Mat-1.150 Real Analysis: 4th exercise, Wed 2002-10-9, U322 (4 pages)

`http://www.math.hut.fi/teaching/realanal/`

1st mid-term examination (1. välikoe) 11.11. (11th Nov) at 9–12,
2nd mid-term examination (2. välikoe) 4.12. (4th Dec) at 14–17. Details will be given later.

Exercise 4.1 ([3.7] *a bit difficult, not long*):
[R2:Exe3.23] Suppose that μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu)$, $\|f\|_\infty > 0$, $\alpha_n := \int_X |f|^n \, d\mu$, $n = 1, 2, 3, 4, \ldots$. Show that

$$\frac{\alpha_{n+1}}{\alpha_n} \to \|f\|_\infty \qquad \text{as} \quad n \to \infty.$$

Exercise 4.2 ([3.9] *(a) easy, very short, useful; (c) harder*):
[R2:Exe3.4abe] Let $f : X \to \mathbb{C}$ be measurable and let $\mu \geq 0$ be a measure on X. Define

$$\phi(p) := \int_X |f|^p \, d\mu, \qquad 0 < p < \infty.$$

Let E be defined by $E := \{p | \phi(p) < \infty\}$. Let f be nonzero on a set of strictly positive measure.

(a) Show that if $r, s \in E$, $r < s$, then any $p \in (r, s)$ satisfies $p \in E$.

(b) Next, try to show that $\log \phi$ is convex on $E \setminus \{\infty\}$.[1]

(c) Finally, assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

Exercise 4.3 ([3.9] *mainly average level, not long*):
[R2:Exe3.5abc] Assume, in addition to what was assumed in Exercise 4.2 above, that $\mu(X) < \infty$.

(a) Prove that $\|f\|_r \leq M_{r,s} \|f\|_s$, $0 < r < s \leq \infty$, for some $M_{r,s} < \infty$ that does not depend on f (we can have $M_{r,s} = 1 \; \forall r, s$ if $\mu(X) = 1$).

(b) Prove that $L^s(\mu) \subset L^r(\mu)$, $0 < r < s \leq \infty$.

(c) Can it happen that $L^s(\mu) = L^r(\mu)$, $0 < r < s \leq \infty$?

Exercise 4.4 ([3.7] *quite easy, average length*):
Show that (here we use the Lebesgue measure):

$$\Big[x(\log \frac{1}{x})^2\Big]^{-1} \begin{cases} \in L^1(0, \frac{1}{2}), \\ \notin L^p(0, \frac{1}{2}), \; p > 1; \end{cases} \tag{1}$$

$$\Big[x^{\frac{1}{2}}(1 + |\log x|)\Big]^{-1} \begin{cases} \in L^2(0, \infty), \\ \notin L^p(0, \infty), \; p \neq 2. \end{cases} \tag{2}$$

Solution to 4.1: $0 = \alpha_n = \int_X |f|^n \, d\mu \implies |f|^n = 0$ a.e. $\implies f = 0$ a.e. $\implies \|f\|_\infty = 0$, so we must have $\alpha_n > 0$ (hence $\mu(X) > 0$), for any n. On the other hand, $\alpha_n \leq \int_X \|f\|_\infty^n \, d\mu = \mu(X)\|f\|_\infty^n < \infty$, because $|f(x)| \leq \|f\|_\infty$ for a.e. x.

We will now show that $\limsup_n \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty$ [1°] and then that $\liminf_n \frac{\alpha_{n+1}}{\alpha_n} \geq \|f\|_\infty$ [2°], which shows that these must be equal, hence $\exists \lim_n \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.

1° $\alpha_{n+1} \leq \int_X \|f\|_\infty |f|^n \, d\mu = \|f\|_\infty \alpha_n$,[2] so $\frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty \; \forall n$, hence $\limsup_n \frac{\alpha_{n+1}}{\alpha_n} \leq \|f\|_\infty$.

2°[3] Let $0 < R < \|f\|_\infty$. Set $E := \{|f| \geq R\}$. Now $\mu(E) > 0$ (because $\|f\|_\infty :=$ sup of those R for which this holds, by [R2: 3.7]), and $\alpha_{n+1} \geq \int_E |f|^{n+1} \geq \int_E R^{n+1} = \mu(E) R^{n+1} > 0$, from which we get

$$\frac{\alpha_n}{\alpha_{n+1}} = \frac{\int_E |f|^n}{\alpha_{n+1}} + \frac{\int_{E^c} |f|^n}{\alpha_{n+1}} \leq \frac{\int_E |f|^n}{R \int_E |f|^n} + \frac{\int_{E^c} |f|^n}{\mu(E) R^{n+1}} \tag{3}$$

$$= R^{-1} + \frac{\int_{E^c} (|f|^n / R^n)}{\mu(E) R} \to R^{-1} + 0, \quad \text{as } n \to \infty, \tag{4}$$

by dominated convergence [R2: 1.34].[4] Thus, $\limsup_n \frac{\alpha_n}{\alpha_{n+1}} \leq R^{-1}$, hence

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[1] $\log := \ln$ usually in mathematics.

[2] The values of $|f|$ in the null set $\{|f| > \|f\|_\infty\}$ don't affect integrals.

[3] A second approach for 2°: show first that $\|f\|_n \to \|f\|_\infty$ (not too easy), and then use Hölder with $p = (n+1)/n$ to show that $\alpha_n / \alpha_{n+1} \leq \mu(X)^{1/(n+1)} / \|f\|_{n+1} \to 1/\|f\|_\infty$.

[4] We can use [1.34], because $(|f(x)|/R)^n \to 0 \; \forall x \in E^c$ and $(|f|/R)^n < 1 \; \forall n$ and $1 \in L^1(E)$.

$\liminf_n \frac{\alpha_{n+1}}{\alpha_n} \geq R$. Because $R < \|f\|_\infty$ was arbitrary, $\liminf_n \frac{\alpha_{n+1}}{\alpha_n} \geq \|f\|_\infty$.[5]

Solution to 4.2: Without loss of generality,[6] we assume that $f \geq 0$. We treate below also the case $p = \infty$ (by setting $\infty \in E \iff f \in \Lambda^\infty$), but you may ignore this (and "2°" below).

(a) Set $A := \{f \geq 1\} := \{x \in X \mid f(x) \geq 1\}$.

1° *Case $s < \infty$:* We have $f^p \leq f^s$ on A and $f^p \leq f^r$ on A^c, hence $\int_X f^p\, d\mu \leq \int_X (f^s + f^r)\, d\mu < \infty$, QED.

2° *Case $s = \infty$:* If $0 < r < p < s = \infty$, then $\int f^p = \int_A f^p + \int_{A^c} f^p \leq \int_A \|f\|_\infty^p + \int_{A^c} f^r < \infty$.

Remark: we have shown that $E \subset (0, \infty]$ is an interval.

(b) For any $0 < r < s < \infty$, $0 < \lambda < 1$, we should show that

$$\log \phi(\lambda r + (1-\lambda)s) \leq \lambda \log(\phi(r)) + (1-\lambda)\log(\phi(s)) = \log\left(\phi(r)^\lambda \phi(s)^{1-\lambda}\right),$$

i.e., that $\phi(\lambda r + (1-\lambda)s) \leq \phi(r)^\lambda \phi(s)^{1-\lambda}$ (because log is strictly increasing); equivalently, that

$$\int_X f^{\lambda r + (1-\lambda)s}\, d\mu \leq \left(\int_X f^r\, d\mu\right)^\lambda \left(\int_X f^s\, d\mu\right)^{1-\lambda}.$$

It suffices to set $h := f^{\lambda r}$, $g := f^{(1-\lambda)s}$ in Hölder's Inequality

$$\int_X hg \leq \left(\int_X h^{1/\lambda}\right)^\lambda \left(\int_X g^{1/(1-\lambda)}\right)^{1-\lambda}.$$

Remark: Being convex, $\log\phi$ is continuous on the interior of $E \setminus \{\infty\}$, by [R2:3.2], hence so is ϕ.[7]

[5] If $a > R$ for all $R < b$, then $a \geq b$ (otherwise $R := (a+b)/2$ would satisfy $a < R < b$, a contradiction). This is a standard approach, remember this!

[6] Apply the results to $|f|$.

[7] Actually, ϕ is continuous on E. By (c), we get continuity at ∞ (if $\infty \in E$; this is trivial if $E = \{\infty\}$), and continuity at $\inf E$ (if $r := \inf E \in E$) can be rather easily proved by using, e.g., dominated convergence (use f^r on A^c and f^s on A, for some $s \in E \setminus \{\infty\}$).

(c) [8] Without loss of generality, we assume that $R := \|f\|_\infty \in (1, \infty]$.[9] Set $a := \int_X |f|^p < \infty$, $A_r := \{x \mid |f(x)| \geq r\}$. From 1° and 2° below it follows that $\lim \|f\|_q = R$ (in particular, that this limit exists), QED.

1^o "$\liminf_{q\to\infty} \|f\|_q \geq R$": We shall show below that given $R' < R$, there is $Q \in \mathbb{R}_+$ s.t. $\forall q \geq Q\ [\|f\|_q > R']$. This proves 1°.

Choose $M \in]R', R[$. Then $d := \mu(A_M) > 0$ (otherwise $R \leq M$). But $d < \infty$ (otherwise $a \geq dM^p = \infty$), and, for any $q \in (0, \infty)$,

$$\|f\|_q^q \geq \int_{A_M} |f|^q \geq dM^q, \text{ hence } \|f\|_q \geq d^{1/q} M.$$

If $d \geq 1$, then $\|f\|_q \geq M > R'\ \forall q \geq 1 =: Q$; if $d < 1$, choose $Q > 0$ s.t. $d^{1/Q} > R'/M$, to obtain $\|f\|_q \geq d^{1/q} M \geq d^{1/Q} M > R'\ \forall q \geq Q$.

2^o "$\limsup_{q\to\infty} \|f\|_q \leq R$": If $R = \infty$, we have $\|f\|_q \to R$, by 1^o. Assume then that $R \in (1, \infty)$. Choose $r \in (0, 1)$ to have $d := \mu(A_r) \in (0, \infty)$ (as in 1^o). Now, for all $q \geq p$ we have $\int_{A_r} |f|^q \leq R^q \mu(A_r) = R^q d$ and $\int_{A_r^c} |f|^q \leq \int_{A_r^c} |f|^p \leq a$, hence

$$\|f\|_q = [\int_{A_r} |f|^q + \int_{A_r^c} |f|^q]^{1/q} \leq [R^q d + a]^{1/q} = R[d + a/R^q]^{1/q} \to R, \text{ as } q \to \infty,$$

because $d_q^{b_q} = (e^{\ln d_q})^{b_q} = e^{b_q \ln d_q} \to e^{0 \ln d} = 1$, as $b_q \to 0+$ & $d_q \to d \in \mathbb{R}_+$. Thus, $\limsup \|f\|_q \leq R$, QED.[10] [11]

Solution to 4.3: W.l.o.g., we assume that $f \geq 0$. Set $M := \mu(X) < \infty$.

[8] The assumption $\|f\|_p < \infty$ is not superfluous: $\infty = \|\chi_\mathbb{R}\|_q \not\to 1 = \|\chi_\mathbb{R}\|_\infty$, when $m = \mu$.

[9] If $f = 0$ a.e., the claim is trivial. If $R \in]0, 1]$, set $c := 2/R$ to have $\|cf\|_\infty = 2 > 1$, so that by what we will show, we have $c\|f\|_q = \|cf\|_q \to \|cf\|_\infty = c\|f\|_\infty$, hence $\|f\|_q \to \|f\|_\infty$, as $q \to +\infty$.

[10] Given any $R' > R$, choose $Q > p$ s.t. $[d + a/R^q]^{1/q} < R'/R\ \forall q > Q$, to have $\|f\|_q < R(R'/R) = R'\ \forall q > Q$. Consequently, $\limsup \leq R'$. Since $R' > R$ was arbitrary, we have $\limsup \leq R$.

[11] Note: If you do not feel comfortable with liminf–limsup thinking, think like this: Let $R := \|f\|_\infty \in]1, \infty[$. Let $\epsilon > 0$. By 1^o, there is Q_1 for the value $R - \epsilon$, and, by 2^o, there is Q_2 for the value $R + \epsilon$. Thus, for all $q \geq \max(Q_1, Q_2)$, we have $\|f\|_q \in]R - \epsilon, R + \epsilon[$. Since $\epsilon > 0$ was arbitrary, we have, $\|f\|_q \to \|f\|_\infty$, as $q \to +\infty$. (In case $R = \infty$, part 1^o was enough.)

(a) 1° *Case $s=\infty$:* Since $f \le \|f\|_\infty$ a.e., we have

$$\|f\|_r^r = \int_X f^r \le \int_X \|f\|_\infty^r = \mu(X)\|f\|_\infty^r,$$

i.e., $\|f\|_r \le M_{r,\infty}\|f\|_\infty$, where $M_{r,\infty} := M^{1/r}$.

2° *Case $s<\infty$:* Set $q := (1-r/s)^{-1} > 0$. By the Hölder inequality, we have

$$\|f\|_r^r = \|f^r \cdot 1\|_1 \le \|f^r\|_{s/r}\|1\|_q = (\int_X f^s)^{r/s}(\int_X 1)^{1/q} = \|f\|_s^r M^{1-r/s}.$$

Set $M_{r,s} := M^{1/r-1/s}$ to obtain (a).

(b) If $f \in L^s(\mu)$, then $\|f\|_r \le M_{r,s}\|f\|_s < \infty$, by (a), hence then $f \in L^r(\mu)$.

(c) E.g., if $\mathfrak{M} = \{X,\emptyset\}$, $0<\mu(X)<\infty$, then {measurable functions} = {constant functions}.[12] If $f(x)=z$ for all $x \in X$, then $\|f\|_p^p = \int_X \|f\|^p = \int_X |z|^p = \mu(X)|z|^p$, i.e., $\|f\|_p = \mu(X)^{1/p}|z| < \infty$ for all $p \in (0,\infty]$.[13]

Remark: From (a) and (b) one easily obtains the following: Consider the following two conditions:

(i) there are sets of arbitrarily small positive measure, i.e., $\forall \epsilon > 0\ \exists A \in \mathfrak{M}\ [\mu(A) \in (0,\epsilon)]$;

(ii) there is an infinite measure of disjoint sets of finite measure, i.e., $\exists \{A_n\}_{n\in\mathbb{N}} \subset \mathfrak{M}$ s.t. $\mu(A_n) \in]0,\infty[\ \forall n\ \&\ (i \ne j \Rightarrow A_i \cap A_j = \emptyset)\ \&\ \sum_n \mu(A_n) = \infty$.

Whenever $(X,\mathfrak{M},\mu)$ is a measure space and $\infty \ge t \ge s > r > 0$, the following hold:

(a) (i) $\Longleftrightarrow$ $L^r \not\subset L^t$,

(b) (ii) $\Longleftrightarrow$ $L^s \not\subset L^r$,

(c) $\mu(X) < \infty \iff L^\infty \subset L^r \iff L^\infty \subset L^s \subset L^r$,

[12] Proof: For them $f^{-1}(V) \in \mathfrak{M}$ for all $V \subset \mathcal{T}$. Conversely, if $f(x)=t$, $f(y)=u$, $t \ne u$, $t.u \in \mathbb{C}$, then $f^{-1}(V) \notin \mathfrak{M}$, when $t \in V \not\ni u$.

[13] Obviously, $\|f\|_\infty = |z| = \mu(X)^0|z|$.

For the proof, see exercise 6.6 in Fall 1994:
`http://www.math.hut.fi/~kmikkola/mat/rea/rea94/reav6.ps`

Solution to 4.4: Continuous (or piecewise continuous) functions are Borel-measurable, hence Lebesgue-measurable. Thus, $f \in L^p$ iff $\|f\|_p < \infty$. Obviously, neither function is in L^∞, so we only need to study the case $p<\infty$.

$(*)$ We have $\sum_{n=2}^\infty \frac{1}{n(\log n)^p} < \infty$ iff $p>1$, by Theorem [R1:3.28].

(a) Set $f(x) := [x(\log 1/x)^2]^{-1}$.

1° $f \in L^1(0,1/2)$. Proof: Set $x=1/t$, $dx/dt = -t^{-2}$ to obtain that[14] $\int_0^{1/2} f(x)\,dx = \int_2^\infty \frac{t}{(\log t)^2}t^{-2}\,dt = \int_2^\infty \frac{1}{t(\log t)^2}\,dt = \sum_{n=2}^\infty \int_n^{n+1} \frac{1}{t(\log t)^2}\,dt \le \sum_{n=2}^\infty \int_n^{n+1} \frac{1}{n(\log n)^2}\,dt = \sum_{n=2}^\infty \frac{1}{n(\log n)^2} < \infty$, by $(*)$.[15]

2° $f \notin L^p(0,1/2)$, $p>1$. Proof: Choose $\epsilon \in (0,p-1)$. Choose[16] $M<\infty$ s.t. $(\log t)^{2p} \le Mt^\epsilon$ for all $t \ge 2$. Then $\int_0^{1/2} f^p = \int_0^{1/2} \frac{1}{x^p(\log 1/x)^{2p}} = \int_2^\infty \frac{t^{p-2}}{(\log t)^{2p}} \ge \int_2^\infty \frac{t^{p-2}}{Mt^\epsilon} = \infty$, because $p-2-\epsilon > -1$.

(b) Set $g(x) := [x^{1/2}(1+|\log x|)]^{-1}$.

1° $\int_2^\infty g^2 < \infty$. Proof: $\int_2^\infty g^2 = \int_2^\infty [x(1+\log x)^2]^{-1} \le \sum_{n=2}^\infty [n(\log n)^2]^{-1} < \infty$, by $(*)$.

2° $\int_0^{1/2} g^2 < \infty$. Proof: $\int_0^{1/2} g^2 = \int_0^{1/2} \frac{1}{x(1+|\log x|)^2} = \int_2^\infty \frac{1}{t^{-1}(1+|\log 1/t|)^2}t^{-2} = \int_2^\infty \frac{1}{t(1+|-\log t|)^2} \le \int_2^\infty \frac{1}{t(\log t)^2} < \infty$, by $(*)$.

3° $g \in L^2((0,\infty))$*:* Proof: $\int_{1/2}^2 g^2 < \infty$,[17] hence $\int_0^\infty g =$

[14] For the Lebesgue measure, you can change the variable exactly as for the Riemann integral: we have $\int_{(F(t_1),F(t_2))} f(x)\,dx = \int_{(t_1,t_2)} f(F(t))|F'(t)|\,dt$ whenever F is e.g., monotonic and continuously differentiable, and $f \ge 0$ (or f is integrable), by formula (15) on p. 156. Note that the absolute value "$|F'(t)|$" compensates the fact that if $F' \le 0$, then one should multiply the integral by -1 to exchange upper and lower bounds.

[15] We used here Theorem [1.27] (with $f_n := f\chi_{[n,n+1)}$) and the fact that f is decreasing.

[16] This is possible, because $(\log t)^{2p}/t^\epsilon = (\log t/t^{\epsilon/2p})^{2p} \to 0$, as $t \to +\infty$ (so that $(\log t)^{2p}/t^\epsilon < 1$ for all $t > T$, for some $T \in (2,\infty)$, and, being continuous, $(\log t)^{2p}/t^\epsilon$ has a maximum M' on the compact interval $[2,T]$, and we can set $M := \max\{1,M'\}$), because $t \mapsto t^{2p}$ is continuous and $\log t/t^r \to 0$ for any $r \to 0$, by the L'Hospital's Rule (see p. 109, Theorem 5.13 of [R1]). (Recall that $\lim_t F(h(t)) = F(\lim_t h(t))$ whenever $\lim h(t)$ exists and F is continuous at $\lim h(t)$ (use, e.g., Theorem [R1:4.2]).

[17] If h is continuous on a compact set K, then $\int_K |j|\,dm \le m(K)\max|h| < \infty$.

$\int_0^{1/2} + \int_{1/2}^2 + \int_2^\infty < \infty$.

4° *$g \notin L^p((0,\infty))$ for $p < 2$.* Proof: Choose $\epsilon \in (0, 1 - p/2)$. Choose $M < \infty$ s.t. $(1 + \log t)^p \le M t^\epsilon$ for all $t \ge 2$.

Then $\int_2^\infty g^p = \int_2^\infty \frac{1}{x^{p/2}(1+\log x)^p} \ge \sum_{n=2}^\infty \frac{1}{n^{p/2}(1+\log n)^p} \ge \sum_{n=2}^\infty \frac{1}{n^{p/2} M n^\epsilon} = \infty$, since $p/2 + \epsilon < 1$.

5° *$g \notin L^p((0,\infty))$ for $p > 2$.* Proof: Choose $\epsilon \in (0, p/2 - 1)$. Choose $M' < \infty$ s.t. $(1 + \log t)^p \le M t^\epsilon$ for all $t \ge 2$.

Then $\int_0^{1/2} g^p = \int_2^\infty \frac{t^{p/2-2}}{(1+\log t)^p} \ge \int_2^\infty \frac{t^{p/2-2}}{M' t^\epsilon} = \infty$, because $p/2 - 2 - \epsilon > -1$.