Mat-1.150 Real Analysis: 1st exercise, Wed 2002-9-18, U322 (4 pages)

The exercises are on each Wednesday at 4 pm (16–18) in room U322. The questions are usually delivered a week earlier to the envelope in front of room U313 and to `http://www.math.hut.fi/teaching/realanal/` The answers can be found later in same locations (email kalle.mikkola@hut.fi if they run out).

Feel free to ask the assistant (kalle.mikkola@hut.fi, room U313, (09) 451 3017) about the exercises or anything. Most easily you find him during or after the exercises or at 15.00 on Fridays (U313).

Most of the exercises are followed by brackets containing the section number in the book up to which you should study before trying the exercise. The subsequent comments about the duration and, particularly, difficulty of the exercise are of course subjective.

We strongly recommend that you try to solve almost every exercise yourself, and if you do not succeed, try again when you have read the solution (really, you absorb the solution better by reproducing it yourself). Naturally, for those with tight time limits and less ambition may skip the hardest exercises and just have a glance at them. Remember that there are usually several ways to solve the problems, and the one written here need not be the most elegant one.

Note: by "$A \subset B$" we mean that A is a subset of B; by "$A \subsetneq B$" we mean that A is a proper subset of B, i.e., that $A \subset B$ & $A \neq B$. This is the standard convention. By *countable* we mean countably infinite (i.e., an at most countable set is finite or countable but not both). Naturally, we omit some details (e.g., the verification of measurability, required by some theorem applied) when they are not too difficult for the reader. Part of such details are given in footnotes.

Exercise 1.1 ([1.3] *easy, rather quick*):
Let

$$X = \{a, b, c, d\} \text{ and } \mathcal{T} = \{\emptyset,\ X,\ \{a\},\ \{a, b\},\ \{a, b, c\}\}.$$

(a) Is $\mathcal{T}$ a topology in X?
(b) Is $\mathcal{T}$ a σ-algebra in X?
(c) What is the smallest σ-algebra in X that contains $\mathcal{T}$?
(d) Let $f : X \to \mathbb{R}$ be the function $f(a) = 1$, $f(b) = 2$, $f(c) = 3$, $f(d) = 4$. Is f continuous (assume $\mathbb{R}$ to be equipped with its standard topology)?
(e) Is f Borel-measurable?

Solution: (a) Certainly, since the axioms of definition [R2:1.2][1] are satisfied:
(i) $\emptyset, X \in \mathcal{T}$.
(ii) If $V_1, ..., V_n \subset \mathcal{T}$, then $V_1 \cap ... \cap V_n$ obviously equals the smallest (with respect to inclusion) of the sets V_i, hence it belongs to $\mathcal{T}$.

(So this axiom (ii) (and (iii) below) was easy because the elements of $\mathcal{T}$ form an increasing "sequence".)
(iii) If $\{V_\alpha\}_{\alpha\in\mathcal{A}} \subset \mathcal{T}$, then $\cup_{\alpha\in\mathcal{A}} V_\alpha$ obviously equals the biggest of the sets V_α, hence it belongs to $\mathcal{T}$.

(b) $\{a\} \in \mathcal{T}$ but yet $\{a\}^c = \{b, c, d\} \notin \mathcal{T}$, hence $\mathcal{T}$ is not a σ-algebra (this makes question (c) non-trivial).

(c) Let $\mathfrak{M}$ be the smallest σ-algebra in X (it necessarily exists, by [R2:1.10]). Then $\{b\} = \{a, b\}\setminus\{a\} \in \mathfrak{M}$ [R2:1.6], $\{c\} = \{a, b, c\}\setminus\{a, b\} \in \mathfrak{M}$ ja $\{d\} = \{a, b, c\}^c \in \mathfrak{M}$.

But all subsets of X are finite unions of the sets $\{a\}, \{b\}, \{c\}, \{d\}$, hence all subsets of X belong to $\mathfrak{M}$ [R2:1.6b], i.e., $\mathcal{P}(X) \subset \mathfrak{M}$, where $\mathcal{P}(X) = 2^X$ is the set of all subsets of X ("the power set of X"). On the other hand, $\mathfrak{M} \subset \mathcal{P}(X)$ by the definition of a σ-algebra, hence $\mathfrak{M} = \mathcal{P}(X)$.[2]
(d) The answer is negative [1.2c]: $f^{-1}(3, 5) := \{x \in X \,|\, f(x) \in (3, 5)\} = \{d\} \notin \mathcal{T}$, i.e., the preimage of an open set is not open (recall that $(a, b) := \{r \in \mathbb{R} \,|\, a < r < b\}$ and that ":=" means "equals by the definition of the left-hand-side").

(e) The answer is positive [1.3c]: The preimage of any set $A \subset \mathbb{R}$ is a subset of X, hence elements of $\mathfrak{M}$. In particular, the preimages of open

[1] This refers to section 1.2 of the book [R2] (Walter Rudin: "Real and Complex Analysis").

[2] Note: $\mathfrak{M}$ is, by definition, the Borel-σ-algebra of the topological space $(X, \mathcal{T})$ [R2:1.11].

sets are elements of $\mathfrak{M}$, hence f is measurable w.r.t. $\mathfrak{M}$.[3]

Exercise 1.2 ([1.12] *rather quick and easy*):
Theorem 1.17 in the book can be proved using the functions

$$s_n(x) := \sum_{i=1}^{n2^n} \left(\frac{i-1}{2^n}\right) \chi_{E_{n,i}} + n\chi_{F_n}, \text{ where} \tag{1}$$

$$E_{n,i} := f^{-1}([\frac{i-1}{2^n}, \frac{i}{2^n})), \tag{2}$$

$$F_n := f^{-1}([n, +\infty]), \tag{3}$$

and where $f : X \to [0, +\infty]$ is measurable. Show that s_n is measurable.

Remarks: A proof is given in Section 1.17 of [R2], hence we give here an alternative one. Note that $[0, +\infty]$ is equipped with its standard topology (since no other is given). We denote the σ-algebra on X By $\mathfrak{M}$.

Solution: 1° *Sets of form* $[a, b], [a, b), (a, b], (a, b)$ *are Borel sets in* $[0, \infty)$*:* The sets of form $[0, b)$, (a, b) and $(a, +\infty]$ are open in $[0, +\infty]$ when $0 \le a < b \le +\infty$.[4]

Let $\mathcal{B}$ denote the Borel sets of $[0, +\infty]$, so that $\mathcal{B}$ contains the open sets. Then $[a, b) \in \mathcal{B}$ whenever $0 < a < b \le \infty$, because $[a, b) = [0, b) \setminus [0, a)$; similarly $(a, b], [a, b] \in \mathcal{B}$ whenever $0 \le a \le b \le \infty$.

2° $E_{n,i}$ *and* F_n *are measurable* for all n, i (use 1°), because the preimages of Borel sets are measurable, by [R2:1.12b].

3° *Linear combinations of measurable scalar functions are measurable:* If $g : X \to [0, +\infty]$ is a constant function (i.e., if there is a constant α such that $g(x) = \alpha$ for all $x \in X$), then g is measurable (since $g^{-1}(V) = X \in \mathfrak{M}$ if $\alpha \in V$ and $g^{-1}(V) = \emptyset \in \mathfrak{M}$ if $\alpha \notin V$, for all open sets $V \subset [0, +\infty]$). But products and sums of measurable functions are measurable, by [R2:1.9c], hence a linear combination of measurable scalar-valued functions is measurable ($\sum_{k=1}^{n} \alpha_i f_i$ is measurable whenever α_i is a constant and f_i is measurable for all i).

4° Characteristic functions of measurable sets are measurable, by [R2:1.9d], hence so are $\chi_{E_{n,i}}$ and χ_{F_n} (for all n, i). Combine this with 2° and 3° to observe that s_n is measurable (for each n).

Exercise 1.3 ([1.3] *nice but not easy, not important*):
[R2:Exe1.1] Does there exist an infinite σ-algebra with only countable many members?

Remarks: Thus, we should find out whether there is a set X and a collection $\mathfrak{M}$ of its subsets such that axioms [1.3] are satisfied and $\mathfrak{M}$ is infinite but yet countable.[5]

Solution: We assume that $(X, \mathfrak{M})$ is a measurable space and that $\mathfrak{M}$ is at most countable. We shall show that $\mathfrak{M}$ is finite; this shows that the answer is negative.[6]

Intuitive solution:[7] It follows that X can have "essentially" at most a

[3]Because $\mathfrak{M}$ is the Borel-σ-algebra of $(X, \mathcal{T})$, by (d), f is said to be Borel-measurable (w.r.t. $\mathcal{T}$)) [1.11].

[4]Why? Sometimes the topology is defined by this (thus the open sets are the arbitrary unions of such sets). Sometimes it is said that "we use the topology inherited from $[-\infty, +\infty]$", which leads to the same topology (since the above sets are intersections of $[0, +\infty]$ with open sets in $[-\infty, +\infty]$, by [R2:1.4], they must be open; we omit the rest of the proof (though it is easy)). Choose either definition as you wish, but this (single) topology is the one used on $[0, +\infty]$ unless something else is implied.

If X is a topological space and $A \subset X$, then the *topology of* A *inherited from* X or the *topology of* A *relative to* X is the set $\mathcal{T}_A := \{A \cap V \mid V \in \mathcal{T}_X\}$. The reader can easily verify that this is a topology in A and that $f : Q \to A$ is continuous (resp. measurable) iff f is continuous (resp. measurable) when considered as a function $Q \to X$ (when Q is a topological (resp. measurable) space). Also several other properties are preserved, thus it is usually not important whether we consider f (or s_n in the current exercise) as a function to A or to X; naturally, there is a difference when you ask, e.g., whether f is surjective ("onto").

[5]On this course it usually suffices to remember that as set E is countably infinite iff it has the same cardinality ("mahtavuus") as $\mathbb{N}$ (the set of natural numbers), i.e., iff $E \approx \mathbb{N}$, and that any other infinite set has a higher cardinality ("mahtavampi").

Recall that $E \approx F$ iff there exists a bijection $f : E \to F$, that $E \preceq F$ (E is of lower or equal cardinality than F) iff there is an injection (i.e., 1-1 map into) $f : E \to F$, equivalently, if there is a surjection (a map onto) $g : F \to E$. For any sets E, F, we have $E \prec 2^E := \mathcal{P}(E)$, $E \prec F \iff 2^E \prec 2^F$, and $E \approx F \iff 2^E \approx F$. Naturally, $\approx$, $\prec$ and $\preceq$ are transitive. (See, e.g., [R1], [Kelley] or [Bredon] for the proofs.)

[6]Note: X need not be finite; e.g., $X = \mathbb{R}$ & $\mathfrak{M} = \{\emptyset, \mathbb{R}\}$).

[7]In principle, this kind of intuitive solutions are meaningless; in practice, intuitive

countable number of elements; even if it had uncountably many elements, $\mathfrak{M}$ "does not see more than a countable number of them". Thus, from the point of view of $\mathfrak{M}$, the elements of X consist of an at most countable number of "clots" s.t. the elements of $\mathfrak{M}$ consist of unions of such clots only.

Because $2^{\mathbb{N}}$ is uncountable, there can be at most a finite number of such clots, say n. But then it follows that the number of elements of $\mathfrak{M}$ is at most 2^n, Q.E.D.

Rigorous solution: $0°$ **The relation** $a \sim b \iff \forall A \in \mathfrak{M}\,[a \in A \iff b \in A]$ **is obviously an equivalence relation** ("a and b belong to same clot").[8]

$1°$ $[a] \in \mathfrak{M}$ **for all** $a \in X$**, i.e., each equivalence class is a measurable set.**[9] Proof: $E_a := \cap\{A \in \mathfrak{M} \mid a \in A\} \in \mathfrak{M}$ [1.6c], because E_a is an at most countable intersection of measurable sets (since $\mathfrak{M}$ has at most countable elements). Obviously,[10] $E_a = [a]$ for any $a \in X$, QED.

$2°$ **The family** $\mathcal{E} := \{[a] \mid a \in X\}$ **("of clots") is finite.** Proof: $M := \{\cup Q \mid Q \subset \mathcal{E}\}$ (this is the family of the unions of clots; in $3°$ we shall show that this equals $\mathfrak{M}$).[11] Since $\mathcal{E} \subset \mathfrak{M}$, by $1°$, the set $\mathcal{E}$ is at most countable, hence $M \subset \mathfrak{M}$, by [1.3(iii)], hence $\mathfrak{M} \succeq M$.

But $Q, Q' \in 2^{\mathcal{E}}$ & $Q \neq Q' \implies \cup Q \neq \cup Q'$ (because, e.g., $[a] \in Q \setminus Q' \implies a \in \cup Q \setminus \cup Q'$). Therefore, $Q \mapsto \cup Q$ is a bijection (1-1 and onto) $2^{\mathcal{E}} \to M$, hence $M \approx 2^{\mathcal{E}}$. Consequently, $\mathcal{E}$ is finite, because otherwise we would have $\mathfrak{M} \succeq M \approx 2^{\mathcal{E}} \succeq 2^{\mathbb{N}} \succ \mathbb{N}$, i.e., $\mathfrak{M}$ would be uncountable, contrary to our assumption.

$3°$ $\mathfrak{M} = M \approx 2^{\mathcal{E}}$**, hence $\mathfrak{M}$ is finite.** Proof: If $A \in \mathfrak{M}$ and $a \in A$, then $[a] \subset A$ (by the definition of $[a]$), hence $A = \cup_{a \in A}[a] \in M$. Thus, $\mathfrak{M} \subset M$. but $M \subset \mathfrak{M}$, by $2°$, hence $\mathfrak{M} = M$. By $3°$, $M \approx 2^{\mathcal{E}}$ and $\mathcal{E}$ is finite, hence so are $2^{\mathcal{E}}$ and $M = \mathfrak{M}$, QED.[12]

Exercise 1.4 ([1.24] *short, possibly hard to find*):
Fix $\alpha > 0$. Assume that $f : \mathbb{R} \to \mathbb{C}$ is integrable over $\mathbb{R}$ w.r.t.[13] the Lebesgue measure. Show that for a.e. $x \in \mathbb{R}$, we have

$$\lim_{n \to +\infty} n^{-\alpha} f(nx) = 0.$$

Remarks: Recall that f being integrable means that f is measurable and $\beta := \int_{\mathbb{R}} |f(x)|\,dx < \infty$.

Hints: 1. We have $\int_{\mathbb{R}} f(\gamma x)\,dx = \int_{\mathbb{R}} f(y)\gamma^{-1}\,dy$ for any $\gamma \in \mathbb{C} \setminus \{0\}$, as for the Riemann integral (you need not prove this, although it would be easy by scaling the functions s_n correspondingly; an alternative proof is given after formula (15) on page 156 of [R2]).

2. Compute $\sum_{n=1}^{\infty} \int_{\mathbb{R}} |n^{-\alpha} f(nx)|\,dx$.

Solution: Change the variable to obtain

$$\beta_n := \int_{\mathbb{R}} |n^{-\alpha} f(nx)|\,dx = n^{-\alpha} \int_{\mathbb{R}} |f(nx)|\,dx \tag{4}$$

$$= n^{-\alpha} \int_{\mathbb{R}} |f(y)| n^{-1}\,dy = n^{-\alpha-1}\beta, \tag{5}$$

hence $\gamma := \sum_{n=1}^{\infty} \beta_n = \beta \sum_{n=1}^{\infty} n^{-\alpha-1} < \infty$ (since $\alpha + 1 > 1$; see [R1]). But, by [R2:1.27], we have

$$\gamma = \sum_{n=1}^{\infty} \int_{\mathbb{R}} |n^{-\alpha} f(nx)|\,dx = \int_{\mathbb{R}} \sum_{n=1}^{\infty} |n^{-\alpha} f(nx)|\,dx. \tag{6}$$

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parts of inventions and mnemonics for the most important part of a mathematicians work and the rest is just mechanical verification of the intuitive ideas (once you get enough routine). But do not forget the rigorous part, your intuitions are often false.

[8] Recall that this means that (i) $a \sim a$, (ii) $a \sim b \implies b \sim a$, (iii) $a \sim b$ & $b \sim c \implies a \sim c$. We proof (iii) and leave the rest for the reader: Let $a \sim b$ & $b \sim c$. Let $A \in \mathfrak{M}$ be arbitrary. If $a \in A$, then $b \in A$, hence then $c \in A$. If $a \notin A$, then $b \notin A$, hence then $c \notin A$. Thus, $a \in A \iff c \in A$. Because $A \in \mathfrak{M}$ was arbitrary, we have $a \sim c$, Q.E.D.

[9] Recall that $[a] := \{b \mid b \sim a\}$ refers to the equivalence class ("clot") of a.

[10] Proof: we have $[a] \subset E_a$, because $a \sim b \implies b \in E_a$. To complete the proof we must show that $E_a \subset [a]$. Let $b \in E_a$. Then $a \in A \implies b \in A$, and $b \in A \implies b \notin A^c \implies a \notin A^c \implies a \in A$, hence $a \sim b$.

[11] Example: if $X = \mathbb{R}$ and $\mathcal{E} = \{(-\infty, 0), [0, 1), [2, +\infty)\}$, then M has $2^3 = 8$ elements.

[12] An alternative solution to this exercise would be to assume the existence of a bijection $f : \mathbb{N} \to \mathfrak{M}$, and use reasoning similar to the diagonal argument to derive a contradiction.

[13] With respect to.

Since $\gamma < \infty$, it obviously follows that $\sum_{n=1}^{\infty} |n^{-\alpha} f(nx)| < \infty$ for a.e. $x \in \mathbb{R}$.[14] For these x, we must have $|n^{-\alpha} f(nx)| \to 0$, as $n \to +\infty$.

Exercise 1.5 ([1.17] *easy*):
Let $f, g : X \to [0, \infty]$ be measurable. Show that the product fg is measurable.

Solution: We could follow the proof in [1.9c] (for complex-valued functions), but it is more instructive to prove this directly. Choose simple measurable functions $\{f_n\}$ and $\{g_n\}$ s.t. $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$, as $n \to +\infty$, for all $x \in X$ (this is possible, by [R2:1.17]). Then $h_n := f_n g_n$ is a simple measurable function,[15]hence fg is measurable, being the limit of h_n.[16]

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[14]Proof: If $g : X \to [0, +\infty]$ is measurable and $\mu(A) > 0$, where $A := g^{-1}(\{+\infty\})$ (this set is measurable since $\{+\infty\}$ is a Borel set), then, obviously, $\int_X g\, d\mu \geq \int_A g\, d\mu = +\infty$. Therefore, $\int_X g\, d\mu < \infty$ implies that $\mu(A) = 0$, i.e., that $g(x) = 0$ a.e.

[15] *The product of simple measurable functions is simple and measurable.* Proof: Given simple measurable functions $G := \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ and $H := \sum_{j=1}^{m} \beta_j \chi_{F_j}$, we have

$$GH = \sum_{k=1}^{n} \sum_{j=1}^{m} \alpha_k \beta_j \chi_{E_k} \chi_{F_j} = \sum_{k=1}^{n} \sum_{j=1}^{m} (\alpha_k \beta_j) \chi_{E_k \cap F_j},$$

and the sets $E_k \cap F_j$ are measurable and the index set $\{1, \ldots, n\} \times \{1, \ldots, m\}$ is finite.

[16]By [1.14(a)] pointwise limits of measurable functions are measurable, and $(fg)(x) := f(x)g(x) = \lim_{n\to\infty} h_n(x)$ for all x.