Weakly coprime factorization and state-feedback stabilization of discrete-time systems

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September 28, 2008

Abstract: The LQ-optimal state feedback of a finite-dimensional linear time-invariant system determines a coprime factorization NM^{-1} of the transfer function. We show that the same is true also for infinite-dimensional systems over arbitrary Hilbert spaces, in the sense that the factorization is *weakly coprime*, i.e., $Nf, Mf \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ for every function f. The factorization need not be Bézout coprime.

We prove that every proper quotient of two bounded holomorphic operator-valued functions can be presented as the quotient of two bounded holomorphic weakly coprime functions. This result was already known for matrix-valued functions with the classical definition $\gcd(N, M) = I$, which we prove equivalent to our definition.

We give necessary and sufficient conditions and further results for weak coprimeness and for Bézout coprimeness. We then establish a variant of the inner-outer factorization with the inner factor being "weakly left-invertible". Most of our results hold also for continuous-time systems and many are new also in the scalar-valued case.

AMS class: 49N10, 93D15, 93B52; 47A68, 47A56, 47B35.

Keywords: Quasi-coprime factorization, stabilizable realization, stabilizable and detectable realization, LQR problem, optimal state feedback, infinite-dimensional systems theory.

1 Introduction and main results

In this article we shall establish the results mentioned in the abstract and some others on weakly coprime and Bézout coprime factorizations, LQ-optimal control, stabilizable realizations and weakly left-invertible holomorphic functions (possibly Hilbert space operator-valued). We work in discrete time, but the results hold in continuous time too, with certain technical differences in the case of very unbounded control and observation operators. By $\mathcal{H}^\infty$ we denote the set of bounded holomorphic (possibly operator-valued) functions on the unit disc. By I we denote either the identity operator or the corresponding constant function $I \in \mathcal{H}^\infty$.

In a "coprime factorization" $p = \frac{n}{m}$ of a rational number p, any common divisors of n and m (other than the units ± 1) have been canceled out, i.e., n and m are relative primes (coprime). Thus, their greatest common divisor is $\gcd(n, m) = 1$. Similarly, in a "right coprime factorization" $P = NM^{-1}$ of a function P, any common (right) divisors (in $\mathcal{H}^\infty$) of $N, M \in \mathcal{H}^\infty$ "have been canceled out". This means that

$$\text{if } \begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} V \text{ for some } A, B, V \in \mathcal{H}^\infty, \text{ then } V \text{ is a (right) divisor of } I, \qquad (1)$$

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i.e., then $I = LV$ for some $L \in \mathcal{H}^\infty$. This is sometimes expressed as "(a right) $\gcd(N, M) = I$". Condition (1) is equivalent to the *Bézout condition* that $XM - YN \equiv I$ for some $X, Y \in \mathcal{H}^\infty$ (take $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$, $V = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ to prove this).

If N and M are scalar-valued and we only require (1) to hold for scalar-valued functions V, then we get the classical definition of a "weakly right coprime factorization" [Fuh81] [Ino88] [Smi89]. The same holds in the matrix-valued case too if we only require (1) to hold for square-matrix-valued functions V [Mik08, Theorem 2.17(c)].

In the operator-valued case one should only require (1) to hold when $V(0)$ is invertible. That definition is equivalent to the classical one in the matrix-valued case (assuming that $M(0)$ is invertible), but it is "the right definition" in the operator-valued case too, in the sense that all functions of the form NM^{-1} do have weakly coprime factorizations and the classical relations to LQ-optimal state feedback are retained, etc. Most of these relations are new in the scalar-valued case too (for nonrational functions). However, we will use the property (2) below as our definition, because it is more useful in state-feedback contexts. We later prove our definition equivalent to the one above.

The theory on the connection between coprime factorization and different forms of stabilization of finite-dimensional systems became rather mature during the 70s and 80s [Vid85] [Fra87]. Thereafter, coprime factorization has played a major role in control theory, both finite- and infinite-dimensional. Also the infinite-dimensional setting has been studied intensively, but only now the theory is becoming complete.

The connection between dynamic stabilization and (Bézout) coprime factorization has been established also for general nonrational functions in, e.g., [Vid85], [Ino88], [Smi89], [Qua04] in the matrix-valued case, and in the operator-valued case in [CWW01], [Cur06] and [Mik07a]; all these for transfer functions only. Fairly general state-space results are given in [WR00].

In [CO06] and [Sta98], certain connections between coprime factorization and stabilizability and detectability were established. These results will be extended to an equivalence in Theorem 1.3.

In the finite-dimensional case, the coprime factorization of the transfer function of a system is determined by the LQ-optimal state feedback. We shall show that, in the infinite-dimensional case, the factorization defined by that state feedback is "weakly coprime" (not always (Bézout) coprime).

Using this result, we establish algebraic and system-theoretic necessary and sufficient conditions for a (possibly operator-valued) function to have a (state-feedback) stabilizable realization or a weakly coprime factorization (Theorem 1.2). Also similar results on the Bézout coprime factorization are given, as well as further properties on both types of coprimeness.

Before presenting the main results, we need some definitions. Let $\mathtt{U}$, $\mathtt{X}$ and $\mathtt{Y}$ be arbitrary complex Hilbert spaces, and set $\mathbb{D} := \{z \in \mathbb{C} \,\big|\, |z| < 1\}$. Let $\mathcal{B}$ stand for bounded linear operators. By $\mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ (resp., $\mathcal{H}^\infty(\mathtt{U})$) we denote the Banach space of bounded holomorphic functions $\mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ (resp., $\mathbb{D} \to \mathcal{B}(\mathtt{U})$) with the supremum norm. A holomorphic function $f : \mathbb{D} \to \mathtt{U}$ is in the Hilbert space $\mathcal{H}^2(\mathtt{U})$ if and only if $\|f\|_{\mathcal{H}^2} := \sup_{0<r<1} \|f(re^{i\cdot})\|_{\mathrm{L}^2} < \infty$.

The functions that are holomorphic on a neighborhood of the origin are called *proper*. If $N \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$, $M \in \mathcal{H}^\infty(\mathtt{U})$ and $M(0)$ is invertible, then we call NM^{-1} a *right factorization* (*of* P, if P is a proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function and $P = NM^{-1}$ near 0). We call such a factorization a *weakly right coprime factorization (w.r.c.f.)* if, in addition,

$$\begin{bmatrix} N \\ M \end{bmatrix} f \in \mathcal{H}^2 \implies f \in \mathcal{H}^2 \tag{2}$$

for every proper $\mathtt{U}$-valued function f; i.e., if a proper $\mathtt{U}$-valued function f is a restriction of an element of $\mathcal{H}^2(\mathtt{U})$ whenever $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f$ is a restriction of an element of $\mathcal{H}^2(\mathtt{Y} \times \mathtt{U})$.[1] In [Mik08] it will be shown that an equivalent definition is obtained with, e.g., any $\mathcal{H}^p$ space in place of $\mathcal{H}^2$ in (2).

In Section 6 the relations between different forms of coprimeness will be treated (the claims on (1) partially in [Mik08]) and the above will be shown equivalent to the classical definition in the matrix-valued case. The property (2) is very important from the control-theoretic point of view. In the literature that property (of Bézout coprime pairs) has been used to reduce unstable control problems to stable ones. Now that can be done in the general case too, as explained in the notes to Section 3.

A right factorization NM^{-1} is called a *(Bézout) right coprime factorization (r.c.f.)* if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is left-invertible in $\mathcal{H}^\infty$, i.e., if there exist $\tilde{X}, \tilde{Y} \in \mathcal{H}^\infty$ such that $\tilde{X}M - \tilde{Y}N \equiv I$ on $\mathbb{D}$ (the minus sign is used for historical and practical reasons). A r.c.f. is a w.r.c.f. (because then $f = \begin{bmatrix} -\tilde{Y} & \tilde{X} \end{bmatrix} \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f \in \mathcal{H}^2$ in (2)), but the converse is not true (Example 7.1).

A right factorization NM^{-1} is *normalized* if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is *inner*, i.e., if $\left\| \left[\begin{smallmatrix} N(z) \\ M(z) \end{smallmatrix}\right] u_0 \right\|_{\mathtt{Y}} = \|u_0\|_{\mathtt{U}}$ for a.e. $z \in \mathbb{T} := \{z \in \mathbb{C} \,\big|\, |z| = 1\}$ for every $u_0 \in \mathtt{U}$. (Here we used the fact that a Hilbert-space-valued $\mathcal{H}^\infty$ function has an L^∞ boundary function [Nik02] [Mik09].)

Now we can state our first main result: every right factorization can be made weakly coprime. (The proof is given in Section 3.)

Theorem 1.1 *A $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function P has a right factorization if and only if it has a normalized weakly right coprime factorization.*

A normalized w.r.c.f. of P is unique modulo the right-multiplication by a unitary operator in $\mathcal{B}(\mathtt{U})$.

Moreover, if $P = NM^{-1}$ is a w.r.c.f., then all right factorizations of P are parameterized by $P = (NV)(MV)^{-1}$, where $V \in \mathcal{H}^\infty(\mathtt{U})$ and V^{-1} is proper. The w.r.c.f.'s are those for which $V^{-1} \in \mathcal{H}^\infty$ too. In particular, if a function P has a Bézout right coprime factorization, then every w.r.c.f. of P is Bézout right coprime.

Thus, with respect to $\mathcal{H}^\infty / \mathcal{H}^\infty$ fractions (or state-feedback stabilization, see Theorem 1.2), weak coprimeness is the more natural form of coprimeness. A r.c.f. is just the special case of the w.r.c.f. for functions of the form characterized in Theorem 1.3.

Nevertheless, for rational matrix-valued functions (or others continuous on $\overline{\mathbb{D}}$), weak coprimeness is equivalent to coprimeness, as well as to N and M having no common zeros on $\overline{\mathbb{D}}$. Therefore, the difference between a r.c.f. and a w.r.c.f. does not show up in the finite-dimensional systems and control theory (cf. [Fra87]).

From Theorem 1.1 we see that a right factorization $P = NM^{-1}$ is a w.r.c.f. if and only if M divides the denominator of every right factorization of P. It follows that M^{-1} must contain the singularities of P but no others (as in Theorem 6.17); note that in the operator-valued case also injective singularity might exist (and is yet forbidden in V). Further details are provided in Section 6 and a rich algebraic description of w.r.c.f.'s in the matrix-valued case is given in [Qua03a] and in [Qua06].

From Theorem 1.1 we also conclude that if $P = NM^{-1}$ is a normalized w.r.c.f., then all normalized right factorizations of P are exactly those corresponding to an inner $V \in \mathcal{H}^\infty(\mathtt{U})$ such that V^{-1} is proper. This solves a problem studied in [OC05].

[1] To be exact, we misuse the notation in a fairly standard way: we write $f \in \mathcal{H}^2(\mathtt{U})$ whenever $f : D(f) \to \mathtt{U}$ is such that $\mathbb{D} \cap D(f)$ is nonempty and open and $f|_{\mathbb{D} \cap D(f)}$ is a restriction of an element of $\mathcal{H}^2$. Analogous misuse applies to $\mathcal{H}^\infty$ and to other spaces of holomorphic functions in place of $\mathcal{H}^2$.

Now it is the time to explain some of the system- and control-theoretic properties and applications of w.r.c.f.'s.

We call $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$ (together with $\mathtt{X}$) a *realization* of P if $P(z) = D + C(z^{-1} - A)^{-1}B$ near 0 (and $\mathtt{X}$ is a Hilbert space). In Section 3 we shall prove the following "extension" of Theorem 1.1 (the fairly standard definitions of (iii)–(v) will be presented in Sections 2 and 3).

Theorem 1.2 (w.r.c.f.) *The following are equivalent for any proper $\mathcal{B}(\mathtt{U}, \mathtt{Y})$-valued function P:*

(i) *P has a right factorization.*

(ii) *P has a normalized weakly right coprime factorization.*

(iii) *P has an output-stabilizable realization.*

(iv) *P has a stabilizable realization.*

(v) *P has a realization that satisfies the Finite Cost Condition* (10).

Moreover, a realization $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ of P is output-stabilizable if and only if there exists a nonnegative solution $\mathcal{P} \in \mathcal{B}(\mathtt{X})$ of the LQR Riccati equation

$$A^*\mathcal{P}A - \mathcal{P} + C^*C = (C^*D + A^*\mathcal{P}B)(I + D^*D + B^*\mathcal{P}B)^{-1}(D^*C + B^*\mathcal{P}A). \tag{3}$$

Let $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ be such. Then there exists a smallest nonnegative solution $\mathcal{P}_{\min}$. Set

$$S := I + D^*D + B^*\mathcal{P}_{\min}B, \qquad F := -S^{-1}(D^*C + B^*\mathcal{P}_{\min}A). \tag{4}$$

Then

$$\left(\begin{array}{c|c} A + BF & B \\ \hline \begin{bmatrix} C + DF \\ F \end{bmatrix} & \begin{bmatrix} D \\ I \end{bmatrix} \end{array}\right), \tag{5}$$

is a realization of $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$, where NM^{-1} is a weakly right coprime factorization of P. Moreover, if we set $S := I + D^*D + B^*\mathcal{P}_{\min}B \in \mathcal{B}(\mathtt{U})$ then $(NS^{-1/2})(MS^{-1/2})^{-1}$ is a normalized weakly right coprime factorization of P.

One more equivalent condition is that the (generalized) Hankel range of P is contained in the (generalized) Toeplitz range of P plus $\mathcal{H}^2$ (Theorem 5.1). Further equivalent conditions are given in [Mik08].

For example, the proper function $P(z) = \sqrt{z + 1/2}$ does not satisfy (i)–(v), being not meromorphic on $\mathbb{D}$. However, without (i)–(v) most typical control problems on P do not have any solutions.

From Theorem 1.2 we conclude that the normalized w.r.c.f. of any function corresponds to the LQ-optimal state-feedback for some system.

Theorem 1.2 provides a constructive formula for the w.r.c.f. Moreover, the result shows that the factorization determined by the *LQ-optimal* state-feedback operator F is always weakly coprime. It need not be Bézout coprime (Example 7.2). Both facts have previously been unknown (cf. [OC05, p. 1208]), even in the scalar-valued case $\mathtt{U} = \mathbb{C} = \mathtt{Y}$.

In the last result of this section, we shall present a similar equivalence on Bézout coprime factorizations. A map $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{Y} \times \mathtt{U})$ such that $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]^{-1} \in \mathcal{H}^\infty$ and $M(0)$ is invertible is called a *d.c.f.* (doubly coprime factorization) of NM^{-1}. It obviously follows that NM^{-1} is a r.c.f. (If we set $\begin{bmatrix} \tilde M & -\tilde N \\ -\tilde Y & \tilde X \end{bmatrix} := \begin{bmatrix} X & N \\ Y & M \end{bmatrix}^{-1}$, then $\tilde M(0)$ is invertible and $\tilde M^{-1}\tilde N$ is called a l.c.f. of NM^{-1}.) The conditions (iii)–(v) below are defined in Section 4, which also contains the proof.

Theorem 1.3 (r.c.f.) *The following are equivalent for a (proper) function P:*

(i) *P has a r.c.f. (right coprime factorization).*

(ii) *P has a d.c.f. (doubly coprime factorization).*

(iii) *P has an output-stabilizable and input-detectable realization.*

(iv) *P has a stabilizable and detectable realization.*

(v) *P has a jointly stabilizable and detectable realization.*

A sixth equivalent condition is that P is dynamically stabilizable (i.e., $\begin{bmatrix} I & -Q \\ -P & I \end{bmatrix}^{-1} \in \mathcal{H}^\infty$ for some proper Q), as will be shown in [Mik07a] using Theorems 1.1 and 1.3. For matrix-valued functions the sufficiency of dynamic stabilization was established in [Ino88] and [Smi89], the necessity and more in [Tre92] (whose extension to the matrix-valued case is contained in [Vas71], as noted in [Qua04]).

Notes

Naturally, to every "right" definition (e.g., "r.c.f.") or result in this article there exists a corresponding "left" definition or result, by duality (replace P (resp., $M, N, \ldots$) by P^{d} (resp., $M^{\mathrm{d}}, N^{\mathrm{d}}, \ldots$), where $P^{\mathrm{d}}(z) := P(\bar z)^*$). In particular, the existence of a "l.c.f." is one more equivalent condition in Theorem 1.3.

We call here $N, M \in \mathcal{H}^\infty$ *gcd-coprime* if (1) holds for every square-matrix-valued V (if it holds for the other $V \in \mathcal{H}^\infty$ too, then N and M are actually Bézout coprime: set $\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$, $V = \begin{bmatrix} N \\ M \end{bmatrix}$ to observe this). For gcd-coprime factorizations, the first result of Theorem 1.1 was established in [vR77] in the scalar-valued case (by showing that $\mathcal{H}^\infty(\mathbb{C})$ is a greatest common divisor domain) and in [Ino88] and [Smi89] in the matrix-valued case, independently.

In control theory, the word "coprime" has been reserved to Bézout coprimeness. Therefore, the "more natural" coprimeness concept, "gcd-coprime", has been called "weakly coprime" [Ino88] [Smi89]. In the matrix-valued case that coincides with our definition, namely with the "quasi-coprime" of [Mik02] and [Mik06], by Theorem 6.14. In the operator-valued case "gcd-coprime" coincides with "(Bézout) r.c." [Mik08], so the term "weakly coprime" can be reserved, instead, to our definition (of "w.r.c.") also in that case. Moreover, our definition is exactly the one satisfied by the factorization determined by the LQ-optimal control, thus generalizing many classical results on rational transfer functions or on finite-dimensional systems.

The main contribution of Theorem 1.1 is the existence of a w.r.c.f. (in our terms); the rest of the theorem is fairly straight-forward. We are grateful to one of the referees for pointing out to us the existence of [Qua05] and [Qua06] and that one obtains an alternative proof of the existence of a w.r.c.f. (and of Theorem 6.14) in the scalar-valued case from [Qua05, Proposition 8 & Corollary 5], whose proofs can be applied to the matrix-valued case too when combined with [Qua06, Theorem 1]. Our Theorem 1.1 was first reported in [MS04] (which is completely due to the author but was presented by O. Staffans).

The contribution of this article to Theorem 1.2 lies in the three appearances of the term "weakly right coprime". The rest of the theorem is contained in [Mik02], and at least the Riccati equation part has been well known earlier (see, e.g., [CZ95]), being a straightforward extension from the finite-dimensional case (of rational transfer functions). Moreover, for the classical definition of weak coprimeness, the equivalence of (i) and (ii) was already established in [Ino88] and [Smi89] as mentioned above.

In the matrix-valued case, the implication "(i)⇔(ii)" in Theorem 1.3 is a direct consequence of Tolokonnikov's Lemma [Tol81]. The lemma was extended to operator-valued functions in [Tre04] (the nonseparable case in [Mik09]), from which we derive the equivalence in Section 4. The implication "(iii)⇒(ii)" was established in [CO06], and the equivalence "(ii)⇔(v)" in [Sta98] (both in continuous time, but the same ideas apply here too). The remaining implications are trivial.

In Section 2 we present the basic properties of discrete-time systems, and in Section 3 we study the LQR problem and prove Theorems 1.1 and 1.2. In Section 4 we prove Theorem 1.3 and explore its conditions (i)–(v) in further detail. There we also present the "power stabilizability" forms of Theorems 1.2 and 1.3.

The Hankel range condition mentioned below Theorem 1.2 is discussed and established in Section 5. There we also show that every "$\mathcal{H}^2_{\text{strong}}$ function" has a realization as an output-stable system (and that the converse holds). Moreover, constructive algorithms are provided for a w.r.c.f. and for an output-stabilizable realization (resp., for a r.c.f., a d.c.f., a robust stabilizing controller and a stabilizable and detectable realization) of a function satisfying the assumptions of Theorem 1.2 (resp., of Theorem 1.3).

The readers not interested in state-space control theory may skip Sections 2–5.

In Section 6 we present further algebraic and function-theoretic properties of w.r.c.f.'s and a similar factorization for general $\mathcal{H}^\infty$ functions; this variant of the inner-outer factorization guarantees the property (2) ("weak left-invertibility") for the inner factor but weakens the requirement on outerness. We also show how the w.r.c.f. is a strictly stronger tool than the (dual) inner-outer factorization. Certain counter-examples are given in Section 7.

With the exceptions mentioned above and in the notes to Section 6, the results in this article (minus Proposition 3.1, Lemma 3.5 and Example 7.1) seem to be new, although part of the results were reported in proceedings [MS04] without proofs and some minor facts were presented in [Mik02].

The continuous-time counterparts of the results in this article (and more) are provided in [Mik08], which is built on the results in this article, with the class of well-posed linear systems [Sta05] [SW02] [Sal89] [Wei94] as the class of realizations. The main exception is that in continuous time the Riccati condition in Theorem 1.2 (is slightly different and) may become very complicated if B is highly unbounded [Mik06] [WZ98] (yet the rest of Theorem 1.2 holds). Moreover, in that article "proper" means "bounded on some right half-plane", and "some right half-plane" takes the role of "some neighborhood of the origin". Thus, e.g., the invertibility of a function M at the origin is replaced by "M^{-1} is proper"; the details are given in the article. There are also slight additional changes in results 6.11–6.14 and 6.18.

Also further discrete-time results on coprimeness and on weak coprimeness can be found in [Mik08]. For frequency-domain weak coprimeness results on matrix-valued functions, particularly for other algebras in place of $\mathcal{H}^\infty$, the reader can consult Quadrat's articles including those mentioned above.

Notation. We define the following terminology in the following order. Section 1: I, $\mathtt{U}$, $\mathtt{X}$, $\mathtt{Y}$, $\mathbb{D}$, $\mathcal{B}$, $\mathcal{H}^\infty$, $\mathcal{H}^2$, proper, right factorization, w.r.c.f., r.c.f., normalized, inner, $\mathbb{T}$, realization, Riccati equation, LQ-optimal d.c.f. Section 2: discrete-time system, $\left(\frac{A \mid B}{C \mid D}\right)$, $\mathbb{N}$, input u, state x, output y, transfer function $\hat{\mathscr{D}}$, Z-transform $\widehat{u}$, state feed-

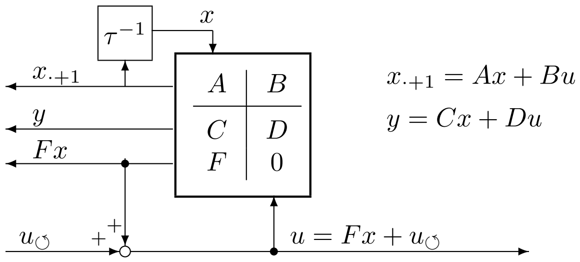

Figure 1: State-feedback connection

back, closed-loop system, output stable, stable, [output-]stabilizable. Section 3: LQR problem, Finite Cost Condition. Section 4: $\mathcal{GB}$, $\mathcal{G}\mathcal{H}^\infty$, [input-]detectable, dual, jointly stabilizable and detectable, power stabilizing, power stable. Section 5: $\mathcal{H}^\infty_r$, $\mathcal{H}^2_r$, $\mathbb{Z}$, ℓ^2_r, π_+, π_-, $\mathbb{Z}_-$, τ, $\mathscr{D}_P$, $\widehat{\pi_+}$, $\widehat{\pi_-}$, $\mathcal{H}^2_{1/r,-}$, $\mathrm{c_c}$, e_k, ℓ^1, $G * u$, $\mathcal{H}^2_{\text{strong}}$. Section 6: weakly left-invertible, coercive, left-invertible, α-proper, α-weakly left-invertible, α-w.r.c., α-right factorization, α-r.c.f., α-w.r.c.f., outer, irreducible, gcd-coprime, dual inner-outer factorization.

2 Discrete-time systems

In this section we recall some well-known details on linear, time-invariant discrete-time systems and state feedback.

A *discrete-time system* on $(\mathtt{U},\mathtt{X},\mathtt{Y})$ is a quadruple $\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right) \in \mathcal{B}(\mathtt{X}\times\mathtt{U},\mathtt{X}\times\mathtt{Y})$. Set $\mathbb{N} := \{0,1,2,\ldots\}$. By $\ell^2(\mathbb{N};\mathtt{U})$ we denote sequences $u:\mathbb{N}\to\mathtt{U}$ such that $\|u\|^2_{\ell^2} := \sum_{k\in\mathbb{N}}\|u_k\|^2_{\mathtt{U}} < \infty$. For each *input* $u\in\ell^2(\mathbb{N};\mathtt{U})$ and *initial state* $x_0\in\mathtt{X}$ we associate the *state* trajectory $x:\mathbb{N}\to\mathtt{X}$ and the *output* $y:\mathbb{N}\to\mathtt{Y}$ through

$$\begin{cases} x_{k+1} = Ax_k + Bu_k, \\ \quad y_k = Cx_k + Du_k, \end{cases} \quad k\in\mathbb{N}. \tag{6}$$

The *transfer function* $\hat{\mathscr{D}}(z) := D + C(z^{-1}-A)^{-1}B = D + zC(I-zA)^{-1}B$ of $\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right)$ is holomorphic $r^{-1}\mathbb{D}\to\mathcal{B}(\mathtt{U},\mathtt{Y})$, where $r^{-1}\mathbb{D} = \{z\in\mathbb{C} \mid |z| < r^{-1}\}$ and $r := r(A)$ is the spectral radius of A. The *Z-transform* $\widehat{u}$ of $u:\mathbb{N}\to\mathtt{U}$ is defined by $\widehat{u}(z) := \sum_n z^n u_n$ (for those z for which the sum converges absolutely). For $x_0 = 0$ we have $\widehat{y} = \hat{\mathscr{D}}\widehat{u}$ on $\mathbb{D}\cap r^{-1}\mathbb{D}$ for every $u\in\ell^2(\mathbb{N};\mathtt{U})$, hence the name transfer function.

State feedback means that, for some *state-feedback operator* $F\in\mathcal{B}(\mathtt{X},\mathtt{U})$, we use the function $u := Fx + u_{\circlearrowleft}$ as the input, where $u_{\circlearrowleft}:\mathbb{N}\to\mathtt{U}$ denotes an exogenous input (or disturbation) $u_{\circlearrowleft}$, as in Figure 1. Thus, equation (6) together with $u = Fx + u_{\circlearrowleft}$ defines the "closed-loop system" that maps x_0 and $u_{\circlearrowleft}$ to x and $\left[\begin{smallmatrix} y \\ u\end{smallmatrix}\right]$; we just have to replace $\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right)$ by (5). By the above definition, the transfer function of the *closed-loop system* (5) is given by

$$\begin{bmatrix}\hat{\mathscr{N}}(z) \\ \hat{\mathscr{M}}(z)\end{bmatrix} := \begin{bmatrix} D \\ I\end{bmatrix} + \begin{bmatrix} C+DF \\ F\end{bmatrix}(z^{-1}-A-BF)^{-1}B. \tag{7}$$

It follows that $\hat{\mathscr{D}} = \hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$, and $\hat{\mathscr{M}}^{-1} = I - F(\cdot^{-1}-A)^{-1}B$.

The system $\left(\begin{array}{c|c} A & B \\ \hline C & D\end{array}\right)$ is called *output stable* if $y\in\ell^2$ whenever $x_0\in\mathtt{X}$ and $u=0$ (or equivalently, if there exists $K<\infty$ such that $\|CA^{\cdot}x_0\|_2 \le K\|x_0\|_{\mathtt{X}}$ $(x_0\in\mathtt{X})$); *stable* if

$y \in \ell^2$ and x is bounded whenever $x_0 \in \mathtt{X}$ and $u \in \ell^2(\mathbb{N};\mathtt{U})$, or equivalently, if there exists $K < \infty$ such that

$$\|x_n\|_{\mathtt{X}} + \|y\|_2 \leq K\left(\|x_0\|_{\mathtt{X}} + \|u\|_2\right) \qquad (n \geq 0,\ x_0 \in \mathtt{X},\ u \in \ell^2(\mathbb{N};\mathtt{U})). \tag{8}$$

The system $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is called *output-stabilizable* (resp., *stabilizable*) if, for some $F \in \mathcal{B}(\mathtt{X},\mathtt{U})$, the closed-loop system (5) becomes output-stable (resp., stable).

3 The LQR problem

In this section we prove Theorems 1.1 and 1.2. This requires us to first present the solution of the LQR problem and to show that its solution possesses certain interesting properties.

The *LQR problem* (Linear Quadratic Regulator problem) refers to the minimization of the cost function

$$\sum_{k=0}^{\infty}\left(\|y\|_{\mathtt{Y}}^2 + \|u\|_{\mathtt{U}}^2\right) = \|y\|_2^2 + \|u\|_2^2. \tag{9}$$

over all $u \in \ell^2(\mathbb{N};\mathtt{U})$ for a fixed initial state $x_0 \in \mathtt{X}$. For the problem to be solvable, we must obviously assume the *Finite Cost Condition*:

$$\text{for each } x_0 \in \mathtt{X} \text{ there exists } u \in \ell^2(\mathbb{N};\mathtt{U}) \text{ such that } y \in \ell^2. \tag{10}$$

The following is well known (see, e.g., [OC05, Corollary 5.8] or [Mik02, Theorem 9.9.1(g2)]):

Proposition 3.1 (LQR) *The Finite Cost Condition holds if and only if the system is output-stabilizable, or equivalently, if and only if there exists a nonnegative solution of* (3).

Assume the Finite Cost Condition. Set (4)*, where* $\mathcal{P}_{\min}$ *stands for the smallest nonnegative solution of* (3)*. Then F is output-stabilizing,* $\hat{\mathscr{D}} = \hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$ *is a right factorization, and* $\hat{\mathscr{D}} = (\hat{\mathscr{N}}S^{-1/2})(\hat{\mathscr{M}}S^{-1/2})^{-1}$ *is a normalized right factorization. Moreover, the state-feedback* $u_n = Fx_n$ *(or equivalently,* $u_n = F(A+BF)^n x_0$*) minimizes the cost* $\|y\|_2^2 + \|u\|_2^2$*, the minimum being* $\langle x_0, \mathcal{P}_{\min} x_0\rangle$*.*

Next we define the time-domain counterparts of $\hat{\mathscr{D}}$ and $\hat{\mathscr{M}}$ (the latter depending on the $F \in \mathcal{B}(\mathtt{X},\mathtt{U})$ chosen). For $x_0 = 0$ and $u \in \ell^2(\mathbb{N};\mathtt{U})$, we denote by $\mathscr{D}u$ (resp., $\mathscr{M}^{-1}u$) the signal $y = Cx + Du$ (resp., $u_{\circlearrowleft} = u - Fx$). (We may have $\mathscr{D}u, \mathscr{M}^{-1}u \notin \ell^2$.) By $\mathscr{M}$ we denote the inverse of the map $\mathscr{M}^{-1}$. Obviously, $\widehat{\mathscr{M}^{-1}u} = \hat{\mathscr{M}}^{-1}\widehat{u}$ near 0 for any $u \in \ell^2(\mathbb{N};\mathtt{U})$.

Now we show that $\mathscr{M}^{-1}$ maps "admissible" inputs into ℓ^2.

Lemma 3.2 *Let* $x_0 = 0$*. With the assumption and notation of Proposition 3.1, for any* $u \in \ell^2(\mathbb{N};\mathtt{U})$ *such that* $y \in \ell^2$*, we have* $\langle x_n, \mathcal{P}_{\min} x_n\rangle_{\mathtt{X}} \to 0$*, as* $n \to +\infty$*, and* $\mathscr{M}^{-1}u \in \ell^2$ *and* $\langle \mathscr{M}^{-1}u, S\mathscr{M}^{-1}u\rangle_{\ell^2(\mathbb{N};\mathtt{U})} = \|y\|_2^2 + \|u\|_2^2$*.*

Proof: 1° Given $n \in \mathbb{N}$, define $\tilde{u} \in \ell^2(\mathbb{N};\mathtt{U})$ and $\tilde{y} \in \ell^2(\mathbb{N};\mathtt{Y})$ by $\tilde{u}_k := u_{n+k}$, $\tilde{y}_k := y_{n+k}$ $(k \in \mathbb{N})$. Then $\tilde{y}$ equals the output with initial state x_n and input $\tilde{u}$. Therefore, $\|\tilde{y}\|_2^2 + \|\tilde{u}\|_2^2 \geq \langle x_n, \mathcal{P}_{\min} x_n\rangle$, by Proposition 3.1. But $\|\tilde{y}\|_2^2 + \|\tilde{u}\|_2^2 \to 0$, as $n \to \infty$, hence $\langle x_n, \mathcal{P}_{\min} x_n\rangle \to 0$.

2° Set $v := \mathscr{M}^{-1}u = u - Fx$. Then, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\langle v_n, Sv_n\rangle &= \langle u_n, Su_n\rangle + \langle Fx_n, SFx_n\rangle - 2\operatorname{Re}\langle u_n, SFx_n\rangle && (11)\\ &= \|u_n\|^2 + \|Du_n\|^2 + \langle Bu_n, \mathcal{P}Bu_n\rangle && (12)\\ &\quad + \langle Ax_n, \mathcal{P}Ax_n\rangle - \langle x_n, \mathcal{P}x_n\rangle + \|Cx_n\|^2 + 2\operatorname{Re}\langle u_n, (D^*C + B^*\mathcal{P}A)x_n\rangle && (13)\\ &= \|u_n\|^2 + \|y_n\|^2 + \langle x_{n+1}, \mathcal{P}x_{n+1}\rangle - \langle x_n, \mathcal{P}x_n\rangle. && (14)\end{aligned}$$

Thus, $\sum_{n=0}^k \langle v_n, Sv_n\rangle = \sum_{n=0}^k \|u_n\|^2 + \sum_{n=0}^k \|y_n\|^2 + \langle x_{k+1}, \mathcal{P}x_{k+1}\rangle - \langle x_0, \mathcal{P}x_0\rangle$. Since $x_0 = 0$, by 1° we get that $\sum_{n=0}^\infty \langle v_n, Sv_n\rangle = \|u\|_2^2 + \|y\|_2^2 < \infty$, hence $v \in \ell^2$ (because $S \geq I$). □

Finally, we observe that domain of the part of $\mathscr{D}$ in ℓ^2 equals $\mathscr{M}[\ell^2]$.

Lemma 3.3 *Set $\mathcal{U} := \{u \in \ell^2(\mathbb{N};\mathtt{U}) \,|\, \mathscr{D}u \in \ell^2\}$. With the assumption and notation of Proposition 3.1, $\mathscr{M}^{-1}[\mathcal{U}] = \ell^2(\mathbb{N};\mathtt{U})$. Moreover, $\hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$ is a w.r.c.f. and $\hat{\mathscr{N}}S^{-1/2}(\hat{\mathscr{M}}S^{-1/2})^{-1}$ is a normalized w.r.c.f.*

Proof: 1° By Lemma 3.2, $\mathscr{M}^{-1}$ maps $\mathcal{U}$ into ℓ^2. Set $\mathscr{N} := \mathscr{D}\mathscr{M}$. If $v \in \ell^2(\mathbb{N};\mathtt{U})$ and we set $u := \mathscr{M}v$, then $\mathscr{D}u = \mathscr{N}v \in \ell^2$ (because $\widehat{\mathscr{N}v} = \hat{\mathscr{N}}\widehat{v} \in \mathcal{H}^2$), hence then $u \in \mathcal{U}$. Thus, $\mathscr{M}^{-1}$ maps $\mathcal{U}$ onto ℓ^2.

2° Assume that $\widehat{v}$ is proper, $\widehat{u} := \hat{\mathscr{M}}\widehat{v} \in \mathcal{H}^2$ and $\hat{\mathscr{N}}\widehat{v} \in \mathcal{H}^2$. Then $\widehat{\mathscr{D}u} = \hat{\mathscr{D}}\widehat{u} = \hat{\mathscr{N}}\widehat{v} \in \mathcal{H}^2$, hence $u \in \mathcal{U}$, hence $v = \mathscr{M}^{-1}u \in \ell^2$, i.e., $\widehat{v} \in \mathcal{H}^2$. Thus, $\hat{\mathscr{N}}\hat{\mathscr{M}}^{-1}$ is a w.r.c.f.

By Lemma 3.2, we have $\|\mathscr{N}v\|_{\ell^2}^2 + \|\mathscr{M}v\|_{\ell^2}^2 = \langle v, Sv\rangle_{\ell^2}$. Set $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] := \left[\begin{smallmatrix} \hat{\mathscr{N}}S^{-1/2} \\ \hat{\mathscr{M}}S^{-1/2} \end{smallmatrix}\right]$. For any $u_0 \in \mathtt{U}$ and $\phi \in \ell^2(\mathbb{N};\mathbb{C})$ we have $\|\phi\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]u_0\|_2 = \|\phi u_0\|_2$ (set $v := \phi u_0$), hence $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is inner. □

It is straight-forward and well known that any $\mathcal{H}^\infty$ function has a stable shift-semigroup realization.

Lemma 3.4 *Let $P \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$. Then a stable realization of P is given by*

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := \left(\begin{array}{c|c} S_L & z^{-1}\cdot \\ \hline \pi_0 P & P(0) \end{array}\right) \in \mathcal{B}(\tilde{\mathtt{X}} \times \mathtt{U}, \tilde{\mathtt{X}} \times \mathtt{Y}), \tag{15}$$

where $\tilde{\mathtt{X}} := \mathcal{H}^2_-(\mathtt{Y}) := \{\sum_{k=-\infty}^{-1} a_k z^k \,|\, a \in \ell^2(\mathbb{Z}_-;\mathtt{U})\} = \widehat{\ }\ell^2(\mathbb{Z}_-;\mathtt{U})$, $\pi_0 \sum_k z^k a_k := a_0$ and S_L is the left shift $\sum_{k=-\infty}^{-1} z^k a_k \mapsto \sum_{k=-\infty}^{-2} z^k a_{k+1}$.

Here $\pi_0 Pf := \pi_0(Pf)$, where P is the multiplication operator $\mathcal{H}^2_-(\mathtt{U}) \ni f \mapsto Pf \in \mathrm{L}^2(\mathbb{T};\mathtt{Y})$ (it is well-known that P extends to a bounded multiplication operator on $\mathcal{H}^2_-$, even on $\mathrm{L}^2(\mathbb{T};\mathtt{U}) = \{\sum_{k=-\infty}^\infty a_k z^k \,|\, a \in \ell^2(\mathbb{Z}_-;\mathtt{U})\}$).

Further results on realizations are provided in Theorems 5.2 and 5.5 and in Remark 5.6.

Proof of Theorem 1.2: 1° *"(i)⇒(iv)":* Assume (i) and redefine N and M to have $M(0) = I$. Let $\left(\begin{array}{c|c} A_\circlearrowleft & B \\ \hline \left[\begin{smallmatrix} C_\circlearrowleft \\ F \end{smallmatrix}\right] & \left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right] \end{array}\right)$ be a stable realization of $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ (e.g., use Lemma 3.4). Set $A := A_\circlearrowleft - BF$, $C := C_\circlearrowleft - DF$ to define a system $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$. This system is stabilizable (by F) and its transfer function is NM^{-1}, as noted below (7).

2° Implications "(ii)⇒(i)" and "(iv)⇒(iii)" are trivial and "(iii)⇒(ii)" follows from Lemma 3.3. From Proposition 3.1 we obtain the equivalence "(v)⇔(iii)" and all claims below (v) except the fact that the N and M defined by (5) (i.e., by (7))) are w.r.c., which follows from Lemma 3.3. □

The stabilizable realization in the above proof was constructed in terms of N and M. For an output-stabilizable realization in terms of P only, see Theorem 5.2.

Bounded multiplication operators on $\mathcal{H}^2$ are $\mathcal{H}^\infty$ functions:

Lemma 3.5 *If V is a proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function and $Vf \in \mathcal{H}^2(\mathtt{U})$ for each $f \in \mathcal{H}^2(\mathtt{X})$, then $V \in \mathcal{H}^\infty(\mathtt{X},\mathtt{U})$.*

(Actually, the same lemma and proof apply to $\mathcal{H}^p$ in place of $\mathcal{H}^2$ when $1 \le p \le \infty$.)

Proof: Note that $\mathcal{H}^2(\mathtt{X}) \ni f \mapsto f(z)$ is bounded for every $z \in \mathbb{D}$ [HP57, Theorem 6.4.2]. From this and the closed-graph theorem we observe that $M_V : f \mapsto Vf$ is bounded $\mathcal{H}^2(\mathtt{X}) \to \mathcal{H}^2(\mathtt{Y})$ (if $f_n \to f$ and $Vf_n \to g$ in $\mathcal{H}^2$, as $n \to \infty$, then $g = \lim_n Vf_n = Vf$ pointwise near the origin). We conclude that the formula $R(z)x_0 := (Vx_0)(z)$ defines a function $R : \mathbb{D} \to \mathcal{B}(\mathtt{X},\mathtt{U})$ and $R = V$ (near the origin). By [HP57, Theorem 3.10.1], R is holomorphic. Trivially, $\|M_V\| \le \|R\|_{\mathcal{H}^\infty} \le \infty$, and it is not difficult to verify that $\|R\|_{\mathcal{H}^\infty} \le \|M_V\| < \infty$. □

The property (2) works for $\mathcal{H}^\infty$ too:

Lemma 3.6 *If NM^{-1} is a w.r.c.f. with $M \in \mathcal{H}^\infty(\mathtt{U})$, R is a proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function, and $NR, MR \in \mathcal{H}^\infty$, then $R \in \mathcal{H}^\infty$.*

(By Theorem 6.13, the above property is also sufficient for weak coprimeness.)

Proof: For each $f \in \mathcal{H}^2$ we have $NRf, MRf \in \mathcal{H}^2$, hence $Rf \in \mathcal{H}^2$. By Lemma 3.5, $R \in \mathcal{H}^\infty$. □

It is well known that invertible inner functions are constants [Sta97, Lemma 18(iii)]:

Lemma 3.7 *If $V, V^{-1} \in \mathcal{H}^\infty(\mathtt{U})$ and V is inner, then $V \in \mathcal{B}(\mathtt{U})$.*

Proof of Theorem 1.1: The equivalence follows from Theorem 1.2. If $P = N_0M_0^{-1}$ is a right factorization and we set $V := M^{-1}M_0$, then $N_0 = PM_0 = PMV = NV$, hence $NV, MV \in \mathcal{H}^\infty$, hence $V \in \mathcal{H}^\infty$, by Lemma 3.6.

By the above, $V^{-1} \in \mathcal{H}^\infty$ if also $N_0M_0^{-1}$ is a w.r.c.f. If, in addition, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} N_0 \\ M_0 \end{smallmatrix}\right]$ are inner, then so is V, hence then $V \in \mathcal{B}(\mathtt{U})$, by Lemma 3.7.

Conversely, if $V, V^{-1} \in \mathcal{H}^\infty$, then $(NV)(MV)^{-1}$ is obviously a w.r.c.f. (even a Bézout r.c.f. if N and M are Bézout coprime). □

Notes for Section 3

In Lemmata 3.2 and 3.3 we showed that the "LQ-optimal" N and M form a w.r.c.f. In an alternative proof one can show that if the elements $(u_\circlearrowleft)_0, (u_\circlearrowleft)_1, \ldots, (u_\circlearrowleft)_{n-1} \in \mathtt{U}$ of the closed-loop input (see Section 2) are fixed, with initial state $x_0 = 0$, then the minimal cost $\|y\|_2^2 + \|u\|_2^2$ is achieved by setting $0 = (u_\circlearrowleft)_n, (u_\circlearrowleft)_{n+1}, (u_\circlearrowleft)_{n+2}, \ldots$. (This must be shown to all sequences $u_\circlearrowleft : \mathbb{N} \to \mathtt{U}$ for which $u, y \in \ell^2$.)

Indeed, whenever $G := \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is inner, then, to minimize $\|Gf\|_{\mathcal{H}^2(\mathtt{Y})}$ over $f \in \mathcal{H}^2(\mathtt{U})$ with $f(z) = \sum_{k=0}^\infty z^k f_k$ for fixed $f_0, f_1, \ldots, f_n \in \mathtt{U}$, we obviously have to take $0 = f_{n+1}, f_{n+2}, \ldots$. If we drop the requirement $f \in \mathcal{H}^2(\mathtt{U})$, then the same holds if and only if, in addition, N and M are weakly right coprime (it is not difficult to show this).

Moreover, as mentioned in the introduction, the LQ-optimal state feedback reduced regulation problems to the stable case without loss of generality. We explain this below.

Given an output-stabilizable system, we can use the state feedback (4) to (output- and I/O-)stabilize it. For any initial state $x_0 \in \mathtt{X}$, the inputs $u_{\circlearrowleft} \in \ell^2$ to the closed-loop system (5) correspond one-to-one to the set of *admissible inputs* for the original system $\left(\frac{A \mid B}{C \mid D}\right)$, i.e., to those inputs $u \in \ell^2$ that result in an $y \in \ell^2$ output (see (6)), because of weak coprimeness. Also any other (output- and I/O-)stabilizing state feedback that makes N and M w.r.c. allows one to reduce any control problem over admissible inputs to a control problem for a "stable" system (over $u_{\circlearrowleft} \in \ell^2(\mathbb{N};\mathtt{U})$). Moreover, such "weakly coprime stabilization" is the weakest form of stabilization with this reduction property—this was the motivation behind the concept in [Mik02] before the equivalence and the other results of this article were known and behind its wide implicit use in the literature. However, sometimes one wants to optimize over power-stabilizing (or exponentially stabilizing) inputs only; for such control problems an analogous result holds without any coprimeness assumptions [Mik02, Lemma 8.4.5(e)].

Finally, we mention that the LQ-optimal state feedback F maximizes the robustness margin (to $\|S\|^{-1/2}$) for state-feedback controllers (even for the feedback $\widehat{u} = K\widehat{x}$ for arbitrary proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function K) with normalized right weakly coprime factor uncertainty in the sense of [KS94]. This generalizes [KS94] to output-stabilizable and infinite-dimensional plants (the proof is analogous and hence omitted).

4 Doubly coprime factorizations and joint stabilizability

In this section we define the concepts in Theorem 1.3 and present its proof.

We set $\mathcal{G}\mathcal{V} = \{G \in \mathcal{V} \mid \text{ there exists } G^{-1} \in \mathcal{V}\}$ when $\mathcal{V} = \mathcal{B}$ or $\mathcal{V} = \mathcal{H}^\infty$.

Lemma 4.1 *If $N \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ and $M \in \mathcal{H}^\infty(\mathtt{U})$ are r.c., then there exist a closed subspace $\mathtt{Y}_2 \subset \mathtt{Y}$ and functions $X, Y \in \mathcal{H}^\infty$ such that $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \in \mathcal{G}\mathcal{H}^\infty(\mathtt{Y}_2 \times \mathtt{U}, \mathtt{Y} \times \mathtt{U})$.*

If $\dim \mathtt{U} < \infty$ or $M(z) \in \mathcal{G}\mathcal{B}(\mathtt{U})$ for some $z \in \mathbb{D}$, then we can choose X and Y above so that $\mathtt{Y}_2 = \mathtt{Y}$.

In particular, any r.c.f. can be extended to a d.c.f.

Proof: 1° The first claim follows from [Tre04]. ([Mik09, Theorem 4.3] in the nonseparable case).

Note that here $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \in \dots$ means that there exists $G \in \mathcal{H}^\infty(\mathtt{Y} \times \mathtt{U}, \mathtt{Y}_2 \times \mathtt{U})$ such that $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] G = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{Y} \times \mathtt{U})$ and $G \left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{Y}_2 \times \mathtt{U})$.

2° We first note that if there exists some operator $T \in \mathcal{G}\mathcal{B}(\mathtt{Y},\mathtt{Y}_2)$, then we can replace $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] \left[\begin{smallmatrix} T & 0 \\ 0 & I \end{smallmatrix}\right] \in \mathcal{G}\mathcal{H}^\infty(\mathtt{Y} \times \mathtt{U})$ to satisfy the also second sentence of the lemma.

3° Assume that $\dim \mathtt{U} < \infty$. We have $\dim \mathtt{Y}_2 \times \mathtt{U} = \dim \mathtt{Y} \times \mathtt{U}$ (because $\mathcal{G}\mathcal{B}(\mathtt{Y}_2 \times \mathtt{U}, \mathtt{Y} \times \mathtt{U})$ is nonempty). Therefore, $\dim \mathtt{Y}_2 = \dim \mathtt{Y}$ (by $\dim \mathtt{Y}$ we refer to the (necessarily unique) cardinality of an arbitrary orthonormal basis of $\mathtt{Y}$). Consequently, $\mathcal{G}\mathcal{B}(\mathtt{Y},\mathtt{Y}_2)$ is nonempty ([Mik02, Lemma A.3.1(a5)]), and hence the trick in 2° applies.

4° Assume, instead, that $M(z) \in \mathcal{G}\mathcal{B}(\mathtt{U})$ for some $z \in \mathbb{D}$. Then $f := X - NM^{-1}Y$ is defined and invertible at z (because $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & N \\ 0 & M \end{smallmatrix}\right] \begin{bmatrix} X - NM^{-1}Y & 0 \\ M^{-1}Y & I \end{bmatrix}$ at z, and $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]$

and $\left[\begin{smallmatrix} I & N \\ 0 & M \end{smallmatrix}\right]$ are invertible at z), i.e., $T := f(z)^{-1} \in \mathcal{GB}(\mathtt{Y}, \mathtt{Y}_2)$, hence the trick in 2° applies.

5° If NM^{-1} is a r.c.f. of a function P, then $M(0) \in \mathcal{GB}(\mathtt{U})$ and hence we can extend it to a d.c.f. of P, by the first two sentences of the lemma. □

(The assumption $\dim \mathtt{U} < \infty$ in is not extraneous: if $\dim \mathtt{Y} \leq \dim \mathtt{U} = \infty$, then there exists an invertible $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] \in \mathcal{B}(\mathtt{U}, \mathtt{U} \times \mathtt{Y})$, thus making $\mathtt{Y}_2 = \{0\}$ the only possible choice of $\mathtt{Y}_2$.)

In [Mik02] and [Sta05] it is explained how to extend a fixed l.c.f. and a fixed r.c.f. to a d.c.f. or how to obtain all r.c.f.'s or d.c.f.'s of a given function P.

Output-stabilizability (Section 2) of a system and its dual leads to a doubly coprime factorization of the transfer function:

Lemma 4.2 *If* $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ *and* $\left(\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right)$ *are output stabilizable, then* $\hat{\mathscr{D}} := D + C(\cdot^{-1} - A)^{-1}B$ *has a d.c.f.*

By Theorem 1.1, the "optimal LQR feedback" factorization NM^{-1} of Theorem 1.2 (or of Proposition 3.1) is then a r.c.f. of $\hat{\mathscr{D}}$.

Proof of Lemma 4.2: By [OC05, Lemma 6.7], the Hankel norm of $\left[\begin{smallmatrix} NS^{-1/2} \\ MS^{-1/2} \end{smallmatrix}\right]$ is strictly smaller than one (the separability assumption in [OC05] is not needed there). By [Mik09, Corollary 4.5], this implies that $MS^{-1/2}$ and $NS^{-1/2}$ are r.c., hence so are M and N. By Lemma 4.1, $\hat{\mathscr{D}}$ has a d.c.f. □

Also the converse result holds, by Theorem 1.3.

Now we present some terminology needed in this section (and Theorem 1.3) only. A system $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in \mathcal{B}(\mathtt{X} \times \mathtt{U}, \mathtt{X} \times \mathtt{Y})$ is called *[input-]detectable* if its *dual* $\left(\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right)$ is [output-]stabilizable (Section 2). The system is called *jointly stabilizable and detectable* if there exist $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$ and $H \in \mathcal{B}(\mathtt{Y}, \mathtt{X})$ such that the systems

$$\left(\begin{array}{c|cc} A+BF & H & B \\ \hline C+DF & 0 & D \\ F & 0 & 0 \end{array}\right) \quad \text{and} \quad \left(\begin{array}{c|cc} A+LC & L & B+LD \\ \hline C & 0 & D \\ F & 0 & 0 \end{array}\right) \tag{16}$$

are stable. By removing the middle row or the bottom column, we observe that this means that $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ and $\left(\begin{array}{c|c} A^* & C^* \\ \hline B^* & D^* \end{array}\right)$ are stabilizable, that the state-feedback operator F also stabilizes the additional input corresponding to L, and that L^* satisfies the dual condition.

If F and L^* are *power stabilizing* to the two systems, respectively, (i.e., $r(A+BF) < 1$ and $r(A+LC) < 1$, where r stands for the spectral radius), then they are necessarily jointly stabilizing and detecting in the above sense. In that case also the standard formulae yield a d.c.f. of $\hat{\mathscr{D}}$. However, it is not known whether the functions defined by the standard formulae for $X, Y, \tilde{X}, \tilde{Y}$ are bounded on $\mathbb{D}$ in the setting of Lemma 4.2 [CO08] [CO07] [CO06]. Nevertheless, in [CO08], nonstandard formulae for (alternative, bounded) Bézout factors $X, Y, \tilde{X}, \tilde{Y}$ are provided (partially also in [CO07], continuous-time in [CO06]).

From the proof below one sees that, given a d.c.f. of P, we can actually choose a realization $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ of P so as to have the two systems in (16) strongly stable.

Proof of Theorem 1.3: Implication "(iii)⇒(ii)" is Lemma 4.2, implication "(i)⇒(ii)" is from Lemma 4.1, and implications "(v)⇒(iv)⇒(iii)" and "(ii)⇒(i)" are trivial, so we assume (ii) and prove (v).

Let the first system in (16) be some stable realization (Lemma 3.4) of $\left[\begin{smallmatrix} I-X & N \\ -Y & M-I \end{smallmatrix}\right]$ (in particular, set $A := A_{\circlearrowleft} - BF$, $C := C_{\circlearrowleft} - DF$, where $A_{\circlearrowleft}$, $C_{\circlearrowleft}$ and F constitute the first column of this realization). As in the proof of [Sta98, Theorem 4.4] (or in that of [Mik02, Theorem 6.6.28]), one verifies that the second system in (16) is a realization of $\left[\begin{smallmatrix} \tilde{M}-I & \tilde{N} \\ -\tilde{Y} & I-\tilde{X} \end{smallmatrix}\right]^{-1}$ and that both realizations are stable. □

Notes for Section 4

A system $\left(\frac{A \mid B}{C \mid D}\right)$ on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$ is called *power stable* if $r(A) < 1$ and *power stabilizable* if $r(A + BF) < 1$ for some $F \in \mathcal{B}(\mathtt{X}, \mathtt{U})$. We call an element of $\mathcal{H}^\infty$ *power stable* if it has a holomorphic extension to an open set containing $\overline{\mathbb{D}}$. We call a right factorization NM^{-1} *power stable* if N and M are power stable.

Theorem 1.3 also holds in its "power form": P has a power stable r.c.f. if and only if it has a power stable d.c.f. (i.e., one with $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} X & N \\ Y & M \end{smallmatrix}\right]^{-1}$ power stable), or equivalently, P has a [jointly] power-stabilizable and power-detectable realization. (The proof is essentially the same.)

Also most of Theorem 1.2 holds in its "power form". In particular, P has a power-stable right factorization if and only if it has a power-stabilizable realization. Similarly, in (10) we have to replace y by x to have (v) equivalent to the existence of a power-stabilizable realization. Power-stabilizability of $\left(\frac{A \mid B}{C \mid D}\right)$ is equivalent to (3) with $C = I$ and $D = 0$ having a nonnegative solution $\mathcal{P}$ (which is necessarily unique). With the original C and D in (4) this $\mathcal{P}$ defines a power-stable right factorization (of $P = D + C(\cdot^{-1} - A)^{-1}B$), which need not be a w.r.c.f.

The scalar function $P := NM^{-1}$ in Example 7.1 has the w.r.c.f. NM^{-1} and a "w.l.c.f." (namely $M^{-1}N$) but no d.c.f. In particular, to have a d.c.f. it is not sufficient to have a stabilizable realization and a (different) detectable realization. In the operator-valued case this holds also for power-stabilizability and power-detectability. However, a matrix-valued "power w.r.c.f." is a "power r.c.f.", by Theorem 6.6(i)&(iii) below.

5 Realizations and Hankel operators

In Theorem 5.1 we shall extend Theorem 1.2 by further equivalent conditions. In Theorem 5.2 we shall construct an output-stabilizable realization of the function P in Theorem 1.2 without reference to a factorization. If P has a r.c.f., then the algorithm in Remark 5.3 yields a r.c.f., a d.c.f., a robust stabilizing controller and a stabilizable and detectable realization of P, constructively. In Theorem 5.5 we characterize the functions that have output-stable realizations.

To do the above, we first need a few state-space definitions and well-known results, for this section only.

Given $r > 0$, by $\mathcal{H}^\infty_r(\mathtt{U}, \mathtt{Y})$ we denote the Banach space of bounded holomorphic functions $r\mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ with supremum norm and by $\mathcal{H}^2_r(\mathtt{U})$ the Hilbert space of holomorphic functions $r\mathbb{D} \to \mathcal{B}(\mathtt{U}, \mathtt{Y})$ having $\sup \|f\|_{\mathcal{H}^2_r} := \sup_{0<t<r} \|f(te^{i\cdot})\|_{\mathrm{L}^2} < \infty$. Thus, $\mathcal{H}^\infty = \mathcal{H}^\infty_1$ and $\mathcal{H}^2 = \mathcal{H}^2_1$. Moreover, we define the Hilbert space $\ell^2_r(\mathbb{Z}; \mathtt{U})$ of weighted square-summable sequences on $\mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}$ by setting

$$\|u\|_{\ell_r^2}^2 := \sum_{k=-\infty}^{\infty} r^{-k} \|u_k\|_{\mathtt{U}}^2. \tag{17}$$

We set $\ell^2 := \ell_1^2$, $(\pi_+ u)_k := \begin{cases} u_k, & k \geq 0; \\ 0, & k < 0 \end{cases}$, $\pi_- := I - \pi_+$, $\mathbb{Z}_- := \mathbb{Z} \setminus \mathbb{N}$, $(\tau u)_k := u_{k+1}$. The Z-transform $u \mapsto \widehat{u} := \sum_k z^k u_k$ is an isomorphism of $\ell_r^2(\mathbb{N}; \mathtt{U})$ onto $\mathcal{H}_{1/r}^2(\mathtt{U})$. Therefore, every $P \in \mathcal{H}_{1/r}^\infty(\mathtt{U}, \mathtt{Y})$ corresponds (isometrically) to a unique $\mathscr{D}_P \in \mathcal{B}(\ell_r^2(\mathbb{N}; \mathtt{U}), \ell_r^2(\mathbb{N}; \mathtt{U}))$ through $\widehat{\mathscr{D}_P u} = P\widehat{u}$. Moreover, $\mathscr{D}_P$ has a unique shift-invariant (i.e., $\tau \mathscr{D}_P = \mathscr{D}_P \tau$) extension $\mathscr{D}_P \in \mathcal{B}(\ell_r^2(\mathbb{Z}; \mathtt{U}), \ell_r^2(\mathbb{Z}; \mathtt{U}))$. This extension satisfies $\pi_- \mathscr{D}_P \pi_+ = 0$ (causality). We set $\widehat{\pi}_+ \sum_{k=-\infty}^{\infty} u_k z^k := \sum_{k=0}^{\infty} u_k z^k$, $\widehat{\pi}_- := I - \widehat{\pi}_+$. Note that $\widehat{\pi_+ u} = \widehat{\pi}_+ \widehat{u}$ and that the functions in the space $\mathcal{H}_{1/r,-}^2(\mathtt{U}) := \{\sum_{k=-\infty}^{-1} a_k z^k \mid a \in \ell_r^2(\mathbb{Z}_-; \mathtt{U})\}$ are holomorphic on $r^{-1}\overline{\mathbb{D}}^c$. We equip $\mathcal{H}_{1/r,-}^2(\mathtt{U})$ with the topology coinduced from ℓ_r^2.

Now we can show that Theorem 1.2(i) holds if and only if the "generalized Hankel range" $\pi_+ \mathscr{D}_P \pi_-$ lies in the "generalized Toeplitz range" $\pi_+ \mathscr{D}_P \pi_+$ plus $\ell^2(\mathbb{N}; \mathtt{U})$, i.e., if and only if "the Toeplitz range of P stabilizes the Hankel range of P":

Theorem 5.1 *Let $P \in \mathcal{H}_{1/\gamma}^\infty(\mathtt{U}, \mathtt{Y})$ for some $\gamma \geq 1$. Then also each of the following conditions is equivalent to the existence of a right factorization of P:*

(vi) *There exists $r \geq \gamma$ such that $\pi_+ \mathscr{D}_P \pi_- [\ell_r^2] \subset \pi_+ \mathscr{D}_P \pi_+ [\ell^2] + \ell^2(\mathbb{N}; \mathtt{Y})$.*

(vi') *There exists $r \geq \gamma$ such that for any $v \in \ell_r^2(\mathbb{Z}_-; \mathtt{U})$ there exists $u \in \ell^2(\mathbb{N}; \mathtt{U})$ such that $\mathscr{D}_P(v + u) \in \ell^2$.*

(vi") *There exists $r \geq \gamma$ such that for any $v \in \ell_r^2(\mathbb{Z}_-; \mathtt{U})$, there exists $u \in \ell^2(\mathbb{N}; \mathtt{U})$ such that $\pi_+ \mathscr{D}_P(v + u) \in \ell^2$.*

(vi"') *There exists $r \geq \gamma$ such that for every $w \in \mathcal{H}_{1/r,-}^2(\mathtt{U})$ there exists $f \in \mathcal{H}^2(\mathtt{U})$ such that $\widehat{\pi}_+ P(f + w) \in \mathcal{H}^2(\mathtt{Y})$.*

(In (vi"'), for every $w \in \mathcal{H}_{1/r,-}^2(\mathtt{U})$ and $f \in \mathcal{H}^2(\mathbb{N}; \mathtt{U})$ we have $f + w \in \mathrm{L}^2(r^{-1}\mathbb{T}; \mathtt{U})$ and $P(f + w) \in \mathrm{L}^2(r^{-1}\mathbb{T}; \mathtt{Y})$, hence $\widehat{\pi}_+ P(f + w) \in \mathcal{H}_{1/r}^2(\mathtt{Y})$.)

Proof: 1° *(vi)–(vi"'), (i):* Now $\pi_- \mathscr{D}_P(v + u) = \pi_- \mathscr{D}_P v \in \ell_r^2(\mathbb{Z}_-; \mathtt{Y}) \subset \ell^2$ (because $r \geq 1$), hence one easily observes that (vi)–(vi"') are equivalent (even with the same r). By Theorem 5.2 below, (vi"') implies Theorem 1.2(iii) (which is equivalent to (i)).

2° *(i)⇒(vi'):* Assume (i). Take $r \geq 1$ such that M and M^{-1} are bounded on $r^{-1}\mathbb{D}$. Let $v \in \ell_r^2(\mathbb{Z}_-; \mathtt{U})$ be arbitrary. Then $\tilde{v} := \pi_- \mathscr{D}_M^{-1} v \in \ell_r^2(\mathbb{Z}_-; \mathtt{U}) \subset \ell^2(\mathbb{Z}_-; \mathtt{U})$, hence $\tilde{u}, \tilde{y} \in \ell^2$, where $\tilde{u} := \pi_+ \mathscr{D}_M \tilde{v}$, $\tilde{y} := \pi_+ \mathscr{D}_N \tilde{v}$. But $v = \pi_- \mathscr{D}_M \mathscr{D}_M^{-1} v = \pi_- \mathscr{D}_M \tilde{v}$, hence

$$\pi_+ \mathscr{D}_P(v + \tilde{u}) = \pi_+ \mathscr{D}_P(\pi_- \mathscr{D}_M \tilde{v} + \pi_+ \mathscr{D}_M \tilde{v}) = \pi_+ \mathscr{D}_P \mathscr{D}_M \tilde{v} = \tilde{y} \in \ell^2. \tag{18}$$

Therefore, (vi') holds (set $u := \tilde{u}$). □

In the proof of Theorem 1.2, a stabilizable realization of P was constructed using a right factorization of P. Even if no such factorization is given, we can use the following formula to obtain an output-stabilizable realization.

Theorem 5.2 *If the condition (vi"') in Theorem 5.1 holds, then the realization* (15) *with state space $\mathtt{X} := \mathcal{H}_{1/r,-}^2(\mathtt{U})$ is output-stabilizable.*

This follows from Proposition 3.1, because the condition (vi"') is equals the Finite Cost Condition for this realization.

For any function P having a right factorization (resp. a r.c.f.), we remark below constructive formulae of 1. a stabilizable realization, 2. normalized w.r.c.f. (resp. 3. normalized r.c.f., 4. stabilizable and detectable realization, 5. d.c.f. and robust stabilizing controllers). In 2., 3. and 5., one can also start from any fixed realization that satisfies certain weak stabilizability conditions.

Remarks 5.3 (Constructive formulae)

1. **(Stabilizable realization)** If a function P has a right factorization, then Theorem 5.2 provides the formula for an output-stabilizable realization of P, and the proof of Theorem 1.2 provides the formula for a stabilizable realization of P.

2. **(Normalized w.r.c.f.)** Given an output-stabilizable realization of P, a normalized w.r.c.f. of P is constructed in Theorem 1.2 (the corresponding nonnormalized w.r.c.f. is given by (7)).

3. **(Normalized r.c.f.)** If P has a r.c.f., then 1.–2. provide a normalized r.c.f. of P, by Theorem 1.1.

4. **(Stabilizable and detectable realization)** If (and only if) P has a r.c.f., then also the dual of the stabilizable realization mentioned in 1. is stabilizable (this will be shown in [Mik07b]).

5. **(D.c.f. and robust controllers)** Assume that a system $\Sigma := \left(\frac{A \mid B}{C \mid D}\right)$ and its dual are output-stabilizable (e.g., that Σ is the realization constructed in 4. above). Then constructive formulae for a d.c.f. (and its inverse) of the transfer function of Σ are provided in [CO08] (partially already in [CO07]; cf. [CO06]). Moreover, their assumption $D = 0$ can always be made, we just have to later replace $\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} I & 0 \\ D & I \end{smallmatrix}\right]\left[\begin{smallmatrix} M & Y \\ N & X \end{smallmatrix}\right]$.)

Finally, constructive formulae for *robust* stabilizing controllers for the transfer function of Σ can be found in [CO07] and in [CO08]. $\triangleleft$

In a follow-up article (summarized in [Mik07c]) the author shall show that if, in 1. above, we start with a function P that is *real-symmetric*, i.e., whose Fourier coefficients are real, then we obtain a real system $\left(\frac{A \mid B}{C \mid D}\right)$. If we start with a real system in 2., then, in 2.–5., all operators are real and all functions are real-symmetric (e.g., if Σ is real, then so are the operators $\mathcal{P}$, F and S of Theorem 1.2). Thus, the numerous control and factorization problems associated with Remark 5.3 and with Theorems 1.1–1.3 all have real/real-symmetric solutions (if they have any solutions at all) provided that the original data is real.

We also need the following lemma, which is of independent interest too. It requires the following additional notation. By $\mathrm{c_c}(\mathbb{Z};\mathtt{U})$ we denote the sequences $\mathbb{Z} \to \mathtt{U}$ with compact support. By e_k's we denote the canonical basis of $\ell^2(\mathbb{Z})$ determined by $(e_k)_j := \delta_{j,k}$. We set $\|u\|_{\ell^1} := \sum_{j\in\mathbb{Z}} \|u_j\|$, $(G*u)_k := \sum_{j\in\mathbb{Z}} G_{k-j}u_j$. We write $P \in \mathcal{H}^2_{\mathrm{strong}}(\mathtt{U},\mathtt{Y})$ if $P : \mathbb{D} \to \mathcal{B}(\mathtt{U},\mathtt{Y})$ is such that $Pu_0 \in \mathcal{H}^2(\mathtt{Y})$ for every $u_0 \in \mathtt{U}$.

If $P \in \mathcal{H}^\infty$, then $\mathscr{D}_P$ maps $\ell^2 \to \ell^2$. If $P \in \mathcal{H}^2_{\mathrm{strong}}$, then P is "almost $\mathcal{H}^\infty$", namely $\mathscr{D}_P\pi_-$ maps ℓ^2_r into ℓ^2 for any $r > 1$:

Lemma 5.4 *Let P be a proper $\mathcal{B}(\mathtt{U},\mathtt{Y})$-valued function and set $\mathscr{D} := \mathscr{D}_P$. Assume that $P \in \mathcal{H}^2_{\mathrm{strong}}(\mathtt{U},\mathtt{Y})$ (or equivalently, that $\mathscr{D}(u_0e_0) \in \ell^2(\mathbb{Z};\mathtt{Y})$ for all $u_0 \in \mathtt{U}$).*

Then the function $Gu_0 := \mathscr{D}(u_0e_0)$ satisfies $G \in \mathcal{B}(\mathtt{U},\ell^2(\mathbb{N};\mathtt{Y}))$, hence for any $s, r > 0$ such that $s < 1 < r$ we have the following: $P \in \mathcal{H}^\infty_r$, and $\mathscr{D}u = G*u$ for every $u \in \mathrm{c_c}$; moreover, $\mathscr{D} : \ell^1 \to \ell^2$, $\mathscr{D}\pi_+ : \ell^2_s \to \ell^2$, and $\mathscr{D}\pi_- : \ell^2_r \to \ell^2$, continuously.

(The condition $G \in \mathcal{B}(\mathtt{U},\ell^2(\mathbb{N};\mathtt{Y}))$, is obviously equivalent to $P \in \mathcal{H}^2_{\mathrm{strong}}$.)

Proof: Pick t such that $P \in \mathcal{H}^\infty_{1/t}$, i.e., that $\mathscr{D} \in \mathcal{B}(\ell^2_t, \ell^2_t)$. Then $G : \mathtt{U} \to \ell^2_t$ continuously and $G[\mathtt{U}] \subset \ell^2$, hence $G : \mathtt{U} \to \ell^2$ continuously, by the closed-graph theorem. Obviously, $\mathscr{D}u = G * u$ and $\|\mathscr{D}u\|_2^2 \leq \sum_k \|\mathscr{D}(u_k e_k)\|^2 \leq \|G\|^2 \|u\|_1^2$ for $u \in \mathrm{c_c}$. The remaining claims follow easily, because $\pi_+ : \ell^2_s \to \ell^1$, and $\pi_- : \ell^2_r \to \ell^1$, continuously. $\square$

Further properties of such functions and an alternative proof are given in [Mik02, Lemma 13.1.3(d)].

Using the notation of Lemma 3.4 we now can characterize the class of transfer functions of output-stable realizations as $\mathcal{H}^2_{\text{strong}}$:

Theorem 5.5 *The transfer function of any output-stable system lies in $\mathcal{H}^2_{\text{strong}}$. Conversely, any $P \in \mathcal{H}^2_{\text{strong}}(\mathtt{U}, \mathtt{Y})$ has the output-stable realization* (15) *with state-space $\mathtt{X} := \mathcal{H}^2_{1/r,-}(\mathtt{U})$ for any $r > 1$.*

Proof: If $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is output-stable on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$, then $\mathcal{H}^2(\mathtt{Y}) \ni f := \widehat{\ } CA^{\cdot} Bu_0 = C(I - \cdot A)^{-1} Bu_0 = z^{-1}(\hat{\mathscr{D}} - D)u_0$, hence $\hat{\mathscr{D}} u_0 = (D + zf)u_0 \in \mathcal{H}^2(\mathtt{Y})$, for every $u_0 \in \mathtt{U}$.

Conversely, if $P \in \mathcal{H}^2_{\text{strong}}(\mathtt{U}, \mathtt{Y})$, then Σ is output-stable, by Lemma 5.4, because $CA^{\cdot} x_0 = \widehat{\pi}_+ P(\widehat{\pi}_-)x_0 \in \widehat{\ }\ell^2(\mathbb{N}; \mathtt{Y}) = \mathcal{H}^2(\mathtt{Y})$ for every $x_0 \in \mathtt{X} = \widehat{\ }\ell^2_r(\mathbb{Z}_-; \mathtt{U})$. $\square$

Theorem 5.5 will be applied in [Mik08] to show that if $M, N \in \mathcal{H}^2_{\text{strong}}$ and $M(0) \in \mathcal{GB}$, then NM^{-1} has a w.r.c.f. This is a formally weaker but equivalent condition to Theorem 1.2(i). Also the Nevanlinna class can be used in place of $\mathcal{H}^2_{\text{strong}}$.

The following facts on shift-semigroup realizations [Fuh81] are more or less well known, but we record them here for easy reference:

Remark 5.6 (Realization) *If $P \in \mathcal{H}^\infty_r(\mathtt{U}, \mathtt{Y})$, $r > 0$, then* (15) *defines a (possibly unstable) realization of P with $\mathtt{X} = \mathcal{H}^2_{r,-}(\mathtt{U})$. Conversely, the transfer function of any system is proper, i.e., $\mathcal{H}^\infty_r$ for some $r > 0$.*

Moreover, if $P \in \mathcal{H}^\infty_r(\mathtt{U}, \mathtt{Y})$, $r > 0$, then also the system $\left(\begin{array}{c|c} S_L & S_L P \\ \hline \pi_0 & P(0) \end{array}\right)$ is a realization of P on $(\mathtt{U}, \mathtt{X}, \mathtt{Y})$, where $\mathtt{X} := \mathcal{H}^2_r(\mathtt{Y}) = \widehat{\ }\ell^2_{1/r}(\mathbb{N}; \mathtt{U})$. With $r = 1$ this realization becomes strongly stable. *This means that it is stable and $x_k \to 0$, as $k \to +\infty$, for any $x_0 \in \mathtt{X}$ when $u = 0$.*

As obvious from the proofs, in the same sense the word "strongly" could be added to Theorem 1.2(iv) and to Theorem 1.3(iv)&(v) ("[jointly] strongly stabilizable and strongly detectable" [Sta05] [Mik02]). More on discrete- and continuous-time shift-semigroup realizations can be found in, e.g., [Sta98] [Sta05] [Fuh81] [Sal89] [Mik07b].

6 Weak left-invertibility and w.r.c.

In this section we present further properties of weak right-coprimeness and of its generalization, weak left-invertibility.

We call $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ *weakly left-invertible* if $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$ for every proper $\mathtt{U}$-valued f, and $F(0)$ is *coercive* (i.e., $F(0)^*F(0) \geq \epsilon I$ for some $\epsilon > 0$). Note that a right factorization NM^{-1} is a w.r.c.f. (resp., r.c.f.) if and only if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is weakly left-invertible (resp., *left-invertible*, i.e., $GF = I$ for some $G \in \mathcal{H}^\infty(\mathtt{Y}, \mathtt{U})$). Thus, all our results for weak left-invertibility trivially lead to analogous corollaries on weak coprimeness, although those of Corollaries 6.3 and 6.4 and Theorem 6.11 are not interesting.

A weakly left-invertible function is one-to-one on $\mathbb{D}$ and coercive on the boundary:

Theorem 6.1 (No zeros) *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is weakly left-invertible, then there exists $\epsilon > 0$ such that, for every $u_0 \in \mathtt{U} \setminus \{0\}$, we have $\|Fu_0\| \geq \epsilon \|u_0\|$ a.e. on $\mathbb{T}$ and $F(z)u_0 \neq 0$ for every $z \in \mathbb{D}$.*

Thus, w.r.c. functions do not have "common zeros" on $\mathbb{D}$. The converse is not true; e.g., $F(z) = \exp(-(1-z)/(1+z))$ is coercive on $\mathbb{D}$ and inner but not weakly left-invertible (since $FF^{-1} \in \mathcal{H}^2$ but $F^{-1} \notin \mathcal{H}^2$). Moreover, the radial limit of an inner weakly left-invertible function may be zero at some points on the boundary $\mathbb{T}$, by Example 7.1. A Tauberian converse to Theorem 6.1 is given in Theorem 6.6.

Proof of Theorem 6.1: If $a \in \mathbb{D}\setminus\{0\}$ and $F(a)u_0 = 0$, then $f(z) := (z+a)(z-a)^{-1}u_0$ is proper and $Ff \in \mathcal{H}^2$, hence $f \in \mathcal{H}^2$ (if F is weakly left-invertible), hence $u_0 = 0$. Also $F(0)$ is one-to-one, being coercive, by assumption.

Assume then that the $\epsilon > 0$ in the lemma does not exist. Then there exist polynomials $f^k \in \mathcal{H}^2(\mathtt{U})$ such that $\|f^k\|_2 = 1$ but $\|Ff^k\|_2 < 2^{-k}$, for each $k \in \mathbb{N}$. Write $f^k =: \sum_{j=0}^{m_k} f_j^k z^j$. Set $n_0 := 0$. For each $k \geq 1$, pick $n_k > n_{k-1} + m_{k-1}$ such that $\|z^{n_k} f^k(z)\| \leq 2^{-k}$ when $|z| < 1 - 1/k$. Set $g^l(z) := \sum_{k=1}^{l} z^{n_k} f^k(z)$ $\forall l$. Then $g^l(z) \to f(z) := \sum_{k=1}^{\infty} z^{n_k} f^k(z)$, as $l \to +\infty$, for each $z \in \mathbb{D}$, and the function f is holomorphic $\mathbb{D} \to \mathtt{U}$. But $\|g^l\|_2^2 = l$ and hence $\|f\|_2 = \infty$ (since their coefficients consist of those of each f^k, possibly with zeros in between). However, $\|Fg^l\|_2 < 1$ and the sum Fg^l converges in $\mathcal{H}^2$, as $l \to +\infty$, hence the limit equals Ff (pointwise on $\mathbb{D}$), so $\|Ff\|_2 \leq 1$. This shows that then F is not weakly left-invertible. $\square$

Lemma 6.2 *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is weakly left-invertible and R is proper and $\mathcal{B}(\mathtt{X}, \mathtt{U})$-valued, then $FR \in \mathcal{H}^\infty \Leftrightarrow R \in \mathcal{H}^\infty$. Moreover, then FR is weakly left-invertible if and only if R is weakly left-invertible.*

(Here the holomorphicity assumption on R could be removed. Recall that $\mathtt{U}$ can be identified with $\mathcal{B}(\mathbb{C}, \mathtt{U})$. A converse to the lemma is given in Theorem 6.13.)

Proof: If $FR \in \mathcal{H}^\infty$, then $FRf \in \mathcal{H}^2$ for all $f \in \mathcal{H}^2$, hence then $Rf \in \mathcal{H}^2$ for all $f \in \mathcal{H}^2$, hence $R \in \mathcal{H}^\infty$, by Lemma 3.5. The proof of the second claim is similar. $\square$

Recall that by $\mathcal{G}$ we denote the subset of invertible elements. (Left-)invertibility in $\mathcal{H}^\infty$ obviously implies weak left-invertibility. We get the equivalence by assuming that F is invertible at the origin:

Corollary 6.3 *If $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is weakly left-invertible and $F(0) \in \mathcal{G}\mathcal{B}(\mathtt{U}, \mathtt{Y})$, then $F \in \mathcal{G}\mathcal{H}^\infty$.*

(Indeed, then $FF^{-1} \in \mathcal{H}^\infty$, hence $F^{-1} \in \mathcal{H}^\infty$, by Lemma 6.2.)

If $F(0)$ is a square matrix, then the second assumption becomes redundant, by Theorem 6.1:

Corollary 6.4 *If $F \in \mathcal{H}^\infty(\mathbb{C}^n)$ is weakly left-invertible, then $F \in \mathcal{G}\mathcal{H}^\infty$.*

However, weak left-invertibility does not imply left-invertibility for non-square functions, by Example 7.1, nor for elements of $\mathcal{H}^\infty(\mathtt{U})$ with $\dim \mathtt{U} = \infty$, by Example 7.4.

The "Corona condition" lies between weak and usual left-invertibility:

Lemma 6.5 (Corona) *Let $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$.*

(a) *If $GF = I$ for some $G \in \mathcal{H}^\infty(\mathtt{Y},\mathtt{U})$, then there exists $\epsilon > 0$ such that $F^*F \geq \epsilon I$ on $\mathbb{D}$. The converse holds if* $\dim \mathtt{U} < \infty$.

(b) *If $F^*F \geq \epsilon I$ on $\mathbb{D}$, then F is weakly left-invertible.*

Proof: (a) Take $\epsilon := 1/\|G\|^2$. The converse is Vasunin's Corona Theorem [Tol81].

(b) Let $f \notin \mathcal{H}^2(\mathtt{U})$ be proper. Then there exists $g : \mathbb{N} \to \mathtt{U}$ be such that $\widehat{g} = f$, where $\widehat{g}(z) = \sum_{k=0}^\infty z^k g_k$. By the Monotone Convergence Theorem, there exist $r_1 < r_2 < r_3 < \cdots$ such that $0 < r_n < 1$ and $\infty > \|r_n^\cdot g_\cdot\|_{\ell^2} > n$ for every n. But then $\|(Ff)(r_n e^{i\cdot})\|_{\mathrm{L}^2} \geq \sqrt{2\pi\epsilon} n$ for every n, hence $Ff \notin \mathcal{H}^2$. □

If $\dim \mathtt{U} = \infty$, then the converse in Lemma 6.5(a) is no longer true [Tre89]. The converse to (b) is not true at all, by (a) and Example 7.1.

For functions in the (matrix-valued) disc algebra, hence for all rational functions, weak left-invertibility is equivalent to left-invertibility as well as to F having no zeros on $\overline{\mathbb{D}}$:

Theorem 6.6 (Disc algebra) *Assume that* $\dim \mathtt{U} < \infty$ *and $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$. If F is continuous on $K := \overline{\mathbb{D}}$, or F is continuous on some (other) compact $K \subset \overline{\mathbb{D}}$ and there exists $\epsilon > 0$ such that $F^*F \geq \epsilon I$ on $\mathbb{D} \setminus K$, then the following are equivalent:*

(i) *$GF = I$ for some $G \in \mathcal{H}^\infty$.*

(ii) *For any open $\Omega \subset \mathbb{D}$ and any function $f : \Omega \to \mathtt{U}$ we have $Ff \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$.*

(iii) *F is weakly left-invertible.*

(iv) *$F(z)u_0 \neq 0$ for all $z \in K$ and all $u_0 \in \mathtt{U} \setminus \{0\}$.*

(Recall in (ii) that "$\in \mathcal{H}^2$" means being a restriction of an $\mathcal{H}^2$ function.)

Because of this fact, the difference between a r.c.f. and a w.r.c.f. becomes redundant in the finite-dimensional systems and control theory (cf. [Fra87]).

Proof of Theorem 6.6: 1° *"(i)⇒(ii)":* Assume (i). Now $g := GFf \in \mathcal{H}^2(\mathtt{U})$ and $g = f$ on Ω, hence "$f \in \mathcal{H}^2(\mathtt{U})$".

2° *"(ii)⇒(iii)⇒(iv)":* The first implication is trivial (except that coercivity at 0 follows as in the proof of Theorem 6.1), the second follows from Theorem 6.1 and continuity.

3° *"(iv)⇒(i)":* Assume (iv). Then we have $\epsilon_K := 1/\min_K \|(F^*F)^{-1}F^*\| > 0$, and $\|F(z)u_0\| \geq \epsilon_K \|u_0\|$ $(z \in K,\ u_0 \in \mathtt{U})$. By Lemma 6.5(a), (i) follows. □

Now we repeat our definitions with an arbitrary but fixed $\alpha \in \mathbb{D}$ taking the role of the origin $0 \in \mathbb{D}$. (This generalization essentially changes nothing as shown below.) Thus, an *α-proper* function means a holomorphic function on a neighborhood of α. We call $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ *α-weakly left-invertible* if $F(\alpha)$ is coercive and we have $Ff \in \mathcal{H}^2(\mathtt{Y}) \implies f \in \mathcal{H}^2(\mathtt{U})$ for every α-proper $\mathtt{U}$-valued function f. Let $N \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ and $M \in \mathcal{H}^\infty(\mathtt{U})$. We call N and M *α-w.r.c.* if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is α-weakly left-invertible. We call NM^{-1} an *α-right factorization* if $M(\alpha) \in \mathcal{GB}(\mathtt{U})$. If, in addition, N and M are r.c. (resp., α-w.r.c.), then we call NM^{-1} an *α-r.c.f.* (resp., *α-w.r.c.f.*).

Lemma 6.7 *Let $\alpha \in \mathbb{D}$. The function $\phi_\alpha : \mathbb{D} \to \mathbb{D}$ given by $\phi_\alpha(z) := \frac{z+\alpha}{1+z\bar{\alpha}}$ is an inner conformal map, and $\phi(0) = \alpha$. Moreover, $f \mapsto f \circ \phi_\alpha$ is an isomorphism of $\mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ onto $\mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ and of $\mathcal{H}^2(\mathtt{U})$ onto $\mathcal{H}^2(\mathtt{U})$, because $\epsilon \leq \|\phi_\alpha'\| \leq \epsilon^{-1}$ on $\overline{\mathbb{D}}$ for certain $\epsilon > 0$. Furthermore, $f \circ \phi_\alpha \circ \phi_{-\alpha} = f$.*

Note that NM^{-1} is an α-w.r.c.f. if and only if $(N \circ \phi_\alpha)(M \circ \phi_\alpha)^{-1}$ is a (0-)w.r.c.f. A similar claim applies to the other definitions too. (Moreover, "α-left-invertible" and "α-right coprime" are independent of α, whereas "α-r.c.f." requires that $M(\alpha) \in \mathcal{GB}$.) This implies that any "0-result" provides an "α-result" too:

Remark 6.8 (α-w.r.c.) Let $\alpha \in \mathbb{D}$. Then all above results in this section also hold if we make the following substitutions: "proper"$\mapsto$"α-proper", "weakly"$\mapsto$"α-weakly", "w.r.c."$\mapsto$"α-w.r.c.", "$F(0)$"$\mapsto$"$F(\alpha)$", "r.c.f."$\mapsto$"α-r.c.f." and "right factorization"$\mapsto$"α-right factorization".

Naturally, the same holds also for Theorems 1.1, 6.13 and 6.15–6.17 and Lemmata 3.5 and 3.6 i.e., to all applicable results in this article (see also Remark 6.18). $\triangleleft$

We omit the simple proof (just apply the original statements to $M \circ \phi_\alpha$, $N \circ \phi_\alpha$ etc.). We shall use Remark 6.8 without further mention.

These α-variants form a bridge between the discrete- and continuous-time results but they are also important themselves when we do not have (or know about) invertibility/coercivity at the point 0.

An α-w.r.c.f. is a z-w.r.c.f. for any reasonable $z \in \mathbb{D}$:

Theorem 6.9 *Let $\alpha \in \mathbb{D}$, $M \in \mathcal{H}^\infty(\mathtt{U})$. Then the following hold:*

(a) *Assume that NM^{-1} is an α-right factorization, $N_0M_0^{-1}$ is an α-w.r.c.f. of NM^{-1}, $\Omega \subset \mathbb{D}$ is open and connected and $\alpha \in \Omega$. If M is invertible on Ω, then so is M_0; if M^{-1} is uniformly bounded on Ω, then so is M_0^{-1}.*

If $\dim \mathtt{U} < \infty$, then Ω need not be connected above.

(b) *If NM^{-1} is an α-w.r.c.f. and M is invertible on an open and connected $\Omega \subset \mathbb{D}$ such that $\alpha \in \Omega$, then NM^{-1} is a z-w.r.c.f. for every $z \in \Omega$.*

If $\dim \mathtt{U} < \infty$, then Ω need not be connected above and, in addition, N and M are z-w.r.c. for every $z \in \mathbb{D}$.

Proof: (a) 1° Now $N = N_0V$, $M = M_0V$ for $V := M_0^{-1}M \in \mathcal{H}^\infty$, by Theorem 1.1. But the function VM^{-1} is holomorphic on Ω and it equals the inverse of M_0 near α, hence on the whole Ω.

2° Even if Ω is not connected, we have $M = M_0V$ on $\mathbb{D}$. If $\dim \mathtt{U} < \infty$, this implies that V and M_0 must be invertible on Ω.

(b) Let $N_0M_0^{-1}$ be a z-w.r.c.f. of NM^{-1} (which is z-proper). Then $N = N_0V$, $M = M_0V$ (on $\mathbb{D}$) for some $V \in \mathcal{H}^\infty$ (hence $V := M_0^{-1}M$ near z and wherever M_0^{-1} exists), by Theorem 1.1. By (a), $N_0M_0^{-1}$ is an α-right factorization, hence $V^{-1} \in \mathcal{H}^\infty$ (being equal to $M^{-1}M_0$ near α), by Theorem 1.1, hence also NM^{-1} is a z-w.r.c.f.

If $\dim \mathtt{U} < \infty$, $z \in \mathbb{D}$, f is z-proper and $\mathtt{U}$-valued, and $Nf, Mf \in \mathcal{H}^2$, then $\det M \neq 0$ a.e., hence $\det M(z') \neq 0$ for some z' such that f is z'-proper, hence $f \in \mathcal{H}^2$ (because NM^{-1} is a z'-w.r.c.f.). $\square$

From Theorem 6.1 it follows that the outer factor of F is invertible (by [Sta97, Lemma 18(ii)] or [Mik09, Theorem 5.11]). Therefore, F can be normalized as follows:

Lemma 6.10 (Inner) *If $\alpha \in \mathbb{D}$ and $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ is α-weakly left-invertible, then there exists $S \in \mathcal{G}\mathcal{H}^\infty(\mathtt{U})$ such that FS is α-weakly left-invertible and inner.*

A function $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is called *outer* if and only if $\{Ff \mid f \in \mathcal{H}^2(\mathtt{U})\}$ is dense in $\mathcal{H}^2(\mathtt{Y})$. This is equivalent to the classical scalar-valued definition if $\mathtt{U} = \mathbb{C} = \mathtt{Y}$ [Nik02, p. 43].

Any $F \in \mathcal{H}^\infty$ that is bounded below at some point $\alpha \in \mathbb{D}$ can be factorized as follows.

Theorem 6.11 ($\mathbf{F{=}F_wF_r}$) **(a)** *If $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$, and $F(\alpha)$ is coercive for some $\alpha \in \mathbb{D}$, then $F = F_wF_r$, where $F_w \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ is inner and α-weakly left-invertible, $F_r \in \mathcal{H}^\infty(\mathtt{U})$ and $F_r(\alpha) \in \mathcal{GB}(\mathtt{U})$.*

All such factorizations are given by $F = (F_wV)(V^{-1}F_r)$, where $V \in \mathcal{B}(\mathtt{U})$ is unitary (or $V \in \mathcal{GH}^\infty(\mathtt{U})$ if we do not require F_w to be inner). In particular, if some left factor F_wV is left-invertible in $\mathcal{H}^\infty$, then so is every F_wV.

(b) *If $F \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Y})$, then there exist $\alpha \in \mathbb{D}$ and $m \le n$ such that $F = F_wF_rF_o$, where $F_o \in \mathcal{H}^\infty(\mathbb{C}^n,\mathbb{C}^m)$ is outer, $F_w \in \mathcal{H}^\infty(\mathbb{C}^m,\mathtt{Y})$ is inner and z-weakly left-invertible for every $z \in \mathbb{D}$, $F_r \in \mathcal{H}^\infty(\mathbb{C}^m)$ is inner, and $F_r(\alpha) \in \mathcal{GB}(\mathbb{C}^m)$.*

All such factorizations are given by $F = (F_wV)(V^{-1}F_rW^{-1})(WF_o)$, where $V, W \in \mathcal{B}(\mathbb{C}^m)$ are unitary.

(c) *In (a) we have $F_w = J\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$, where $J \in \mathcal{B}(\mathtt{Y}_1 \times \mathtt{U},\mathtt{Y})$ is unitary, $\mathtt{Y}_1 \subset \mathtt{Y}$ is a closed subspace, $\mathtt{Y}_1^\perp$ is isometric to $\mathtt{U}$ and NM^{-1} is a $\mathcal{B}(\mathtt{U},\mathtt{Y}_1)$-valued α-w.r.c.f.*

Proof of Theorem 6.11: (a) 1° Set $\mathtt{Y}_2 := F(\alpha)[\mathtt{U}]$, $\mathtt{Y}_1 := \mathtt{Y}_2^\perp$. Fix some unitary $J_2 \in \mathcal{B}(\mathtt{U},\mathtt{Y}_2)$. Then $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right] := \left[\begin{smallmatrix} I & 0 \\ 0 & J_2^* \end{smallmatrix}\right] F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y}_1 \times \mathtt{U})$, and $M_1(\alpha)$ is coercive and onto, hence $M_1(\alpha) \in \mathcal{GB}(\mathtt{U})$. By Theorem 1.1 (and Remark 6.8), $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] V$, where NM^{-1} is a α-w.r.c.f., $V \in \mathcal{H}^\infty(\mathtt{U})$ and V^{-1} is α-proper. Thus, $F = \left[\begin{smallmatrix} I & 0 \\ 0 & J_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] V = WV$, where $W := \left[\begin{smallmatrix} I & 0 \\ 0 & J_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is α-weakly left-invertible.

By Theorem 6.1 and Lemma 6.10, $W = F_wS^{-1}$, where $S \in \mathcal{GH}^\infty(\mathtt{U})$ and F_w is inner and α-weakly left-invertible. Set $F_r := SV$.

2° *Uniqueness:* If $F = F_wF_r = F_w'F_r'$, where both factorizations are as in (a), then $F_w = F_w'V$ and $F_w' = F_wV^{-1}$, where $V := F_r'F_r^{-1}$, hence $V, V^{-1} \in \mathcal{H}^\infty(\mathtt{U})$, by Lemma 6.2. But V and V^{-1} are inner (since F_w and F_w' are inner), hence $V \in \mathcal{B}(\mathtt{U})$ (hence it is unitary), by Lemma 3.7.

(b) Let $F = F_iF_o$ be an inner-outer factorization [RR85]. Then $F_o \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Z})$, $F_i \in \mathcal{H}^\infty(\mathtt{Z},\mathtt{Y})$ for some Hilbert space $\mathtt{Z}$, and $F_o(z)$ has a dense range for each z, so $\mathtt{Z}$ can be replaced by $\mathbb{C}^m$, $m \le n$. Pick $z \in \mathbb{T}$ such that $F_i(z)^*F_i(z) = I$. Then $F_i(\alpha)$ is coercive for any $\alpha \in \mathbb{D}$ near z, so $F_i = F_wF_r$ as in (a). Since F_i and F_w are inner, so is F_r. The inner-outer factorization is unique modulo a unitary W, so we get the uniqueness as in (a) (and m is unique).

(c) This follows from the above (by uniqueness) by setting $J := \left[\begin{smallmatrix} I & 0 \\ 0 & J_2 \end{smallmatrix}\right]$ and replacing $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} NS \\ MS \end{smallmatrix}\right]$. N.B.: $N(\alpha) = 0$. □

Remarks: If, in (a), we form an inner-outer factorization $F_r = F_iF_o$ (or $F_r^{\mathrm{d}} = F_iF_o$), then $F_i(\alpha), F_o(\alpha) \in \mathcal{GB}(\mathtt{U})$. In (b) we might, instead, have F_r co-outer and F_o co-inner.

If $\dim \mathtt{U} < \infty$ (not in general, by Example 7.3), then the point $\alpha \in \mathbb{D}$ does not have a special role:

Lemma 6.12 (Every α) *If $F \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Y})$ is α-weakly left-invertible for some $\alpha \in \mathbb{D}$, then F is z-weakly left-invertible for every $z \in \mathbb{D}$.*

Proof of Lemma 6.12: By Theorem 6.11(a)&(c), $F = J\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ (right-multiplied by $F_r = V \in \mathcal{GH}^\infty$, by (a)), where NM^{-1} is an α-w.r.c.f. (hence z-w.r.c.f. for every $z \in \mathbb{D}$, by Theorem 6.9(b)) and J is unitary. $\square$

In Lemma 6.12, a third equivalent condition is Theorem 6.6(ii) (or the same condition with $\mathcal{H}^\infty$ in place of $\mathcal{H}^2$).

Weak left-invertibility (or w.r.c.f.'s) could be defined with $\mathcal{H}^\infty$ in place of $\mathcal{H}^2$:

Theorem 6.13 *Assume that $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$, $F(0)$ is coercive and $\mathtt{X} \neq \{0\}$. Then F is weakly left-invertible if and only if $FR \in \mathcal{H}^\infty \Rightarrow R \in \mathcal{H}^\infty$ for every proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function R.*

(The coercivity assumption is not redundant (take $F(z) = z$). Moreover, an analogous result holds with $\mathcal{H}^p$ or $\mathcal{H}^p_{\text{strong}}$ in place of $\mathcal{H}^\infty$.)

Proof of Theorem 6.13: 1° "Only if" is from Lemma 6.2.

2° Conversely, assume that $FR \in \mathcal{H}^\infty \Rightarrow R \in \mathcal{H}^\infty$ for every proper $\mathcal{B}(\mathtt{X},\mathtt{U})$-valued function R, hence for every proper $\mathtt{U}$-valued function R (if R is such and $FR \in \mathcal{H}^\infty(\mathtt{Y})$, pick $x \in \mathtt{X}$ and $L \in \mathtt{X}^*$ such that $Lx = 1$; then $FRL \in \mathcal{H}^\infty$, hence then $RL \in \mathcal{H}^\infty(\mathtt{X},\mathtt{U})$, hence then $R = RLx \in \mathcal{H}^\infty(\mathbb{C},\mathtt{U})$), hence for $R = \tilde{R}u_0$ whenever $u_0 \in \mathtt{U}$ and $\tilde{R}$ is proper and $\mathcal{B}(\mathtt{U})$-valued, hence for every proper $\mathcal{B}(\mathtt{U})$-valued function R, by the uniform boundedness theorem (use $\tilde{R}$ in place of R).

Let $F = F_w F_r$ be as in Theorem 6.11(a) with $\alpha = 0$. Then $F_r^{-1} \in \mathcal{H}^\infty$ (because $FF_r^{-1} \in \mathcal{H}^\infty$), hence F is weakly left-invertible (since so is F_w). $\square$

Now we present the classical definition of weak coprimeness (of matrix-valued functions) [Fuh81] [Ino88] [Smi89] and show that it is equivalent to ours. We call $F \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Y})$ *irreducible* if $Ff \in \mathcal{H}^\infty \implies f \in \mathcal{H}^\infty$ holds for every function of the form $f = g^{-1}G$, where $0 \not\equiv g \in \mathcal{H}^\infty(\mathbb{C})$, $G \in \mathcal{H}^\infty(\mathbb{C},\mathbb{C}^n)$. If $\dim \mathtt{Y} < \infty$, then F is irreducible if and only if 1 is a gcd of all highest order minors of F [Smi89, Lemma 4]. We call functions $N \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Y})$ and $M \in \mathcal{H}^\infty(\mathbb{C}^n)$ *gcd-coprime* if and only if $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is irreducible. That is the classical definition of (right) "weak coprimeness".

The following theorem shows that the factorization of [Smi89, Lemma 5] is a special case of that in Theorem 6.11(a). Consequently, the weakly coprime right factorization of [Smi89, p. 1007] is the same as a w.r.c.f. (when $\dim \mathtt{U}, \dim \mathtt{Y} < \infty$, as [Smi89] assumes).

Theorem 6.14 (gcd-coprime) *Let $F \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Y})$. Then F is irreducible if and only if for some (hence every) $\alpha \in \mathbb{D}$ the function F is α-weakly left-invertible.*

In particular, functions $M \in \mathcal{H}^\infty(\mathbb{C}^n)$ and $N \in \mathcal{H}^\infty(\mathbb{C}^n,\mathtt{Y})$ are gcd-coprime if and only if they are α-w.r.c. for some (hence every) $\alpha \in \mathbb{D}$.

Proof of Theorem 6.14: The other direction follows from Lemmata 6.2 and 6.12, so assume that F is irreducible. It easily follows that F is injective everywhere, hence so is $F_r \in \mathcal{H}^\infty(\mathbb{C}^n)$, where $F = F_w F_r$ as in Theorem 6.11(a). But $F_r^{-1} \in \mathcal{H}^\infty$ (because $FF_r^{-1} \in \mathcal{H}^\infty$), hence F is α-weakly left-invertible for all $\alpha \in \mathbb{D}$ (since so is F_w, by Lemma 6.12). $\square$

We set $F^{\mathrm{d}}(z) := F(\bar{z})^*$. Every $F \in \mathcal{H}^\infty(\mathtt{U},\mathtt{Y})$ has a *dual inner-outer factorization* $F = F_o^{\mathrm{d}} F_i^{\mathrm{d}}$ (or $F^{\mathrm{d}} = F_i F_o$), where $F_i \in \mathcal{H}^\infty(\mathtt{U}_0,\mathtt{Y})$ is inner, $F_o \in \mathcal{H}^\infty(\mathtt{U},\mathtt{U}_0)$ is outer and $\mathtt{U}_0$ is a closed subspace of $\mathtt{U}$ [RR85] [Nik02] [Mik09]. At least when $\dim \mathtt{U} < \infty$, that factorization (and its dual) is a strictly weaker factorization than that of Theorem 6.11(a) in what comes to the left factor:

Theorem 6.15 (Outer) *If $F \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{Y})$ is weakly left-invertible, then F^{d} is outer.*

The converse does not hold, because the function $F(z) = 1 - z = F^{\mathrm{d}}(z)$ is outer but F is not weakly left-invertible.

Similarly, the w.r.c.f. is a strictly stronger tool than that provided by the dual inner-outer factorization. Indeed, if $M(z) = 1 - z = N(z)$ ($z \in \mathbb{D}$), then a dual inner-outer factorization of $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is given by $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \cdot I$, whereas in a w.r.c.f. $N_0 M_0^{-1}$ of NM^{-1} the maps N_0 and M_0 necessarily become invertible in $\mathcal{H}^\infty$ (and identical), by Theorem 1.1. As in that example, a dual inner-outer factorization $\left[\begin{smallmatrix} N_0 \\ M_0 \end{smallmatrix}\right] V$ of $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ removes the common zeros of N and M inside the disc but not those on the boundary, whereas a w.r.c.f. removes both (and more) in the sense of Theorem 6.1.

Proof of Theorem 6.15: Let $F^{\mathrm{d}} = F_i F_o$ be an inner-outer factorization. Then $F = F_o^{\mathrm{d}} F_i^{\mathrm{d}}$, hence $F_i^{\mathrm{d}} \in \mathcal{H}^\infty(\mathbb{C}^n, \mathtt{U}_0)$ is weakly left-invertible (here $\mathtt{U}_0 \subset \mathbb{C}^n$), because $F_i^{\mathrm{d}}(0)$ must be coercive (since $F(0)$ is) and $F_i^{\mathrm{d}} f \in \mathcal{H}^2 \implies Ff = F_o^{\mathrm{d}} F_i^{\mathrm{d}} f \in \mathcal{H}^2 \implies f \in \mathcal{H}^2$. Because $F_i^{\mathrm{d}}(0)$ is coercive, we have $\dim \mathtt{U}_0 = n$, hence F_i^{d} is invertible in $\mathcal{H}^\infty$, by Corollary 6.4. Therefore, F^{d} is outer too. $\square$

One can easily verify that F^{d} is outer if and only if the anti-Toeplitz operator ($\widehat{\pi_-} F \widehat{\pi_-}$ in terms of Section 5) of F is injective. But F is left-invertible if and only if the anti-Toeplitz operator of F is coercive [Arv75] [SF76]. Thus, weak left-invertibility is strictly between these two conditions (at least when $\dim \mathtt{U} < \infty$).

As mentioned above, all above results on weak left-invertibility contain analogous results on weak right coprimeness. Now we go on with right-factorization-specific results.

We identify $M \in \mathcal{H}^\infty(\mathtt{U})$ with the multiplication operator $M : f \mapsto Mf$ on $\mathcal{H}^2(\mathtt{U})$. If(f) NM^{-1} is a w.r.c.f., then $M[\mathcal{H}^2] = D_P := \{f \in \mathcal{H}^2 \,|\, Pf \in \mathcal{H}^2\}$:

Theorem 6.16 (Graph) *Let $P = NM^{-1}$ be a right factorization and $M \in \mathcal{H}^\infty(\mathtt{U})$. Then N and M are w.r.c. if and only if the graph $\left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right][D_P]$ equals $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right][\mathcal{H}^2(\mathtt{U})]$.*

This is why a w.r.c.f. allows one to reduce optimization problems to the stable case [Mik02].

Proof of Theorem 6.16: If $g \in \mathcal{H}^2(\mathtt{U})$, then $f := Mg \in D_P$, because $Pf = Ng \in \mathcal{H}^2$. Thus, $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right][\mathcal{H}^2(\mathtt{U})] \subset \left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right][D_P]$. Conversely, if $f \in D_P$, then $g := M^{-1} f$ satisfies $h := \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right] g = \left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right] f \in \mathcal{H}^2$. All such h's are in $\left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right][\mathcal{H}^2(\mathtt{U})]$ if and only if all such g's are in $\mathcal{H}^2(\mathtt{U})$, i.e., if and only if N and M are w.r.c. $\square$

For $P = NM^{-1}$ to be an α-right factorization, any poles (and essential singularities) of P on $\overline{\mathbb{D}}$ must be contained in M^{-1} (to have $N \in \mathcal{H}^\infty$). For N and M to be w.r.c., the function M^{-1} must not contain any other poles, i.e., the functions $N = PM$ and M may not have common zeros (this necessary condition is not sufficient for general non-rational functions). If $\mathtt{U}$ is finite-dimensional, then the poles of M^{-1} on $\mathbb{D}$ are isolated and hence then we can formulate that part simply:

Theorem 6.17 *If NM^{-1} is a w.r.c.f. and $M \in \mathcal{H}^\infty(\mathbb{C}^n)$, $n \in \mathbb{N}$, then the nonremovable singularities of M^{-1} on $\mathbb{D}$ are the same as those of $P := NM^{-1}$.*

Proof of Theorem 6.17: Obviously, any singularity of $P = NM^{-1}$ on $\mathbb{D}$ must be a singularity of M^{-1}. Assume then that $z \in \mathbb{D}$ and $u_0 \in \mathtt{U} \setminus \{0\}$ are such that $M(z)u_0 = 0$. Let $f \in \mathcal{H}^2(\mathbb{C})$ be such that $f(z) \neq 0$.

Set $g := Mfu_0 \in D_P$ (see Theorem 6.16). Then $g(z) = f(z)M(z)u_0 = 0$ but $(Pg)(z) = f(z)N(z)u_0 \neq 0$, by Theorem 6.1, hence P must have a singularity at z

(otherwise we would have $(Pg)(z) = P(z)g(z) = 0$). $\square$

The coercivity assumption on $F(\alpha)$ seems somewhat artificial (and it has not been used explicitly before this article). For an α-right factorization, it is redundant, but in general it is needed to avoid labeling the function $F(s) := s - \alpha$ as α-weakly left-invertible. However, even if we dropped this requirement, most results would still hold:

Remark 6.18 Redefine weak and α-weak left-invertibility by dropping the coercivity requirement. Then all above results hold (with the same proofs) except that Corollary 6.4, Lemma 6.12 and Theorems 6.6, 6.11(b), 6.14 and 6.15 become false and in Theorem 6.1 we must require that $z \neq 0$ (or that $z \neq \alpha$ in the setting of Remark 6.8). $\triangleleft$

Proof: The positive claim is self-explaining. The "except" part holds strictly, because in this weaker sense the function $F(z) := z - \alpha$ is α-weakly left-invertible but not irreducible, invertible, co-outer nor z-weakly left-invertible for any $z \neq \alpha$ (take $f(z) = (z - \alpha)^{-1}$ to have $Ff \in \mathcal{H}^2 \cap \mathcal{H}^\infty$). $\square$

This weaker concept was called "quasi-left-invertibility" in [Mik02, Chapter 4, Sections 6.4–6.5; Theorem 13.3.13], where further results are given (in the continuous- and discrete-time settings). Note that weak coprimeness and the above "quasi coprimeness" of N and M are equivalent when M^{-1} is proper.

One can easily show that a constant $F \in \mathcal{B}(\mathtt{U}, \mathtt{Y})$ is quasi-left-invertible if and only if it is coercive (left-invertible).

Notes for Section 6

For the conditions in Theorem 6.6, we always have (i)⇒(ii)⇒(iii) (see Theorem 6.1 for (iv)). However, if we dropped the assumption $\dim \mathtt{U} = \infty$ or continuity, then (ii)⇒(i) would become false. Conditions (ii) and (iii) are equivalent when $\dim \mathtt{U} < \infty$, by Lemma 6.12. Without continuity, being inner and without zeros on $\mathbb{D}$ is no longer sufficient for (iii) (take $F(z) = \mathrm{e}^{-(1-z)/(1+z)}$). It is well known that (iv) is equivalent to (i) in Theorem 6.6 when $K = \overline{\mathbb{D}}$, but the general case covers much more general systems, because in control-theoretic applications often M is boundedly invertible outside some compact set (particularly in continuous-time parabolic PDE systems).

In Theorem 6.9(a) the function M_0 need not have a radial limit at, e.g., $1 \in \mathbb{T}$ even if N, M and M^{-1} were entire and scalar-valued. Indeed, let $N = M = I$, take $F \in \mathcal{H}^\infty(\mathbb{C})$ without limit at 1, and set $N_0 := M_0 := F + 2\|F\|_\infty$ to obtain these. Moreover, the assumption $\dim \mathtt{U} = \infty$ in the last claim of (b) is not superfluous, by Example 7.3.

In the matrix-valued case ($\dim \mathtt{U}, \dim \mathtt{Y} < \infty$), part of Theorem 6.11 would follow from the results of [Smi89] through Theorem 6.14, which, however, is a corollary of Theorem 6.11.

Lemma 6.12 is not true for $F \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ in general (when $\dim \mathtt{U} = \infty$). However, we do not know if the analogies of Theorems 6.17, 6.15 and 6.11(b) can be extended to the operator-valued case.

Theorem 6.16 (which was reported in [Mik06]) is related to [GS93, Proposition 1], which, in the matrix-valued case, establishes the identity $\left[\begin{smallmatrix} I \\ P \end{smallmatrix}\right][D_P] = \left[\begin{smallmatrix} M \\ N \end{smallmatrix}\right][\mathcal{H}^2]$ but not the fact that M can be chosen to be square under our assumptions.

The algebraic approach illustrated by [Vid85], [Ino88], [Smi89], [GS93] and, most up-to-date, several recent articles by Alban Quadrat, including [Qua03a], [Qua03b] and [Qua04], provide additional results and insight on many forms of factorizations, particularly with other algebras (e.g., the disc algebra, the Wiener class, the Callier–Desoer class, absolutely converging Taylor series) in place of $\mathcal{H}^\infty$.

Reward. Does a holomorphic function $V : \mathbb{D} \to \mathcal{B}(\mathtt{U})$ exist such that $V(1/2)$ is invertible and $V(0)$ is coercive ($V(0)^*V(0) \geq \epsilon I$) but not invertible? The author pays \$10 for a proof or a disproof (at least in the case of separable $\mathtt{U}$). A disproof would extend some of our results from the matrix-valued to the operator-valued case.

7 Counter-examples

In this section we provide examples that show it impossible to extend or reverse certain implications presented above.

By the (Cayley) substitution $z \mapsto \frac{1-z}{1+z}$ (and some modifications) the example of [Smi89] becomes a scalar-valued w.r.c.f. that is not a r.c.f.:

Example 7.1 (Scalar w.r.c.f. $\not\Rightarrow$ r.c.f.) The functions $N(z) := (1-z)\mathrm{e}^{-\frac{1-z}{1+z}}$ and $M(z) := 1+z$ form a w.r.c.f., by [Smi89] and Theorem 6.14 (the gcd's of the inner and outer factors are both equal to the identity), but not a r.c.f., because $N(-1+) = 0 = M(-1)$. ◁

As mentioned in the introduction, the w.r.c.f. determined by the LQ-optimal feedback need not be Bézout coprime:

Example 7.2 (LQ-optimal feedback is not coprime) (a) By Theorem 1.2, the function $P := NM^{-1}$ of Example 7.1 has an output-stabilizable realization. The LQ-optimal state-feedback for this realization determines a w.r.c.f. $N_1M_1^{-1}$ of P, by Theorem 1.2, but that w.r.c.f. is not a r.c.f., because P does not have an r.c.f., by the last claim in Theorem 1.1.

(b) If, in (a), we use Example 7.3 in place of Example 7.1, then the same conclusions hold and, in addition, we have $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right](1/5)$ noncoercive, because $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] V$ for some $V \in \mathcal{G}\mathcal{H}^\infty$, by Theorem 1.1. ◁

(Note that the above is true whether by N_1 and M_1 we refer to the "N" and "M" or to the "$NS^{-1/2}$" and "$MS^{-1/2}$" of Theorem 1.2.)

By Theorem 6.1, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ is injective on $\mathbb{D}$ for any w.r.c.f. NM^{-1}. It need not be coercive on $\mathbb{D}$ even if N, M are continuous on $\overline{\mathbb{D}}$, unlike in the finite-dimensional case (cf. Theorem 6.6):

Example 7.3 (Continuous w.r.c.f $\not\Rightarrow$ r.c.f.) (a) Let $\mathtt{U} := \ell^2(\mathbb{N}) =: \mathtt{Y}$. There exists a normalized w.r.c.f. NM^{-1} such that $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](1/5)e_k \to 0$, as $k \to \infty$ (hence $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](1/5)$ is not coercive), and N and M are continuous $\overline{\mathbb{D}} \to \mathcal{B}(\mathtt{U})$ but not r.c.

(b) For $F := \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y} \times \mathtt{U})$, conditions (ii)–(iv) of Theorem 6.6 hold (with $K = \overline{\mathbb{D}}$) but not (i), and NM^{-1} is not a 1/5-w.r.c.f.

(c) In fact, N and M are α-w.r.c. for every $\alpha \neq 1/5$. Moreover, $M(z) \in \mathcal{GB}(\mathtt{U}) \Leftrightarrow z \in E := \{0\} \cup \{\frac{5+9k}{1+45k} \mid k \in \mathbb{N}\}$. In particular, $P := NM^{-1}$ is an α-w.r.c.f. for every $\alpha \in \mathbb{D} \setminus E$ but P does not have an α-w.r.c.f. for any $\alpha \in E$.

(d) If we replace $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \circ \phi_{-1/5}$, we get a normalized 1/5-w.r.c.f. NM^{-1} such that $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](0)$ is not coercive and the conditions (ii) and (iv) of Theorem 6.6 hold but not (i) nor (iii).

(e) Moreover, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \circ \phi_{1/5}$ are holomorphic on $3\mathbb{D}$, hence they belong to the operator-valued "disc algebra", even to the "Wiener class" (of uniformly absolutely converging sequences). ◁

Proof: We construct the 1/5-w.r.c.f. Replace $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right]$ by $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \circ \phi_{1/5}$ to obtain the 0-w.r.c.f. (Actually, we make N and M inner, so they should be divided by $\sqrt{2}$ to make them normalized.)

1° Set $a_k := -1/9k$, $b_k := 1/9k$ ($k \in \mathbb{N}$). Then $a_k \to 0$ and $b_k \to 0$, as $k \to \infty$. Set $N_k := \phi_{a_k}$, $M_k := \phi_{b_k}$ ($k \in \mathbb{N}$), where $\phi_a(z) := (z+a)/(1+z\bar{a})$, and let $N, M \in \mathcal{H}^\infty(\mathtt{U})$ be the (diagonal) inner functions determined by $N(z)e_k = N_k(z)e_k, M(z)e_k = M_k(z)e_k$ ($z \in \overline{\mathbb{D}},\ k \in \mathbb{N}$), where $e_k := \chi_{\{k\}}$. Then $|N_k(0)| = |M_k(0)| = 1/9k \to 0$, as $k \to \infty$, hence $\| \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right](0)e_k\| \to 0$ (hence N and M are not r.c., by Lemma 6.5(a)). Moreover, $M_k(1/5) \geq 1/5/2\ \forall k$, hence $M(1/5)^{-1}$ exists, i.e., NM^{-1} is a 1/5-right factorization. Similarly, $M(z), N(-z) \in \mathcal{GB}(\mathtt{U})$ whenever $z \in E' := \{0\} \cup \{b_k \mid k \in \mathbb{N}\}$.

2° But N_k and M_k are r.c., by Theorem 6.6. Thus, if $\Omega \subset \mathbb{D}$ is open and $f : \Omega \to \mathtt{U}$ satisfies $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f \in \mathcal{H}^2$, then $f_k \in \mathcal{H}^2$ ($k \in \mathbb{N}$), where f_k is the kth component of f. Furthermore, $\| \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f_k e_k\|_2^2 = 2\|f_k\|_2^2\ \forall k \in \mathbb{N}$ (because N_k and M_k are inner), hence $\| \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] f\|_2^2 = 2\|f\|_2^2$, hence $f \in \mathcal{H}^2$. We conclude that N and M are α-w.r.c. for every $\alpha \in \mathbb{D} \setminus \{0\}$ (recall 2°) and NM^{-1} is a α-w.r.c.f. for every $\alpha \neq 0$ such that $\alpha \neq b_k\ \forall k \in \mathbb{N}$.

(c) By Theorem 6.9(a), NM^{-1} cannot have a z-w.r.c.f. for any $z \in E$. The rest of (c) was shown above.

(e) Note that $M(z)$ and $N(-z)$ have poles only at $9k$ ($k \in \mathbb{N}$), hence $F := \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \circ \phi_{1/5}$ has its poles at $z = -5 + 24/(5-9n)$, $n = \pm 1, \pm 2, \pm 3, \ldots$. Therefore, it is holomorphic on $3\mathbb{D}$.

□

By applying different ϕ_α functions (Lemma 6.7) we can have any $\beta \in \mathbb{D}$ in place of 1/5 above. It is not difficult to show that, by using such functions as diagonal elements of bigger operators N_1 and M_1 (with separable input and output spaces), given an open set $\Omega \subset \mathbb{D}$, we can have N_1 and M_1 to be z-w.r.c. for every $z \in \Omega$ and yet $\left[\begin{smallmatrix} N_1 \\ M_1 \end{smallmatrix}\right](z)$ non-coercive for every $z \in \mathbb{D} \setminus \Omega$ (use a countable dense set of β's in $\mathbb{D} \setminus \Omega$).

A common right divisor $V \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{X})$ of $N \in \mathcal{H}^\infty(\mathtt{U}, \mathtt{Y})$ and $M \in \mathcal{H}^\infty(\mathtt{U})$ (see (1)) is *square* if $\mathtt{X} = \mathtt{U}$. If NM^{-1} is a w.r.c.f., then such a divisor V is obviously weakly left-invertible (hence invertible if $\dim \mathtt{U} < \infty$). In the operator-valued case it need not be left-invertible (nor right-invertible):

Example 7.4 (divisor not left-invertible) Let $N, M, \mathtt{U}, \mathtt{Y}$ be as in Example 7.3. Then $V := J^{-1} \left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] \in \mathcal{H}^\infty(\mathtt{U})$ is weakly left-invertible for any $J \in \mathcal{GB}(\mathtt{U}, \mathtt{Y} \times \mathtt{U})$. However, $\left[\begin{smallmatrix} N \\ M \end{smallmatrix}\right] = JV$, but V is not left-invertible; in fact, $V(1/5) \in \mathcal{B}(\mathtt{U})$ is not left-invertible (nor right-invertible). Nevertheless, $V(z)$ is left-invertible for every $z \in \mathbb{D} \setminus \{1/5\}$. ◁

This also shows that Corollary 6.4 cannot be extended to the infinite-dimensional case.

Notes for Section 7

Example 7.1 is adapted from [Smi89], where the first example of a non-r.c. w.r.c.f. was constructed. The first example based on definition (2) was due to Sergei Treil (personal communication, 2003, before the equivalence Theorem 6.14).

Further "counter-examples" are given in the previous sections below certain results and at the end of Section 6.

Acknowledgments. The author wishes to thank Ruth Curtain and Mark Opmeer for the preprints of [CO06] and [OC05] and Olof Staffans and Sergei Treil for useful comments. This work was written with the support of the Academy of Finland under grant #203946.

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