Mat-1.150 Real Analysis: 5th exercise, Wed 2002-10-16, U322 (4 pages)
`http://www.math.hut.fi/teaching/realanal/`

Exercise 5.1 ([3.7] *a bit difficult & long*):
≈[R2:Exe4.3] (a) Show that $L^p(\mathbb{R}^n)$ is separable. (b) Show that $L^\infty(\mathbb{R}^n)$ is not separable (i.e., it does not contain an (at most) countable dense subset).[1]

Exercise 5.2 ([3.14] *quite easy, not long, useful*):
(a) Show that translation is continuous in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.
(b) How about $L^\infty(\mathbb{R}^n)$?

Exercise 5.3 ([4.10] *quite easy, quick*):
A Banach space X is said to be *uniformly convex*[2]if

$$\|x_n\|_X \le 1, \quad \|y_n\| \le 1, \quad \|x_n + y_n\| \overset{n\to\infty}{\to} 2$$

implies that $\|x_n - y_n\| \overset{n\to\infty}{\to} 0$, for all $x, y \in X$.
(a) Show that any Hilbert space is uniformly convex.
(b) Show that L^p is uniformly convex for $p \in [2, \infty)$.
Hint: **(Clarkson's inequality)** Whenever $f, g \in L^p$, $p \in [2, \infty)$, we have

$$\|\frac{1}{2}(f+g)\|_p^p + \|\frac{1}{2}(f-g)\|_p^p \le \frac{1}{2}\|f\|_p^p + \frac{1}{2}\|g\|_p^p$$

Exercise 5.4 ([Chapter4] *rather easy and fast*):
[R2:Exe4.1] Let M be a (vector) subspace of a Hilbert space H. Show that $M^{\perp\perp} := (M^\perp)^\perp = M$ if M is closed. What if M is not closed?

[1] $A \subset X$ is *dense* iff it does not lie outside any nonempty open set, i.e., iff $A \cap V \ne \emptyset$ for all nonempty open sets $V \subset X$.

[2] For *strict convexity*, see Exercise 7.5/1999.

Exercise 5.5 ([4.11] *rather easy and fast*):
[R2:Exe4.5] Let H be a Hilbert space and let L be a continuous linear functional on H. Assume that $L \not\equiv 0$. Show that $M := \{x \in H \mid Lx = 0\}^\perp$ is a vector space of dimension one.

Solution to 5.1:

(a) Sketch of a proof: For $\mathbb{R}^1$, one dense countable subset would be quite simple

$$A := \{\sum_{k=1}^n q_k \chi_{(a_k,b_k)} \mid n \in \mathbb{N},\ q_k \in \mathbb{Q} + i\mathbb{Q},\ a_k, b_k \in \mathbb{Q} \forall k\}.$$

Since we are working with $\mathbb{R}^d$ for arbitrary $d \in \mathbb{N}+1$, we choose

$$A := \{\sum_{k=1}^n q_k \chi_{t_k+E_k} \mid n \in \mathbb{N},\ q_k \in \mathbb{Q}+i\mathbb{Q},\ t_k \in \mathbb{Q}^d,\ E_k = [0, 2^{-j_k}]^d,\ j_k \in \mathbb{N}\,\forall k\}$$

Hints: 1° Show that A is countable.

2° If $f = \chi_V$, where V is open and $m(V) < \infty$, there is $s \in A$ s.t. $\|f - s\|_p < \epsilon$ (show first that V is a countable union of sets of form E_k above).

3° If $f = \chi_E$, where $E \in \mathfrak{M}$, then there is $s \in A$ s.t. $\|f - s\|_p < \epsilon$ (use the outer regularity of m and 2°).

4° If f is a simple measurable function... (use 3°).

5° For any $f \in L^p(\mathbb{R})$ (use the density of simple measurable functions and 4°).

A different proof: Let Q be all functions of the form $c\chi_{[a,b)}$, where $c \in \mathbb{Q} + i\mathbb{Q}$ and

$$[a, b) := [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_d, b_d), \quad a_k, b_k \in \mathbb{Q}\ \forall k \in \{1, ..., d\}.$$

Let $\mathcal{D}$ be the set of all finite sums of functions in Q, so that $\mathcal{D}$ is countable, by Theorems 2.12 and 2.13 of [R1].[3]

[3] 1. The set of all elements of form $(x_1, ..., x_n)$, with $x_k \in X_k$, where each X_k is at

We shall now show that $\mathcal{D}$ is dense in $L^p := L^p(\mathbb{R}^d, m)$, i.e., that for each $f \in L^p$ and $\epsilon > 0$, there is $s \in \mathcal{D}$ s.t. $\|f - s\| < \epsilon$. By density [3.12], we may assume that $f \in \mathcal{C}_c(\mathbb{R}^d)$.

For each $m \in \mathbb{N}$, let P_m consist of all sets of the form $[a, b)$, where $a_i = j2^{-m}$ and $b_i = (j+1)2^{-m}$ for $j \in \mathbb{Z}$. Thus, P_m splits $\mathbb{R}^d$ into a countable number of half-open rectangles with diameter $2^{-m}\sqrt{d}$, we can enumerate them, say $P_m = \{[x_{m,k}, y_{m,k})\}_{k\in\mathbb{N}}$ (where $x_{m,k}, y_{m,k} \in \mathbb{R}^d$ for each k). Then $\mathbb{R}^d = \cup_{i\in\mathbb{N}}[x_{m,k}, y_{m,k})$. Choose $c_{m,k} \in \mathbb{Q} + i\mathbb{Q}$ $(k, m \in \mathbb{N})$ s.t. $|f(x_{m,k}) - c_{m,k}| < 5^{-m}$ and

$$|c_{m,k}| \leq |f(x_{m,k})| \tag{1}$$

Let $s_m := \sum_{i\in\mathbb{N}} c_{m,i}\chi_{[x_{m,i}, y_{m,i})}$. Since $f(x_{m,i}) = 0$ except for finitely many i (since $f \in \mathcal{C}_c(\mathbb{R}^d)$), we obtain from (1) that $s_m \in \mathcal{D}$ (set $c_{m,i} = 0$ whenever $f(x_{m,i}) = 0$). (End of fixed m.)

Since $f \in \mathcal{C}_c$ is uniformly continuous (by [R1]), we have $\lim_{m\to\infty} s_m(x) = f(x)$ uniformly[4] in x, i.e., $\sup|s_m - f| \to 0$, as $m \to \infty$. Therefore, $\|s_m - f\|_p^p \leq \int_E \sup|s_m - f|^p = m(E)\sup|s_m - f|^p \to 0$, as $m \to \infty$ (choose $R > 0$ s.t. $\operatorname{supp} f \subset [-R, R]^d$, and set $E := [-R-1, R+1]^d$, so that $m(E) < \infty$ and $s_m, f = 0$ on E^c for all m), QED.

(b) $\mathbb{N}_+ := \{1, 2, 3, ...\}$, $I_n := [n, n+1]^d \subset \mathbb{R}^d$, $e_n := \chi_{I_n} \in L^\infty(\mathbb{R}^d) =: L^\infty$. For $\alpha \in \{0, 1\}^{\mathbb{N}} =: 2^{\mathbb{N}}$ we define

$$f_\alpha := \sum_{n=1}^\infty \alpha_n e_n \in L^\infty.$$

Then $\alpha_n \neq \beta_n \implies \|f_\alpha - f_\beta\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} |f_\alpha(x) - f_\beta(x)| \geq \operatorname{ess\,sup}_{x\in I_n} |f_\alpha(x) - f_\beta(x)| = \operatorname{ess\,sup}_{x\in I_n} |\alpha_n - \beta_n| = 1$, so the balls $B(f_\alpha, \frac{1}{2})$ are disjoint (for $\alpha \in 2^{\mathbb{N}}$). If $\mathcal{G} \subset L^\infty$ is countable, then some of the ("$2^{\mathbb{N}}$") balls $B(f_\alpha, \frac{1}{2}) := \{f \in L^\infty \mid \|f - f_\alpha\|_\infty < \frac{1}{2}\}$ does not contain any points of $\mathcal{G}$ (because $2^{\mathbb{N}} \succ \mathbb{N}$ and each $g \in \mathcal{G}$ can be in at most one $B(f_\alpha, \frac{1}{2})$). $\square$

Another proof: "Diagonal argument": Let $\{g_n\}_{n=1}^\infty \subset L^\infty$. We will choose $\alpha \in 2^{\mathbb{N}}$ s.t. $\|g_n - f_\alpha\|_\infty \geq 1/2$ $\forall n$.

$$\alpha_n := \begin{cases} 1, & \text{when } \operatorname{ess\,sup}_{x\in I_n} |g_n(x)| < \frac{1}{2}; \\ 0, & \text{otherwise.} \end{cases} \qquad (n \in \{1, 2, 3, \dots\})$$

Then $\|f_\alpha - g_n\|_\infty := \operatorname{ess\,sup}_{x\in[0,1]} |f_\alpha(x) - g_n(x)| \geq \operatorname{ess\,sup}_{x\in I_n} |f_\alpha(x) - g_n(x)| = \operatorname{ess\,sup}_{x\in I_n} |\alpha_n - g_n(x)| \geq \frac{1}{2}$ $\forall n$. $\square$

Remarks: We conclude from (a) that $L^p(A, m)$ is separable for any measurable $A \subset \mathbb{R}^d$ (the set $\mathcal{D}_A := \{s_{|_A} \mid s \in \mathcal{D}\}$ is dense and at most countable). By slightly modifying the proof of (b) one can show that $L^\infty(A, \mu)$ is unseparable when, e.g., $\mu = m$ and $m(A) > 0$ (it suffices that A has a countable number of disjoint subsets having a positive measure).

Solution to 5.2: (a) Let $1 \leq p < \infty$. The translation of f is the operation $h \mapsto \tau^h f$, where $\tau^h f$ is the function $x \mapsto f(x+h)$. Thus, $\tau^\cdot f : \mathbb{R} \to L^p$, when $f \in L^p$. (We are asked to show that this is continuous for (any) fixed f, although the wording is not quite exact.)[5]

It obviously[6] suffices to show continuity at zero, i.e., that $\lim_{h\to 0} \tau^h f = f$ in $L^p(\mathbb{R})$; equivalently, that $\|\tau^h f - f\|_p \to 0$, equivalently, that

$$\lim_{h\to 0} \int_{-\infty}^{\infty} |f(x+h) - f(x)|^p \, dx = 0$$

for any $f \in L^p(\mathbb{R})$.

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most countable, is at most countable. 2. At most countable unions of at most countable sets are at most countable.

[4] Given $\epsilon' > 0$, choose M s.t. $|x - y| < 2^{-M} \Rightarrow |f(x) - f(y)| < \epsilon'/3$, and $5^{-M} < \epsilon'/2$, so that for any $m \geq M$ we have $|f(x) - s(x)| = |f(x) - c_{m,i}| \leq |f(x) - f(x_{m,i})| + |f(x_{m,i}) - c_{m,i}| < \epsilon'/3 + 5^{-m} < \epsilon'$ (where i is the one for which $x \in [x_{m,i}, y_{m,i})$), for any $x \in \mathbb{R}^d$.

[5] It is very easy to show that $\tau^h : L^p \to L^p$ is continuous, even an isometric isomorphism.

[6] We have $\tau^{h_0+h} f = \tau^h \tau^{h_0} f$ for all $f \in L^p$. Thus, when continuity at zero ($h_0 = 0$) is established, and we are given any $f \in L^p(\mathbb{R})$ and $h_0 \in \mathbb{R}$, we obtain that $\lim_{h\to h_0} \tau^h f = \lim_{h\to h_0} \tau^{h-h_0}\tau^{h_0} f = \tau^{h_0} f$ (where the limits are taken in $L^p(\mathbb{R})$), i.e., that $h \mapsto \tau^h f$ is continuous at $h = h_0$.

If you are bothered by the above, just replace 0 by h_0 and f by $\tau^{h_0} f$ in several places in the proof.

1° *Case* $f \in \mathcal{C}_c(\mathbb{R}) \subset L^p(\mathbb{R})$. Given $\{h_n\}$ s.t. $h_n \to 0$, we have $f_n(x) := f(x+h_n) \to f(x)$ for all $x \in \mathbb{R}$ (because $f \in \mathcal{C}$). A continuous function has a maximum on a compact set, hence $M := \max|f| < \infty$ exists. Choose $R > 0$ s.t. $\operatorname{supp} f \subset [-R, R]$. Set $E := [-R-1, R+1]$. Then $\operatorname{supp} f_n \subset E$ for big n, so that $g := (2M)^p \chi_E \ge |f_n - f|^p$ everywhere for big n. But $g \in L^1$, hence we can apply Lebesgue's Dominated Convergence Theorem to obtain that $\int_{\mathbb{R}} |f_n - f|^p \to 0$, i.e., that $f_n \to f$ in $L^p(\mathbb{R})$ (equivalently, that $\tau^{h_n} f \to f$ in $L^p(\mathbb{R})$) as $n \to \infty$. Since the sequence $\{h_n\}$ was arbitrary, we have $\tau^h f \to f$ as $h \to 0$ (by [R1:4.2]).

2° *General case* $f \in L^p(\mathbb{R})$*:* By density ([R2:3.14]), there is $g \in \mathcal{C}_c(\mathbb{R})$ s.t. $\|f-g\|_p < \epsilon/3$. By 1°, there is $\delta > 0$ s.t. $\|\tau_h g - g\|_p < \epsilon/4$ whenever $|h| < \delta$. Whenever $|h| < \delta$, we have[7]

$$\|\tau_h f - f\|_p \le \|\tau_h f - \tau_h g\|_p + \|\tau_h g - g\|_p + \|g - f\|_p < \epsilon/3 + \epsilon/4 + \epsilon/3 < 3\epsilon/3 = \epsilon.$$

(b) Let $f = \chi_{[0,1]}$. Then $\|f - \tau_h f\|_\infty = \|\chi_{[0,1]} - \chi_{[-h,1-h]}\|_\infty \equiv 1 \ne 0$, as $h \ne 0$, because $\chi_{[0,1]} - \chi_{[-h,1-h]} = 1$ in the set $[0,1] \setminus [-h, 1-h]$ having a positive measure.

Remarks: 1. Of course, the above need not hold for arbitrary measures (in place of the Lebesgue measure).

2. The space $\mathcal{C}_{\mathrm{bu}}(\mathbb{R})$ of bounded uniformly continuous functions is the greatest subspace of L^∞ on which translation is continuous.[8] For the proof, see exercises 6.1 and 9.6 in Fall 1994:
`http://www.math.hut.fi/~kmikkola/mat/rea/rea94/`

3. All of the above holds with $\mathbb{R}^n$ in place of $\mathbb{R}$, with essentially the same proofs (just replace $[-R, R]$ by $[-R, R]^d$ and $[-R-1, R+1]$ by $[-R-1, R+1]^d$.

Solution to 5.3: **(a)** Add $\|x \pm y\|^2 = \langle x \pm y, x \pm y \rangle = \|x\|^2 + \|y\|^2 \pm \langle x, y\rangle \pm \langle y, x \rangle$ to obtain the *parallelogram law* [p.80]

$$\|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 - \|x+y\|^2 \quad (x, y \in X). \tag{2}$$

Substitute $x \mapsto x_n$, $y \mapsto y_n$ and take lim sup of both sides to obtain

$$\limsup_{n\to\infty} \|x_n - y_n\|^2 \le 2 + 2 - 2^2 = 0.$$

Since $\liminf_{n\to\infty} \ge 0$, we conclude that $\exists \lim = 0$, QED.

(b) As in (a), the uniform convexity follows from the fact that (use Clarkson's inequality and note that $\|\frac{1}{2}(f_n - g_n)\|_p^p = 2^{-p} \cdot \|f_n - g_n\|_p^p$)

$$\limsup_{n\to\infty} 2^{-p} \|(f_n - g_n)\|_p^p \le \frac{1}{2} + \frac{1}{2} - 2^{-p} \cdot 2^p = 0.$$

Remarks: Some results of Chapter 4 (e.g., [4.10]) hold for any uniformly convex Banach spaces; see. e.g., [R3:Exe12.32] for details. However, L^1 and L^∞ are not uniformly convex (e.g., take $x_n = \chi_{[0,1]}$, $y_n = \chi_{[1/2,3/2]}$ $\forall n$).

Solution to 5.4: We always have $M^{\perp\perp} = \bar{M}$, hence $M^{\perp\perp} = M$ iff M is closed (i.e., iff $M = \bar{M}$). Let's show this.

0° *Claim:* $M^\perp = (\bar{M})^\perp$. Proof: "$\supset$": trivially $y \perp \bar{M} \Rightarrow y \perp M$. "$\subset$": $(x, y) = 0\ \forall x \in M \Rightarrow (x, y) = 0\ \forall x \in \bar{M}$, because $(\cdot, y)$ is continuous, by [R2:4.6].

1° $(M^\perp)^\perp \supset \bar{M}$. Proof: Let $x \in \bar{M}$. Let $M \ni x_n \to x$, as $n \to \infty$. Then $(x, y) = (\lim_n x_n, y) = \lim_n (x_n, y) = \lim_n 0 = 0\ \forall y \in M^\perp$, hence $x \in (M^\perp)^\perp$.

2° $(M^\perp)^\perp \subset \bar{M}$. Proof: Let $x \in (M^\perp)^\perp$. Then $x = Px + Qx$, where $Px \in \bar{M}$ and $Qx \in \bar{M}^\perp = M^\perp$, by 0° and [R2:4.11a]. Since $x \perp Qx \in \bar{M}^\perp = M^\perp$, by 0°, we have $0 = (x, Qx) = (Px + Qx, Qx) = (Px, Qx) + (Qx, Qx) = (Qx, Qx)$, i.e., $Qx = 0$, hence $x = Px + 0 = Px \in \bar{M}$.

Solution to 5.5: 0° Since $N := \{x \in H \mid Lx = 0\} \subset H$ is closed (since $N = L^{-1}(\{0\})$ and $L : H \to \mathbb{C}$ is continuous), we have $H = M + N$,

[7] Obviously, $\|\tau F\|_p = \|F\|_p$ for any measurable function F, because m is translation invariant.

[8] To be exact, here we refer to the equivalence classes of elements of $\mathcal{C}_{\mathrm{bu}}(\mathbb{R})$.

by [R2:4.11]. Moreover, $M \neq \{0\}$, since $N \neq H$ (since $L \not\equiv 0$).

1° *There is $a \in M$ s.t. $La = 1$.* Proof: Choose $x \in M \setminus \{0\}$ and set $a := (Lx)^{-1}x \in M$, so that $La = L((Lx)^{-1}x) = (Lx)^{-1}L(x) = 1$.

2° *$M = \mathbb{C}a := \{\alpha a \,|\, \alpha \in \mathbb{C}\}$, in particular, M is one-dimensional.* Proof: Since M is a vector space, $\mathbb{C}a \subset M$. Conversely, if $y \in M$, then $y = (Ly)a =: z \in \mathbb{C}a$, because $L(y - z) = Ly - Lz = Ly - (Ly)La = 0$, so that $y - z \in M^\perp$, hence $y - z \in M^\perp \cap M = \{0\}$ (we have $y - z \in M$ because $y, z \in M$), thus, $M \subset \mathbb{C}a$.