Stochastic monotonicity properties in loss networks with repacking[1]

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Centrum voor Wiskunde en Informatica

New Directions in Applied Probability: Stochastic Networks and Beyond
Heriot-Watt University, 14 July 2006

[1] Joint work with M. Jonckheere

Warm-up

Problem

For which integers n,

$$\cos(2\sqrt{2}\pi n) + 1 + \left(-\frac{1}{2}\right)^n \geq 0 \ ?$$

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$$\cos(2\sqrt{2}\pi n) + 1 + \left(-\frac{1}{2}\right)^n \geq 0 \text{ ?}$$

According to Bell & Gerhold (2006)

- The inequality holds for $n \leq 10^5$
- Unknown what happens for large n

Outline

I Loss network with with monoskill and multiskill servers

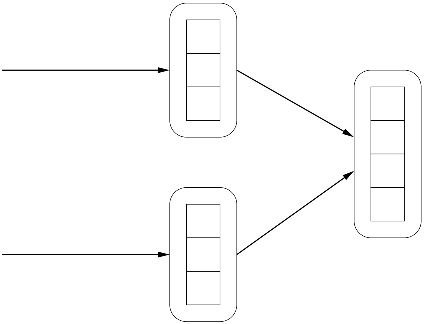
- Repacking vs. no-repacking
- Stochastic comparison of throughput

II Multiclass Erlang loss model

- Time-dependent mean throughput
- Deterministic dynamical system
- Coupling

III Some extensions

Loss network with monoskill and multiskill servers



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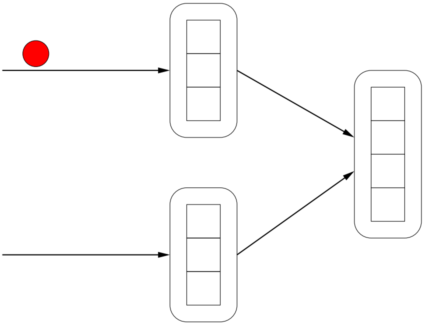

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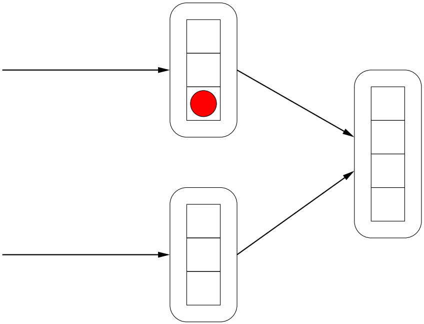

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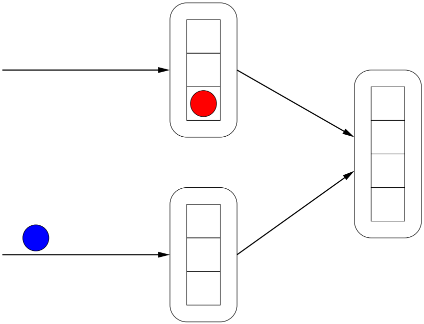

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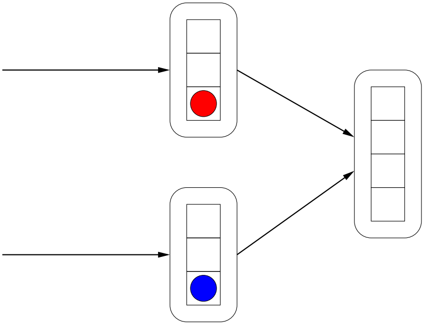

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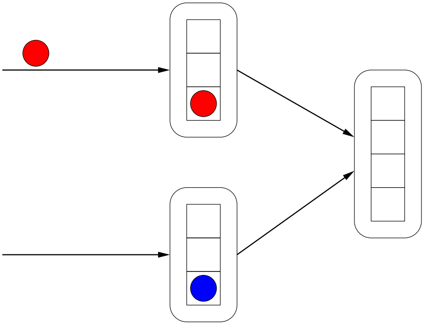

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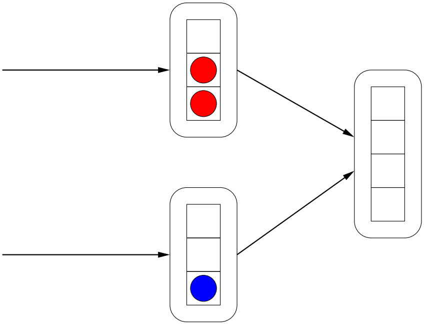

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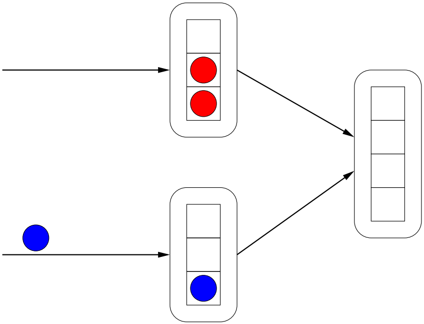

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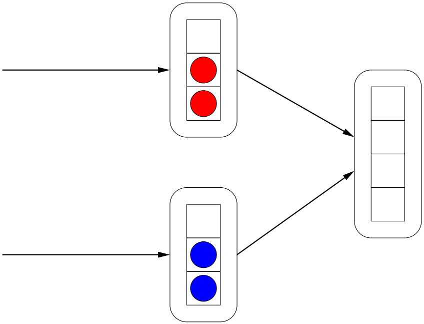

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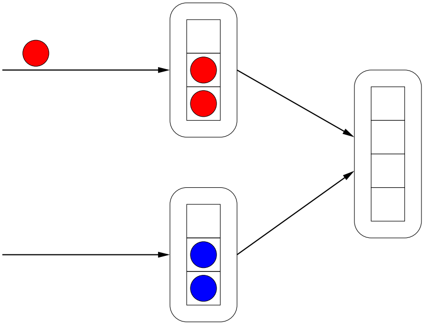

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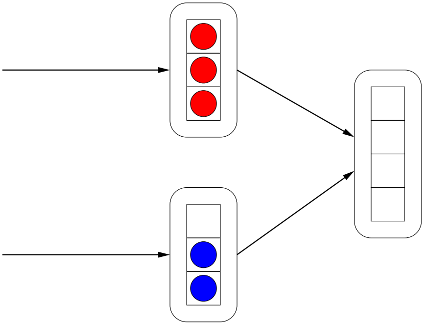

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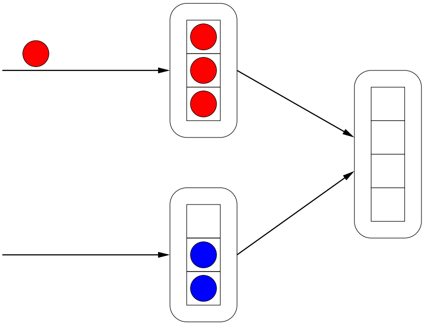

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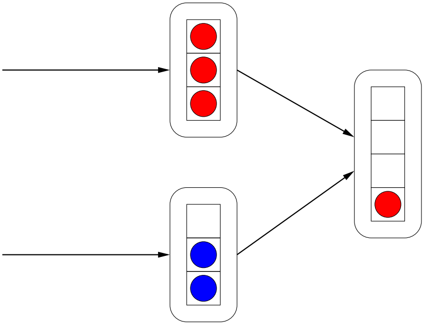

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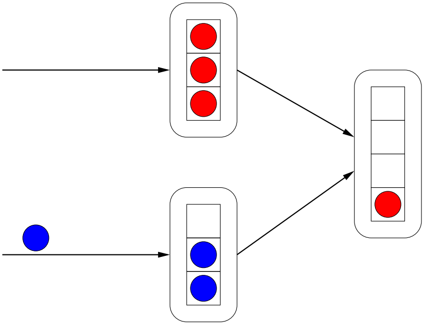

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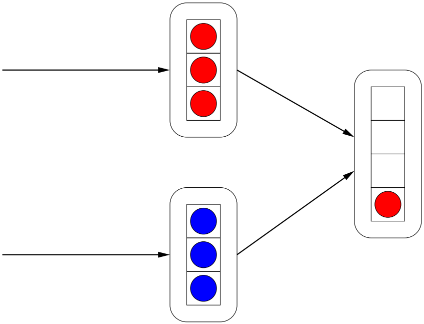

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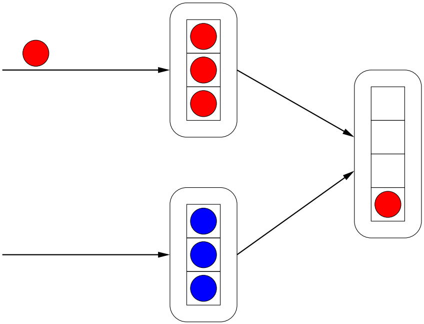

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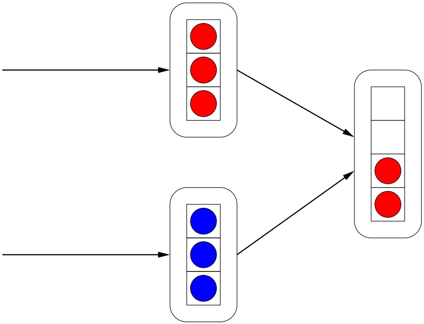

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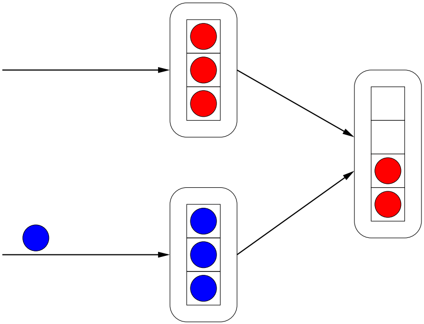

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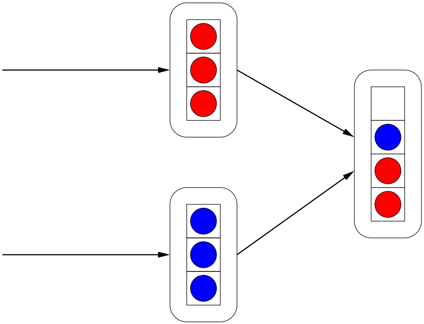

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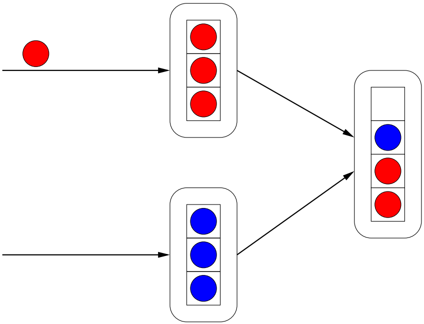

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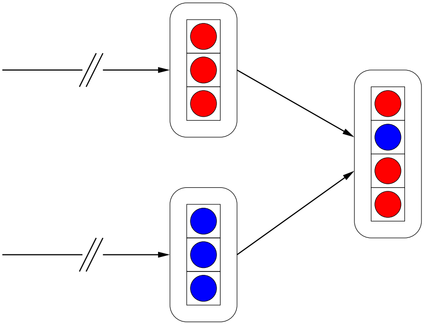

Applications

- Call centers
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- Telecom operators
  - Customer = Fixed bit-rate data stream
  - Monoskill server = Channel of bandwidth in own network
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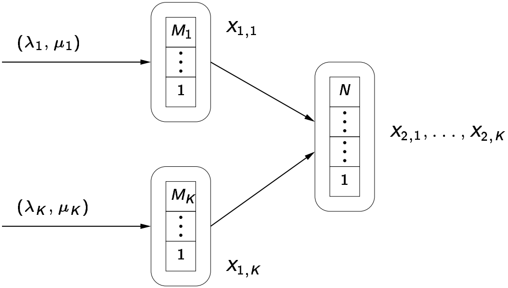
- Other
  - Customer = Any object requesting a single (atomic) resource
  - Monoskill server = Any dedicated resource
  - Multiskill server = Any shared resource

Loss network with K customer classes

- M_k monoskill servers dedicated to class k
- N multiskill servers
- State vector $X = (X_{1,1}, \dots, X_{1,K}; X_{2,1}, \dots, X_{2,K})$



Performance

Measure workload in bits

- Each class-k customer brings $\exp(\mu_k)$ bits of work
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Throughput:

- Rate of processed work (bit/s):

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t |X(s)|\, ds$$

- $|X(t)| := \sum_k (X_{1,k}(t) + X_{2,k}(t))$

Steady-state analysis

Assume

- Interarrival times $\sim \exp(\lambda_k)$
- Service times $\sim \exp(\mu_k)$
- All independent

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The process $X = (X_{1,1}, \ldots, X_{1,K}; X_{2,1}, \ldots, X_{2,K})$

- is Markov
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$\Rightarrow$ Steady-state distribution of X solvable by matrix inversion

Analytical complexity

Example (Simplest nontrivial case)

- Two traffic classes
- $M_1 = 1$, $M_2 = 0$
- One multiskill server

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$$c_0 = 2\lambda_1^2\mu_1^2\mu_2 + \lambda_1\lambda_2\mu_1^2\mu_2 + 4\lambda_1\mu_1^3\mu_2 + 2\lambda_2\mu_1^3\mu_2 + 2\mu_1^4\mu_2 + 2\lambda_1\mu_1^2\mu_2^2 + 2\mu_1^3\mu_2^2,$$

$$G = \lambda_1^3\lambda_2\mu_1 + \lambda_1^2\lambda_2^2\mu_1 + 5\lambda_1^2\lambda_2\mu_1^2 + 3\lambda_1\lambda_2^2\mu_1^2 + 6\lambda_1\lambda_2\mu_1^3 + 2\lambda_2^2\mu_1^3 + 2\lambda_2\mu_1^4 + \lambda_1^4\mu_2 + \lambda_1^3\lambda_2\mu_2 + 4\lambda_1^3\mu_1\mu_2 + 4\lambda_1^2\lambda_2\mu_1\mu_2 + 7\lambda_1^2\mu_1^2\mu_2 + 7\lambda_1\lambda_2\mu_1^2\mu_2 + 6\lambda_1\mu_1^3\mu_2 + 4\lambda_2\mu_1^3\mu_2 + 2\mu_1^4\mu_2 + \lambda_1^3\mu_2^2 + 3\lambda_1^2\mu_1\mu_2^2 + 4\lambda_1\mu_1^2\mu_2^2 + 2\mu_1^3\mu_2^2$$

Computational complexity

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Number of states proportional to $M_1 \cdots M_K N^K$

Approximative methods

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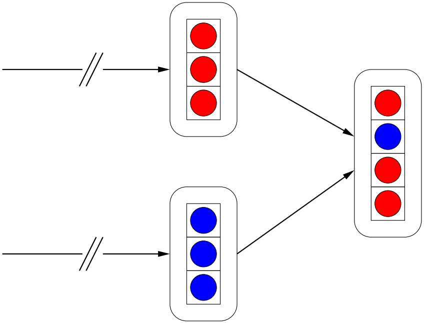
- Find a simpler system that behaves better/worse
- $\Rightarrow$ Upper/lower bound for performance
- Try to perturb the system slightly

Perturbed system



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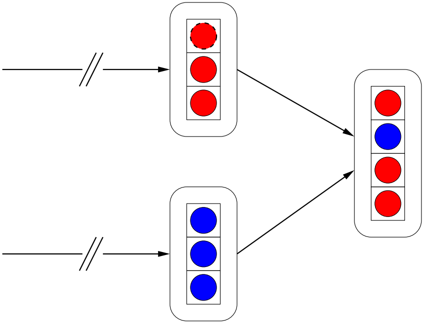

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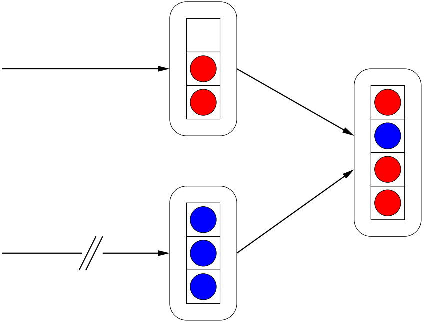

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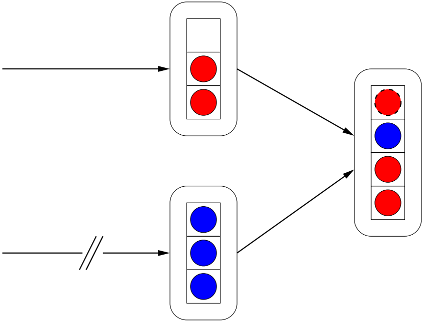

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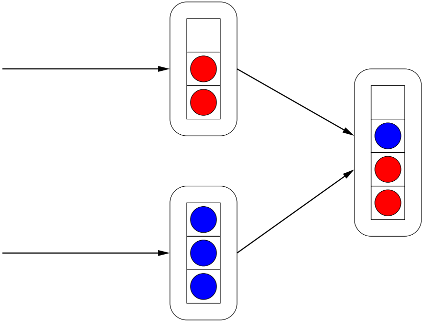

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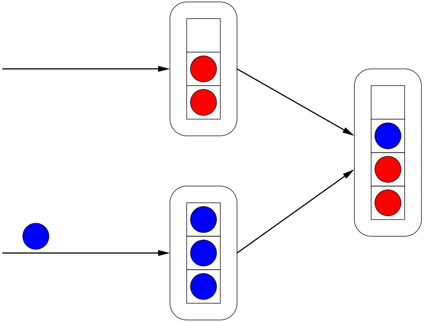

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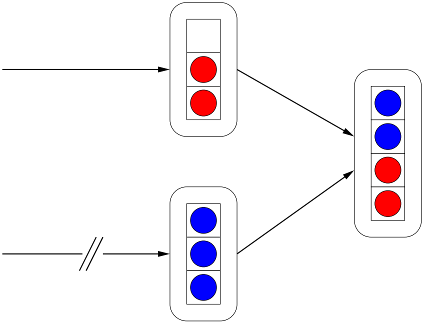

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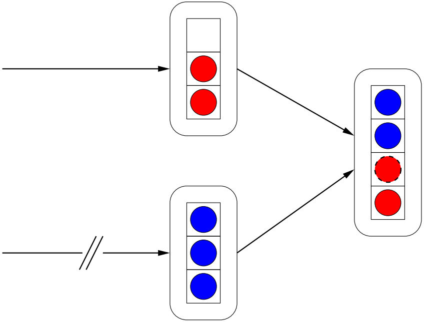

Blocking of blue customers can be avoided by redirecting one red customer

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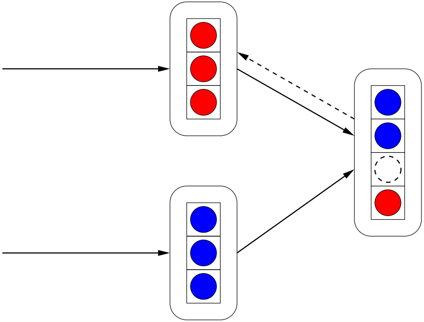

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Repacking policy

Redirect customers from multiskill to monoskill servers, as soon as possible

- Service interruptions (for memoryless customers)

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- Throughput

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |X'(s)|\, ds$$

Steady-state analysis of the system with repacking

Define $Y' = (Y'_1, \ldots, Y'_K)$ with $Y'_k = X'_{1,k} + X'_{2,k}$

- Arriving customer is accepted if and only if

$$|Y'| < M_1 + \cdots + M_K + N$$

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- $\Rightarrow$ Easy numerical computation of throughput:

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t |X'(s)|\, ds = \mathrm{E}\,|X'| = \mathrm{E}\,|Y'|$$

Markov reward approach (1/4)

How to prove $\mathrm{E}\, r(X) \le \mathrm{E}\, r(X')$, that is

$$\sum_x r(x)\, \pi(x) \le \sum_x r(x)\, \pi'(x),$$

without knowing π and π'?

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Markov reward approach (van Dijk; 1998)

- Prove that $\mathrm{E}^x \int_0^t r(X(s))\, ds \leq \mathrm{E}^x \int_0^t r(X'(s))\, ds$ for all t

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- Reduce the problem to discrete time using uniformization

Markov reward approach (2/4)

Uniformization

- Markov chain Y_n with transition matrix $P_\gamma = I + \gamma^{-1} Q$
- Poisson process $\mathcal{N}$ with rate γ

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$$\mathsf{E}^x \, \phi(Y_{\mathcal{N}(t)}) = \sum_{n=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \, \mathsf{E}^x \, \phi(Y_n)$$

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\begin{aligned}
\mathrm{E}^x \phi(Y_{\mathcal{N}(t)}) &= \sum_{n=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \mathrm{E}^x \phi(Y_n) \\
&= e^{-\gamma t} \sum_{n=0}^{\infty} \frac{(\gamma t)^n}{n!} (I + \gamma^{-1} Q)^n \phi(x) \\
&= e^{tQ} \phi(x) = \mathrm{E}^x \phi(X(t))
\end{aligned}
$$

- $\Rightarrow X(t) =_{st} Y_{\mathcal{N}(t)}$

Markov reward approach (3/4)

Let Y'_n be the uniformized Markov chain for $X'(t)$, then

$$\mathrm{E}^x\, r(X(t)) = \sum_{n=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \mathrm{E}^x\, r(Y_n)$$

$$\mathrm{E}^x\, r(X'(t)) = \sum_{n=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \mathrm{E}^x\, r(Y'_n)$$

Markov reward approach (3/4)

Let Y_n' be the uniformized Markov chain for $X'(t)$, then

$$\mathrm{E}^x\, r(X(t)) = \sum_{n=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!}\, \mathrm{E}^x\, r(Y_n)$$

$$\mathrm{E}^x\, r(X'(t)) = \sum_{n=0}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!}\, \mathrm{E}^x\, r(Y_n')$$

Sufficient condition for $\mathrm{E}\, r(X) \leq \mathrm{E}\, r(X')$:

$$\mathrm{E}^x\, r(Y_n) \leq \mathrm{E}^x\, r(Y_n') \quad \text{for all } n$$

Markov reward approach (4/4)

Cumulative reward (similarly for X')

$$\mathsf{E}^x \int_0^t r(X(s))\,ds = \gamma^{-1} \sum_{n=1}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \left(\mathsf{E}^x \sum_{k=0}^{n-1} r(Y_k) \right)$$

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Define

$$V_n(x) = \mathsf{E}^x \sum_{k=0}^{n-1} r(Y_k) \quad \text{and} \quad V_n'(x) = \mathsf{E}^x \sum_{k=0}^{n-1} r(Y_k')$$

Markov reward approach (4/4)

Cumulative reward (similarly for X')

$$\mathrm{E}^x \int_0^t r(X(s))\,ds = \gamma^{-1} \sum_{n=1}^{\infty} e^{-\gamma t} \frac{(\gamma t)^n}{n!} \left(\mathrm{E}^x \sum_{k=0}^{n-1} r(Y_k) \right)$$

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Sufficient condition for $\mathrm{E}\, r(X) \le \mathrm{E}\, r(X')$

$$V_n(x) \le V_n'(x) \quad \text{for all } n$$

Comparison of throughput (1/3)

Theorem (George, Jonckheere, Leskelä; 2005)

Assume

$$\frac{\lambda_1}{\mu_1} + \cdots + \frac{\lambda_K}{\mu_K} \le 1.$$

Then repacking improves the steady-state mean throughput:

$$\mathsf{E}\,|X| \le \mathsf{E}\,|X'|.$$

Comparison of throughput (2/3)

Proof.
Markov reward approach for

$$V_t(x) = \mathsf{E}^x \int_0^t |X(s)|\, ds$$

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2. Show that $V_t(x) \leq V_t(x + e_{2,k})$

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$\square$

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□

CWI

Comparison of throughput (3/3)

Unnatural "stability" condition

$$\frac{\lambda_1}{\mu_1} + \cdots + \frac{\lambda_K}{\mu_K} \leq 1$$

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- Monoskill servers are isolated
- $\Rightarrow$ Focus on the multiskill servers

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Simpler problem:

- Assume no monoskill servers
- $\Rightarrow$ Erlang loss model with N servers

Outline

I Loss network with with monoskill and multiskill servers

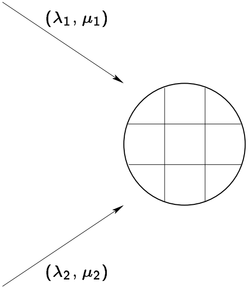
- Repacking vs. no-repacking
- Stochastic performance comparison

II Multiclass Erlang loss model

- Time-dependent mean throughput
- Deterministic dynamical system
- Coupling

III Some extensions

Multiclass Erlang loss model



Shared resource, N units

- Complete sharing
- Interarrival times $\exp(\lambda_k)$
- Holding times $\exp(\mu_k)$
- All independent

Reward rate

$$r(X(t)) = |X(t)| := X_1(t) + X_2(t)$$

Time-dependent analysis

Problem

Mean collected reward

$$V_t(x) = \mathsf{E}^x \int_0^t r(X(s))\, ds$$

- *Is the map $x \mapsto V_t(x)$ increasing?*

Time-dependent analysis

Problem

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Earlier work

- Monotonicity with respect to input rates (Nain; 1990)

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Earlier work

- Monotonicity with respect to input rates (Nain; 1990)
- Monotonicity criteria for *optimal* admission policies (Altman, Jiménez, Koole; 2001) and (van der Wal, Örmeci; 2006)

Special limiting case: No blocking

Theorem

Assume $N = \infty$, and let X and $X^{(k)}$ be versions of the the multiclass Erlang model started at x and $x + e_k$, respectively. Then for all t,

$$|X(t)| \leq_{st} |X^{(k)}(t)|.$$

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- Choose independently $\sigma =_{st} \exp(\mu_k)$

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- Choose independently $\sigma =_{st} \exp(\mu_k)$
- Let

$$\hat{X}(t) = \begin{cases} X(t) + e_k, & t < \sigma, \\ X(t), & t \geq \sigma \end{cases}$$

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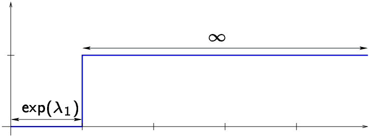
- $\Rightarrow |X(t)| \leq |\hat{X}(t)|$ for all t

Example: One server with $\mu_1 = 0$ and $\lambda_2 = 0$



Typical reward rate with $X(0) = 0$

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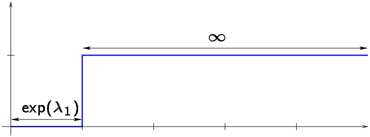

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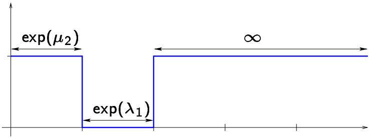

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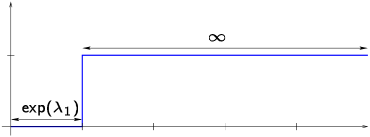

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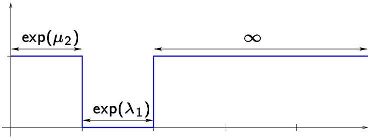

Typical reward rate with $X(0) = e_2$

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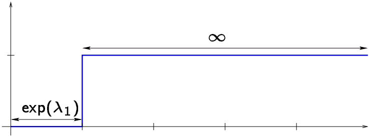

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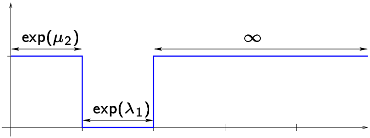

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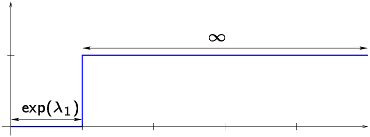

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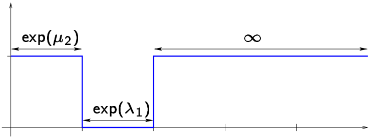

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But anyway for all t,

$$\mathrm{E}^0 \int_0^t |X(s)|\, ds \le \mathrm{E}^{e_2} \int_0^t |X(s)|\, ds$$

Problem as a deterministic dynamical system

Uniformized cumulative mean reward $V_n(x)$

- Define $\delta_k V_n(x) = V_n(x + e_k) - V_n(x)$

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$$\delta_k V^{n+1}(x) =$$

$$\begin{cases} 1 + (1 - \lambda \cdot 1 - \mu_k - \mu \cdot x)\delta_k V_n(x) \\ \quad + \sum_j \lambda_j \delta_k V_n(x+e_j) + \sum_j \mu_j x_j \delta_k V_n(x-e_j), & |x| < N-1, \\ 1 + (1 - \mu_k - \mu \cdot x)\delta_k V_n(x) - \sum_j \lambda_j \delta_j V_n(x) \\ \quad + \sum_j \mu_j x_j \delta_k V_n(x-e_j), & |x| = N-1. \end{cases}$$

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How to prove $\delta_k V_n(x) \geq 0$ for all k and x?

Positive trajectory of an affine dynamical system

Problem

Given a positive vector b in $\mathbb{R}^d$, determine the set of matrices $A \in \mathbb{R}^{d\times d}$ such that the system

$$x(0) = 0,$$
$$x(t+1) = Ax(t) + b,$$

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Positive linear systems theory (Farini and Rinaldi; 2000)

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Positive linear systems theory (Farini and Rinaldi; 2000)

- Restrict to A, b such that $x(t)$ is positive for an arbitrary positive initial state
- $\Rightarrow$ all entries of A must be positive
- $\Rightarrow$ not helpful in Markov context

Monotonicity for "stable" system

Assume

$$\frac{\lambda_1}{\mu_1} + \cdots + \frac{\lambda_K}{\mu_K} \leq 1$$

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$$\frac{\lambda_1}{\mu_1} + \cdots + \frac{\lambda_K}{\mu_K} \leq 1$$

- Not hard to verify that $\delta_k V_n(x) \leq \frac{1}{\mu_k}$
- Apply induction to

$$\delta_k v^{n+1}(x) =$$

$$\begin{cases} 1 + (1 - \lambda \cdot 1 - \mu_k - \mu \cdot x)\delta_k V_n(x) \\ \quad + \sum_j \lambda_j \delta_k V_n(x + e_j) + \sum_j \mu_j x_j \delta_k V_n(x - e_j), & |x| < N - 1, \\ 1 + (1 - \mu_k - \mu \cdot x)\delta_k V_n(x) - \sum_j \lambda_j \delta_j V_n(x) \\ \quad + \sum_j \mu_j x_j \delta_k V_n(x - e_j), & |x| = N - 1. \end{cases}$$

Natural coupling (1/3)

Find a stochastic process $\tilde{X} = (X, \hat{X})$

- State space $\{(x_1, x_2, \hat{x}_1, \hat{x}_2) : |x| \leq N, |\hat{x}| \leq N\}$
- X Markov with generator Q and initial state x
- $\hat{X}$ Markov with generator Q and initial state $x + e_2$

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Natural construction

- Dynamical evolution map $F(x, \cdot) : A \mapsto X$
- Arrival point process $A = \{(T_n, S_n)\}_{n \geq 1}$
- $X(t) = F(x, A)(t)$
- $\hat{X}(t) = F(x + e_2, A)(t)$

Natural coupling (2/3)

$(X, \hat{X})$ is Markov with generator $\tilde{Q}$

$$\tilde{q}(x, \hat{x}; y, \hat{y}) =$$

$$\begin{cases} \lambda_k 1(|x| < N, |\hat{x}| < N), & (y, \hat{y}) = (x, \hat{x}) + (e_k, e_k) \\ \lambda_k 1(|x| < N, |\hat{x}| = N), & (y, \hat{y}) = (x, \hat{x}) + (e_k, 0) \\ \lambda_k 1(|x| = N, |\hat{x}| < N), & (y, \hat{y}) = (x, \hat{x}) + (0, e_k) \\ \mu_k (x_k + \hat{x}_k) 1(x_k > 0, \hat{x}_k > 0), & (y, \hat{y}) = (x, \hat{x}) - (e_k, e_k) \\ \mu_k x_k 1(x_k > 0, \hat{x}_k = 0), & (y, \hat{y}) = (x, \hat{x}) - (e_k, 0) \\ \mu_k \hat{x}_k 1(x_k = 0, \hat{x}_k > 0), & (y, \hat{y}) = (x, \hat{x}) - (0, e_k) \end{cases}$$

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For all x and $\hat{x}$,

$$\sum_{\hat{y}} \tilde{q}(x, \hat{x}; y, \hat{y}) = q(x, y)$$

$$\sum_{y} \tilde{q}(x, \hat{x}; y, \hat{y}) = q(\hat{x}, \hat{y})$$

Natural coupling (3/3)

Recall that

$$V_t(x) = \mathsf{E}^x \int_0^t |X(s)|\, ds,$$

so in terms of the coupling $(X, \hat{X})$,

$$V_t(x + e_2) - v_t(x) = \mathsf{E} \int_0^t \Big(|\hat{X}(s)| - |X(s)| \Big)\; ds$$

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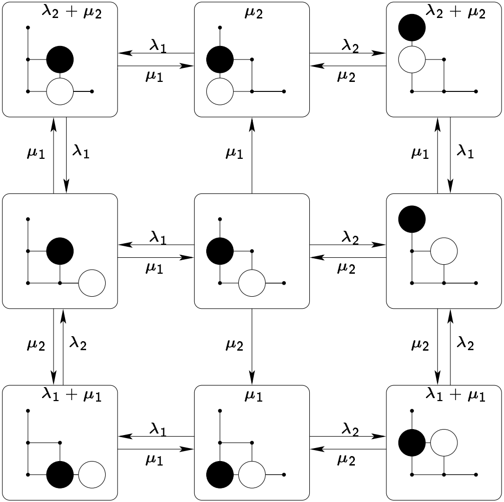
Because $\mathcal{D} = \{(x, \hat{x}) : x = \hat{x}\}$ is absorbing,

$$V_t(x + e_2) - V_t(x) = \mathsf{E} \int_0^{t \wedge T_{\mathcal{D}}} \left(|\hat{X}(s)| - |X(s)| \right)\, ds,$$

where $T_{\mathcal{D}}$ is the entry time of $(X, \hat{X})$ into $\mathcal{D}$

Natural coupling for $N = 2$



Consequences of the natural coupling

Theorem

Let X and $\hat{X}$ be versions of the multiclass Erlang process started at x and $x + e_k$, respectively. Then for all t,

$$|\hat{X}(t)| - 1 \leq_{st} |X(t)| \leq_{st} |\hat{X}(t)| + 1,$$

and especially,

$$|V_t(x + e_k) - V_t(x)| \leq t.$$

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Proof.

The set

$$\mathcal{D}' = \Big\{(x, \hat{x}) : \hat{x} - x \in \{0, \pm e_1, \pm e_2, \pm(e_2 - e_1)\}\Big\}$$

is absorbing for the natural coupling of X and $\hat{X}$. □

Asymmetric coupling (1/4)

Assume $\mu_1 \leq \mu_2$

- Class-1 customers stay longer
- $x \mapsto x - e_1$ is less probable than $x \mapsto x - e_2$

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Split the faster rate exponential

- $\exp(\mu_2) =_{st} \exp(\mu_1) \wedge \exp(\mu_2 - \mu_1)$

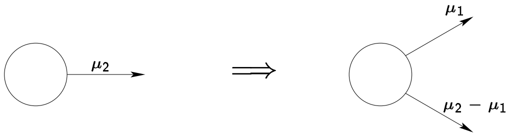
Asymmetric coupling (1/4)

Assume $\mu_1 \leq \mu_2$

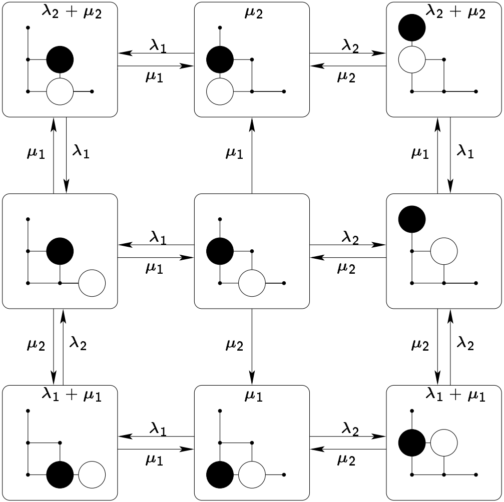
- Class-1 customers stay longer
- $x \mapsto x - e_1$ is less probable than $x \mapsto x - e_2$

Split the faster rate exponential

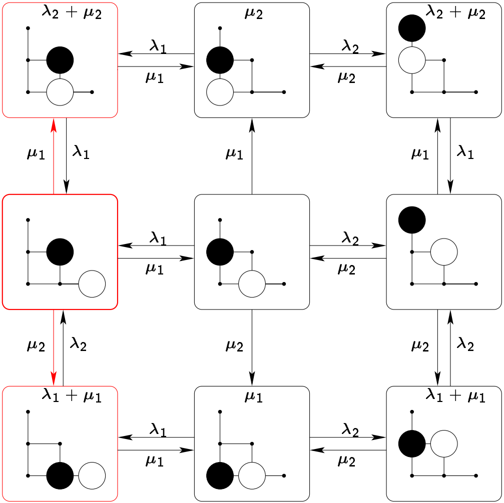
- $\exp(\mu_2) =_{st} \exp(\mu_1) \wedge \exp(\mu_2 - \mu_1)$



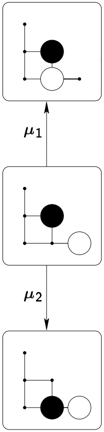
Asymmetric coupling (2/4)



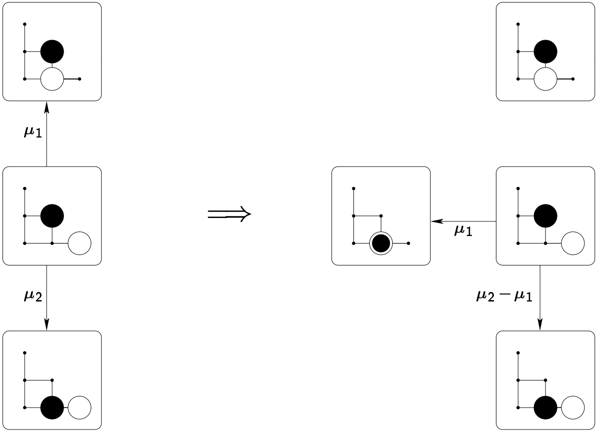
Asymmetric coupling (2/4)



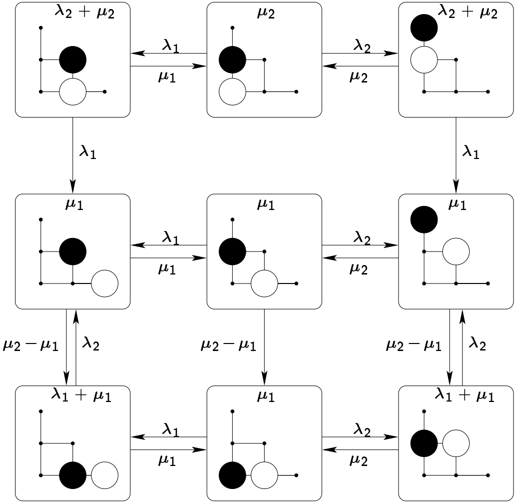
Asymmetric coupling (3/4)



Asymmetric coupling (3/4)



Asymmetric coupling (4/4)



Consequences of the asymmetric coupling

Theorem

Assume $\mu_1 \leq \mu_2$. Let X and $\hat{X}$ be versions of the multiclass Erlang process started at x and $x + e_1$, respectively. Then for all t,

$$|X(t)| \leq_{st} |\hat{X}(t)|,$$

and especially,

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Proof.

The set

$$\mathcal{D}_1^+ = \Big\{(x, \hat{x}) : \hat{x} - x \in \{0, e_1, e_1 - e_2\}\Big\}$$

is absorbing for the asymmetric coupling of X and $\hat{X}$. □

Special case with $\mu_1 = \mu_2$

Corollary

Assume $\mu_1 = \mu_2$, let $k \in \{1, 2\}$. Let X and $\hat{X}$ be versions of the multiclass Erlang process started at x and $x + e_k$, respectively. Then for all t,

$$|X(t)| \leq_{st} |\hat{X}(t)|,$$

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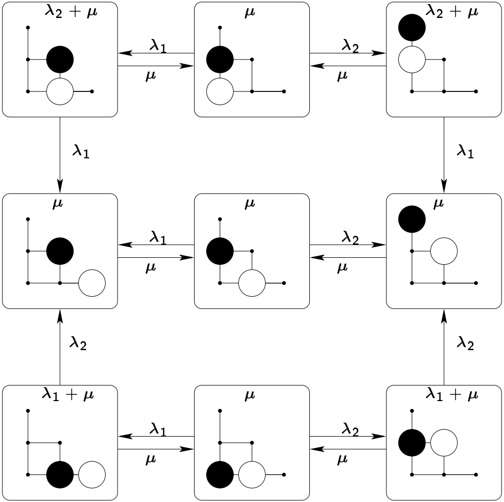
Asymmetric coupling for $\mu_1 = \mu_2$



Strong monotonicity of the one-server model (1/2)

Theorem

Assume $N = 1$, and X and $X^{(k)}$ be versions of the multiclass Erlang process starting at 0 and e_k, respectively. Then for all k and t,

$$\int_0^t |X(s)| \, ds \leq_{st} \int_0^t |X^{(k)}(s)| \, ds.$$

Strong monotonicity of the one-server model (2/2)

Proof.
Non-Markov coupling

- Choose a version of X
- Let $\sigma =_{st} \exp(\mu_k)$ be independent of X
- Construct $\hat{X}$ by

$$\hat{X}(t) = \begin{cases} e_k, & t < \sigma \\ X(t-\sigma), & t \geq \sigma \end{cases}$$

Strong monotonicity of the one-server model (2/2)

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Then

- $\hat{X}$ is a version of $X^{(k)}$
- $\int_0^t |\hat{X}(s)|\, ds = \sigma \wedge t + \int_0^{(t-\sigma)^+} |X(s)|\, ds \geq \int_0^t |X(s)|\, ds$

$\square$

Outline

I Loss network with with monoskill and multiskill servers

- Repacking vs. no-repacking
- Stochastic performance comparison

II Multiclass Erlang loss model

- Time-dependent mean throughput
- Deterministic dynamical system
- Coupling

III Some extensions

Comparison of overall blocking probability

Theorem

Assume $\mu_1 = \cdots = \mu_K$. Then the overall blocking probability is smaller in the system with repacking.

Proof.

By the Little's law,

$$\mathsf{E}(X_{1,k} + X_{2,k}) = (1 - b_k)\lambda_k/\mu_k.$$

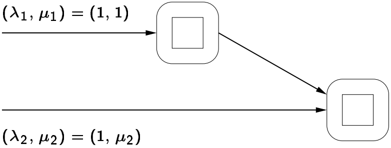
Hence

$$b = 1 - \frac{\sum_{k=1}^{K} \mu_k \, \mathsf{E}(X_{1,k} + X_{2,k})}{\sum_{k=1}^{K} \lambda_k}.$$

Likewise, $b' = \ldots$ □

What if $\mu_j \neq \mu_k$ for some j, k?

Example: no monoskill servers for class 2



Repacking $\Rightarrow$

- class-1 blocking *increases*
- class-2 blocking decreases
- overall blocking probability *increases* if $\mu_2 < 2/5$

Summary

The time-dependent mean throughput $x \mapsto V_t(x)$ in the multiclass Erlang model is monotone

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Applications

- Computable performance bounds
- Optimality criteria for admission policies

Eitan Altman, Tania Jiménez, and Ger Koole.
On optimal call admission control in a resource-sharing system.
IEEE Trans. Comm., 49(9):1659–1668, 2001.

Jason P. Bell and Stefan Gerhold.
On the positivity set of a linear recurrence sequence.
Israel J. Math., 2006.
To appear.

Nico M. van Dijk.
Bounds and error bounds for queueing networks.
Ann. Oper. Res., 79:295–319, 1998.

Geert Jan Franx, Ger Koole, and Auke Pot.
Approximating multi-skill blocking systems by hyperexponential decomposition.
Perform. Evaluation, 2006.
To appear.

Lorenzo Farina and Sergio Rinaldi.

Positive Linear Systems: Theory and Applications.
Wiley, 2000.

A. Fredericks.
Congestion in blocking systems – a simple approximation technique.
Bell Syst. Tech. J., 59(6):805–827, 1980.

Lise George, Matthieu Jonckheere, and Lasse Leskelä.
Does repacking improve performance of multiclass loss networks with overflow routing?
In *Proc. 19th International Teletraffic Congress*, 2005.

Philippe Nain.
Qualitative properties of the Erlang blocking model with heterogeneous user requirements.
Queueing Syst., 6:189–206, 1990.

E. Lerzan Örmeci and Jan van der Wal.
Admission policies for a two class loss system with general interarrival times.
Stoch. Mod., 22:37–53, 2006.