{v})) - (p,\operatorname{div}\mathbf{v}) + (1-\beta)(\sigma,\boldsymbol{\epsilon}(\mathbf{v})) - (\operatorname{div}\mathbf{u},q) - \frac{1-\beta}{2\eta_p}(\boldsymbol{\sigma},\boldsymbol{\tau}) + (1-\beta) \times (\boldsymbol{\epsilon}(\mathbf{u}),\boldsymbol{\tau}) - \frac{\alpha}{2\eta_p}\sum_{K\in\mathscr{T}_h} h_K^2(\mathscr{L}_1\mathbf{w}_1, \nabla_q - \operatorname{div}\boldsymbol{\tau})_K.$$

Using the discrete Cauchy–Schwarz inequality we have

$$B_h(\mathbf{w}_1,\mathbf{w}_0) \leqslant \Bigg((2\eta_s + 2\eta_p\beta)\|\boldsymbol{\epsilon}(\mathbf{u})\|^2 + \frac{1}{2\eta}\|p\|^2 + \frac{1-\beta}{2\eta_p}\|\boldsymbol{\sigma}\|^2 + 2\eta\|\operatorname{div}\mathbf{u}\|^2 + \frac{1-\beta}{2\eta_p}\|\boldsymbol{\sigma}\|^2 + 2\eta_p(1-\beta)\|\boldsymbol{\epsilon}(\mathbf{u})\|^2 + 2\frac{\alpha}{2\eta_p}\sum_{K\in\mathscr{T}_h} h_K^2\|\mathscr{L}_1\mathbf{w}_1\|_K^2\Bigg)^{1/2}$$
$$\times \Bigg((2\eta_s + 2\eta_p\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + 2\eta\|\operatorname{div}\mathbf{v}\|^2 + 2\eta_p(1-\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \frac{1}{2\eta}\|q\|^2 + \frac{1-\beta}{2\eta_p}\|\boldsymbol{\tau}\|^2 + \frac{1-\beta}{2\eta_p}\|\boldsymbol{\tau}\|^2 + \frac{\alpha}{2\eta_p}\sum_{K\in\mathscr{T}_h} h_k^2\Big(\|\nabla_q\|_K^2 + \|\operatorname{div}\boldsymbol{\tau}\|_K^2\Big)\Bigg)^{1/2}.$$

We need now the following inverse inequality for the pressure. Since the mesh is regular in the sense of [23], there exists a positive constant C_I^p independent of the mesh size, such that

$$C_I^p \sum_{K\in\mathscr{T}_h} h_K^2\|\nabla_q\|_K^2 \leqslant \|q\|^2 \quad \forall q\in Q_h. \tag{27}$$

Using Korn's inequality and the inverse inequalities for the stress (18) and for the pressure (27) we obtain

$$|B_h(\mathbf{w}_1,\mathbf{w}_0)| \leqslant \Bigg(2\eta(1+C^K)\|\boldsymbol{\epsilon}(\mathbf{u})\|^2 + \frac{1}{2\eta}\|p\|^2 + \frac{1-\beta}{\eta_p}\|\boldsymbol{\sigma}\|^2 + \frac{\alpha}{\eta_p}\sum_{K\in\mathscr{T}_h} h_K^2\|\mathscr{L}_1\mathbf{w}_1\|_K^2\Bigg)^{1/2} \times \Bigg(2\eta(1+C^K)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \frac{1+\alpha\eta/C_I^p\eta_p}{2\eta}\|q\|^2 + \frac{1-\beta+\alpha/C_I}{2\eta_p}\|\boldsymbol{\tau}\|^2\Bigg)^{1/2},$$

where C^K is the constant of the Korn inequality. The above estimate and the definition of norm $\|\cdot\|_W$ yield (25) with C proportional to

$$\max\left(1, C^K, \alpha, \frac{\alpha}{C_I^p}, \frac{\alpha}{C_I}\right)\left(1 + \frac{\eta_s}{\eta_p}\right).$$

Proof of Theorem 6. Let us set $\mathbf{w} = (\mathbf{u}, p, \boldsymbol{\sigma}), \mathbf{w}_h = (\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ and choose $\rho_h\mathbf{w}$ the interpolant defined by

$$\rho_h\mathbf{w} = (\mathbf{r}_h\mathbf{u}, R_h p, \mathbf{R}_h\boldsymbol{\sigma}),$$

where $\mathbf{r}_h$ denotes the standard Lagrange interpolant on vectors, R_h and $\mathbf{R}_h$ are the interpolant of Clément [25,26] on scalars and tensors, respectively. Using Young's inequality one can write

$$\|\mathbf{w}-\mathbf{w}_h\|_W^2 \leqslant 2\|\mathbf{w}-\rho_h\mathbf{w}\|_W^2 + 2\|\rho_h\mathbf{w}-\mathbf{w}_h\|_W^2. \tag{28}$$

By Lemma 5, there exists a constant C_1 (independent of h, η_s and η_p, but depending on Ω, α, β and on the shape of the triangles) and $\mathbf{w}_0 \in \mathbf{W}_h$ satisfying

$$B_h(\rho_h\mathbf{w}-\mathbf{w}_h, \mathbf{w}_0) \geqslant C_1\|\rho_h\mathbf{w}-\mathbf{w}_h\|_W\|\mathbf{w}_0\|_W.$$

Using the consistency, Eq. (16), we have

$$B_h(\rho_h\mathbf{w} - \mathbf{w}, \mathbf{w}_0) \geqslant C_1\|\rho_h\mathbf{w} - \mathbf{w}_h\|_W\|\mathbf{w}_0\|_W.$$

On the other hand, by Lemma 7 there exists a constant C_2 (independent of h, η_s and η_p, but depending on Ω, α, β and the shape of the triangles) such that

$$B_h(\rho_h\mathbf{w} - \mathbf{w}, \mathbf{w}_0) \leqslant C_2\left(1 + \frac{\eta_s}{\eta_p}\right)|G_h(\mathbf{w} - \rho_h\mathbf{w})|\|\mathbf{w}_0\|_W.$$

By putting together the last two estimates we obtain

$$\|\rho_h\mathbf{w} - \mathbf{w}_h\|_W \leqslant \frac{C_2}{C_1}\left(1 + \frac{\eta_s}{\eta_p}\right)|G_h(\mathbf{w} - \rho_h\mathbf{w})|. \tag{29}$$

Estimate (29) into (28) yields

$$\|\mathbf{w} - \mathbf{w}_h\|_W^2 \leqslant 2\|\mathbf{w} - \rho_h\mathbf{w}\|_W^2 + 2\left(\frac{C_2}{C_1}\right)^2\left(1 + \frac{\eta_s}{\eta_p}\right)^2|G_h(\mathbf{w} - \rho_h\mathbf{w})|^2. \tag{30}$$

From the definitions of $\mathbf{w}, \rho_h\mathbf{w}, \|\cdot\|_W$ and $G_h(\cdot)$, we have

$$\|\mathbf{w} - \mathbf{w}_h\|_W^2 \leqslant 2\left(2 + \frac{C_2^2}{C_1^2}\right)\left(1 + \frac{\eta_s}{\eta_p}\right)^2\left(\eta\|\epsilon(\mathbf{u} - \mathbf{r}_h\mathbf{u})\|^2 + \frac{1}{\eta}\|p - R_hp\|^2 + \frac{1}{\eta_p}\|\boldsymbol{\sigma} - \mathbf{R}_h\boldsymbol{\sigma}\|^2 \right.$$
$$\left. + \frac{1}{\eta}\sum_{K\in\mathscr{T}_h} h_K^2\|\mathscr{L}_1(\mathbf{w} - \rho_h\mathbf{w})\|_K^2\right).$$

Using standard interpolation estimates (see for instance [23] for $\mathbf{r}_h$, and [25,26] for $\mathbf{R}_h$ and $\mathbf{R}_h$), and considering the forms of the constants C_1 and C_2, we obtain (24) with C independent of η_s and η_p.

Finally, convergence can be proved in the L^2 norm for the velocity by using the Aubin–Nitsche trick. Let $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W$ be the solution of (3) and assume that it is regular enough, in the sense that there exists a positive constant C, independent of h, η_s and η_p, but depending on Ω, such that

$$\eta\|\mathbf{u}\|_{H^2(\Omega)^2}^2 + \frac{1}{\eta}\|p\|_{H^1(\Omega)}^2 + \frac{1}{\eta_p}\|\boldsymbol{\sigma}\|_{H^1(\Omega)^4}^2 \leqslant C\frac{1}{\eta}\|\mathbf{f}_1, f_2, \mathbf{f}_3\|_*^2. \tag{31}$$

Let $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ be the solution of (14), where B_h is defined by (15) with $\alpha > 0$ and $0 < \beta < 1$. Then, there exists a positive constant $\tilde{C}$, independent of h, η_s and η_p, but depending on Ω, α, β and on the shape of the triangles, such that

$$\|\mathbf{u} - \mathbf{u}_h\| \leqslant \tilde{C}\left(1 + \frac{\eta_s}{\eta_p}\right)^4 h^2\left(\|\mathbf{u}\|_{H^2(\Omega)^2} + \frac{1}{\eta}\|p\|_{H^1(\Omega)} + \frac{1}{\eta_p}\|\boldsymbol{\sigma}\|_{H^1(\Omega)^4}\right). \tag{32}$$

4. A reduced GLS method

The GLS method introduced in the previous section is stable and convergent, but requires several stabilization terms to be added. If an Oldroyd-B model was used instead of the Stokes model, the number of stabilization terms would be much greater since convective and tensor derivatives would have to be added. Our aim is thus to find which of the GLS terms are important for the stability, without loosing on consistency. For this purpose, the following reduced GLS method is proposed. Find $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ such that

$$B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) = F_h(\mathbf{v}, q, \boldsymbol{\tau}) \tag{33}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$, where the (non-symmetric) bilinear form B_h is defined by

$$\begin{aligned} B_h(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) &= B(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h; \mathbf{v}, q, \boldsymbol{\tau}) - \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_{\mathrm{p}}} (- 2\eta_{\mathrm{s}} \operatorname{div} \boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \operatorname{div} \boldsymbol{\sigma}_h, \nabla q)_K \\ &\quad + 2\eta_{\mathrm{p}}\beta \left(\frac{1}{2\eta_{\mathrm{p}}} \boldsymbol{\sigma}_h - \boldsymbol{\epsilon}(\mathbf{u}_h), - \boldsymbol{\epsilon}(\mathbf{v}) \right) \end{aligned} \tag{34}$$

and the form F_h by

$$F_h(\mathbf{v}, q, \boldsymbol{\tau}) = F(\mathbf{v}, q, \boldsymbol{\tau}) - \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_{\mathrm{p}}} (\mathbf{f}_1, \nabla q)_K + 2\eta_{\mathrm{p}}\beta(\mathbf{f}_3, \boldsymbol{\epsilon}(\mathbf{v}))$$

with the dimensionless parameters $\alpha > 0$ and $0 < \beta < 2$. Let us point out that scheme (33) is consistent in the sense that, if the solution $(\mathbf{u}, p, \boldsymbol{\sigma}) \in W$ of the weak formulation (3) is regular enough and if $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h) \in W_h$ is the solution of (33), then Eq. (16) holds for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$.

In the case of this reduced GLS scheme, we shall first prove stability in the discrete norm $\|\cdot\|_h$. The stability and convergence in the norm $\|\cdot\|_W$ will simply be recalled without proofs.

Lemma 8 (Stability in norm $\|\cdot\|_h$). *Let $\eta_{\mathrm{s}} \geqslant 0$ and $\eta_{\mathrm{p}} > 0$. Let B_h be defined by* (34) *with $0 < \beta < 2$ and $0 < \alpha < 2C_I$, where C_I is the largest positive constant that satisfies the inverse inequality* (18). *Then, there exists a constant $C > 0$, independent of h, η_{s} and η_{p}, but depending on Ω, α, β and the shape of the triangles, such that, for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$ we have*

$$B_h(\mathbf{v}, q, \boldsymbol{\tau}; \mathbf{v}, -q, -\boldsymbol{\tau}) \geqslant C\|\mathbf{v}, q, \boldsymbol{\tau}\|_h^2. \tag{35}$$

Therefore (33) *has a unique solution.*

Proof of Lemma 8. By definition of B_h we have

$$\begin{aligned} B_h(\mathbf{v}, q, \boldsymbol{\tau}; \mathbf{v}, -q, -\boldsymbol{\tau}) &= (2\eta_{\mathrm{s}} + 2\eta_{\mathrm{p}}\beta)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \frac{1}{2\eta_{\mathrm{p}}}\|\boldsymbol{\tau}\|^2 - \beta(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v})) \\ &\quad + \sum_{K \in \mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_{\mathrm{p}}} \left(\|\nabla q\|_K^2 - (\operatorname{div} \boldsymbol{\tau}, \nabla q)_K \right). \end{aligned}$$

Using Young's inequality, we have

$$-(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\mathbf{v})) \geqslant -\frac{2\eta_{\mathrm{p}}\gamma_1}{2}\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 - \frac{1}{2\eta_{\mathrm{p}}2\gamma_1}\|\boldsymbol{\tau}\|^2, \quad -(\operatorname{div} \boldsymbol{\tau}, \nabla q)_K \geqslant -\frac{\gamma_2}{2}\|\nabla q\|_K^2 - \frac{1}{2\gamma_2}\|\operatorname{div} \boldsymbol{\tau}\|_K^2,$$

where γ_1 and γ_2 are positive constants to be chosen. Using (18) and the above two inequalities we obtain

$$\begin{aligned} B_h(\mathbf{v}, q, \boldsymbol{\tau}; \mathbf{v}, -q, -\boldsymbol{\tau}) &\geqslant \left(2\eta_{\mathrm{s}} + 2\eta_{\mathrm{p}}\beta\left(1 - \frac{\gamma_1}{2}\right)\right)\|\boldsymbol{\epsilon}(\mathbf{v})\|^2 + \frac{1}{2\eta_{\mathrm{p}}}\left(1 - \frac{\tilde{\alpha}}{2\gamma_2} - \frac{\beta}{2\gamma_1}\right)\|\boldsymbol{\tau}\|^2 \\ &\quad + \frac{\alpha}{2\eta_{\mathrm{p}}}\left(1 - \frac{\gamma_2}{2}\right) \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla q\|_K^2, \end{aligned} \tag{36}$$

where we have set $\tilde{\alpha} = \alpha / C_I$. Clearly, we have to choose γ_1 and γ_2 such that

$$0 < \gamma_1 < 2, \quad 0 < \gamma_2 < 2 \quad \text{and} \quad 1 - \frac{\tilde{\alpha}}{2\gamma_2} - \frac{\beta}{2\gamma_1} > 0$$

for the three terms in the right-hand side of (36) to be positive. A possibility is to choose γ_1 and γ_2 such that

$$\frac{1}{2}-\frac{\beta}{2\gamma_1}>0 \quad \text{and} \quad \frac{1}{2}-\frac{\tilde{\alpha}}{2\gamma_2}>0,$$

i.e.,

$$0<\beta<\gamma_1<2 \quad \text{and} \quad 0<\tilde{\alpha}<\gamma_2<2.$$

Choosing for instance $\gamma_1 = 1+(\beta/2)$ and $\gamma_2 = 1+(\tilde{\alpha}/2)$, a simple calculation shows that

$$1-\frac{\gamma_1}{2}=\frac{2-\beta}{4}>0, \quad 1-\frac{\gamma_2}{2}=\frac{2-\tilde{\alpha}}{4}>0 \quad \text{and} \quad 1-\frac{\tilde{\alpha}}{2\gamma_2}-\frac{\beta}{2\gamma_1}=\frac{1}{2}\left(\frac{2-\beta}{2+\beta}+\frac{2-\tilde{\alpha}}{2+\tilde{\alpha}}\right)>0. \tag{37}$$

Finally (37) in (36) yields (35), with C proportional to

$$\min\left(\beta(2-\beta), 1-\frac{\alpha}{2C_I}, \alpha\left(1-\frac{\alpha}{2C_I}\right)\right).$$

The same argument as for Lemma 3 ensures existence and uniqueness of the solution of (33).

Remark 9. As for the previous GLS method, our numerical interpretation of Lemma 8 is the following. The velocity is controlled for any η_s, even when $\eta_s = 0$. Moreover, a careful analysis of the constant of (35) shows that the control on the velocity is maximum when $\beta = 1$ and that the control on the pressure is maximum when $\alpha = C_I$. When β approaches 2 or when α approaches $2C_I$, the method can be expected to fail. These predictions are compared to experiments in Section 7.

As in the previous section, stability and convergence can be proved for the reduced GLS scheme (33) in the norm $\|\cdot\|_W$ defined in (5). Using definition (34) instead of (15) for the bilinear form B_h and assuming that $0 < \alpha < 2C_I$ and $0 < \beta < 2$, the stability result (23) of Lemma 5 and the convergence estimate (24) of Theorem 6 hold unaltered for the reduced GLS scheme of this section. The proofs are not given here, since they are very similar to those of Section 3.

5. Link between reduced GLS and modified EVSS methods

In this section, a modified version of the so-called modified EVSS method of [1] is presented. Then the link with the reduced GLS method of the previous section is presented.

The modified EVSS method analysed in [1] consists in adding to the problem an extra unknown for stability purposes. More precisely, the method consists in solving, instead of (1), the following set of partial differential equations:

$$\begin{aligned} &-2(\eta_s+\delta)\operatorname{div}\boldsymbol{\epsilon}(\mathbf{u})+\nabla p-\operatorname{div}\boldsymbol{\sigma}+2\delta\operatorname{div}\mathbf{d}=\mathbf{f}_1, \quad \operatorname{div}\mathbf{u}=f_2,\\ &\frac{1}{2\eta_p}\boldsymbol{\sigma}-\boldsymbol{\epsilon}(\mathbf{u})=\mathbf{f}_3, \quad \mathbf{d}-\boldsymbol{\epsilon}(\mathbf{u})=0. \end{aligned} \tag{38}$$

Here $\mathbf{d}$ is the rate of deformation introduced for stability purposes, and δ is a positive parameter that must be chosen carefully to ensure stability. Again, for viscoelastic flows, the third equation of (38) should be replaced by the extra-stress constitutive equation, for instance (2) if the Oldroyd-B model is considered.

In [1], it was proved that the modified EVSS scheme was stable and convergent. The finite element spaces for the velocity and pressure had to satisfy the inf–sup condition (8), but there was no additional condition on the finite element space for the stresses. This discussion aims at showing that the reduced GLS stabilization method of Section 4 is related to some modified EVSS scheme.

More precisely, let V_h, Q_h, T_h be the continuous, piecewise linear finite element spaces defined in (12), the following method is considered for solving the four-field Stokes problem (38):

Find $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \mathbf{d}_h) \in V_h \times Q_h \times T_h \times T_h$ such that

$$\begin{aligned} & 2(\eta_s + \delta)(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v})) - (p_h, \operatorname{div}\mathbf{v}) + (\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v})) - 2\delta(\mathbf{d}_h, \boldsymbol{\epsilon}(\mathbf{v})) - (\operatorname{div}\mathbf{u}_h, q) \\ & \quad - \sum_{K \in \mathscr{T}_h} \frac{\alpha h_K^2}{2\eta_p}(- 2\eta_s \operatorname{div}\boldsymbol{\epsilon}(\mathbf{u}_h) + \nabla p_h - \operatorname{div}\boldsymbol{\sigma}_h, \nabla q)_K - \frac{1}{2\eta_P}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau}) \\ & \quad + 2\delta(\mathbf{d}_h, \mathbf{e}) - 2\delta(\boldsymbol{\epsilon}(\mathbf{u}_h), \mathbf{e}) = (\mathbf{f}_1, \mathbf{v}) - (f_2, q) - (\mathbf{f}_3, \boldsymbol{\tau}) - \sum_{K \in \mathscr{T}_h} \frac{\alpha h_K^2}{2\eta_p}(\mathbf{f}_1, \nabla q)_K \end{aligned} \tag{39}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}, \mathbf{e}) \in V_h \times Q_h \times T_h \times T_h$.

Note that in (39) a stabilizing term corresponding to the first equation of (38) has been added, aiming at avoiding spurious oscillations in the pressure by circumventing (8), as in GLS methods for the two-field Stokes problem. This is the main difference between method (39) and the EVSS method of [1,2], which must still satisfy (8) in the finite element subspaces. According to [27], (39) can also be seen as a continuous, piecewise linear approximation of (38) with bubble stabilization and static condensation. Note also that the stabilizing terms are not added in a fully consistent way, since the term $2\delta \operatorname{div}\mathbf{d}_h$ is missing.

Since no stabilization terms are added explicitly for the velocity-stress discretization, method (39) can be considered as a modified EVSS method, in the sense that attention is paid only to the velocity–pressure discretization and not to the velocity-stress discretization.

We show here that the modified EVSS formulation (39) is equivalent to the reduced GLS method of Section 4 in the sense of the following lemma.

Lemma 10. *Assume that $\mathbf{f}_3 \in T_h$. Let $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ be the solution of* (33) *and let $\mathbf{d}_h \in T_h$ be defined by*

$$(\mathbf{d}_h - \boldsymbol{\epsilon}(\mathbf{u}_h), \mathbf{e}) = 0 \tag{40}$$

for all $\mathbf{e}$ in T_h. Then $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \mathbf{d}_h)$ is also the solution of (39), *provided we set $\delta = \eta_p\beta$. Conversely, let $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \mathbf{d}_h)$ be the solution of* (39) *with $\delta = \eta_p\beta$. Then, $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h,)$ is solution of* (33).

Proof of Lemma 10. Let $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ be the solution of (33). If we choose $\mathbf{v} = 0, q = 0$ in (33), we obtain

$$\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau}\right) = (\mathbf{f}_3, \boldsymbol{\tau})$$

for all $\boldsymbol{\tau}$ in T_h. Using the definition of $\mathbf{d}_h$, Eq. (40), we clearly have

$$\mathbf{d}_h = \frac{1}{2\eta_p}\boldsymbol{\sigma}_h - \mathbf{f}_3. \tag{41}$$

Now, since $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ is the solution of (33), we have

$$\begin{aligned} & (2\eta_s + 2\eta_p\beta)(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v})) - (p_h, \operatorname{div}\mathbf{v}) + (1-\beta)(\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v})) - (\operatorname{div}\mathbf{u}_h, q) - \sum_{K \in \mathscr{T}_h} \frac{\alpha h_K^2}{2\eta_p}(\nabla p_h - \operatorname{div}\boldsymbol{\sigma}_h, \nabla q)_K \\ & \quad - \frac{1}{2\eta_p}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau}) = (\mathbf{f}_1, \mathbf{v}) - (f_2, q) - (\mathbf{f}_3, \boldsymbol{\tau}) - \sum_{K \in \tau_h} \frac{\alpha h_K^2}{2\eta_p}(\mathbf{f}_1, \nabla q)_K + 2\eta_p\beta(\mathbf{f}_3, -\boldsymbol{\epsilon}(\mathbf{v})) \end{aligned}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}) \in W_h$. Using relation (41), we replace the term $\beta(\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v}))$ by $2\eta_p\beta(\mathbf{d}_h + \mathbf{f}_3, \boldsymbol{\epsilon}(\mathbf{v}))$ in the above equation, and add (40). Then $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h, \mathbf{d}_h)$ satisfies

$$\begin{aligned} & (2\eta_s + 2\eta_p\beta)(\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\epsilon}(\mathbf{v})) - (p_h, \operatorname{div}\mathbf{v}) + (\boldsymbol{\sigma}_h, \boldsymbol{\epsilon}(\mathbf{v})) - 2\eta_p\beta(\mathbf{d}_h, \boldsymbol{\epsilon}(\mathbf{v})) - (\operatorname{div}\mathbf{u}_h, q) \\ & \quad - \sum_{K \in \mathscr{T}_h} \frac{\alpha h_K^2}{2\eta_p}(\nabla p_h - \operatorname{div}\boldsymbol{\sigma}_h, \nabla q)_K - \frac{1}{2\eta_p}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\boldsymbol{\epsilon}(\mathbf{u}_h), \boldsymbol{\tau}) + 2\eta_p\beta(\mathbf{d}_h, \mathbf{e}) - 2\eta_p\beta(\boldsymbol{\epsilon}(\mathbf{u}_h), \mathbf{e}) \\ & \qquad = (\mathbf{f}_1, \mathbf{v}) - (f_2, q) - (\mathbf{f}_3, \boldsymbol{\tau}) - \sum_{K \in \mathscr{T}_h} \frac{\alpha h_K^2}{2\eta_p}(\mathbf{f}_1, \nabla_q)_K \end{aligned}$$

for all $(\mathbf{v}, q, \boldsymbol{\tau}, \mathbf{e}) \in V_h \times Q_h \times T_h \times T_h$. This is nothing but Eq. (39), provided we set $\delta = \eta_p\beta$.

6. A stable iterative GLS method

In this section, a simple iterative algorithm is presented for solving (33), similar to the method presented in [1,2]. The interest of such an algorithm is to de-couple velocity–pressure computations from stress computations, thus leading to smaller linear systems.

Let $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$ be the known approximation of $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ after n steps. Roughly speaking, step $(n+1)$ of the algorithm consists in first computing $(\mathbf{u}_h^{n+1}, p_h^{n+1})$ by solving the incompressible Navier–Stokes equations using the previous stress $\boldsymbol{\sigma}_h^n$ in the right-hand side of the linear system, and then computing σ_h^{n+1} by using its constitutive equation, with the new velocity $\mathbf{u}_h^{n+1}$ in the right-hand side. Thus iteration $(n+1)$ consists in

1. Seeking $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that for all $(\mathbf{v}, q) \in V_h \times Q_h$

$$
\begin{aligned}
&(2\eta_s + 2\eta_p\beta)\big(\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \boldsymbol{\epsilon}(\mathbf{v})\big) - (p_h^{n+1}, \operatorname{div}\mathbf{v}) - (\operatorname{div}\mathbf{u}_h^{n+1}, q) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} (\nabla p_h^{n+1}, \nabla q)_K \\
&\quad = (\mathbf{f}_1, \mathbf{v}) - (f_2, q) - (1-\beta)(\boldsymbol{\sigma}_h^n, \boldsymbol{\epsilon}(\mathbf{v})) - \sum_{K\in\mathcal{T}_h} \frac{\alpha h_K^2}{2\eta_p} (\mathbf{f}_1 + \operatorname{div}\boldsymbol{\sigma}_h^n, \nabla q)_K + 2\eta_p\beta(\mathbf{f}_3, -\boldsymbol{\epsilon}(\mathbf{v})).
\end{aligned}
\tag{42}
$$

2. Seeking $\boldsymbol{\sigma}_h^{n+1} \in T_h$ such that for all $\boldsymbol{\tau} \in T_h$

$$
\left(\frac{1}{2\eta_p}\boldsymbol{\sigma}_h^{n+1}, \boldsymbol{\tau}\right) = (\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \boldsymbol{\tau}) + (\mathbf{f}_3, \boldsymbol{\tau}). \tag{43}
$$

The following stability result holds for this algorithm.

Lemma 11 (Stability). *Let $\mathbf{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\sigma}_h^{n+1}$ be the solution of* (42) *and* (43) *with $\mathbf{f}_1 = f_2 = \mathbf{f}_3 = 0$, $\alpha = C_I$ and $\beta = 1$, where C_I be the largest positive constant that satisfies* (18). *Then for all $n \geqslant 0$, we have*

$$
(2\eta_s + \eta_p)\|\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1})\|^2 + \frac{C_I}{4\eta_p}\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla p_h^{n+1}\|_K^2 + \frac{1}{4\eta_p}\|\boldsymbol{\sigma}_h^{n+1}\|^2 \leqslant \frac{1}{4\eta_p}\|\boldsymbol{\sigma}_h^n\|^2. \tag{44}
$$

Remark 12. From (44) we deduce that the sequence $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)_{n\geqslant 0}$ is a lower bounded decreasing sequence. Thus the solution of (42) and (43) converges towards the unique solution of (33) when $\mathbf{f}_1 = f_2 = \mathbf{f}_3 = 0$, which is 0.

Remark 13. This lemma ensures stability of the decoupling algorithm hereabove for the optimal values of stabilization parameters α and β discussed in Remark 9. As will be seen in Section 7, numerical experiments confirm these predictions.

Remark 14. The usual mass lumping procedure can be used to integrate numerically the mass matrix in Eq. (43). In this case, the scheme remains stable and (43) reduces to an explicit formula to compute the stress at the nodes of the finite element mesh.

Proof of Lemma 11. We set $\alpha = C_I$ and $\beta = 1$ in (42) and (43) and choose $(\mathbf{v}, q, \boldsymbol{\tau}) = \big(\mathbf{u}_h^{n+1}, -p_h^{n+1}, -\boldsymbol{\sigma}_h^{n+1}\big)$ to obtain

$$
\begin{aligned}
&(2\eta_s + 2\eta_p)\|\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1})\|^2 + \frac{C_I}{2\eta_p}\sum_{K\in\mathcal{T}_h} h_K^2 \|\nabla p_h^{n+1}\|_K^2 + \frac{1}{2\eta_p}\|\boldsymbol{\sigma}_h^{n+1}\|^2 - \big(\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \boldsymbol{\sigma}_h^{n+1}\big) \\
&\quad = \frac{C_I}{2\eta_p}\sum_{K\in\mathcal{T}_h} h_K^2 \big(\operatorname{div}\boldsymbol{\sigma}_h^n, \nabla p_h^{n+1}\big)_K.
\end{aligned}
\tag{45}
$$

Young's inequality yields

$$\left|\left(\operatorname{div}\boldsymbol{\sigma}_h^n, \nabla p_h^{n+1}\right)_K\right| \leqslant \frac{\gamma_1}{2}\|\nabla p_h^{n+1}\|_K^2 + \frac{1}{2\gamma_1}\|\operatorname{div}\boldsymbol{\sigma}_h^n\|_K^2,$$

$$\left|\left(\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1}), \boldsymbol{\sigma}_h^{n+1}\right)\right| \leqslant 2\eta_{\mathrm{p}}\frac{\gamma_2}{2}\|\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1})\|^2 + \frac{1}{2\eta_{\mathrm{p}}2\gamma_2}\|\boldsymbol{\sigma}_h^{n+1}\|^2.$$

By introducing these inequalities into (45) we obtain

$$\left(2\eta_{\mathrm{s}} + 2\eta_{\mathrm{p}}\left(1 - \frac{\gamma_2}{2}\right)\right)\|\boldsymbol{\epsilon}(\mathbf{u}_h^{n+1})\|^2 + \frac{C_I}{2\eta_{\mathrm{p}}}\left(1 - \frac{\gamma_1}{2}\right)\sum_{K\in\mathscr{T}_h} h_K^2\|\nabla p_h^{n+1}\|_K^2 + \frac{1}{2\eta_{\mathrm{p}}}\left(1 - \frac{1}{2\gamma_2}\right)\|\boldsymbol{\sigma}_h^{n+1}\|^2$$
$$\leqslant \frac{1}{2\eta_{\mathrm{p}}}\frac{1}{2\gamma_1}\|\boldsymbol{\sigma}_h^n\|^2.$$

Then, setting $\gamma_1 = \gamma_2 = 1$ yields (44).

7. Numerical results

Numerical results of computations of the three-field Stokes problem (1) in the 4:1 abrupt contraction flow case are presented and compared to [1]. The symmetry of the geometry is used to reduce the computation domain by half, as shown in Fig. 1. The computations were performed on three meshes: the rather coarse, structured mesh `M1` shown with the geometry definitions in Fig. 1, the coarse, unstructured mesh `M2` and the refined, unstructured mesh `M3`, both shown in Fig. 2. The authors of [2] found coarse, structured meshes like `M1` to be more likely to reveal stability problems, whereas unstructured meshes like `M2` or more refined, unstructured meshes like `M3` tended to mask the instabilities.

In every computation, zero Dirichlet boundary conditions are imposed on the walls, the Poiseuille velocity profile $u_x(y) = V_u(1 - (y/R_u)^2)$ is imposed at the inlet, natural boundary conditions on the symmetry axis and at the outlet of the domain. We choose $V_u = 0.15$ so the flow rate is $D = 0.1$, all the physical parameters being expressed in SI units.

The results corresponding to the GLS formulations (14) and (33) are presented first, in a case where the stabilization of the third equation is of importance, that is when η_{s} is small compared to η_{p}. Then the results for the iterative algorithm (42) and (43) are presented.

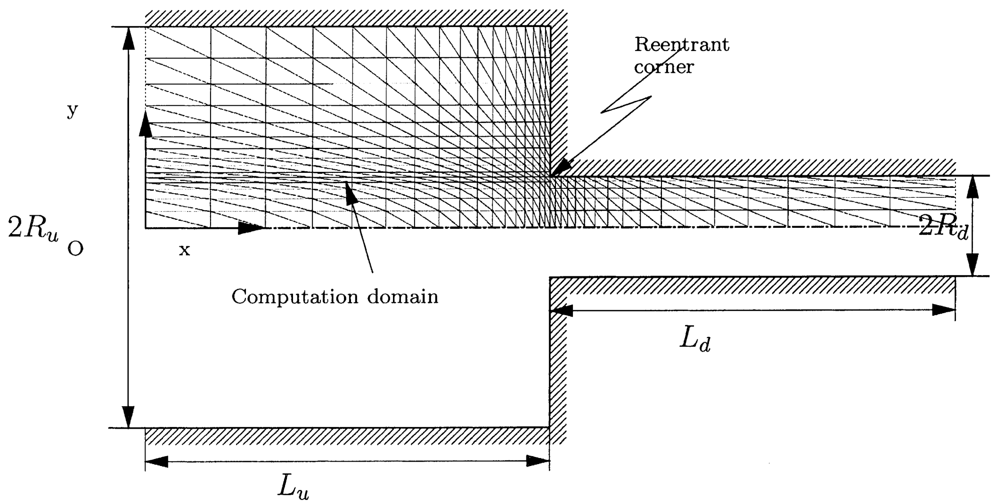

Fig. 1. Geometry and mesh `M1` used for the 4:1 abrupt contraction flow simulations, with the values $R_u = 1$, $R_d = 0.25$ and $L_u = L_d = 2$.

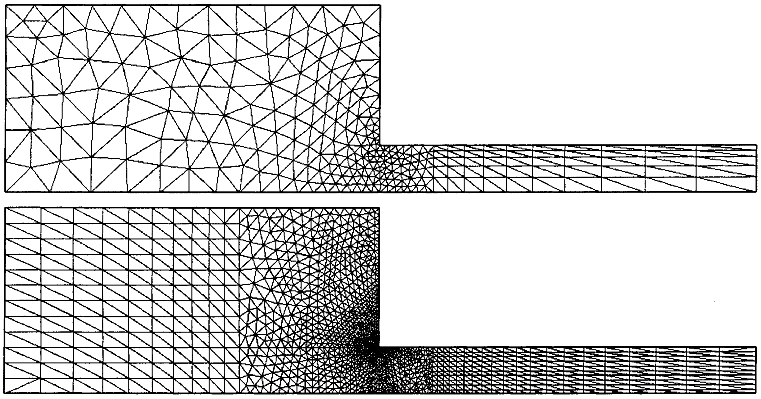

Fig. 2. Meshes M2 (top) and M3 (bottom) used for the 4:1 abrupt contraction flow simulations.

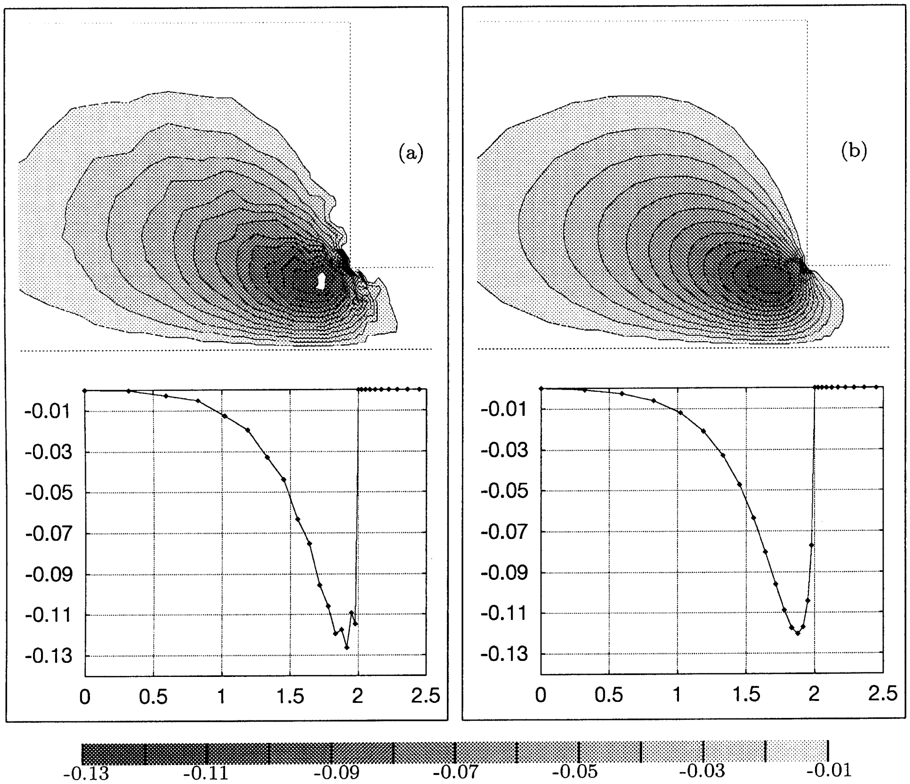

Fig. 3. Isolines and cross-section at $y = 0.25$ of the vertical velocity component u_y for the coupled resolution of the three-field Stokes problem with $\eta_s = 0.01$, $\eta_p = 1$, $\alpha = 0.01$, on mesh M1 and with the reduced GLS method (33): (a) $\beta = 0$, and (b) $\beta = 1$.

7.1. Coupled resolution

For the computations presented in this section, the linear system corresponding to the GLS schemes (14) and (33) was solved in one block, yielding the velocity, pressure and stress in one step.

The viscosity constants were set to $\eta_s = 0.01$ and $\eta_p = 1$, which are realistic values in the case of non-newtonian fluids. Both stabilization schemes are used, with parameter $\alpha = 0.01$ in all the coupled computations presented here. The reasons for this choice are the following. Many computations on other test cases (e.g., quadratic exact solution on unit square) showed the optimal values of α to be approximately

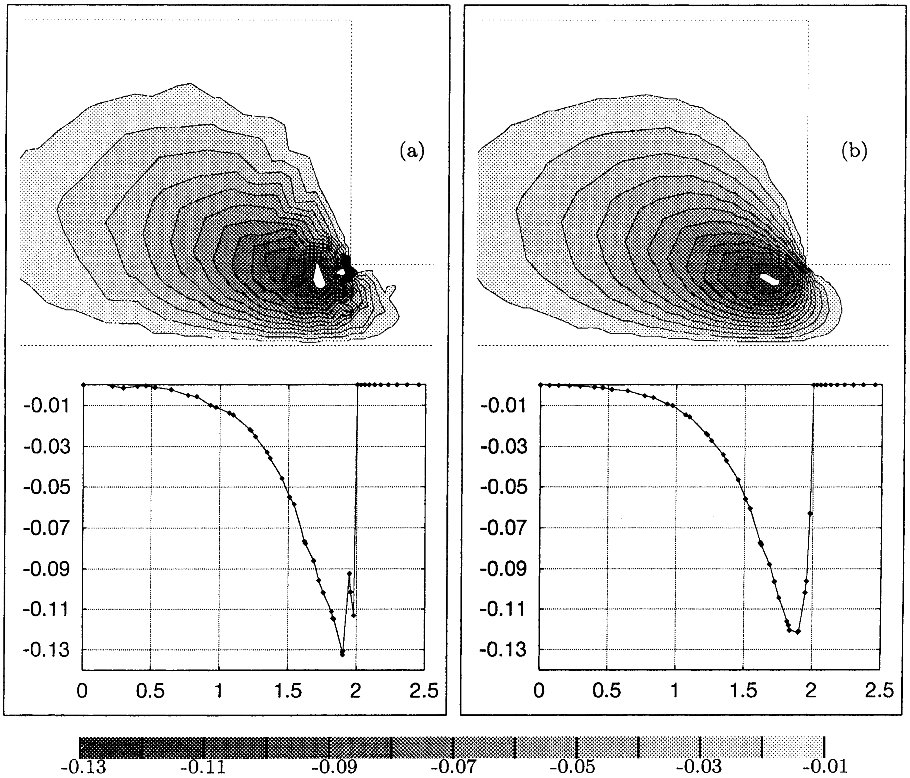

Fig. 4. Isolines and cross-section at $y = 0.25$ of the vertical velocity component u_y for the coupled resolution of the three-field Stokes problem with $\eta_s = 0.01$, $\eta_p = 1$, $\alpha = 0.01$, on mesh M2 and with the reduced GLS method (33): (a) $\beta = 0$, and (b) $\beta = 1$.

$0.01 \leqslant \alpha \leqslant 0.02$, whereas $\alpha = 0$ always yielded a singular linear system (a zero pivot was encountered). Remember that, according to Remarks 4 and 9, the optimal value for the reduced scheme (33) should be $\alpha = C_I$, where C_I is the (unknown) constant involved in (18). This interval can be compared to the value of the inverse inequality constant for elasticity problems on triangular meshes, which was analytically computed in [28] for straight-edged triangles, assuming a specific definition of h_K: the authors found a value of $1/42 \approx 0.023809$ for the stress inverse inequality constant, which is of the same order as our choice for α.

Figs. 3–5 show the isolines and cross-sections at abscissa $y = 0.25$ of the vertical component of the velocity u_y for the three-field Stokes problem, on meshes M1, M2 and M3, respectively. In each figure, the results are those obtained when using the reduced GLS method (33) with (a) $\beta = 0$, and (b) $\beta = 1$. The part of the domain represented is defined by $1 \leqslant x \leqslant 2.25$, $0 \leqslant y \leqslant 1$. The numerical results corresponding to the GLS method (14) with $\beta = 0$ and $\beta = 0.5$ are not represented here since they are almost identical to those of Figs. 3–5. Note that these results compare well to those of [1] in the frame of the modified EVSS method.

These results agree with the theoretical predictions of Lemma 8: $\beta > 0$ ensures stabilization when η_s is nought or too small. If $\beta = 0$ with small or nought values of η_s, spurious oscillations appear in the stress $\boldsymbol{\sigma}_h$, the pressure p_h and the velocity $\mathbf{u}_h$, especially near the reentrant corner. When setting $\beta = 2$ with the reduced GLS method, the linear system was singular (a zero pivot was encountered; also note that the constant C in (35) becomes large when β approaches 2). When α or β are out of the ranges predicted by Lemma 8, spurious oscillations appear. The same observations hold for the GLS method (14). However, there is no upper bound to α for this method but the error clearly increases when α increases.

Fig. 4 shows that a coarse, unstructured mesh like M2 does not mask the instabilities, nor does a refined mesh like M3. Actually, making $\beta \to 0$ or $\alpha \to 0$ caused spurious modes to appear, no matter the mesh. The only difference we have observed is that, as these modes appear at element level, refined meshes give a better global impression than coarse meshes.

Finally, let us stress that the reduced GLS scheme (33) behaves as well as the full GLS scheme (14), even though only a selected number of stabilization terms have been added.

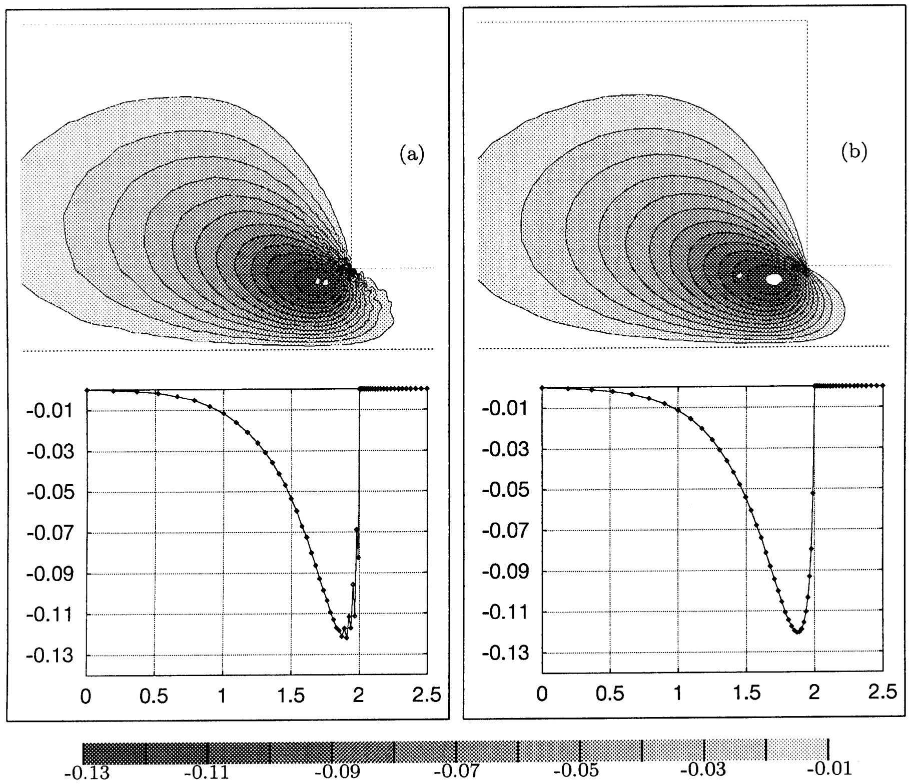

Fig. 5. Isolines and cross-section at $y = 0.25$ of the vertical velocity component u_y for the coupled resolution of the three-field Stokes problem with $\eta_s = 0.01$, $\eta_p = 1$, $\alpha = 0.01$, on mesh M3 and with the reduced GLS method (33): (a) $\beta = 0$, and (b) $\beta = 1$.

7.2. Decoupled resolution

In order to ensure convergence of the decoupling algorithms (42) and (43) presented in Section 6, the values of parameters α and β have to be tuned, a priori within the intervals given by Lemma 8.

Fig. 6 shows the number of iterations n for algorithms (42) and (43) to converge in the sense that the following criteria are satisfied as from step n:

$$\frac{\|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n\|}{\|\mathbf{u}_h^{n+1}\|} < 10^{-14}, \quad \frac{\|p_h^{n+1} - p_h^n\|}{\|p_h^{n+1}\|} < 10^{-14}, \quad \frac{\|\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n\|}{\|\boldsymbol{\sigma}_h^{n+1}\|} < 10^{-14}$$

on mesh M1 and with $\eta_s = 0.01$, $\eta_p = 1$. In other words, the discrepancies here above have to be $\sim 10^{-15}$, i.e., nearly of the order of the machine precision.

Clearly, the fastest convergence properties were found for $\alpha \leqslant 0.02$ and $\beta = 1$, where our convergence criteria were satisfied as from the eighth iteration. For $\beta \leqslant 0.485$, the algorithm was always divergent.

Note that, contrary to what happens with the iterative algorithm presented in [1,2], our algorithm does not converge in one step when $\beta = 1$. This is due to the presence of the term $h_K(\mathrm{div}\,\boldsymbol{\sigma}_h^n, \nabla q)_K$ among the stabilization terms. Indeed, even if Fig. 6 seems to show that when $\alpha \to 0$ the number of iterations for convergence tends towards 1, it is not so. In fact, since the velocity–pressure linear system is singular for $\alpha = 0$, too small values of α cause the rounding errors to become dominant and hinder the algorithm to converge, while allowing spurious modes for the pressure to appear. This is the case as from $\alpha \leqslant 0.001$. Hence choosing $0.01 \leqslant \alpha \leqslant 0.02$, guarantees an optimal compromise between the control on the pressure and the convergence speed.

When other values of the viscosities are set, the minimum value of β to ensure convergence changes, whereas the optimal α interval remains unchanged. For instance, with $\eta_s = 0.5$ and $\eta_p = 1$, convergence was

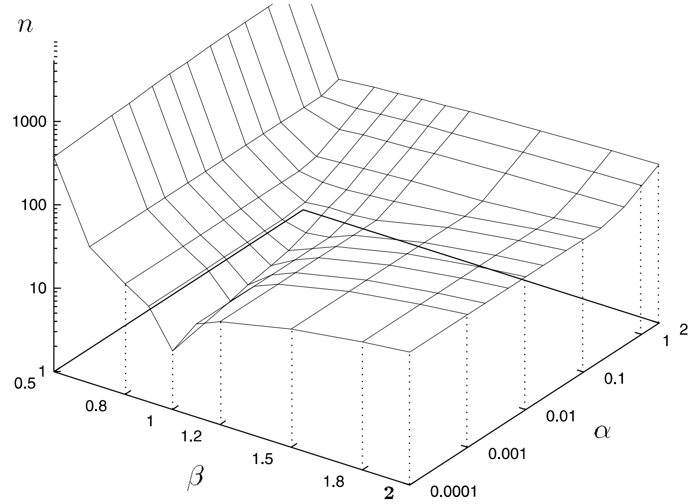

Fig. 6. Dependence on parameters α and β of the number of iterations n for convergence of the decoupled algorithms (42) and (43) with the reduced GLS stabilization, on mesh M1 and with $\eta_s = 0.01$, $\eta_p = 1$.

obtained for $\beta \geqslant 0.25$, whereas with $\eta_s \geqslant \eta_p = 1$, convergence was obtained for $\beta \geqslant 0$. Actually, the following empirical convergence criterion on β was observed:

$$\eta_s + \eta_p \beta \geqslant \frac{\eta_s + \eta_p}{2}, \quad \text{i.e.,} \quad \beta \geqslant \frac{1}{2}\left(1 - \frac{\eta_s}{\eta_p}\right).$$

When $\eta_s = 0$, this criterion reduces to that of [1,2]. Moreover, in every case the optimal value remained $\beta = 1$.

Computations have been performed on meshes coarser and finer than M1, the minimum mesh spacing varying from 0.075 to 0.006. With α and β within the optimal range it resulted that the number of iterations does not depend on the mesh size.

As a last remark, let us mention that in the computations presented in this paper the mass matrices issued from terms of the form $(\boldsymbol{\sigma}_h, \boldsymbol{\tau})$ were always lumped. However, when these mass matrices were not lumped, the results were very similar to those presented in Figs. 3–6, with the same optimal values for parameters α and β.

8. Conclusions

Throughout this paper, we have presented two continuous, piecewise linear finite element methods for solving the three-field Stokes problem (1). The reduced GLS method is put into relation with the modified EVSS method. Stability and convergence of the methods have been established, as well as the convergence of a simple decoupling algorithm. The method is designed to work even when the solvent viscosity η_s is much smaller that the polymer viscosity η_p. Optimal values of stabilization parameters α and β have been proposed and justified.

This approach with continuous, piecewise linears for all finite element spaces (velocity, pressure and stress), stabilized through the addition of GLS-like terms, opens large perspectives for developing codes easy to implement, in order to compute viscoelastic flows. Moreover, it should be recalled that triangular elements offer many advantages, such as meshing of complicated domains or mesh adapting.

We have applied the methods of this paper to the Oldroyd-B model (2), as well as to molecular models for polymeric liquids. In such models, the dynamics of the polymer chains are modelled using dumbbells (see [29,30]), that is two beads connected by a spring of length **Q**, with a linear (Hookean dumbbells) or non-linear (e.g., FENE dumbbells) force **F(Q)**. The evolution of the connection vectors **Q** obeys a set of coupled stochastic partial differential equations. Their interaction with the solvent induces an extra-stress $\boldsymbol{\sigma}$ defined by

$$\boldsymbol{\sigma} = nkT(\mathbb{E}(\mathbf{Q} \otimes \mathbf{F}(\mathbf{Q})) - \mathbf{I}),$$

see [29–31] for modelling details. Numerical methods inspired from this theory have been developed, see [32–37]. The iterative algorithm presented in Section 5 is particularly well suited for these microscopic models, since it allows the extra-stress to be computed separately from the velocity and the pressure. Actually, the two-dimensional computations presented by the authors in [36] were performed on the basis of that stabilized decoupled scheme.

References

[1] M. Fortin, R. Guénette, R. Pierre, Numerical analysis of the modified EVSS method, Comput. Methods Appl. Mech. Engrg. 143 (1997) 79–95.
[2] R. Guénette, M. Fortin, A new mixed finite element method for computing viscoelastic flows, J. Non-Newtonian Fluid Mech. 60 (1995) 27–52.
[3] J.M. Marchal, M.J. Crochet, A new mixed finite element for calculating viscoelastic flows, J. Non-Newtonian Fluid Mech. 26 (1987) 77–114.
[4] M. Behr, L. Franca, T. Tezduyar, Stabilized finite element methods for the velocity–pressure-stress formulation of incompressible flows, Comput. Methods Appl. Mech. Engrg. 104 (1993) 31–48.
[5] Y. Fan, R.I. Tanner, N. Phan-Thien, Galerkin/least-square finite element methods for steady viscoelastic flows, J. Non-Newtonian Fluid Mech. 84 (1999) 233–256.
[6] M. Fortin, A. Fortin, A new approach for the FEM simulations of viscoelastic flows, J. Non-Newtonian Fluid Mech. 32 (1989) 295–310.
[7] F.P.T. Baaijens, S.H.A. Selen, H.P.W. Baaijens, G.W.M. Peters, H.E.H. Meijer, Viscoelastic flow past a confined cylinder of a LDPE melt, J. Non-Newtonian Fluid Mech. 60 (1997) 173–203.
[8] J. Baranger, D. Sandri, Finite element approximation of viscoelastic fluid flow, Numer. Math. 63 (1992) 13–27.
[9] M. Fortin, D. Esselaoui, A finite element procedure for viscoelastic flows, Int. J. Numer. Methods Fluids 7 (1987) 1035–1052.
[10] J. Baranger, C. Guillopé, J.-C. Saut, Mathematical analysis of differential models for viscoelastic fluids, in: J.-M. Piau, J.-F. Agassant (Eds.), Rheology for Polymer Melt Processing, Elsevier, Amsterdam, 1996, pp. 199–236.
[11] M. Fortin, R. Pierre, On the convergence of the mixed method of Crochet and Marchal for viscoelastic flows, Comput. Methods Appl. Mech. Engrg. 73 (1989) 341–350.
[12] L. Franca, R. Stenberg, Error analysis of some GLS methods for elasticity equations, SIAM J. Numer. Anal. 28 (6) (1991) 1680–1697.
[13] D. Rajagopalan, R.C. Armstrong, R.A. Brown, Finite element methods for calculation of steady, viscoelastic flow using constitutive equations with a newtonian viscosity, J. Non-Newtonian Fluid Mech. 36 (1990) 159–192.
[14] F. Yurun, A comparative study of the discontinuous Galerkin and continuous SUPG finite element methods for computation of viscoelastic flows, Comput. Methods Appl. Mech. Engrg. 141 (1997) 47–65.
[15] F.P.T. Baaijens, Mixed finite element methods for viscoelastic flow analysis: A review, J. Non-Newtonian Fluid Mech. 79 (1998) 361–385.
[16] J. Sun, N. Phan-Thien, R.I. Tanner, An adaptive viscoelastic stress splitting scheme and its applications: AVSS/SI and AVSS/SUPG, J. Non-Newtonian Fluid Mech. 65 (1996) 75–91.
[17] L.P. Franca, T.J.R. Hughes, Convergence analyses of Galerkin least-squares methods for symmetric advective–diffusive forms of the stokes and incompressible Navier–Stokes equations, Comput. Methods Appl. Mech. Engrg. 105 (1993) 285–298.
[18] T.J.R. Hughes, L. Franca, M. Balestra, A new finite element formulation for computational fluid dynamics, V. Circumventing the Babuška–Brezzi condition: A stable Petrov–Galerkin formulation of the Stokes problem accommodating equal-order interpolations, Comput. Methods Appl. Mech. Engrg. 59 (1986) 85–99.
[19] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, Springer, Berlin, 1994.
[20] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer, Berlin, 1991.
[21] L. Franca, S. Frey, T.J.R. Hughes, Stabilized finite element methods: Application to the advective–diffusive model, Comput. Methods Appl. Mech. Engrg. 95 (1992) 253–276.
[22] V. Ruas, Finite element methods for the three-field stokes system, RAIRO Modell. Math. Anal. Numer. 30 (1996) 489–525.
[23] P.G. Ciarlet, The Finite Element Method for Elliptic Problems, Academic Press, London, 1990.
[24] L.P. Franca, J.R. Hughes, R. Stenberg, Stabilized finite element methods, in: M.D. Gunzburger, R.A. Nicolaides (Eds.), Incompressible Computational Fluid Dynamics – Trends and Advances, Cambridge University Press, Cambridge, 1993, pp. 87–107.
[25] P. Clément, Approximation by finite elements using local regularization, RAIRO Série Rouge 8 (1975) 77–84.
[26] C. Bernardi, Optimal finite element interpolation on curved domains, SIAM J. Numer. Anal. 26 (5) (1989) 1212–1240.
[27] R. Pierre, Simple $\mathscr{C}^0$ approximations for the computation of incompressible flows, Comput. Methods Appl. Mech. Engrg. 68 (1988) 205–227.

[28] I. Harari, T.J.R. Hughes, What are *C* and *h*?, Comput. Methods Appl. Mech. Engrg. 97 (1992) 157–192.
[29] R. Bird, C. Curtiss, R. Armstrong, O. Hassager, Dynamics of Polymeric Liquids, vol. 2, Wiley, New York, 1987.
[30] H.C. Öttinger, Stochastic Processes in Polymeric Fluids, Springer, Berlin, 1996.
[31] R.B. Bird, J.M. Wiest, Constitutive equations for polymeric liquids, Ann. Rev. Fluid Mech. 27 (1995) 169–193.
[32] M. Laso, H.C. Öttinger, Calculation of viscoelastic flow using molecular models, J. Non-Newtonian Fluid Mech. 47 (1993) 1–20.
[33] M. Laso, M. Picasso, H.C. Öttinger, 2D time-dependent viscoelastic flow calculations using CONNFFESSIT, AIChE. J. 43 (4) (1997) 877–892.
[34] K. Feigl, M. Laso, H.C. Öttinger, The CONNFFESSIT approach for solving a two-dimensional viscoelastic fluid problem, Macromolecules 28 (1995) 3261–3274.
[35] M.A. Hulsen, A.P.G. van Heel, B.H.A.A. van den Brule, Simulation of viscoelastic flows using Brownian configuration fields, J. Non-Newtonian Fluid Mech. 70 (1997) 79–101.
[36] J.C. Bonvin, M. Picasso, Variance reduction methods for CONNFFESSIT-like simulations, J. Non-Newtonian Fluid Mech. 84 (1999) 191–215.
[37] P. Halin, G. Lielens, R. Keunings, V. Legat, The Lagrangian particle method for macroscopic and micro-macro viscoelastic flow computations, Comput. Methods Appl. Mech. Engrg. 180 (1999) 345–364.