Mat-1.150 Real Analysis: 11th exercise, Wed 2002-11-27, U322 (3 pages)

`http://www.math.hut.fi/teaching/realanal/`

2nd mid-term examination 4.12. (4th Dec) at 14–17. in hall D. From [R2]: 6.1-6.13, 6.15-6.18, 7.1-7.11, 7.16-7.20. Moreover, 6.19 without its proof. Understand Fubini's Theorem (not proof). From [ABHN]: Sections 1.1 & 1.2 (except Corollary 1.1.3 and the proof of Th. 1.1.1).

Final examination 19.12. (19th Dec) at 9–13 in hall L (you need not take this if you take the two mid-term examinations; registration at webtopi.hut.fi recommended but not obligatory).

[ABHN] The second course book "Wolfgang Arendt, CJK Batty, M Hieber, F Neubrander: "Vector-valued Laplace Transforms and Cauchy Problems", 2001 (cf. Exercises 10.4 and 11.1–11.4). Ask kalle.mikkola@iki.fi if you do not have access to [ABHN].

Notes: *measurable* in [ABHN] is usually called *Bochner measurable* or *strongly measurable* (the latter is ambiguous when the values are in $\mathcal{L}(X,Y)$). Almost all that is true for Lebesgue integration, is also true for Bochner integration, with the same proofs (use $\|\cdot\|$ in place of $|\cdot| = \|\cdot\|_{\mathbb{C}}$). The main exception is that in non-Radon–Nikodym spaces AC functions need not have a derivative (a.e.).[1] When (during your later studies) you need such results, see e.g., Appendix B of [Mikkola2002] for details:
`http://www.math.hut.fi/~kmikkola/research/thesis/contents/contents.html`

Exercise 11.1 ([ABHN:1.1.6/1.1.10] *not tough nor long*):
Prove Proposition 1.1.6, i.e., $f \in L^1(I;X)$ & $T \in \mathcal{L}(X,Y) \implies Tf \in L^1(I;Y)$ & $\int_I (Tf)(t)\,dt = T\int_I f(t)\,dt$.

Exercise 11.2 ([ABHN:1.1.6/1.1.10] *easy and quick*):
In the proof of Proposition 1.1.7 you find the statement "Moreover, $\int_I g(t)\,dt \in G(A)$". Prove this statement.

[1] From this it follows that $L^p \neq (L^q)^*$ is possible etc. Note from Corollary 1.2.7 that reflexive spaces (hence all Hilbert spaces) do have the Radon–Nikodym property.

Exercise 11.3 ([ABHN:1.1.4/1.1.10] *average*):
Show that if $T : \mathbb{R}_+ \to \mathcal{L}(X,Y)$ is strongly continuous, $R := \int_0^\infty \|T(t)\|\,dt < \infty$ and $f \in BUC(\mathbb{R}_+;X)$, then[2] $T*f \in BUC(\mathbb{R}_+;Y)$. Here $(T*f)(t) := \int_0^t T(t-s)f(s)\,ds = \int_0^t T(s)f(t-s)\,ds$.

Exercise 11.4 ([ABHN:p. 14] *average*):
Let $f \in L^p(I;X)$ for some $p \in [1,\infty)$. Define

$$f_t(s) := \begin{cases} f(s-t), & \text{if } s-t \in I; \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Show that $t \mapsto f_t$ is continuous from $\mathbb{R}^n$ to $L^p(I;X)$.

Solution to 11.1: 1° $Tf \in L^1(I;Y)$.[3] <u>Proof:</u> Since f is Bochner integrable, there are simple measurable functions (SMFs) g_n ($n \in \mathbb{N}$) s.t. $g_n \to f$ pointwise a.e. and in L^1, as $n \to \infty$ (by definition, see p. 9 of [ABHN]). But then $Tg_n := T \circ g_n$ are SMFs,[4]

$$\|Tg_n - Tf\|_1 := \int_I \|Tg_n - Tf\|_Y\,dm \le \int_I \|T\|_{\mathcal{L}(X,Y)}\|g_n - f\|_X\,dm = \|T\|_{\mathcal{L}}\|g_n - f\|_1 \to 0, \tag{2}$$

and $\|Tg_n(t) - Tf(t)\|_Y \le \|T\|_{\mathcal{L}}\|g_n(t) - f(t)\|_X \to 0$, as $n \to \infty$, hence

[2] $BUC := \mathcal{C}_{\mathrm{bu}}$:="Bounded and uniformly continuous".

[3] In an alternative proof we would obtain the measurability of $T \circ f$ from Pettis' Theorem 1.1.1, whenever T is continuous and f measurable (1. $\Lambda \circ \mathbf{T} \circ f = (\Lambda \circ T) \circ f$ is measurable for all $\Lambda \in Y^*$, by [R2:1.7b], because $\Lambda \circ T$ is continuous; 2. $T(E)$ is separable whenever E is separable and T continuous (if D is dense in E, then $T(D)$ is dense in $T(E)$)).

[4] Note that the integrals in (2) are ordinary Lebesgue integrals. Indeed, $L^p(I;X)$ spaces have nothing to do with the Bochner integral (except that the Bochner integral is a bounded linear functional $L^1(I;X) \to X$ of norm 1 (unless $I = \emptyset$ or $X = \{0\}$)).

$Tf \in L^1(I;Y)$.

2° $\int_I Tf\,dm = T\int_I f\,dm$. Proof: By 1° and definition (p. 9), we have $\int_I Tf := \lim_n \int_I Tg_n \overset{\dagger}{=} \lim_n T\int_I g_n \overset{\ddagger}{=} T\lim_n \int_I g_n =: T\int_I f$, QED.[5]

Solution to 11.2: Since $g \in L^1(I;G(A))$, we have[6] $\int_I g\,dm \in G(A)$ (recall that $G(A)$ is a closed subspace of $X \times X$, by p. 461 (or our exercise 6.3), hence a Banach space).[7]

Solution to 11.3: 1° *If $S : I \to \mathcal{L}(X,Y)$ is s.t. Sx is measurable for all $x \in X$, and $f \in L^1(I;X)$, then $Sf : I \to Y$ is measurable.*[8] Proof:

1.1° If $f = x\chi_E$, $E \in \mathfrak{M}$, $x \in X$, then $Sf = \chi_E Sx$ is measurable by p. 6.

1.2° By linearity, Sf is measurable for any simple measurable function (SMF) f.

1.3° General measurable f: Let SMF $\ni f_n \to f$ a.e. Then $S(s)f_n(s) \to S(s)f(s)$ for a.e. $s \in (0,t)$, QED.

2° *$(T*f)(t)$ exists and $\|(T*f)(t)\| \le R\|f\|_\infty$ for any $t \ge 0$.* Proof: Obviously, $(T*f)(0) = 0$. Fix $t > 0$. We must first show that $T(t-\cdot)f(\cdot) \in L^1([0,t];Y)$, so that $(T*f)(t)$ exists.

Obviously, $T(t-\cdot)x$ is measurable for all $x \in H$,[9] hence also the function $s \mapsto T(t-s)f(s)$ is measurable, by 1°. Set $M := \|f\|_\infty$ (we assume that $M > 0$; case $f = 0$ a.e. is trivial). Then, $\int_0^t \|T(t-s)f(s)\|_Y \le \int_0^t \|T(t-s)\|_{\mathcal{L}(X,Y)}\|f(s)\|_X \le \int_0^t \|T(t-s)\|_{\mathcal{L}(X,Y)}M \le RM < \infty$, hence $T(t-\cdot)f(\cdot) \in L^1([0,t];Y)$ and $\|(T*f)(t)\|_Y \le RM$, by Theorem 1.1.4.

3° $T*f \in BUC(\mathbb{R}_+;Y)$. Proof: By 2°, $T*f$ is bounded, hence it suffices to find, for arbitrary $\epsilon > 0$, a number $\delta > 0$ s.t. for any $t, t+h \in \mathbb{R}_+$ with $0 < h < \delta$ we have $\|(T*f)(t+h) - (T*f)(h)\|_Y < \epsilon$.

By [R2:6.11] (or 1° of our exercise 10.4), there is $\delta' > 0$ s.t. $\int_E \|T\|_{\mathcal{L}} < \epsilon/2M$ when $m(E) < \delta'$. Choose $\delta'' > 0$ s.t. $\|f(s+h) - f(s)\| < \epsilon/3R$ when $0 \le s \le s+h$ & $h < \delta''$. Then $\delta := \min\{\delta', \delta''\}$ is as required:

$$\|(T*f)(t+h) - (T*f)(t)\|_Y \tag{3}$$

$$\le \Big\| \int_0^{t+h} T(s)f(t+h-s)\,ds - \int_0^t T(s)f(t-s)\,ds \Big\|_Y = \Big\| \int_0^t + \int_t^{t+h} - \int_0^t \Big\|_Y$$

$$\le \Big\| \int_t^{t+h} T(s)f(t+h-s)\,ds \Big\|_Y + \Big\| \int_0^t T(s)[f(t+h-s) - f(t-s)]\,ds \Big\|_Y \tag{4}$$

$$\le \int_t^{t+h} \|T(s)\|_{\mathcal{L}(X,Y)}M\,ds + \int_0^t \|T(s)\|_{\mathcal{L}(X,Y)}\epsilon/3R\,ds \le \epsilon/2 + \epsilon/3 < \epsilon. \tag{5}$$

Solution to 11.4: (See our exercise 5.2 for a different proof (which should be somewhat modified to this problem), using the Dominated Convergence theorem.)

Assume that $I \subset \mathbb{R}^n$ is open.[10] We shall show that $t \mapsto f_t$ is actually *uniformly continuous* $\mathbb{R}^n \to L^p(I;X)$. Let $\epsilon > 0$ be arbitrary.

1° *Case $f \in \mathcal{C}_c(\mathbb{R}^n;X) \subset L^p(\mathbb{R}^n;X)$ and $I = \mathbb{R}^n$.* Choose $R > 0$ s.t. $\operatorname{supp} f \subset B_R$. Set $K := \overline{B_{R+1}}$. Being continuous on the compact set K, f is uniformly continuous (by [R1:4.19]), hence there is[11] $\delta \in (0,1)$ s.t.

$$s, h \in \mathbb{R}^n,\ |h| < \delta \implies \|f(s+h) - f(s)\|_X^p < \epsilon^p/m(K). \tag{6}$$

[5] †: This is obvious from the definition of the Bochner integral of a SMF (p. 9). ‡: $F(\lim_n t_n) = \lim_n F(t_n)$ when F is continuous and $\lim_n t_n$ exists (in fact, F is continuous at a iff $\lim_{t\to a} F(t) = F(a)$ [R1:Th4.6]).

[6] It is obvious that if $F : I \to X$ has its values in a closed subspace Z of X, then $h : I \to Z$ defined by $h(t) := F(t)\ \forall t \in I$ is measurable iff F is and $h \in L^1(I;Z) \iff F \in L^1(I;Z)$. Moreover, $\int_E h\,dm = \int_E F\,dm$, as one observes from the definition.

[7] An alternative proof would show that $f(E) \subset Z \implies \int_E f \in Z$ for any closed subspace (obvious for SMFs, use limits of SMFs for other L^1 functions).

[8] Actually, any measurable $f : I \to X$ would do, but then we should add the (limit) case of countably-valued measurable functions before 1.3°. See Lemma F.1.3(a) of [Mikkola2002]. Similarly, TS is strongly measurable whenever T and S are.

[9] **Proof:** If $g_n \to T(t-\cdot)x$, then $g_n(t-\cdot) \to T(t-\cdot)x$ and $\{g_n(t-\cdot)\}$ are also countably-valued and measurable (if $\{g_n\}$ are).

[10] Note that $L^p(I^o;X) = L^p(\bar{I};X)$ when $m(\partial I) = 0$ and that $m(\partial I) = 0$ for I's appearing in [ABHN], so that this does not reduce generality.

[11] Originally for $s, s+h \in K$, but if $s \in K^c$ or $s+h \in K^c$, then $s, h \in B_R^c$.

It obviously follows that for all $t, h \in \mathbb{R}^n$ with $|h| < \delta$ we have $\|f_{t+h} - f_t\|_p^p := \int_{\mathbb{R}^n} \|f_{t+h} - f_t\|_X^p \, dm = \int_{\mathbb{R}^n} \|f(s-t-h) - f(s-t)\|^p \, ds = \int_{(t+K)^c} 0 + \int_{t+K} \|f(s-t-h) - f(s-t)\|^p \, ds \le m(t+K)\epsilon^p / m(K) = \epsilon^p$, QED.

2° *General case* $f \in L^p(I; X)$*:* By density (p. 14 of [ABHN]), there is $g \in \mathcal{C}_c(I; X))$ s.t. $\|f - g\|_p < \epsilon/3$. By 1°, there is $\delta > 0$ s.t. $\|g(\cdot - t - h) - g(\cdot - t)\|_{L^p(\mathbb{R}^n; X)} < \epsilon/4$ whenever $t, h \in \mathbb{R}^n$, $|h| < \delta$. Whenever $t, h \in \mathbb{R}^n$, $|h| < \delta$, we have[12]

$$\|f_{t+h} - f_t\|_p \le \|f_{t+h} - g_{t+h}\|_p + \|g_{t+h} - g_t\|_p + \|g_t - f_t\|_p < \epsilon/3 + \epsilon/4 + \epsilon/3 < 3\epsilon/3 = \epsilon.$$

[12] Obviously, $\|g_{t+h} - g_t\|_{L^p(I;X)}^p := \int_I \|g_{t+h} - g_t\|^p \le \int_{\mathbb{R}^n} \|g(s-t-h) - g(s-t)\|^p \, ds < \epsilon/4$; similar reasoning applies to the other two terms.