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Kenrick Bingham

Mathematics of Local X-Ray Tomography

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Työn ohjaaja professori Erkki Somersalo

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Työssä käydään läpi lokaalin röntgentomografian matemaattista teoriaa. Tämä teoria osoittaa, milloin ja miten n-ulotteinen kappale ($n \geq 2$) voidaan rekonstruoida käyttämällä mittaustuloksia siitä, kuinka paljon röntgensäteet vaimenevat kappaleen läpi kulkiessaan. Matemaattisesti kyse on kompaktikantajaisen funktion $f : \mathbb{R}^n \to \mathbb{R}$ määrittämisestä sen integraaleista eri suoria pitkin.

Itse funktion f rekontruktio on numeerisesti epästabiili toimitus, mutta työssä johdetaan lähteen [SK] mukaisesti kaavat, joiden avulla $e*f$ ja $\Lambda e*f$ voidaan rekonstruoida stabiilisti. Tässä $e*f$ ja $\Lambda e*f$ ovat eräiden oletusten vallitessa funktioiden f ja $\Lambda f = \mathcal{F}^{-1}\big(|\xi|\,\hat{f}(\xi)\big)$ approksimaatioita, joissa pienimmät yksityiskohdat ovat sumentuneet. Rekonstruktiokaavoja johdettaessa tarvittavaa Calderón-Zygmund-teoriaa käydään myös läpi, pitkälti teoksen [Ner] esitystapaa seuraten.

Funktio Λf antaa käyttökelpoista tietoa kappaleen sisäisestä rakenteesta, sillä Calderónin pseudodifferentiaalioperaattori Λ säilyttää epäjatkuvuuskohtien sijainnin. Tämä osoitetaan lähteestä [RK] löytyvän hahmotelman mukaisesti näyttämällä, että funktioiden f ja Λf aaltorintamajoukot ovat samat. Todistuksessa ei tarvita pseudodifferentiaalioperaattoreiden teoriaa.

Eräs syy funktion Λf tarkastelemiseen on, että sen likimääräinen rekonstruktio onnistuu paikallisesti: jos kiinnostuksen kohteena on vain osa kappaleesta, rekonstruointia varten tarvitaan mittaukset ainoastaan niitä suoria pitkin, jotka kulkevat kiinnostavan alueen läpi tai aivan sen läheltä. Funktioita f ja $e*f$ ei voida rekonstruoida paikallisesti.

Kaksi stabiilisuustulosta, joita ei ole suoraan esitetty kirjallisuuslähteissä, todistetaan myös. Niiden mukaan mittausvirheen L^2-normi rajoittaa tasaisesti funktioiden $e*f$ ja $\Lambda e*f$ rekonstruktioiden virheitä.

Lukijalta edellytetään reaalianalyysin, distribuutioteorian ja Fourier-analyysin perustietoja; näiden keskeisimpiä kohtia luetellaan liitteessä. Työssä esitetään todistukset kaikille tuloksille, joita käytetään teorian johtamisessa, lukuunottamatta edellä mainittuja esitietoja sekä kahta Rieszin muunnosten jatkuvuutta koskevaa lausetta, joiden osalta viitataan lähteeseen [Zie]. Näistä lauseista seuraa myös se, että lähteessä [SK] esiintyvä funktioavaruus D_{xr} onkin itse asiassa vain neliöintegroituvien funktioiden avaruus. Tästä seikasta ei löydy mainintoja aikaisemmassa kirjallisuudessa.

Avainsanat: tomografia, tietokonetomografia, CT, lokaali, inversio-ongelmat, singulaariset integraalit, Calderón-Zygmund-teoria

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This thesis provides a self-contained account of the basic mathematical theory of local x-ray tomography. The theory shows when and how an n-dimensional object ($n \geq 2$) can be reconstructed, using attenuation measurements of x-rays passing through the object. Mathematically speaking, the problem is to determine a compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$ from its line integrals.

The reconstruction of f itself is a numerically instable operation, but formulae are derived for the stable reconstruction of $e*f$ and $\Lambda e*f$, following [SK]. Here $e*f$ and $\Lambda e*f$ are, under certain assumptions, blurred approximations of f and $\Lambda f = \mathcal{F}^{-1}\big(|\xi|\,\hat{f}(\xi)\big)$, respectively. The Calderón-Zygmund theory needed for deriving the reconstruction formulae is also presented, largely according to [Ner].

The function Λf provides meaningful information about the internal structure of the object, since the Calderón pseudodifferential operator Λ preserves the locations of discontinuities. This is shown by proving that the wave front sets of f and Λf are the same. The elementary proof of this fact, which does not use the theory of pseudodifferential operators, is presented as outlined in [RK].

One reason for considering Λf is that its approximate reconstruction can be done locally. This is to say that if the region of interest is only part of the object examined, measurements are needed only along lines through the region of interest, or very close to it. The functions f and $e * f$ cannot be reconstructed locally.

Two stability results not directly given in the references are proved. They state that the errors in the reconstructed $e * f$ and $\Lambda e * f$ are uniformly bounded by the L^2 norm of the error in the measurements.

The presentation is self-contained in the sense that only some basic knowledge of real analysis, distribution theory and Fourier analysis is required; some of the most central results of this background theory are listed in an appendix for reference. Apart from these prerequisites, proofs of all results used are given, except for two theorems concerning the continuity of the Riesz transform, for whom the reader is referred to [Zie]. These theorems also imply that the function space D_{xr}, considered in [SK], is just the space of square-integrable functions. Earlier statements of this fact were not found in literature.

Keywords: tomography, computed tomography, CT, local, inverse problems, singular integrals, Calderón-Zygmund theory

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Preface

The process of writing this thesis has been rewarding and enriching. I wish to express my gratitude to Professor Erkki Somersalo for his expert guidance, and to Professor Olavi Nevanlinna for providing me with the time needed and an interesting and stimulating environment.

Numerous colleagues have made this task a pleasant and instructive one. Lic.Tech. Samuli Siltanen deserves particular recognition for reviewing my manuscript and always being ready to engage in helpful discussions. I am also much obliged to Ph.D. Juha Kinnunen for his kind advice regarding the derivatives of the Riesz potential.

Last but not least, I thank my beloved wife Ella for being an excellent companion on the way towards our Master's degrees.

Espoo, 18^{th} August, 1998

Kenrick Bingham

Contents

Chapter 1

Introduction

The need to find out the internal structure of an object without opening it physically arises in many situations. One such situation is medical imaging, where the objective is to look inside the patient's body without performing surgery.

A way to satisfy the need for noninvasive imaging was found in 1895 when x-rays were discovered by Wilhelm Röntgen. They made it possible to see through the softer parts of different objects and thus to produce a silhouette of, *e.g.*, the skeleton of a living person. The full three-dimensional structure remained, however, hidden from the outside observer for nearly a century.

The invention of *computed tomography* (CT)[1] in the beginning of the 1970's finally made it possible to reconstruct the three-dimensional structure of the object with a computer, using digitized x-ray pictures taken from different directions. The intensity resolution with which the x-ray attenuation could be analysed also increased substantially. This made it possible to characterize different types of materials, or tissues, by the x-ray attenuation coefficient, which is also called the Hounsfield number in honour of Sir Godfrey Hounsfield who developed the first CT device in 1972.

Some of the mathematical foundations needed for the reconstruction have been known for a relatively long time. The reconstruction formula for a two-dimensional slice was derived by Johann Radon as early as 1917 [Rad], but it was only with the development of computers that this reconstruction could be realized in practice.

Computed tomography presents an *inverse problem*. If the three-dimensional structure of the object is known, calculating the two-dimensional projection in a particular direction is a straightforward computation: one just needs to add up the x-ray attenuation coefficients at all points along the straight line along which the x-ray travels, to find out how much the ray will be attenuated when it reaches the detector, photographic film or fluorescent plate. In the other direction, the task of reconstructing the three-dimensional structure from the x-ray pictures, is a more complicated problem.

It is often the case that the region of interest is only a small part of the object examined, for instance a particular organ of a living person. In this case, it would be convenient to be able to do the reconstruction *locally*, that is, only using x-rays

[1]The names *computerized tomography*, *computer aided tomography* (CAT), *computerized axial tomography* (CAT) and *CAT-scan* are also used.

passing through the region of interest or very close to it. This would decrease both the x-ray dose and the amount of data that must be processed (see Figure 1.1). It turns out that the x-ray attenuation coefficient f itself cannot be reconstructed locally, but another function $\Lambda f = \mathcal{F}^{-1}\big(|\xi|\,\hat{f}(\xi)\big)$ can, and that for many practical purposes, Λf yields significant information about the internal structure of the object.

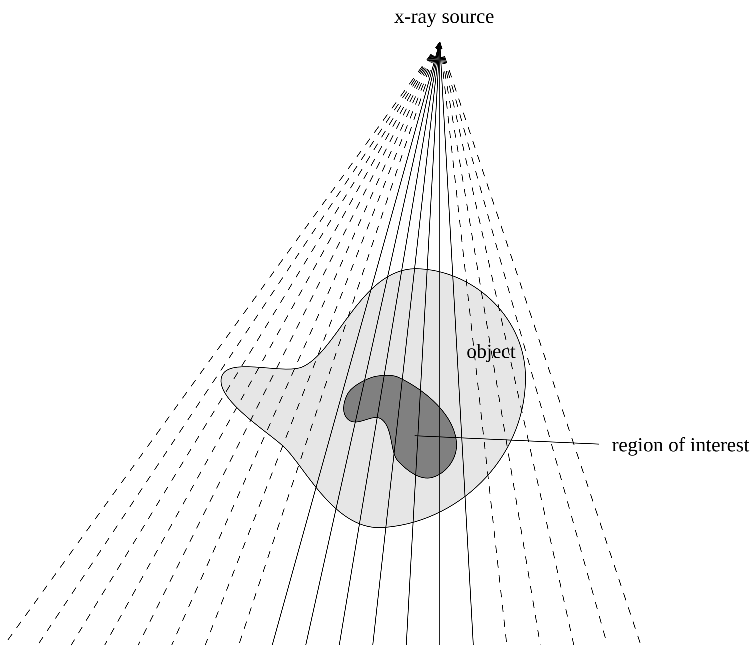


Figure 1.1: Local tomography only uses data from x-rays passing through the region of interest or very close to it.

This thesis attempts to give a self-contained account of the basic mathematical theory showing when and how the reconstruction can be done. The reader will be assumed to know some real analysis, distribution theory and Fourier analysis, some of the most central results of which will be revised in Appendix A. Apart from these basic results, two theorems concerning the continuity of the Riesz potential operators will also be stated without proof in Section A.2 of Appendix A. The reason for omitting the proofs is that they require a comparatively lengthy treatment which does not directly support the rest of the work. The proofs can be found in [Zie].

The text is organised as follows. Chapter 2 introduces the notation that will be used. Chapter 3 derives the main results of this work in a descriptive and non-rigorous way. Chapter 4 calculates the Fourier transforms of the singular Riesz kernel, presents some Calderón-Zygmund theory according to [Ner] and applies it to calculate the Fourier transform of the Riesz transform, which is a singular convolution operator.

These Fourier transforms are used in Chapter 5 for deriving the reconstruction formulae, following the treatment in [SK]. As the exact reconstruction of the x-ray attenuation coefficient f turns out to be an instable operation, an approximate reconstruction formula for $e * f$, where e is a blurring kernel, is derived too, as well as a local reconstruction formula for $\Lambda e * f$, which can be seen as an approximation of

Λf. Stability results for these approximate reconstructions are derived at the end of the chapter from the reconstruction formulae given in [SK]. It is also noted that the space D_{xr}, appearing in the assumptions of many of the theorems, is actually just the space of square-integrable functions, as follows from the two theorems in [Zie] mentioned above.

Chapter 6 shows that f and Λf have the same discontinuities, following the treatment in [RK]. Chapter 7 lists some further results describing the properties of Λf, presents a few extensions to the theory and proposes possible subjects for further study.

Computed tomography of course involves many other, both theoretical and practical issues, which we shall not go into but rather leave as possible subjects for further study. Such issues relate, for instance, to the numerical implementation of the reconstruction, using some algorithm like the algebraic reconstruction technique (ART) or the filtered backprojection algorithm, and statistical questions relating to the measurement process. The choice of the blurring kernel e involves important numerical considerations, too. Convolution with e can usually be seen as a low-pass filtering process on the image, and other digital image processing operations may be expedient for making the reconstructed image as informative as possible. [CJS, Nat]

We shall also not consider other types of tomography, like magnetic resonance imaging (MRI, NMR), electric impedance tomography (EIT), ultrasound imaging, magnetoencephalography (MEG) or nuclear emission tomography, all of which are lively research fields. On the other hand our treatment covers imaging using any beams of rays or particles that travel along a straight line and whose attenuation is measured behind the object, because the mathematical model describing them is identical to that of x-ray imaging. Such beams include gamma rays, which are in wide use in medical imaging, and electron beams.

Electron beams are used in electron microscopy, where the object examined is normally a planar specimen through which the beam must pass more or less transversally. Therefore the attenuation of the beam can be measured only in a limited range of angles, which presents an incomplete data problem. Other types of incomplete data problems are encountered when an opaque implant prevents measurements through an area within the region of interest, or when only part of the object can be x-rayed, for example if the object is too big. Questions concerning the well-posedness of incomplete data problems and the uniqueness of their solutions will also be left as topics for further study.

The "colour spectrum" — the dependence of the x-ray attenuation coefficient on the energy (wavelength) of x-rays used — will also be neglected, as well as all physical, technical and medical issues.

Chapter 2

Notational Conventions

Sets of Numbers and Euclidean Spaces

We shall use the notation

$$\begin{aligned} \mathbb{N} &= \{0, 1, 2, \dots\} \\ \mathbb{Z} &= \{\dots, -2, -1, 0, 1, 2, \dots\} \\ \mathbb{Z}_+ &= \{1, 2, 3, \dots\} \end{aligned}$$

for the sets of natural numbers, integers and positive integers, respectively. The real field will be denoted by $\mathbb{R}$ and the complex field by $\mathbb{C} = \{x + iy \,|\, x, y \in \mathbb{R}\}$.

The n-dimensional Euclidean space will be denoted by

$$\mathbb{R}^n = \Big\{ x = (x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} x_j e_j \;\Big|\; x_j \in \mathbb{R} \Big\}. \tag{2.1}$$

The natural basis is $\{e_j\}_{j=1}^n$, where $e_j = (0, 0, \dots, 0, 1, 0, \dots, 0)$ with the 1 at the j^{th} place. Analogously, $\mathbb{N}^n$ will be the set of n-tuples of natural numbers,

$$\mathbb{N}^n = \Big\{ \alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) = \sum_{j=1}^{n} \alpha_j e_j \;\Big|\; \alpha_j \in \mathbb{N} \Big\}. \tag{2.2}$$

Inner products in $\mathbb{R}^n$ will be denoted by

$$x \cdot y = \sum_{j=1}^{n} x_j y_j \tag{2.3}$$

and norms by

$$|x| = \sqrt{x \cdot x}. \tag{2.4}$$

The letters r, θ, φ and ω will often be used without separate mention to denote the polar and spherical coordinates introduced in Section A.3 of Appendix A.

All integrals appearing will be Lebesgue integrals and all functions will be Lebesgue measurable. If the domain of integration is omitted, it is assumed to be the entire space. The Lebesgue measure of the obvious dimension will be denoted by m; in the zero-dimensional case, we shall use the counting measure. By saying that a condition $P(x)$ holds "*for almost all* x" or "*almost everywhere*" (abbreviated "*a.e.*"), we mean that there exists a set N such that $m(N) = 0$ and that $P(x)$ holds for all $x \in \mathbb{R}^n \setminus N$.

For all $n \in \mathbb{Z}_+$, we shall denote the unit ball in $\mathbb{R}^n$ by B^n and its boundary, the unit sphere in $\mathbb{R}^n$, by S^{n-1}:

$$B^n = \{x \in \mathbb{R}^n \mid |x| < 1\}, \qquad S^{n-1} = \{x \in \mathbb{R}^n \mid |x| = 1\} = \partial B^n \tag{2.5}$$

(the notation ∂A is used for the boundary of the set A). Their n-dimensional and $n-1$-dimensional Lebesgue measures, respectively, are calculated in Lemma A.10:

$$|B^n| := m(B^n) = \frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)} \quad \text{and} \quad |S^{n-1}| := m(S^{n-1}) = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} = n\,|B^n|. \tag{2.6}$$

The unit hemisphere closest to $y \in \mathbb{R}^n \setminus \{0\}$ will be denoted by

$$S_+^{n-1}(y) = \{x \in S^{n-1} \mid x \cdot y > 0\}. \tag{2.7}$$

More generally, the open ball and sphere with radius R and centre at x_0 will be denoted, respectively, by

$$B^n(x_0, R) = \{x \in \mathbb{R}^n \mid |x - x_0| < R\} \tag{2.8}$$

and

$$S^{n-1}(x_0, R) = \{x \in \mathbb{R}^n \mid |x - x_0| = R\}. \tag{2.9}$$

The dimension n will in most cases be omitted from the notation when it is obvious. For $k \leq n$, we shall often think of $\mathbb{R}^k$ as being embedded into $\mathbb{R}^n$.

The distance between a point $x \in \mathbb{R}^n$ and a set $B \subset \mathbb{R}^n$ and the distance between two sets A, $B \subset \mathbb{R}^n$ are denoted by

$$\operatorname{dist}(x, B) = \inf_{y \in B} |x - y| \qquad \text{and} \qquad \operatorname{dist}(A, B) = \inf_{x \in A} \operatorname{dist}(x, B), \tag{2.10}$$

respectively. The characteristic function of a subset $X \in \mathbb{R}^n$ is the function $\chi_X : \mathbb{R}^n \to \{0, 1\}$ defined by

$$\chi_X(x) = \begin{cases} 1, & x \in X \\ 0, & x \notin X. \end{cases} \tag{2.11}$$

The symbol $:=$ will be used in definitions in the middle of the text, for emphasizing that we are making a definition and not asserting an equality. For example $a := b$ will mean that a is defined to be equal to b. Analogously, $a =: b$ will mean that b is defined to be equal to a.

Multi-indices

Multi-indices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}^n$ will often be used to write monomials of $x \in \mathbb{R}^n$:

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} = \prod_{j=1}^{n} x_j^{\alpha_j}. \tag{2.12}$$

The total degree of a multi-index is

$$|\alpha| = \sum_{j=1}^{n} |\alpha_j|. \tag{2.13}$$

An analogous notation will be used for classical partial derivatives:

$$\partial^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}. \tag{2.14}$$

Hence, the total degree coincides here with the order of the partial derivative. In first-order partial derivatives, ∂_{e_j} is abbreviated as ∂_j. For distribution derivatives (weak derivatives), the letter D will be used instead of ∂.

The Leibniz rule for multi-indices takes the form

$$\partial^\alpha (uv) = \sum_{\beta \leq \alpha} c_{\alpha\beta}\, \partial^\beta u\, \partial^{\alpha-\beta} v, \tag{2.15}$$

where $c_{\alpha\beta} \in \mathbb{N}$ are constants and $\beta \leq \alpha$ means the partial order $\beta_j \leq \alpha_j$ for all $j \in \{1, \ldots, n\}$.

Function Spaces and Distributions

The most important mathematical tools that will be used are the basic results of real analysis, the theory of distributions and Fourier analysis. Some of the most central ones are presented for reference in Appendix A. The theory can be found in references [Fri, HS, Hör, Rau, Ru1, Ru2].

We shall be dealing with the following function spaces, all of which have a vector space structure:

- $C^k(X)$, the space of k times continuously differentiable functions, $k \in \mathbb{N} \cup \{\infty\}$
- $C_0^k(X) = \{f \in C^k(X) \,|\, \operatorname{supp} f \text{ compact}\}$, the space of compactly supported C^k functions
- $L^p(X) = \{f : X \to \mathbb{C} |\, \|f\|_{L^p} < \infty\}$, with norm $\|f\|_{L^p} = \left(\int_X |f(x)|^p\, dx\right)^{1/p}$, $1 \leq p < \infty$ (see Section A.1.1)
- $L^1(X)$, the space of integrable functions
- $L^\infty(X) = \{f : X \to \mathbb{C} |\, \|f\|_{L^\infty} < \infty\}$, the space of essentially bounded functions, with norm $\|f\|_{L^\infty} = \inf\{M \in \mathbb{R} |\, f(x) \leq M \text{ a.e.}\}$

- $L^p_0(X) = \{f \in L^p(X) \,|\, \operatorname{supp} f \text{ compact}\}$, the space of compactly supported L^p functions
- $L^p_{\mathrm{loc}}(X) = \{f : X \to \mathbb{C} \,|\, K \subset X \text{ compact} \Rightarrow \int_K |f(x)|^p \, dx < \infty\}$, $1 \leq p < \infty$
- $L^1_{\mathrm{loc}}(X)$, the space of locally integrable functions
- $\mathcal{D}(X)$, $\mathcal{D}'(X)$, $\mathcal{S}(X)$, $\mathcal{S}'(X)$, the spaces of compactly supported test functions, distributions, rapidly decreasing functions and tempered distributions (see Section A.1.3)
- $H^s(\mathbb{R}^n) = \{f \in \mathcal{S}' \,|\, \|f\|_{H^s} < \infty\}$, with norm $\|f\|_{H^s} = \|(1 + |\xi|^2)^{s/2} \hat{f}(\xi)\|_{L^2}$, Sobolev space of order $s \in \mathbb{R}$ (see Section A.1.5).

If the set $X \subset \mathbb{R}^n$ is omitted, it is assumed to be the whole $\mathbb{R}^n$.

The inner product of $f \in L^2(X)$ and $g \in L^2(X)$ will be denoted by

$$(f, g) = \int_X f(x) \, \overline{g(x)} \, dx. \tag{2.16}$$

The notation $\langle f, \phi \rangle$ will be used for the integral $\int f(x)\phi(x) \, dx$, and more generally for the distribution duality of the distribution f and the test function ϕ. Observe that the complex conjugate of the second argument is taken in (f, g) but not in $\langle f, g \rangle$.

Of the various variants of the definition of the Fourier transform, we choose

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int f(x) \, e^{-ix\cdot\xi} \, dx \tag{2.17}$$

for functions $f \in \mathcal{S}$. For the definition for more general functions and further properties of the Fourier transform, see Section A.1.4.

Various constants whose exact values are not of interest, will be denoted by C_j, $j \in \mathbb{N}$.

For other notation, see the table of notation in Appendix B.

Chapter 3

X-Ray Imaging

3.1 Divergent Beam Radiography

Consider an object in n-dimensional space, the inside structure of which is to be investigated using x-ray imaging. (For the rest of this chapter, $n \geq 2$ can be considered fixed; in practice the value of n is usually, of course, 2 or 3.) Call $f(x)$ the x-ray attenuation coefficient at point $x \in \mathbb{R}^n$; we expect f to give information about the object since the attenuation coefficient depends on the material through which the ray passes.

Suppose that the object is contained in a ball of radius R with the centre at the origin, and that the x-ray attenuation coefficient f is zero outside the object:

$$\operatorname{supp} f \subset B(0, R). \tag{3.1}$$

If the object is x-rayed in a direction $\theta \in S^{n-1}$ from a point $a \in A := S^{n-1}(0, R)$, the attenuation of the x-ray intensity I at each point $a + t\theta$, $t \geq 0$, is

$$-dI = f(a + t\theta)\, I\, dt. \tag{3.2}$$

By solving this differential equation, we see that the intensity of the x-ray measured by a detector situated behind the object is

$$I_{\text{meas}} = I_0 \exp\left(-\int_0^\infty f(a + t\theta)\, dt \right). \tag{3.3}$$

Our aim will be to derive formulae for reconstructing $f(x)$ from the measurements I_{meas}, or equivalently from

$$\int_0^\infty f(a + t\theta)\, dt = \ln\left(\frac{I_0}{I_{\text{meas}}} \right), \tag{3.4}$$

with different combinations of $a \in A$ and $\theta \in S^{n-1}$.

Definition 3.1. *For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the function $\mathcal{D}f : A \times S^{n-1} \to \mathbb{R}$,*

$$(a, \theta) \mapsto \mathcal{D}_a f(\theta) := \int_0^\infty f(a + t\theta)\, dt, \tag{3.5}$$

is called its divergent beam radiograph.

In the case $n = 2$, which is naturally very important in practice, the divergent beam radiograph $\mathcal{D}_a f(\theta)$ coincides with the Radon transform $\mathcal{R}f(\theta, a \cdot \theta)$, which is by definition the integral of f over a hyperplane:

$$\mathcal{R}f(\theta, s) = \int_{x \cdot \theta = s} f(x)\, dx. \tag{3.6}$$

This transform is named after Johann Radon who already in 1917 derived an inversion formula for the two-dimensional Radon transform [Rad]. The theory of computed tomography is often presented using the Radon transform instead of the divergent beam radiograph, and formulae are derived for reconstructing f from $\mathcal{R}f$ [Nat, RK]. We shall, however, do the generalisation into higher dimensions using the divergent beam radiograph, sometimes also called the fan-beam transform in two dimensions and the cone-beam transform in three dimensions.

In practice, only a finite number of measurements are ever made. The attenuation factors in different directions can be measured using a spherical array of detectors on A. The x-ray source can then be turned together with the detectors, and the measurements repeated in each position. (See Figure 3.1.)

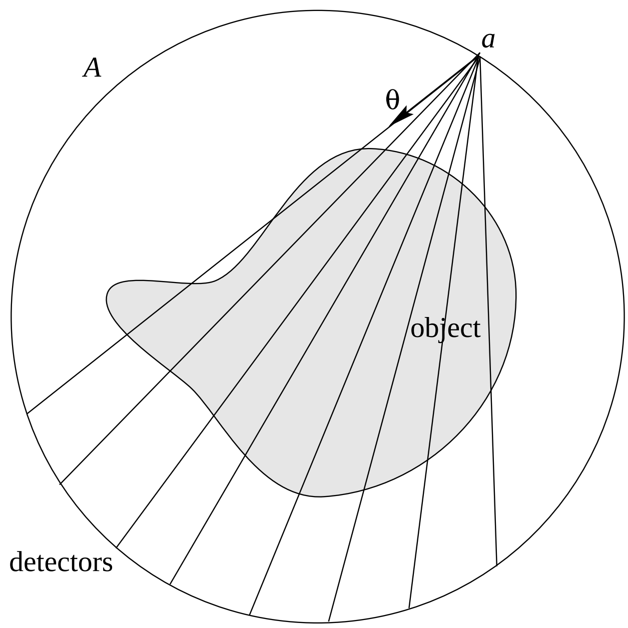


Figure 3.1: Divergent beam radiography.

The precise radiograph function $\mathcal{D}f : A \times S^{n-1} \to \mathbb{R}$ is thus approximated by an interpolation of the measurements. Continuity results will be important in order to ensure that this approximation and slight inaccuracies in the measurements do not cause large errors in the reconstructed x-ray attenuation coefficient.

The structure of a three-dimensional object can be computed by working in $\mathbb{R}^3$, where the x-ray source must be moved about the whole surface of a sphere surrounding the object. Another approach would be to "slice" the object up mathematically in one direction, and to reconstruct the cross sections of the 3-dimensional structure on each 2-dimensional slice, using measurements on a 1-dimensional sphere, which is to say a circle, surrounding the slice in $\mathbb{R}^2$. The latter technique may be simpler to implement in practice, with respect to both the mechanics of the measurement device and the amount of data that must be processed in the computations. However, both approaches have their own advantages.

In deriving the reconstruction formulae, let us first proceed formally, and give rigorous proofs of the theorems in later chapters.

3.2 Exact Reconstruction Formula

Integrating the measurements in all directions with the x-ray source at a fixed point, we get, using polar coordinates $x = t\theta$, $dx = t^{n-1}\, dt\, d\theta$,

$$\int_{S^{n-1}} \mathcal{D}_a f(\theta)\, d\theta = \int_{S^{n-1}} \int_0^\infty f(a + t\theta)\, dt\, d\theta = \int_{\mathbb{R}^n} f(a - x)\, |x|^{1-n}\, dx. \tag{3.7}$$

We shall write this in the form

$$\int_{S^{n-1}} \mathcal{D}_a f(\theta)\, d\theta = \frac{1}{b_n} R_1 * f(a), \tag{3.8}$$

using the Riesz potential:

Definition 3.2. *For $n \geq 2$, the tempered distribution $R_1 \in \mathcal{S}'(\mathbb{R}^n)$, defined by the locally integrable function*

$$R_1(x) = b_n\, |x|^{1-n}, \qquad b_n = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}} = \frac{1}{\pi\, |S^{n-2}|}, \tag{3.9}$$

is called the Riesz kernel. *The function $R_1 * f$ is called the* Riesz potential *of the measurable function f.*

Many texts, including [Nat], [RK], [Ste] and [Zie], use the notation $I_1 f$ or $I^1 f$ instead of $R_1 * f$.

Ignoring for a while that the measurements $\mathcal{D}_a f(\theta)$ are only known for $a \in A$, we perform the Fourier transform with respect to $a \in \mathbb{R}^n$ and get (the Fourier transform of R_1 will be calculated in Section 4.1)

$$\mathcal{F}\left(\int_{S^{n-1}} \mathcal{D}_a f(\theta)\, d\theta\right) = \frac{(2\pi)^{n/2}}{b_n}\, \mathcal{F}(R_1)\, \mathcal{F} f = \frac{1}{b_n}\, |\xi|^{-1} \hat{f}, \tag{3.10}$$

which leads to the reconstruction formula

$$f(x) = b_n\, \mathcal{F}^{-1}\left(|\xi|\, \mathcal{F}\left(\int_{S^{n-1}} \mathcal{D}_x f(\theta) d\theta\right)\right), \tag{3.11}$$

where x is written instead of a. This can be written more compactly as

$$f(x) = b_n\, \Lambda\left(\int_{S^{n-1}} \mathcal{D}_x f(\theta) d\theta\right), \tag{3.12}$$

where Λ is the Calderón operator:

Definition 3.3. *The operator* $\Lambda : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$, $n \in \mathbb{Z}_+$, *defined through its Fourier transform as*

$$\mathcal{F}(\Lambda g)(\xi) = |\xi|\,\widehat{g}(\xi), \tag{3.13}$$

is called the Calderón operator.

The formal calculation

$$\mathcal{F}\big(\Lambda(R_1 * g)\big)(\xi) = |\xi|\,|\xi|^{-1}\widehat{g}(\xi) = \widehat{g}(\xi) \tag{3.14}$$

above, when carried out properly in Chapter 5, yields the following result:

Theorem 3.4. (Repeated later as Theorem 5.18.) *If* $n \geq 2$, $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$ *and* $|\xi|^{-1}\hat{f}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, *then for almost all* x,

$$\Lambda(R_1 * f)(x) = f(x). \tag{3.15}$$

The x-ray domain D_{xr} will be introduced in Definition 5.1.

It will be convenient to know a more direct expression for the Calderón operator Λ, in addition to its Fourier transform. The formal calculation

$$\begin{aligned}\Lambda f &= \mathcal{F}^{-1}\Big(\sum_{j=1}^{n} -i\,\xi_j\,|\xi|^{-1}\,i\,\xi_j\,\hat{f}(\xi)\Big)\\ &= -\sum_{j=1}^{n}(2\pi)^{-n/2}\,\mathcal{F}^{-1}\big(i\,\xi_j\,|\xi|^{-1}\big) * \mathcal{F}^{-1}\big(i\,\xi_j\,\hat{f}(\xi)\big)\\ &= -\sum_{j=1}^{n}\frac{\partial R_1}{\partial x_j} * \frac{\partial f}{\partial x_j}\end{aligned} \tag{3.16}$$

will be justified for $f \in L^2$ in Chapter 4, if the convolution is interpreted as a *principal value* (p. v.) convolution, introduced in Definition 4.3. We then get the following theorem:

Theorem 3.5. (Repeated later as Theorem 4.11.) *For all* $n \in \mathbb{Z}_+$, *the Calderón operator* Λ *can be expressed as*

$$\Lambda = -\sum_{j=1}^{n}\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * D_j. \tag{3.17}$$

It is a continuous operator from H^1 *to* L^2.

The problem of only knowing $\mathcal{D}_a f(\theta)$ when $a \in A$ can now be avoided by using the elementary relationship

$$\int_{S^{n-1}}\int_{-\infty}^{\infty} g(x+t\theta)\,dt\,d\theta = 2\int_{S^{n-1}}\int_{0}^{\infty} g(x+t\theta)\,dt\,d\theta, \tag{3.18}$$

and making the substitution (see Figure 3.2)

$$\theta = \frac{a-x}{|a-x|} \in S^{n-1}, \qquad d\theta = \frac{|(x-a)\cdot a|}{R\,|x-a|^n}\,da \tag{3.19}$$

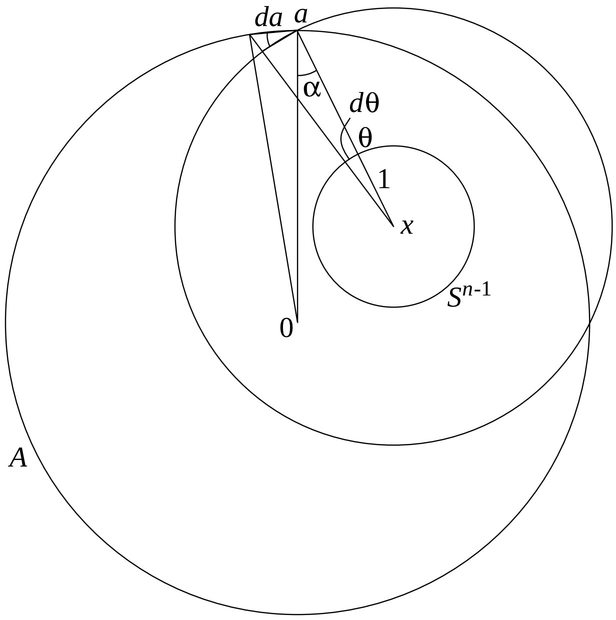


Figure 3.2: The change of variable $\theta = \frac{a-x}{|a-x|}$, $d\theta = \frac{|\cos\alpha|}{|a-x|^{n-1}}\,da = \frac{|\frac{a-x}{|a-x|}\cdot\frac{a}{R}|}{|a-x|^{n-1}}\,da = \frac{|(x-a)\cdot a|}{R\,|x-a|^n}\,da$ *used for deriving (3.20).*

into (3.12). This yields the reconstruction formula

$$\begin{aligned} f(x) &= \tfrac{b_n}{2}\,\Lambda \int_{S^{n-1}} \int_{-\infty}^{\infty} f(x+t\theta)\,dt\,d\theta \\ &= \tfrac{b_n}{2R}\,\Lambda \int_A \int_{-\infty}^{\infty} f(x+t\,\tfrac{a-x}{|a-x|})\,dt\,\tfrac{|(x-a)\cdot a|}{|x-a|^n}\,da \\ &= \tfrac{b_n}{2R}\,\Lambda \int_A \int_{-\infty}^{\infty} f(a+(-|a-x|+t)\,\tfrac{a-x}{|a-x|})\,dt\,\tfrac{|(x-a)\cdot a|}{|x-a|^n}\,da \\ &= \tfrac{b_n}{2R}\,\Lambda \int_A \Big(\mathcal{D}_a f(\tfrac{a-x}{|a-x|}) + \mathcal{D}_a f(-\tfrac{a-x}{|a-x|})\Big)\,\tfrac{|(x-a)\cdot a|}{|x-a|^n}\,da. \end{aligned} \tag{3.20}$$

However, there are problems with using this formula. Since the operator Λ involves differentiation and convolution with the strongly singular kernel $\frac{\partial R}{\partial x_j}$, the reconstruction is not stable with respect to errors in the measurements. The continuity from H^1 to L^2 allows small but rapidly changing errors in the measurements to result in large errors in the x-ray attenuation coefficient reconstructed. An approximate reconstruction can be used instead with more success.

The names of the commonly used *filtered backprojection* and *backprojection filtering* algorithms [Nat, CJS] come from the reconstruction formula (3.15): The calculation of $R_1 * f$ by integrating the measurements over the sphere, as in (3.12), is the *backprojection* operator. Multiplication on the Fourier transformed side, as by $|\xi|$ in Λ, is traditionally called *filtering* in signal processing. Some other filtering can be combined with it to alleviate the effects of noise. This gives an approximate reconstruction $e * f$, where e is a blurring kernel.

3.3 Parallel Beam Radiography

When deriving an approximate reconstruction formula, we shall use the *parallel beam radiograph*:

Definition 3.6. *For a measurable function $f : \mathbb{R}^n \to \mathbb{R}$, the function*

$$\mathcal{P}f : \{(\theta, x) \in S^{n-1} \times \mathbb{R}^n \mid x \in \theta^\perp\} \to \mathbb{R}, \tag{3.21}$$

$$(\theta, x) \mapsto \mathcal{P}_\theta f(x) := \int_{-\infty}^{\infty} f(x + t\theta)\, dt, \tag{3.22}$$

is called the parallel beam radiograph *of f.*

Here

$$\theta^\perp := \{\theta\}^\perp = \{y \in \mathbb{R}^n \mid y \cdot \theta = 0\} \tag{3.23}$$

is the *orthogonal complement* of θ.

As is easily seen, the parallel beam radiograph is related to the divergent beam radiograph through

$$\mathcal{P}_\theta f(E_\theta x) = \mathcal{D}_x f(\theta) + \mathcal{D}_x f(-\theta), \tag{3.24}$$

where

$$E_\theta(x) := x - (x \cdot \theta)\, \theta \tag{3.25}$$

designates the orthogonal projection onto $\theta^\perp$, $\theta \in S^{n-1} \subset \mathbb{R}^n$.

After using relation (3.24) and passing the operator Λ under the integral sign, (3.12) yields the parallel beam reconstruction formula

$$f(x) = \frac{b_n}{2} \int_{S^{n-1}} \Lambda \mathcal{P}_\theta f(E_\theta x)\, d\theta. \tag{3.26}$$

The parallel beam and divergent beam radiographs carry the same information because $\mathcal{D}_x f(\theta)$, $x \in A$, can be nonzero only when θ points towards the interior of the sphere A, so that always at least one of the members of (3.24) in known to vanish. Therefore $\mathcal{P}f$ can be seen just as an alternative notation that is more convenient in some cases. However, the parallel beam radiograph operator $\mathcal{P}$, sometimes also called the *x-ray transform*, was perhaps more natural in the historical setting where information about the object was collected in a different way than that described above in Section 3.1.

In parallel beam radiography, the object is exposed to a beam of parallel x-rays covering the whole object. (See Figure 3.3.) The attenuation factor is measured at one point for each ray, and the measurements are repeated for all different directions $\theta \in S^{n-1}$ of the beam.

Another possibility would be to use a single x-ray source and a single detector behind the object in direction θ, to scan over $\theta^\perp$ by moving the source and the detector

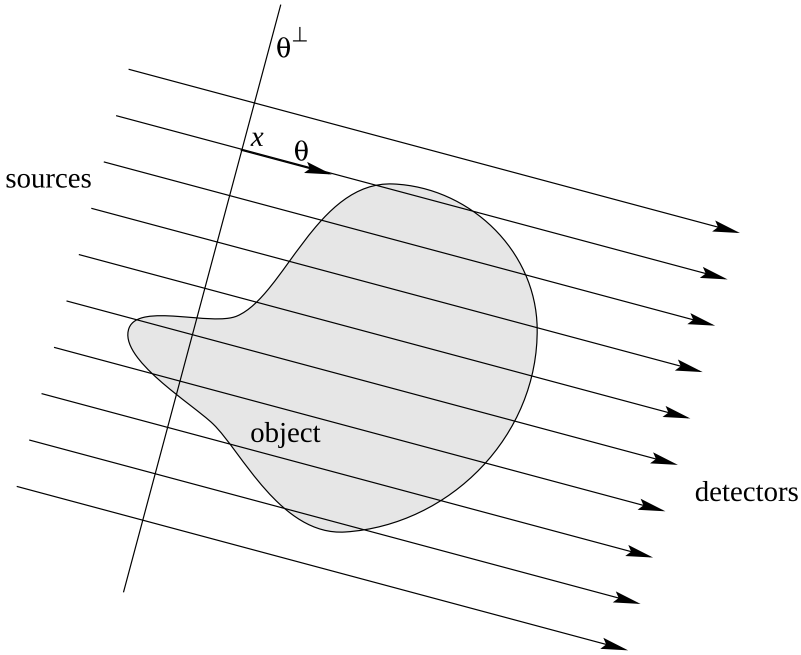


Figure 3.3: Parallel beam radiography.

together, and to repeat this for all θ. First generation CT scanners were based on this technique.

Since parallel x-rays are difficult to produce in practice, and the latter method of scanning over each hyperplane $\theta^\perp$ can be quite time-consuming, divergent beam radiography is nowadays more widely used in computed tomography.

3.4 Approximate Reconstruction Formula

The attenuation coefficient $f = \delta * f$ can be approximated by $e * f$, where e is an approximate delta function, also called a *blurring kernel* or *point spread function*, for instance the one defined in (A.48). For deriving a reconstruction formula for $e * f$, we shall use the following formal calculations:

The substitutions $z = y + t\theta \in \theta^\perp \oplus \mathbb{R}\theta = \mathbb{R}^n$, and $dz = dt\,dy$ and $\tau = t + s$, $d\tau = ds$ yield

$$\begin{aligned}(\mathcal{P}_\theta e * \mathcal{P}_\theta f)\,(x) &= \int_{\theta^\perp}\int_{-\infty}^{\infty} e(y+t\theta)\,dt \int_{-\infty}^{\infty} f(x+s\theta-y)\,ds\,dy \\ &= \int_{\mathbb{R}^n} e(z)\int_{-\infty}^{\infty} f\big(x-z+(t+s)\theta\big)\,ds\,dz \qquad (3.27)\\ &= \int_{-\infty}^{\infty}\int_{\mathbb{R}^n} e(z) f(x+\tau\theta-z)\,dz\,d\tau = \mathcal{P}_\theta(e*f)(x),\end{aligned}$$

which in turn gives

$$
\begin{aligned}
\Lambda\big(\mathcal{P}_\theta(e*f)\big) &= -\sum_{j=1}^{n} \mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * \frac{\partial}{\partial x_j}(\mathcal{P}_\theta e * \mathcal{P}_\theta f) \\
&= -\sum_{j=1}^{n} \mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * \Big(\frac{\partial \mathcal{P}_\theta e}{\partial x_j} * \mathcal{P}_\theta f\Big) \\
&= \Big(-\sum_{j=1}^{n} \mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * \frac{\partial \mathcal{P}_\theta e}{\partial x_j}\Big) * \mathcal{P}_\theta f = (\Lambda \mathcal{P}_\theta e) * \mathcal{P}_\theta f.
\end{aligned}
\tag{3.28}
$$

If we now replace f by $e * f$ in Equation (3.26), we get the approximate parallel beam reconstruction formula

$$
\begin{aligned}
e * f(x) &= \frac{b_n}{2}\int_{S^{n-1}} \Lambda \mathcal{P}_\theta (e*f)(E_\theta x)\, d\theta \\
&= \frac{b_n}{2}\int_{S^{n-1}} (\Lambda \mathcal{P}_\theta e) * \mathcal{P}_\theta f(E_\theta x)\, d\theta \\
&= \frac{b_n}{2}\int_{S^{n-1}}\int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(E_\theta x - y)\, \mathcal{P}_\theta f(y)\, dy\, d\theta.
\end{aligned}
\tag{3.29}
$$

We then perform the change of variable $y = E_\theta a$, $dy = \frac{|a\cdot\theta|}{R} da$. Here y runs twice over the set $B(0,R)\cap\theta^\perp$ (containing $\operatorname{supp}\mathcal{P}_\theta f$) as a runs over the two hemispheres of A (see Figure 3.4). This gives the approximate divergent beam reconstruction formula

$$
\begin{aligned}
e*f(x) &= \frac{b_n}{4R}\int_{S^{n-1}}\int_A \big(\Lambda \mathcal{P}_\theta e(E_\theta(x-a))\big)\, \mathcal{P}_\theta f(E_\theta a)\, |a\cdot\theta|\, da\, d\theta \\
&= \frac{b_n}{4R}\int_A\int_{S^{n-1}} \big(\Lambda \mathcal{P}_\theta e(E_\theta(x-a))\big)\, \big(\mathcal{D}_a f(\theta) + \mathcal{D}_a f(-\theta)\big)\, |a\cdot\theta|\, d\theta\, da.
\end{aligned}
\tag{3.30}
$$

These formulae will be derived in Chapter 5 under the following assumptions:

Theorem 3.7. (Repeated later as Corollary 5.19.) *Let $n \geq 2$ and $e \in D_{\mathrm{xr}}(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n)$ be such that $|\xi|^{-1}\widehat{e}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $f \in L^2_0(\mathbb{R}^n)$ and $\operatorname{supp} f \subset B(0,R)$, then*

$$
\begin{aligned}
e*f(x) &= \frac{b_n}{2}\int_{S^{n-1}}\int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(E_\theta x - y)\, \mathcal{P}_\theta f(y)\, dy\, d\theta \\
&= \frac{b_n}{4R}\int_A\int_{S^{n-1}} \Lambda \mathcal{P}_\theta e\big(E_\theta(x-a)\big)\, \big(\mathcal{D}_a f(\theta) + \mathcal{D}_a f(-\theta)\big)\, |a\cdot\theta|\, d\theta\, da,
\end{aligned}
\tag{3.31}
$$

where $A = S^{n-1}(0,R)$.

In practice, it is of fundamental importance that this reconstruction is stable with respect to errors in the measurements:

Theorem 3.8. (Repeated later as Theorem 5.21.) *If $n \geq 2$ and $e \in D_{\mathrm{xr}}(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n)$ is such that $|\xi|^{-1}\widehat{e}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$
|e * f(x)| \leq C\, \|\mathcal{D} f\|_{L^2(S^{n-1}(0,R)\times S^{n-1})} \tag{3.32}
$$

for all $f \in L^2_0(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B(0,R)$ and all $x \in \mathbb{R}^n$.

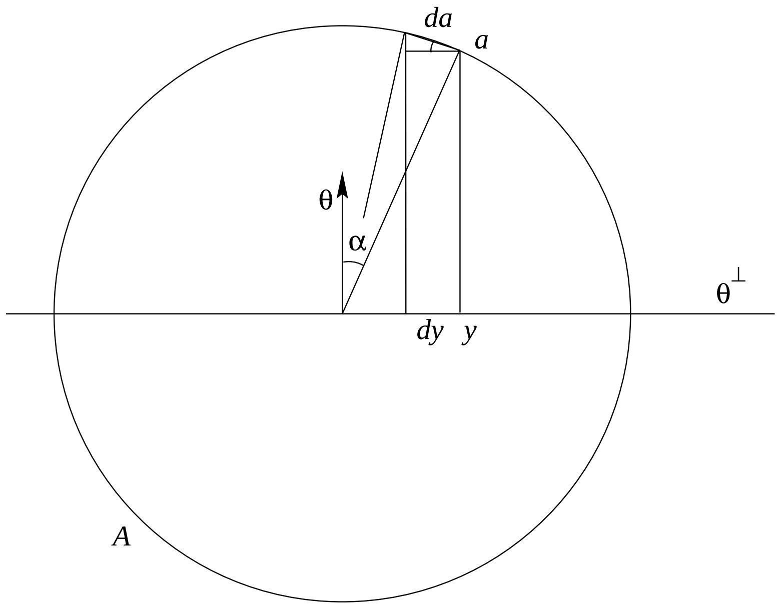


Figure 3.4: Substitution $y = E_\theta a$, $dy = |\cos\alpha|\, da = \frac{|a\cdot\theta|}{|a|\,|\theta|}\, da = \frac{|a\cdot\theta|}{R}\, da$.

3.5 Reconstruction Formula for $\Lambda e * f$

For extracting information about the internal structure of the object, $\Lambda e * f$ can also be reconstructed instead of $e * f$. In that case, we have the following reconstruction formula, which will also be derived in Chapter 5.

Theorem 3.9. (Repeated later as Theorem 5.20.) *Let $n \geq 2$ and let $e \in H^{3/2}(\mathbb{R}^n)$ be such that $\Lambda e \in D_{\mathrm{xr}}(\mathbb{R}^n)$. If $f \in L_0^2(\mathbb{R}^n)$, then*

$$\begin{aligned} \Lambda(e*f)(x) &= (\Lambda e) * f(x) \\ &= -\frac{b_n}{2}\int_{S^{n-1}} \big(\triangle\mathcal{P}_\theta e\big) * \big(\mathcal{P}_\theta f\big)(E_\theta x)\, d\theta \\ &= -\frac{b_n}{4R}\int_A\int_{S^{n-1}} \triangle\mathcal{P}_\theta e(E_\theta(x-a))\,(\mathcal{D}_a f(\theta) + \mathcal{D}_a f(-\theta))\,|a\cdot\theta|\, d\theta\, da \end{aligned} \tag{3.33}$$

for all $x \in \mathbb{R}^n$. If, in addition, e has compact support, then

$$\Lambda(e * f) = e * \Lambda f. \tag{3.34}$$

This reconstruction, too, is a stable operation:

Theorem 3.10. (Repeated later as Theorem 5.22.) *If $n \geq 2$ and $e \in H^{3/2}(\mathbb{R}^n)$ is such that $\Lambda e \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$|\Lambda e * f(x)| \leq C\, \|\mathcal{D}f\|_{L^2(S^{n-1}(0,R)\times S^{n-1})} \tag{3.35}$$

for all $f \in L_0^2(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B(0,R)$ and all $x \in \mathbb{R}^n$.

Of course, $\Lambda e * f$ looks different from $e * f$, but in many practical applications it gives relevant information. Most importantly, $e * \Lambda f$ is nearly singular at points where Λf is singular. These are precisely the same points at which f is singular, as will be shown in Chapter 6:

Theorem 3.11. (Repeated later as Theorem 6.5.) *If $n \geq 2$, $f \in L^2(\mathbb{R}^n)$ and $\Lambda f \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then $WF(f) = WF(\Lambda f)$.*

The *wave front* set $WF(f)$, describing the discontinuities of the distribution f, will be introduced in Definition 6.1.

This gives hope for the visibility of discontinuities of f in $\Lambda e * f$. The fact that the discontinuities of these two functions are located at the same points does not, of course, prevent the jumps of $\Lambda e * f$ from being much smaller and therefore undetectable in practice. Fortunately, the situation is actually the opposite: discontinuities of f become emphasized in Λf.

This fact and other further results will be described in the concluding Chapter 7.

3.6 Local Tomography

Reconstructing $e * \Lambda f(x)$ rather than $e * f(x)$ has the advantage that it can be done using data only from beams passing very close to x, in which case the process is called *local tomography*. This is done by choosing the convolution kernel e in such a way that for a fixed x, the support of the kernel $\triangle \mathcal{P}_\theta e$ in the reconstruction formula (3.33) is small. Then $\mathcal{D}_a f(\theta)$ is only needed for such a and θ that $E_\theta(x-a) \in \operatorname{supp} \triangle \mathcal{P}_\theta e$.

An approximation of the x-ray attenuation coefficient f itself cannot be reconstructed locally, because in the $e * f$ reconstruction formula (3.30), the support of the kernel $\Lambda \mathcal{P}_\theta e$ may not be bounded even if the support of e is. This is a consequence of the fact that all pseudodifferential operators

$$\mathcal{B} : f(x) \mapsto \mathcal{F}^{-1}\big(p(x,\xi)\hat{f}(\xi)\big) \tag{3.36}$$

that are *local* in the sense that

$$\operatorname{supp} \mathcal{B} f \subset \operatorname{supp} f \tag{3.37}$$

are differential operators [Pee1, Pee2]. Equivalently, for $\mathcal{B}$ to be local, the function $p : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ in (3.36), called the *symbol* of $\mathcal{B}$, must be a polynomial in ξ, possibly with functions of x as coefficients. In the case of Λ,

$$p(x,\xi) = |\xi| = \sqrt{\sum_{j=1}^{n} \xi_j^2} \tag{3.38}$$

is clearly not a polynomial.

There are many practical benefits if the area of interest is only part of the object x-rayed, for example, the heart of a living person.

Because the x-ray beam can be kept narrower, the overall x-ray dose to which the object is exposed can be reduced, which is beneficial since x-ray imaging is not *completely* noninvasive; adverse effects of x-ray exposure to both living creatures and inanimate objects are well known.

Fewer x-ray beams also imply a decrease in the amount of data that must be processed for reconstructing an image with a given resolution. This speeds up the imaging process, which makes it possible to monitor the time evolution, for example, of a beating heart. Alternatively, the resolution can be increased, since concentrating on a smaller region reduces the technological restrictions arising from the design of the detectors and the optical demagnification needed.

The x-ray attenuation coefficient f itself could now be computed as $e*f = R_1*\Lambda e*f$ from the locally reconstructed $\Lambda e * f$. However, this is usually not done because the great amplitude of $\Lambda e * f$ near f's discontinuities would substantially increase the instability of reconstructing $e * f$. In practice, $\Lambda e * f$ also often gives enough information.

Chapter 4

Fourier Transforms of Singular Integrals

4.1 Fourier Transform of the Riesz Kernel

For working with the Riesz kernel R_1, we shall first show that it is a tempered distribution and calculate its Fourier transform $\mathcal{F}R_1$, largely according to [Ner].

Theorem 4.1. *Let $n \geq 2$. Then R_1 is a tempered distribution and*

$$\mathcal{F}R_1(\xi) = (2\pi)^{-n/2}\,|\xi|^{-1}. \tag{4.1}$$

Proof. The function $h(x) = |x|^{n-1} = \frac{1}{b_n}\,R_1(x)$ is locally integrable, because for any compact $K \subset B^n(0,R) \subset \mathbb{R}^n$,

$$\begin{aligned}\int_K |x|^{1-n}\,dx &\leq \int_{B^n(0,R)} |x|^{1-n}\,dx \\ &= |S^{n-1}|\int_0^R r^{1-n}\,r^{n-1}\,dr = R\,|S^{n-1}| < \infty.\end{aligned} \tag{4.2}$$

Write $h = g_1 + g_2$, where $g_1 = h\chi_{\overline{B(0,1)}} \in L^1 \subset \mathcal{S}'$ and $g_2 = h\chi_{\mathbb{R}^n\setminus\overline{B(0,1)}} \in L^2 \subset \mathcal{S}'$:

$$\begin{aligned}\|g_1\|_{L^1} &= |S^{n-1}|\int_0^1 r^{1-n}\,r^{n-1}\,dr = |S^{n-1}| < \infty, \\ \|g_2\|_{L^2} &= |S^{n-1}|\int_1^\infty r^{2-2n}\,r^{n-1}\,dr = \frac{|S^{n-1}|}{2-n} < \infty.\end{aligned}$$

Thus $h \in \mathcal{S}'$, so it makes sense to speak of $\mathcal{F}h = \widehat{g_1} + \widehat{g_2}$, where $\widehat{g_1} \in L^\infty$ and $\widehat{g_2} \in L^2$. The distribution $\mathcal{F}h$ is in fact a locally integrable function, since using Hölder's inequality,

$$\begin{aligned}\int_K |\mathcal{F}h(\xi)|\,d\xi &\leq \int_K |\widehat{g_1}| + |\widehat{g_2}|\,d\xi \leq \|\widehat{g_1}\|_{L^\infty}\,m(K) + \|\widehat{g_2}\|_{L^2}\,\sqrt{m(K)} \\ &\leq (2\pi)^{-n/2}\|g_1\|_{L^1}\,m(K) + \|g_2\|_{L^2}\,\sqrt{m(K)} < \infty.\end{aligned}$$

Because $\sigma_\lambda h(x) = |\lambda x|^{1-n} = \lambda^{1-n} h(x)$,

$$\mathcal{F}(\sigma_\lambda h)(\xi) = \lambda^{1-n}\, \widehat{h}(\xi). \tag{4.3}$$

On the other hand by Equation (A.58),

$$\mathcal{F}(\sigma_\lambda h)(\xi) = \lambda^{-n} \sigma_{1/\lambda} \widehat{h}(\xi), \tag{4.4}$$

and therefore, with $\mu = 1/\lambda$,

$$\sigma_\mu \widehat{h} = \mu^{-1} \widehat{h}. \tag{4.5}$$

Hence $\widehat{h}$ can be written as

$$\widehat{h}(\xi) = \widehat{h}(|\xi| \tfrac{\xi}{|\xi|}) = |\xi|^{-1} \widehat{h}(\tfrac{\xi}{|\xi|}). \tag{4.6}$$

The function $h(x) = |x|^{1-n}$ is clearly radial, which is to say that its value at a point x only depends on $|x|$. We shall next show that $\widehat{h} = \widehat{g_1} + \widehat{g_2}$ is radial, too, and hence $\widehat{h}(\xi/|\xi|) = C \in \mathbb{C}$ is a constant. An equivalent characterization of the radiality of a function ϕ is that $\phi(Tx) = \phi(x)$ whenever T is a rotation, *i.e.* whenever $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator for which $|\det T\,| = 1$.

Let T be a rotation. Then its transpose T' and the inverse of its transpose T'^{-1} are rotations, as well, since

$$\det T' = \det T = 1 \qquad \text{and} \qquad \det T'^{-1} = \frac{1}{\det T'} = \frac{1}{\det T} = 1. \tag{4.7}$$

The change of variable $y = T'x$, $x = T'^{-1}y$, $dx = |\det T'^{-1}|\, dy = dy$ yields

$$\begin{aligned} \widehat{g_1}(T\xi) &= (2\pi)^{-n/2} \int g_1(x)\, e^{-ix\cdot T\xi}\, dx = (2\pi)^{-n/2} \int g_1(x)\, e^{-i(T'x)\cdot\xi}\, dx \\ &= (2\pi)^{-n/2} \int g_1(T'^{-1}y)\, e^{-iy\cdot\xi}\, dy = \widehat{g_1}(\xi), \end{aligned}$$

because $g_1(T'^{-1}y) = g_1(y)$. Analogously,

$$\begin{aligned} \widehat{g_2}(T\xi) &= \lim_{R\to\infty} \int_{|x|\le R} g_2(x)\, e^{-ix\cdot T\xi}\, dx \\ &= \lim_{R\to\infty} \int_{|y|\le R} g_2(T'^{-1}y)\, e^{-iy\cdot\xi}\, dy = \widehat{g_2}(\xi). \end{aligned} \tag{4.8}$$

Thus $\widehat{h}$ is radial and can be written as $\widehat{h}(\xi) = C\,|\xi|^{-1}$. The value of the constant C will be calculated using the function $v(x) = e^{-|x|^2/2} \in \mathcal{S}$, which has the property that $\widehat{v} = v$, as the following calculations show:

$$\begin{aligned} \widehat{v}(\xi) &= (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}} e^{-ix\cdot\xi}\, dx \\ &= e^{-\frac{|\xi|^2}{2}} (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-\frac{(x+i\xi)^2}{2}}\, dx \\ &= e^{-\frac{|\xi|^2}{2}} (2\pi)^{-n/2} \prod_{j=1}^{n} \int_{-\infty}^{\infty} e^{-\frac{(x_j+i\xi_j)^2}{2}}\, dx_j. \end{aligned} \tag{4.9}$$

For evaluating this integral, note that since $e^{-z^2/2}$ is analytic everywhere, its closed line integral along $\mathcal{C}$, the rectangle with vertices at R, $-R$, $-R+i\xi_j$ and $R+i\xi_j$, vanishes:

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-\frac{(x_j+i\xi_j)^2}{2}}\,dx_j \\
&= \lim_{R\to\infty}\left(\oint_{\mathcal{C}} e^{-\frac{z^2}{2}}\,dz + \int_{-R}^{R} e^{-\frac{x_j^2}{2}}\,dx_j + \int_0^{\xi_j} e^{-\frac{(R+it)^2}{2}} - e^{-\frac{(-R+it)^2}{2}}\,dt\right) \qquad (4.10)\\
&= 0 + \int_{-\infty}^{\infty} e^{-\frac{x_j^2}{2}}\,dx_j + 0 = \sqrt{2}\int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{2\pi}.
\end{aligned}$$

Therefore

$$\widehat{v}(\xi) = e^{-\frac{|\xi|^2}{2}} = v(\xi). \qquad (4.11)$$

For calculating C, we evaluate $\langle h, v\rangle$ in two different ways. Firstly, using the Plancherel formula (A.62) and the substitution $t = r^2/2$, $r = (2t)^{1/2}$, $dt = r\,dr$, we get

$$\begin{aligned}
\langle h(x), v(x)\rangle = \langle C\,|\xi|^{-1}, v(\xi)\rangle &= C\,|S^{n-1}| \int_0^\infty r^{-1} e^{-\frac{r^2}{2}}\, r^{n-1}\,dr \\
&= C\,|S^{n-1}|\, 2^{\frac{n-3}{2}} \int_0^\infty r^{\frac{n-3}{2}} e^{-t}\,dt = C\,|S^{n-1}|\,2^{\frac{n-3}{2}}\Gamma(\tfrac{n-1}{2}),
\end{aligned} \qquad (4.12)$$

and secondly,

$$\langle h(x), v(x)\rangle = |S^{n-1}| \int_0^\infty r^{1-n}\, e^{\frac{-r^2}{2}}\, r^{n-1}\,dr = |S^{n-1}|\sqrt{\frac{\pi}{2}}. \qquad (4.13)$$

Hence,

$$C = \frac{|S^{n-1}|\,\pi^{1/2}\,2^{-1/2}}{|S^{n-1}|\,2^{\frac{n-3}{2}}\,\Gamma(\frac{n-1}{2})} = \frac{\pi^{1/2}}{2^{\frac{n-2}{2}}\,\Gamma(\frac{n-1}{2})}. \qquad (4.14)$$

Therefore, as claimed,

$$\mathcal{F}R_1(\xi) = b_n\widehat{h}(\xi) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\,\frac{\pi^{1/2}}{2^{\frac{n-2}{2}}\,\Gamma(\frac{n-1}{2})}\,|\xi|^{-1} = (2\pi)^{-n/2}\,|\xi|^{-1}. \qquad (4.15)$$

$\square$

4.2 Calderón-Zygmund Theory

We shall next aim at justifying the formal calculation in (3.14) that leads to the result that the Calderón operator Λ, which can be expressed as

$$\Lambda = -\sum_{j=1}^{n} \mathrm{p.v.}\,\frac{\partial R_1}{\partial x_j} * D_j, \qquad (4.16)$$

inverts the Riesz potential (Theorems 3.4 and 3.5). We also want to show that when calculating the partial derivatives of the Riesz potential, the derivative can be taken under the integral sign,

$$D_j(R_1 * f) = \mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * f. \tag{4.17}$$

In Equations (4.16) and (4.17) appears the Riesz transform, defined as follows:

Definition 4.2. *For $n \geq 2$, the operator*

$$\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * = b_n\,(1-n)\,\mathrm{p.\,v.}\,\frac{x_j}{|x|^{n+1}} *, \qquad j \in \{1, 2, \dots, n\}, \tag{4.18}$$

is called the j^{th} Riesz transform. *In one dimension,*

$$\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_1} * := -\mathrm{p.\,v.}\,\frac{1}{\pi x} * \tag{4.19}$$

is called the Riesz transform *or the* Hilbert transform.

The name Hilbert transform is sometimes used instead of the Riesz transform also in $\mathbb{R}^n$, $n \geq 2$. Texts that use the notation $I_1 f$ instead of $R_1 * f$, often write $R_j f$ for $\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * f$.

Here, "p. v." stands for principal value convolution, defined as follows:

Definition 4.3. *Let $n \in \mathbb{Z}_+$ and let $K : \mathbb{R}^n \to \mathbb{C}$ be of the form*

$$K(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n}. \tag{4.20}$$

where $\Omega : S^{n-1} \to \mathbb{C}$ is an odd, bounded function. For $f \in L^2(\mathbb{R}^n)$, the principal value convolution *associated with K is the function*

$$\mathrm{p.\,v.}\,K * f(x) = L^2\text{-}\lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} K(y) f(x-y)\,dy. \tag{4.21}$$

The kernel K is called a singular convolution kernel, *and the operator $f \mapsto \mathrm{p.\,v.}\,K * f$ a* singular convolution operator.[1]

Observe that for $n \geq 2$,

$$\begin{aligned} \frac{\partial R_1}{\partial x_j}(x) &= b_n\,\frac{\partial}{\partial x_j}\left(\sum_{i=1}^n x_i^2\right)^{\frac{1-n}{2}} \\ &= b_n\,\frac{1-n}{2}\left(\sum_{i=1}^n x_i^2\right)^{\frac{-1-n}{2}} 2x_j^2 \\ &= b_n(1-n)\,\frac{x_j}{|x|^{n+1}} = b_n(1-n)\,\frac{\theta_j}{r^n} \end{aligned} \tag{4.22}$$

[1]The usual definition of a singular convolution kernel admits somewhat more general functions, but to simplify the treatment, we shall confine our study to functions whose radial part is odd and bounded.

is just the partial derivative of the Riesz kernel, but for $n = 1$, (4.22) does not make sense, so (4.19) must be seen as the definition of $\mathrm{p.v.}\,\frac{\partial R_1}{\partial x_1}*$.

We shall now first derive some results of the Calderón-Zygmund theory, concerning the continuity properties of singular convolutions operators and their Fourier transforms. They will be applied in Section 4.3 for finding answers to the questions about the Calderón operator and the derivative of the Riesz potential. Our derivation will largely follow [Ner].

The results will mostly be needed when $n \geq 2$. The case $n = 1$ will, be required for two-dimensional computed tomography, since the proof of Theorem 5.17 uses the Calderón-Zygmund theory results in $\mathbb{R}^{n-1}$.

Many of the following facts would be simpler to derive in one dimension. As the main focus is on $n \geq 2$, we shall not treat that situation separately, but only consider general $n \in \mathbb{Z}_+$, which covers $n = 1$ as a special case.

Our first result is that the Fourier transform of a truncated convolution kernel is uniformly bounded, and has a pointwise limit as the truncation radii are taken to zero and infinity:

Lemma 4.4. *For $n \in \mathbb{Z}_+$, let $\Omega : S^{n-1} \to \mathbb{C}$ be an odd, bounded function and*

$$K(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n}, \qquad K_{\varepsilon,\eta}(x) = \begin{cases} K(x), & \varepsilon < |x| < \eta \\ 0, & \textit{otherwise.} \end{cases} \tag{4.23}$$

Then there is a constant $A > 0$ such that

$$|\widehat{K}_{\varepsilon,\eta}(\xi)| \leq A \qquad \textit{for all } \xi \in \mathbb{R}^n \setminus \{0\} \textit{ and all } \varepsilon, \eta > 0, \tag{4.24}$$

and the limit

$$\widehat{K}(\xi) = \lim_{\substack{\varepsilon \to 0 \\ \eta \to \infty}} \widehat{K}_{\varepsilon,\eta}(\xi) \tag{4.25}$$

exists for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof. Fix $\xi \in \mathbb{R}^n \setminus \{0\}$ and write $\rho = |\xi|$. Introducing spherical coordinates, with φ measuring the angle between x and ξ, yields for $x \neq 0$

$$r = |x|, \qquad \theta = \frac{x}{|x|}, \qquad dx = r^{n-1}\,dr\,d\theta, \qquad x \cdot \xi = r\rho\cos\varphi, \tag{4.26}$$

and

$$\begin{aligned} \widehat{K}_{\varepsilon,\eta}(\xi) &= (2\pi)^{-n/2} \int_{\varepsilon<|x|<\eta} K(x)\, e^{-ix\cdot\xi}\, dx \\ &= (2\pi)^{-n/2} \int_{S^{n-1}} \Omega(\theta) \int_\varepsilon^\eta e^{-ir\rho\cos\varphi}\, \frac{dr}{r}\, d\theta. \end{aligned} \tag{4.27}$$

In the inner integral, make the substitution $s = \rho r$, $ds = \rho\, dr$ to get

$$\widehat{K}_{\varepsilon,\eta}(\xi) = (2\pi)^{-n/2} \int_{S^{n-1}} \Omega(\theta) \int_{\varepsilon\rho}^{\eta\rho} e^{-is\cos\varphi}\, \frac{ds}{s}\, d\theta. \tag{4.28}$$

Since Ω is an odd function,

$$\int_{S^{n-1}} \Omega(\theta)\, d\theta = 0 \tag{4.29}$$

and

$$\int_{S^{n-1}} \Omega(\theta)\, d\theta \int_{\varepsilon}^{\eta} \chi_{(0,1)}(s)\, ds = 0. \tag{4.30}$$

Thus,

$$\widehat{K}_{\varepsilon,\eta}(\xi) = (2\pi)^{-n/2} \int_{S^{n-1}} \Omega(\theta)\, I_{\varepsilon,\eta}\, d\theta, \tag{4.31}$$

where

$$I_{\varepsilon,\eta}(\theta) = \int_{\varepsilon\rho}^{\eta\rho} \frac{e^{-is\cos\varphi} - \chi_{(0,1)}(s)}{s}\, ds \tag{4.32}$$

It now suffices to find a function $g \in L^1(S^{n-1})$ that dominates the absolute value of the inner integral $I_{\varepsilon,\eta}$:

$$|I_{\varepsilon,\eta}(\theta)| \leq g(\theta) \qquad \text{for all } \varepsilon,\, \eta \geq 0. \tag{4.33}$$

In fact, if we find such a g, we immediately get the estimate

$$|\widehat{K}_{\varepsilon,\eta}(\xi)| \leq (2\pi)^{-n/2} \sup_{\theta\in S^{n-1}} |\Omega(\theta)| \int_{S^{n-1}} g(\theta)\, d\theta =: A, \tag{4.34}$$

proving the first part of the claim, and the second part follows from Lebesgue's theorem of dominated convergence: As $\varepsilon \to 0$ and $\eta \to \infty$,

$$\Omega(\theta) I_{\varepsilon,\eta}(\theta) \to \Omega(\theta) I_{0,\infty}(\theta) \quad \text{pointwise} \tag{4.35}$$

and $|\Omega(\theta) I_{\varepsilon,\eta}(\theta)| \leq |\Omega(\theta)|\, g(\theta) \in L^1(S^{n-1})$, whence

$$\widehat{K}_{\varepsilon,\eta}(\xi) = \int_{S^{n-1}} \Omega(\theta) I_{\varepsilon,\eta}(\theta)\, d\theta \to \int_{S^{n-1}} \Omega(\theta) I_{0,\infty}(\theta)\, d\theta, \tag{4.36}$$

which is finite.

We now choose

$$g(\theta) = -2\ln|\cos\varphi| + C, \tag{4.37}$$

where $C < \infty$ is a constant, and show that $g \in L^1(S^{n-1})$ and that (4.33) holds with an appropriate choice of C.

Since changing the sign of $\cos\varphi$ only changes $I_{\varepsilon,\eta}(\xi)$ to its complex conjugate, we may assume that $\cos\varphi > 0$. We prove (4.33) separately in the three cases

1. $\varepsilon\rho \leq 1 \leq \eta\rho$,
2. $\varepsilon\rho > 1$,
3. $\eta\rho < 1$.

In case 1, $I_{\varepsilon,\eta}(\theta) = I_1 + I_2$, where

$$I_1 = \int_{\varepsilon\rho}^{1} \frac{e^{-is\cos\varphi} - 1}{s}\, ds \qquad \text{and} \qquad I_2 = \int_{1}^{\eta\rho} \frac{e^{-is\cos\varphi}}{s}\, ds. \tag{4.38}$$

Because $|e^{iu} - 1| = |e^{iu/2}(e^{iu/2} - e^{-iu/2})| = |2i\sin\frac{u}{2}| \le |u|$,

$$|I_1| \le \int_{\varepsilon\rho}^{1} \frac{s\cos\varphi}{s}\, ds \le 1. \tag{4.39}$$

The substitution $t = s\cos\varphi$, $dt = ds\cos\varphi$ gives us

$$\begin{aligned} |I_2| &= \left|\int_{\cos\varphi}^{\eta\rho\cos\varphi} \frac{e^{-it}}{t}\, dt\right| \\ &\le \int_{\cos\varphi}^{1} \frac{dt}{t} + \left|\int_{1}^{\eta\rho\cos\varphi} \frac{e^{-it}}{t}\, dt\right| \le -\ln\cos\varphi + C_1, \end{aligned} \tag{4.40}$$

where

$$C_1 = \sup_{v>1} \left|\int_{1}^{v} \frac{e^{-iu}}{u}\, du\right| \tag{4.41}$$

is finite, since it is the supremum of a continuous function defined on $[1,\infty)$ whose limit at infinity is finite:

$$\begin{aligned} \left|\int_{1}^{\infty} \frac{e^{-iu}}{u}\, du\right| &= \left| i \Big/_{1}^{\infty} \frac{e^{-iu}}{u} + i\int_{1}^{\infty} \frac{e^{-iu}}{u^2}\, du\right| \\ &\le |-ie^{-i}| + \int_{1}^{\infty} \frac{du}{u^2} < \infty. \end{aligned} \tag{4.42}$$

Thus in case 1, $|I_{\varepsilon,\eta}(\theta)| \le |I_1| + |I_2| \le -\ln\cos\varphi + C_1 + 1$.

In case 2, $I_{\varepsilon,\eta}(\theta) = I_2 - I_3$, where

$$I_3 = \int_{1}^{\varepsilon\rho} \frac{e^{-is\cos\varphi}}{s}\, ds, \tag{4.43}$$

which is of the same form as I_2 above, with η replaced by ε. Thus, we get the same estimate for I_3 as for I_2, and $|I_{\varepsilon,\eta}(\theta)| \le |I_2| + |I_3| \le -2\ln\cos\varphi + 2C_1$.

In case 3,

$$\begin{aligned} |I_{\varepsilon,\eta}(\theta)| &\le \int_{\varepsilon\rho}^{\eta\rho} \left|\frac{e^{-is\cos\varphi} - 1}{s}\right| ds \\ &\le \int_{0}^{1} \left|\frac{e^{-is\cos\varphi} - 1}{s}\right| ds \le \int_{0}^{1} \frac{s\cos\varphi}{s}\, ds \le 1. \end{aligned} \tag{4.44}$$

Therefore with $C = 2C_1 + 1$,

$$|I_{\varepsilon,\eta}(\theta)| \le -\ln\cos\varphi + C = g(\theta) \tag{4.45}$$

in all three cases, proving (4.33). The fact that $g \in L^1(S^{n-1})$ can easily be verified using the spherical coordinates of Lemma A.13: If $n > 2$,

$$
\begin{aligned}
\int_{S^{n-1}} -\ln|\cos\varphi|\,d\theta &= -2\int_{S_+^{n-1}(\xi)} \ln\cos\varphi\,d\theta \\
&= -2\,|S^{n-2}|\int_0^{\pi/2} \ln\cos\varphi\,\sin^{n-2}\varphi\,d\varphi \\
&\leq 2\,|S^{n-2}|\int_0^{\pi/2} -\sin\varphi\,\ln\cos\varphi\,d\varphi \\
&= 2\,|S^{n-2}|\Big/_0^{\pi/2} \cos\varphi\,\ln\cos\varphi - \cos\varphi \\
&= 2\,|S^{n-2}| < \infty,
\end{aligned}
\tag{4.46}
$$

and if $n = 2$,

$$
\int_{S^1} -\ln|\cos\varphi|\,d\theta = -4\int_0^{\pi/2} \frac{\varphi^2}{2} + \varphi^4\,b(\varphi)\,d\varphi < \infty, \tag{4.47}
$$

where b is a bounded function.

The case $n = 1$ would be quite simple to handle all the way on its own, but this would be unnecessary, as the treatment above covers it as well. Now $S^0 = \{-1, +1\}$ and

$$
\int_{S^0} f(x)\,dx = f(-1) + f(1). \tag{4.48}
$$

The angle between x and ξ is either 0 or π, and $g(\pm 1) = \ln|\pm 1| + C = C$, so $\|g\|_{L^1(S^0)} = C + C < \infty$. $\square$

Not only do we know that $\widehat{K}$ exists, we know how to calculate it:

Lemma 4.5. *Let $n \in \mathbb{Z}_+$ and let K be of the form*

$$
K(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n}, \tag{4.49}
$$

where $\Omega : S^{n-1} \to \mathbb{C}$ is an odd, bounded function, and let $\widehat{K}$ be as in Lemma 4.4. Then for all $\xi \neq 0$,

$$
\widehat{K}(\xi) = -i\pi\,(2\pi)^{-n/2}\int_{S_+^{n-1}(\xi)} \Omega(\theta)\,d\theta. \tag{4.50}
$$

Proof. We use the same notation as in the proof of Lemma 4.4:

$$
\rho = |\xi|, \quad r = |x|, \quad \theta = \frac{x}{|x|}, \quad dx = r^{n-1}\,dr\,d\theta, \quad x\cdot\xi = r\rho\cos\varphi \tag{4.51}
$$

and

$$
\widehat{K}_{\varepsilon,\eta}(\xi) = (2\pi)^{-n/2}\int_{S^{n-1}} \Omega(\theta)\int_{\varepsilon\rho}^{\eta\rho} \frac{e^{-is\cos\varphi}}{s}\,ds\,d\theta. \tag{4.52}
$$

Lemma 4.4 shows that when $\xi \neq 0$, the limit

$$\widehat{K}(\xi) = \lim_{\delta\to 0} (2\pi)^{-n/2} \int_{S^{n-1}} \Omega(\theta) \int_{\delta}^{\infty} \frac{e^{-is\cos\varphi}}{s}\, ds\, d\theta \tag{4.53}$$

exists. We now use the fact that Ω is odd and note that if we think of $\varphi \in [0,\pi]$ as a function of $\theta \in S^{n-1}$, $\cos\varphi(-\theta) = \cos(\pi - \varphi(\theta)) = -\cos\varphi(\theta)$. Hence

$$\begin{aligned} \int_{S^{n-1}} \Omega(\theta) \int_{\delta}^{\infty} \frac{e^{-is\cos\varphi}}{s}\, ds\, d\theta &= \int_{S_+^{n-1}(\xi)} \Omega(\theta) \int_{\delta}^{\infty} \frac{e^{-is\cos\varphi} - e^{is\cos\varphi}}{s}\, ds\, d\theta \\ &= \int_{S_+^{n-1}(\xi)} \Omega(\theta) \int_{\delta}^{\infty} \frac{-2i\sin(s\cos\varphi)}{s}\, ds\, d\theta. \end{aligned} \tag{4.54}$$

The change of variable $t = 2\pi s\cos\varphi$, $dt = 2\pi\cos\varphi\, ds$ leads to

$$\begin{aligned} \widehat{K}(\xi) &= -2i\,(2\pi)^{-n/2} \lim_{\delta\to 0} \int_{S_+^{n-1}(\xi)} \Omega(\theta) \int_{\delta}^{\infty} \frac{\sin t}{t}\, dt\, d\theta \\ &= -i\pi\,(2\pi)^{-n/2} \int_{S_+^{n-1}(\xi)} \Omega(\theta)\, d\theta, \end{aligned} \tag{4.55}$$

as claimed, since the inner integral evaluates to $\pi/2$, which can be seen as follows: If $g \in L^2(\mathbb{R})$ is the triangular function $g(x) = \max(0, 1 - |x|/2)$, then

$$\begin{aligned} \sqrt{2\pi}\,\widehat{g}(\xi) &= \int_{-2}^{0} (1 + \tfrac{x}{2})\, e^{-ix\xi}\, dx + \int_{0}^{2} (1 - \tfrac{x}{2})\, e^{-ix\xi}\, dx \\ &= \int_{0}^{2} (1 - \tfrac{y}{2})\, e^{iy\xi}\, dy + \int_{0}^{2} (1 - \tfrac{x}{2})\, e^{-ix\xi}\, dx \\ &= \int_{0}^{2} (1 - \tfrac{x}{2})\, 2\cos x\xi\, dx \\ &= 2\Big/_{0}^{2} \frac{\sin x\xi}{\xi} - \Big/_{0}^{2} \frac{x\sin x\xi}{\xi} + \int_{0}^{2} \frac{\sin x\xi}{\xi}\, dx \\ &= \frac{1 - \cos 2\xi}{\xi^2}, \end{aligned} \tag{4.56}$$

and the inverse Fourier transform gives

$$\begin{aligned} \frac{\pi}{2} &= \frac{\pi}{2}\, g(0) \\ &= \frac{\pi}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{g}(\xi)\, e^{i0\xi}\, d\xi \\ &= \frac{\pi}{2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1 - \cos 2\xi}{\xi^2}\, d\xi \\ &= -\frac{1}{4} \Big/_{-\infty}^{\infty} \frac{1 - \cos 2\xi}{\xi} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin 2\xi}{\xi}\, d\xi \\ &= 0 + \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx \\ &= \int_{0}^{\infty} \frac{\sin x}{x}\, dx. \end{aligned} \tag{4.57}$$

□

Aiming at establishing continuity results for the operator p. v. $K*: L^2 \to L^2$ and showing that a certain form of the convolution theorem (A.60) holds, we shall first prove the existence of the convolution $K_{\varepsilon,\infty} * f$ truncated from below:

Lemma 4.6. *Assume that* $n \in \mathbb{Z}_+$, $f \in L^2(\mathbb{R}^n)$, $\Omega \in L^1(S^{n-1})$ *and let* K *and* $K_{\varepsilon,\eta}$ *be as in Lemma 4.4. Then*

$$K_{\varepsilon,\infty} * f(x) = \int_{|y|>\varepsilon} K(y) f(x-y)\, dy \tag{4.58}$$

exists almost everywhere.

Proof. Because

$$|K_{\varepsilon,\infty} * f(x)| \leq \int_{|y|>\varepsilon} |K(y)||f(x-y)|\, dy =: I(x), \tag{4.59}$$

it suffices to show that $I(x)$ is locally integrable, because in that case it must be finite almost everywhere. Let $B \subset B(0, R_B) \subset \mathbb{R}^n$ be compact. Fubini's theorem yields

$$\begin{aligned}\int_B |I(x)|\, dx &= \int_B \int_{|y|>\varepsilon} \frac{|\Omega(\frac{y}{|y|})|}{|y|^n} |f(x-y)|\, dy\, dx \\ &= \int_B \int_{S^{n-1}} |\Omega(\theta)| \int_\varepsilon^\infty |f(x-r\theta)|\, r^{-1}\, dr\, d\theta\, dx \\ &= \int_{S^{n-1}} |\Omega(\theta)| \int_B \int_\varepsilon^\infty |f(x-r\theta)|\, r^{-1}\, dr\, dx\, d\theta\end{aligned} \tag{4.60}$$

since the integrand is non-negative. We use Hölder's inequality twice, first in the innermost integral to get

$$\begin{aligned}\int_\varepsilon^\infty |f(x-r\theta)|\, r^{-1}\, dr &\leq \sqrt{\int_\varepsilon^\infty r^{-2}\, dr} \sqrt{\int_\varepsilon^\infty |f(x-r\theta)|^2\, dr} \\ &= \frac{1}{\sqrt{\varepsilon}} \sqrt{\int_\varepsilon^\infty |f(x-r\theta)|^2\, dr}\end{aligned} \tag{4.61}$$

and then in the middle integral to get

$$\begin{aligned}\int_B \int_\varepsilon^\infty |f(x-r\theta)|\, r^{-1}\, dr\, dx &\leq \frac{1}{\sqrt{\varepsilon}} \int_B \sqrt{\int_\varepsilon^\infty |f(x-r\theta)|^2\, dr}\, dx \\ &\leq \sqrt{\frac{m(B)}{\varepsilon} \int_B \int_\varepsilon^\infty |f(x-r\theta)|^2\, dr\, dx} \\ &\leq \sqrt{\frac{m(B)}{\varepsilon} \int_B \int_{-\infty}^\infty |f(x-r\theta)|^2\, dr\, dx}.\end{aligned} \tag{4.62}$$

Because $B \subset \{x' + t\theta \in \mathbb{R}^n \mid x' \cdot \theta = 0,\ |t| < R_B\}$, the change of variable $x = x' + t\theta$,

$dx = dx'\,dt$ gives us

$$\begin{aligned}
&\sqrt{\frac{m(B)}{\varepsilon}\int_B\int_{-\infty}^{\infty}|f(x-r\theta)|^2\,dr\,dx} \\
&\leq \sqrt{\frac{m(B)}{\varepsilon}\int_{-R_B}^{R_B}\int_{\theta^\perp}\int_{-\infty}^{\infty}|f(x'+t\theta-r\theta)|^2\,dr\,dx'\,dt} \qquad (4.63)\\
&= \sqrt{\frac{m(B)}{\varepsilon}\int_{-R_B}^{R_B}\|f\|_{L^2}^2\,dt} = \sqrt{\frac{2R_B\,m(B)}{\varepsilon}}\,\|f\|_{L^2},
\end{aligned}$$

whence we conclude that

$$\int_B |I(x)|\,dx \leq \|\Omega\|_{L^1}\sqrt{\frac{2R_B\,m(B)}{\varepsilon}}\,\|f\|_{L^2} < \infty, \tag{4.64}$$

as claimed. □

We are now ready to show that p. v. $K*$ is a continuous operator on L^2:

Theorem 4.7 (Calderón-Zygmund). *Assume that $n \in \mathbb{Z}_+$, let $\Omega : S^{n-1} \to \mathbb{C}$ be an odd, bounded function and let*

$$K(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n}, \qquad K_{\varepsilon,\eta}(x) = \begin{cases} K(x), & \varepsilon < |x| < \eta \\ 0, & \textit{otherwise} \end{cases} \tag{4.65}$$

and

$$\widehat{K}(\xi) = \lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} \widehat{K}_{\varepsilon,\eta}(\xi). \tag{4.66}$$

Then for all $f \in L^2(\mathbb{R}^n)$,

(a) there exists a constant $B > 0$, independent of ε, η, and f, such that

$$\|K_{\varepsilon,\eta} * f\|_{L^2} \leq B\|f\|_{L^2} \tag{4.67}$$

(b) the limit

$$\tilde{f} := L^2\text{-}\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} K_{\varepsilon,\eta} * f = \mathrm{p.\,v.}\,K * f \tag{4.68}$$

exists

(c) the Fourier transform of $\tilde{f}$ satisfies

$$\mathcal{F}(\tilde{f}) = (2\pi)^{n/2}\,\widehat{K}\hat{f} \qquad \textit{and} \qquad \|\tilde{f}\|_{L^2} \leq M\,\|f\|_{L^2}, \tag{4.69}$$

where $M = (2\pi)^{n/2}\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}|\widehat{K}(\xi)| < \infty$

(d) if f is in addition Hölder-α-continuous for some $\alpha > 0$, i.e. for all $x \in \mathbb{R}^n$ there exist constants $\delta, C > 0$ such that

$$|f(x) - f(y)| < C\,|x-y|^\alpha \quad \textit{whenever} \quad |x-y| \leq \delta, \tag{4.70}$$

then the pointwise limit

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} K_{\varepsilon,\eta} * f(x) \tag{4.71}$$

exists for all $x \in \mathbb{R}^n$ and it equals $\tilde{f}(x)$ almost everywhere.

Proof. Lemma 4.4 asserts that $|\widehat{K}_{\varepsilon,\eta}| \leq A$ for some fixed $A > 0$, whence the Parseval formula (A.61) and the Convolution theorem (A.60) imply part (a):

$$\begin{aligned} \|K_{\varepsilon,\eta} * f\|_{L^2} &= \|\mathcal{F}(K_{\varepsilon,\eta} * f)\|_{L^2} = (2\pi)^{n/2}\,\|\widehat{K}_{\varepsilon,\eta}\hat{f}\|_{L^2} \\ &\leq (2\pi)^{n/2}\,A\,\|\hat{f}\|_{L^2} = B\,\|f\|_{L^2} \end{aligned} \tag{4.72}$$

with $B = (2\pi)^{n/2}\,A$. If we set $\tilde{f} = \mathcal{F}^{-1}\big((2\pi)^{n/2}\,\widehat{K}\hat{f}\big)$, part (c) is obviously true, since

$$\|\tilde{f}\|_{L^2} = \|\mathcal{F}(\tilde{f})\|_{L^2} = (2\pi)^{n/2}\,\|\widehat{K}\hat{f}\|_{L^2} \leq M\|\hat{f}\|_{L^2} = M\|f\|_{L^2}. \tag{4.73}$$

where $M = (2\pi)^{n/2}\,\sup_{\xi\in\mathbb{R}^n\setminus\{0\}}|\widehat{K}(\xi)| \leq B < \infty$. This $\tilde{f}$ satisfies (b), too, since

$$\begin{aligned} \|K_{\varepsilon,\eta} * f - \tilde{f}\|_{L^2} &= \|\mathcal{F}(K_{\varepsilon,\eta} * f - \tilde{f})\|_{L^2} \\ &= \|(\widehat{K}_{\varepsilon,\eta} - \widehat{K})\hat{f}\|_{L^2} \xrightarrow{\varepsilon\to 0,\,\eta\to\infty} 0, \end{aligned} \tag{4.74}$$

as Lebesgue's theorem of dominated convergence implies: Because $|\widehat{K}_{\varepsilon,\eta} - \widehat{K}| \leq 2A$,

$$|(\widehat{K}_{\varepsilon,\eta} - \widehat{K})^2\hat{f}^2| \leq 4A^2|\hat{f}|^2 \tag{4.75}$$

which is in L^1 since $\|4A^2\,|\hat{f}|^2\|_{L^1} = 4A^2\|\hat{f}\|_{L^2}^2 = 4A^2\|f\|_{L^2}^2 < \infty$. As, in addition,

$$(\widehat{K}_{\varepsilon,\eta}(\xi) - \widehat{K}(\xi))^2\hat{f}(\xi)^2 \to 0 \qquad \text{for almost all } \xi \tag{4.76}$$

by Lemma 4.4, Lebesgue's theorem asserts that

$$\|(\widehat{K}_{\varepsilon,\eta} - \widehat{K})\hat{f}\|_{L^2}^2 = \|(\widehat{K}_{\varepsilon,\eta} - \widehat{K})^2\hat{f}^2\|_{L^1} \xrightarrow{\varepsilon\to 0,\,\eta\to\infty} \|0\|_{L^1} = 0. \tag{4.77}$$

Part (d) remains to be proved. Fix $x \neq 0$ and let $0 < \varepsilon < \delta$. By Lemma 4.6, $K_{\delta,\infty} * f$ exists. Because Ω is odd and bounded,

$$\int_{\varepsilon\leq|y|\leq\delta} \frac{\Omega(\frac{y}{|y|})}{|y|^n}\,f(x)\,dy = f(x)\int_\varepsilon^\delta \frac{dr}{r}\int_{S^{n-1}}\Omega(\theta)\,d\theta = 0, \tag{4.78}$$

whence

$$\begin{aligned} |K_{\varepsilon,\infty} * f(x) - K_{\delta,\infty} * f(x)| &= \Big|\int_{\varepsilon\leq|y|\leq\delta} \frac{\Omega(\frac{y}{|y|})}{|y|^n}\,f(x-y)\,dy\Big| \\ &= \Big|\int_{\varepsilon\leq|y|\leq\delta} \frac{\Omega(\frac{y}{|y|})}{|y|^n}\,\big(f(x-y) - f(x)\big)\,dy\Big| \\ &\leq \int_{\varepsilon\leq|y|\leq\delta} \frac{|\Omega(\frac{y}{|y|})|}{|y|^n}\,\big|f(x-y) - f(x)\big|\,dy. \end{aligned} \tag{4.79}$$

By the assumption about Hölder continuity, $|f(x-y) - f(x)| \leq C\,|y|^\alpha$ and therefore

$$\begin{aligned} |K_{\varepsilon,\infty} * f(x) - K_{\delta,\infty} * f(x)| &\leq \int_\varepsilon^\delta\int_{S^{n-1}}|\Omega(\theta)|\,d\theta\,r^{-n}\,Cr^\alpha\,r^{n-1}\,dr \\ &\leq C\,\|\Omega\|_{L^\infty}\,|S^{n-1}|\,\frac{\delta^\alpha - \varepsilon^\alpha}{\alpha} \\ &\leq C\,\|\Omega\|_{L^\infty}\,|S^{n-1}|\,\frac{\delta^\alpha}{\alpha} \xrightarrow{\delta\to 0} 0. \end{aligned} \tag{4.80}$$

This shows that if $\delta_k \xrightarrow{k\to\infty} 0$, then $\big(K_{\delta_k,\infty} * f(x)\big)_{k\in\mathbb{N}}$ is a Cauchy sequence, so that the limit

$$\lim_{k\to\infty} K_{\delta_k,\infty} * f(x) \tag{4.81}$$

exists. Since this is true for all sequences (δ_k) converging to 0, this limit is also the limit $\lim_{\delta\to 0} K_{\delta,\infty} * f(x)$. Therefore the pointwise limit exists.

Now if $\varepsilon_k \to 0$, $\eta_k \to \infty$ and $f_k = K_{\varepsilon_k,\eta_k} * f$, the sequence f_k converges to $\tilde{f}$ in L^2. Hence there exists a subsequence (f_{k_j}) such that for almost all x, $f_{k_j}(x) \to \tilde{f}(x)$. On the other hand, since the pointwise limit $\lim_{k\to\infty} f_k(x)$ exists, it must be equal to the limit of the subsequence. Consequently

$$\lim_{k\to\infty} f_k(x) = \lim_{j\to\infty} f_{k_j}(x) = \tilde{f}(x) \tag{4.82}$$

for almost all x. $\square$

It is also true that the pointwise limit

$$\text{p.v.}\, K * f(x) = \lim_{\varepsilon\to 0} \int_{|y|>\varepsilon} K(y)\, f(x-y)\, dy \tag{4.83}$$

exists for almost all $x \in \mathbb{R}^n$ even without the condition of Hölder continuity, and under somewhat looser assumptions about Ω. Since we shall not need this result, we only refer to [Ner, Theorem I.IV.3.8, page 113].

The function $\tilde{f}$ is defined above in (4.68) as a principal value limit of truncated convolutions. It would be pleasant to be able to "pass the limit under the integral", *i.e.*, to view $\tilde{f}$ as a convolution of f with some distribution. The following theorem does this for $f \in \mathcal{D}$.

Theorem 4.8. *Let $n \in \mathbb{Z}_+$, let $\Omega : S^{n-1} \to \mathbb{C}$ be an odd, bounded function and let K, $K_{\varepsilon,\eta}$ and $\widehat{K}$ be as in Theorem 4.7. Then* p.v. K, *defined by*

$$\langle \text{p.v.}\, K, \phi\rangle = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} K(x)\phi(x)\, dx, \tag{4.84}$$

is a tempered distribution and $\mathcal{F}(\text{p.v.}\, K) = \widehat{K} \in L^\infty(\mathbb{R}^n)$. Furthermore for $f \in \mathcal{D}$,

$$\tilde{f} := L^2\text{-}\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} K_{\varepsilon,\eta} * f = (\text{p.v.}\, K) * f \tag{4.85}$$

and

$$\mathcal{F}((\text{p.v.}\, K) * f) = (2\pi)^{n/2}\, \mathcal{F}(\text{p.v.}\, K)\, \mathcal{F}(f). \tag{4.86}$$

Proof. We begin by showing that p.v. K is a distribution: We shall first show that for all $\phi \in \mathcal{D}$, $\langle \text{p.v.}\, K, \phi\rangle = I_1 + I_2$ is finite, where

$$I_1 = \lim_{\varepsilon\to 0} \int_{\varepsilon<|x|<1} K(x)\phi(x)\, dx, \qquad I_2 = \int_{|x|\geq 1} K(x)\phi(x)\, dx. \tag{4.87}$$

Since Ω is odd,

$$\int_{\varepsilon<|x|<1} K(x)\phi(0)\,dx = \phi(0)\int_{S^{n-1}} \Omega(\theta)\,d\theta \int_{\varepsilon}^{1} \frac{dr}{r} = 0 \tag{4.88}$$

and hence we can write

$$I_1 = \lim_{\varepsilon\to 0}\int_{\varepsilon<|x|<1} g(x)\,dx, \tag{4.89}$$

where $g(x) = K(x)(\phi(x)-\phi(0)) = K(x)\nabla\phi(tx)\cdot x$ for some $t\in[0,1]$. Now, if

$$C = \sup_{|t|\le 1}|\nabla\phi(tx)| < \infty \qquad \text{and} \qquad M_0 = \sup_{\theta\in S^{n-1}}|\Omega(\theta)|, \tag{4.90}$$

we see that $|g(x)| \le |\Omega(\theta)|\,|x|^{-n}\,|\nabla\phi(tx)|\,|x| \le CM_0\,|x|^{1-n}$ and hence

$$|I_1| \le CM_0\int_{S^{n-1}} d\theta \int_0^1 r^{1-n}r^{n-1}\,dr = CM_0\,|S^{n-1}| < \infty. \tag{4.91}$$

Since also

$$|I_2| \le \int_{|x|\ge 1}|K(x)\phi(x)|\,dx \le \sup_{|x|\ge 1}|K(x)|\,\|\phi\|_{L^1} < \infty, \tag{4.92}$$

we conclude that $\langle \mathrm{p.v.}\,K,\phi\rangle$ is finite for all $\phi\in\mathcal{D}$.

Since $\mathrm{p.v.}\,K$ is clearly linear, it remains to show that it is continuous. Let $\phi_k\to\phi$ in $\mathcal{D}$. By the definition of convergence in $\mathcal{D}$, there exists an $R>0$ such that for all $k\in N$, $\operatorname{supp}\phi_k\subset\overline{B(0,R)}$, and

$$\sup_{|x|\le R}|\nabla(\phi_k(x)-\phi(x))| \le \sqrt{\sum_{j=1}^{n}\sup_{|x|\le R}\Big|\frac{\partial\phi_k}{\partial x_j}(x)-\frac{\partial\phi}{\partial x_j}(x)\Big|^2} \xrightarrow{k\to\infty} 0. \tag{4.93}$$

Hence as above, for some $t\in[0,1]$,

$$\begin{aligned}
|\langle \mathrm{p.v.}\,K,\phi_k-\phi\rangle| &= \Big|\lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\Omega(\frac{x}{|x|})}{|x|^n}\big((\phi_k(x)-\phi(x))-(\phi_k(0)-\phi(0))\big)\,dx\Big| \\
&\le \lim_{\varepsilon\to 0}\int_{\varepsilon}^{R}\int_{S^{n-1}}\frac{|\Omega(\theta)|}{r^n}\,|\nabla(\phi_k-\phi)(tx)|\,|x|\,r^{n-1}\,dr\,d\theta \\
&\le M_0R\,|S^{n-1}|\sup_{|x|\le R}|\nabla(\phi_k-\phi)(x)| \xrightarrow{k\to\infty} 0,
\end{aligned}$$

which shows that $\mathrm{p.v.}\,K$ is continuous and consequently a distribution.

To show that this distribution is tempered, first consider the distribution $u = (1+|x|^2)^{-1}(\mathrm{p.v.}\,K)$. Let $\zeta\in\mathcal{D}$ be such that $\zeta(x)=1$ for $|x|\le 1$; for instance the function $\tilde{\chi}_{1,2}$ of (A.34). Write $u_1=\zeta u$, $u_2=u-u_1$. Now $u_1\in\mathcal{S}'$, which is seen as follows: If $\phi\in\mathcal{S}$, $\zeta\phi\in\mathcal{D}$ and $\langle u_1,\phi\rangle=\langle u,\zeta\phi\rangle$ is finite. For continuity, let $\phi_k\to\phi$ in $\mathcal{S}$. Thus also $\zeta\phi_k\to\zeta\phi$ in $\mathcal{D}$, since $\operatorname{supp}\zeta\phi_k\subset\operatorname{supp}\zeta$ is compact, and for all $m\in\mathbb{N}$,

$$\begin{aligned}
&\sup_{\substack{x\in\operatorname{supp}\zeta\\ |\alpha|\le m}}\big|\big(\partial^\alpha(\zeta\phi_k)-\partial^\alpha(\zeta\phi)\big)(x)\big| \\
&\qquad\le \sup_{\substack{x\in\operatorname{supp}\zeta\\ |\alpha|\le m}}\sum_{\beta\le\alpha}c_{\alpha,\beta}\,|\partial^\beta\zeta(x)|\,|\partial^{\alpha-\beta}(\phi_k-\phi)(x)| \xrightarrow{k\to\infty} 0,
\end{aligned}\tag{4.94}$$

because $|\partial^{\alpha-\beta}(\phi_k-\phi)(x)| \le \|\phi_k-\phi\|_{0,\alpha-\beta} \to 0$. Therefore $\langle u_1, \phi_k-\phi\rangle = \langle u, \zeta\phi_k - \zeta\phi\rangle \to 0$.

Also u_2 is a tempered distribution because

$$\|u_2\|_{L^1} \le \int_{|x|\ge 1} \frac{|K(x)|}{1+|x|^2}\,dx = \int_{S^{n-1}} |\Omega(\theta)|\,d\theta \int_1^\infty \frac{dr}{r+r^3} < \infty. \tag{4.95}$$

Therefore $u = u_1 + u_2 \in \mathcal{S}'$. Now since

$$\langle \mathrm{p.\,v.}\,K, \phi\rangle = \langle (1+|x|^2)u, \phi\rangle = \langle u, (1+|x|^2)\phi\rangle \tag{4.96}$$

and $(1+|x|^2)\phi \in \mathcal{S}$ whenever $\phi \in \mathcal{S}$, we see that $\mathrm{p.\,v.}\,K$ is a tempered distribution.

The estimate

$$|\langle \tilde f - K_{\varepsilon,\infty} * f, \phi\rangle| \le \|\tilde f - K_{\varepsilon,\infty} * f\|_{L^2}\|\phi\|_{L^2} \xrightarrow{\varepsilon\to 0} 0, \tag{4.97}$$

yields

$$\begin{aligned}
\langle \tilde f, \phi\rangle &= \lim_{\varepsilon\to\infty} \langle K_{\varepsilon,\infty} * f, \phi\rangle \\
&= \lim_{\varepsilon\to\infty} \langle K_{\varepsilon,\infty}, \check f * \phi\rangle \\
&= \lim_{\varepsilon\to\infty} \int_{|y|>\varepsilon} K(y)\, \check f * \phi(y)\,dy \\
&= \langle \mathrm{p.\,v.}\,K, \check f * \phi\rangle \\
&= \langle \mathrm{p.\,v.}\,K * f, \phi\rangle.
\end{aligned} \tag{4.98}$$

This shows that (4.85) holds.

For proving (4.86), choose a sequence of functions $\zeta_k \in \mathcal{S}$ such that $\zeta_k \to \mathrm{p.\,v.}\,K$ in $\mathcal{S}'$. Using the definition of convergence in $\mathcal{S}'$ and the convolution theorem (A.60) for two functions both in $\mathcal{D} \subset L^1$, we get that for all $\phi \in \mathcal{S}$,

$$\begin{aligned}
\langle \mathcal{F}((\mathrm{p.\,v.}\,K) * f), \phi\rangle &= \langle \mathrm{p.\,v.}\,K * f, \widehat{\phi}\rangle = \langle \mathrm{p.\,v.}\,K, \check f * \widehat{\phi}\rangle = \lim_{k\to\infty} \langle \zeta_k, \check f * \widehat{\phi}\rangle \\
&= \lim_{k\to\infty} \langle \mathcal{F}(\zeta_k * f), \phi\rangle = \lim_{k\to\infty} \langle (2\pi)^{n/2}\widehat{\zeta_k}\hat f, \phi\rangle \\
&= \lim_{k\to\infty} \langle (2\pi)^{n/2}\zeta_k, \mathcal{F}(\hat f\phi)\rangle = \langle (2\pi)^{n/2}\,\mathrm{p.\,v.}\,K, \mathcal{F}(\hat f\phi)\rangle \\
&= \langle (2\pi)^{n/2}\,\mathcal{F}(\mathrm{p.\,v.}\,K)\,\mathcal{F}(f), \phi\rangle,
\end{aligned} \tag{4.99}$$

as claimed.

Aiming at establishing the fact that $\mathcal{F}(\mathrm{p.\,v.}\,K) = \widehat{K}$, construct a family of functions $g_\rho \in \mathcal{D}$, $\rho > 0$, such that given any $R > 0$,

$$\widehat{g_\rho}(\xi) \ne 0 \text{ for all } \xi \in B(0,R) \tag{4.100}$$

when ρ is sufficiently large. Such functions are given, for example, by

$$g_\rho(x) := \sigma_\rho\,\tilde\chi_{0,1}(x) = \begin{cases} \exp\left(1 + \frac{1}{\rho^2|x|^2-1}\right), & |x| < 1/\rho \\ 0, & |x| \ge 1/\rho, \end{cases} \tag{4.101}$$

which can be seen as follows: Because

$$\widehat{g_1}(0) = \int g_1(x)\,dx > 0, \tag{4.102}$$

and $\widehat{g_1} \in \mathcal{S}$ is continuous, $\widehat{g_1}(\xi) \neq 0$ in some neighbourhood $B(0,\tilde{\rho})$. By (A.58), $\widehat{g_\rho}(\xi) = \rho^{-n}\widehat{g_1}(\xi/\rho) \neq 0$ when $|\xi| < \rho\tilde{\rho}$. Hence, choosing $\rho \geq R/\tilde{\rho}$ ensures that $\widehat{g_\rho}(\xi) \neq 0$ when $\xi \in B(0,R)$.

Now for any $\phi \in \mathcal{D}$, let R_ϕ be such that $\operatorname{supp}\phi \subset B(0,R_\phi)$ and choose ρ so large that $\widehat{g_\rho}(\xi) \neq 0$ for all $\xi \in B(0,R_\phi+1)$. Then by (4.85) and (4.86),

$$\begin{aligned}\langle \mathcal{F}(\mathrm{p.v.}\,K), \phi\rangle &= \Big\langle \mathcal{F}(\mathrm{p.v.}\,K), \frac{\widehat{g_\rho}\,\tilde{\chi}_{R_\phi,R_\phi+1}}{\widehat{g_\rho}}\,\phi\Big\rangle \\ &= \Big\langle \mathcal{F}(\mathrm{p.v.}\,K)\,\widehat{g_\rho}, \frac{\tilde{\chi}_{R_\phi,R_\phi+1}}{\widehat{g_\rho}}\,\phi\Big\rangle \\ &= \Big\langle (2\pi)^{-n/2}\,\mathcal{F}\Big(L^2\text{-}\lim_{\substack{\varepsilon\to 0\\ \eta\to\infty}} K_{\varepsilon,\eta} * g_\rho\Big), \frac{\tilde{\chi}_{R_\phi,R_\phi+1}}{\widehat{g_\rho}}\,\phi\Big\rangle,\end{aligned} \tag{4.103}$$

whence Theorem 4.7(c) implies that

$$\langle \mathcal{F}(\mathrm{p.v.}\,K), \phi\rangle = \Big\langle \widehat{K}\widehat{g_\rho}, \frac{\tilde{\chi}_{R_\phi,R_\phi+1}}{\widehat{g_\rho}}\,\phi\Big\rangle = \langle \widehat{K}, \phi\rangle. \tag{4.104}$$

Consequently $\mathcal{F}(\mathrm{p.v.}\,K) = \widehat{K}$, which is essentially bounded by Theorem 4.7(c). □

The principal results of the previous theorems can be summarized as

Corollary 4.9. *Let $n \in \mathbb{Z}_+$ and let $K : \mathbb{R}^n \to \mathbb{C}$ be of the form*

$$K(x) = \frac{\Omega(\frac{x}{|x|})}{|x|^n} \tag{4.105}$$

where $\Omega : S^{n-1} \to \mathbb{C}$ is an odd, bounded function. Then the essentially bounded function $\mathcal{F}(\mathrm{p.v.}\,K)$ is given almost everywhere by

$$\mathcal{F}(\mathrm{p.v.}\,K)(\xi) = -i\pi\,(2\pi)^{-n/2}\int_{S_+^{n-1}(\xi)} \Omega(\theta)\,d\theta. \tag{4.106}$$

For all $f \in L^2$,

$$\mathcal{F}(\mathrm{p.v.}\,K * f) = (2\pi)^{n/2}\,\mathcal{F}(\mathrm{p.v.}\,K)\,\hat{f} \tag{4.107}$$

and

$$\|\mathrm{p.v.}\,K * f\|_{L^2} \leq (2\pi)^{n/2}\,\|\mathcal{F}(\mathrm{p.v.}\,K)\|_{L^\infty}\,\|f\|_{L^2}. \tag{4.108}$$

4.3 Riesz Transforms

The results of the Calderón-Zygmund theory, derived in the previous section, can now be applied to the Riesz transforms. This will enable us to to calculate the derivatives of the Riesz potential, and to finally show that $\Lambda\,R_1 * f = f$ in Chapter 5.

Theorem 4.10. *Let $n \in \mathbb{Z}_+$. For almost all $\xi \in \mathbb{R}^n$,*

$$\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j}\right)(\xi) = i\,(2\pi)^{-n/2}\,\frac{\xi_j}{|\xi|}. \tag{4.109}$$

The Riesz transform $\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} *$ *is a bounded operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, and*

$$\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * f\right)(\xi) = i\,\frac{\xi_j}{|\xi|}\,\hat{f}(\xi) \tag{4.110}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. For $n \geq 2$, recall that

$$\frac{\partial R_1}{\partial x_j}(x) = b_n(1-n)\,\frac{x_j}{|x|^{n+1}} = b_n(1-n)\,\frac{\theta_j}{|x|^n}. \tag{4.111}$$

First assume that $|\xi| = 1$ and choose an orthonormal basis $(u^1, \ldots, u^n)$ of $\mathbb{R}^n$ with $u^1 = \xi$, so that

$$\theta_j = \sum_{k=1}^{n}(\theta \cdot u^k)u_j^k = (\theta\cdot\xi)\xi_j + \sum_{k=2}^{n}(\theta\cdot u^k)u_j^k. \tag{4.112}$$

Using Corollary 4.9, with $\Omega(\theta) = \theta_j$, which is clearly an odd and bounded function, we have that almost everywhere

$$\begin{aligned}\mathcal{F}&\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j}\right)(\xi)\\ &= -\frac{i\pi\,b_n\,(1-n)}{(2\pi)^{n/2}}\left(\xi_j\int_{S_+^{n-1}(\xi)}\theta\cdot\xi\,d\theta + \sum_{k=2}^{n}u_j^k\int_{S_+^{n-1}(\xi)}\theta\cdot u^k\,d\theta\right).\end{aligned} \tag{4.113}$$

For $k \neq 1$, the integrand $\theta \cdot u^k$ is positive on one half of $S_+^{n-1}(\xi)$ and negative but equal in absolute value on the other half, whence all but the first term in the sum vanish. (See Figure 4.1.)

Using the spherical coordinates of Lemma A.13 with $\theta\cdot\xi = x_1 = \cos\varphi$,

$$\int_{S_+^{n-1}(\xi)}\theta\cdot\xi\,d\theta = |S^{n-2}|\int_0^{\pi/2}\cos\varphi\sin^{n-2}\varphi\,d\varphi = |S^{n-2}|\,\Big/_{0}^{\pi/2}\frac{\sin^{n-1}\varphi}{n-1} = \frac{|S^{n-2}|}{n-1}. \tag{4.114}$$

Hence

$$\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j}\right)(\xi) = -\frac{i\pi}{(2\pi)^{n/2}}\,\frac{1}{\pi\,|S^{n-2}|}\,(1-n)\,\xi_j\,\frac{|S^{n-2}|}{n-1} = i\,(2\pi)^{-n/2}\,\xi_j \tag{4.115}$$

as claimed, since we assumed that $|\xi| = 1$. If then $|\xi| \neq 1$ and $\xi \neq 0$, we see from (4.106) that since $S_+^{n-1}(\xi)$ only depends on the direction of ξ and not on its absolute value,

$$\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j}\right)(\xi) = \mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j}\right)\left(\frac{\xi}{|\xi|}\right) = i\,(2\pi)^{-n/2}\left(\frac{\xi}{|\xi|}\right)_j = i\,(2\pi)^{-n/2}\,\frac{\xi_j}{|\xi|}. \tag{4.116}$$

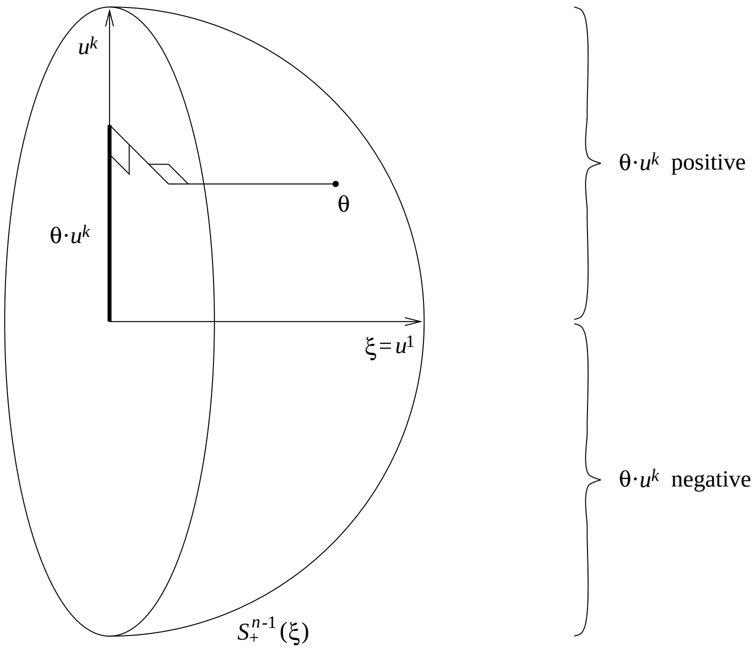


Figure 4.1: The integrand $\theta \cdot u^k$, $k \neq 1$, is positive on one half of $S_+^{n-1}(\xi)$ and negative but equal in absolute value on the other half.

In the special case $n = 1$

$$\frac{\partial R_1}{\partial x}(x) = \frac{\Omega(\frac{x}{|x|})}{|x|} \tag{4.117}$$

where $\Omega(\theta) = -\theta/\pi$, and

$$S_+^0(\xi) = \{\tfrac{\xi}{|\xi|}\} = \begin{cases} \{1\}, & \xi > 0 \\ \{-1\}, & \xi < 0. \end{cases} \tag{4.118}$$

Therefore

$$\mathcal{F}\left(\text{p. v.}\,\frac{\partial R_1}{\partial x}\right)(\xi) = -\frac{i\pi}{\sqrt{2\pi}} \int_{\{\frac{\xi}{|\xi|}\}} -\frac{\theta}{\pi}\, d\theta = \frac{i}{\sqrt{2\pi}} \frac{\xi}{|\xi|} \tag{4.119}$$

by Corollary 4.9, as claimed.

Boundedness and (4.110) follow from (4.108) and (4.107). □

We are now ready to derive the representation of Λf as a sum of Riesz transforms of the derivatives of f.

Theorem 4.11. *For all $n \in \mathbb{Z}_+$, the Calderón operator Λ can be expressed as*

$$\Lambda = -\sum_{j=1}^{n} \text{p. v.}\,\frac{\partial R_1}{\partial x_j} * D_j. \tag{4.120}$$

It is a continuous operator from $H^1(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Proof. Write Λ_0 for the operator on the right hand side of (4.120). If $f \in H^1$, its first order distribution derivatives $D_j f \in L^2 \subset \mathcal{S}'$ satisfy

$$\|D_j f\|_{L^2} = \|\mathcal{F}(D_j f)\|_{L^2} = \|\xi_j \hat{f}\|_{L^2} \leq \|(1+|\xi|^2)^{1/2} \hat{f}\|_{L^2} = \|f\|_{H^1}. \tag{4.121}$$

By Theorem 4.7, $\|\mathrm{p.v.} \frac{\partial R_1}{\partial x_j} * D_j f\|_{L^2} \leq M_j \|D_j f\|_{L^2}$ for some $M_j < \infty$. Therefore,

$$\|\Lambda_0 f\|_{L^2} \leq C\|f\|_{H^1}, \qquad C = \sum_{j=1}^{n} M_j, \tag{4.122}$$

showing that $\Lambda_0 : H^1 \to L^2$ is continuous. By Corollary 4.9, Theorem 4.10 and Equation (A.57),

$$\begin{aligned} \mathcal{F}\left(\mathrm{p.v.} \frac{\partial R_1}{\partial x_j} * D_j f\right) &= (2\pi)^{n/2} \mathcal{F}\left(\mathrm{p.v.} \frac{\partial R_1}{\partial x_j}\right) \mathcal{F}(D_j f) \\ &= i \frac{\xi_j}{|\xi|} i \xi_j \hat{f} = -\frac{\xi_j^2}{|\xi|} \hat{f} \end{aligned} \tag{4.123}$$

and hence

$$\mathcal{F}(\Lambda_0 f) = \frac{\sum_{j=1}^{n} \xi_j^2}{|\xi|} \hat{f} = |\xi| \hat{f}. \tag{4.124}$$

Therefore, $\Lambda_0 = \Lambda$. □

The following theorem shows that in the partial derivatives of the Riesz potential, the derivative can be taken under the integral sign if the integral is understood as a principal value integral.

Its proof uses Theorems A.7 and A.8, for the proofs of which we only refer to [Ste] and [Zie]. This is the only place in this work where a proof relies on outside references, save of course some very basic results.

Theorem 4.12. *If $f \in L^2(\mathbb{R}^n)$, then*

$$D_j(R_1 * f) = \mathrm{p.v.} \frac{\partial R_1}{\partial x_j} * f. \tag{4.125}$$

Proof. Choose a sequence of test functions $\phi_k \in \mathcal{D}$ with $\|f - \phi_k\|_{L^2} \xrightarrow{k\to\infty} 0$. Theorems 4.1 and 4.10 assert that for all test functions $\psi \in \mathcal{D}$,

$$\begin{aligned} \langle D_j(R_1 * \phi_k), \psi\rangle &= \left\langle R_1 * \phi_k, -\frac{\partial \psi}{\partial x_j}\right\rangle \\ &= \left\langle (2\pi)^{n/2} \mathcal{F}R_1(\xi)\widehat{\phi_k}(\xi), -\mathcal{F}^{-1}\left(\frac{\partial \psi}{\partial x_j}\right)\right\rangle \\ &= \left\langle |\xi|^{-1}\widehat{\phi_k}(\xi), i\,\xi_j\, \mathcal{F}^{-1}(\psi)\right\rangle \\ &= \left\langle i \frac{\xi_j}{|\xi|} \widehat{\phi_k}(\xi), \mathcal{F}^{-1}(\psi)\right\rangle \\ &= \left\langle \mathrm{p.v.} \frac{\partial R_1}{\partial x_j} * \phi_k, \psi\right\rangle, \end{aligned} \tag{4.126}$$

so that $D_j(R_1 * \phi_k) = \text{p.v.}\, \frac{\partial R_1}{\partial x_j} * \phi_k$. By Theorem 4.10,

$$\|D_j(R_1 * \phi_k)\|_{L^2} = \left\|\text{p.v.}\, \frac{\partial R_1}{\partial x_j} * \phi_k\right\|_{L^2} \leq C_1 \|\phi_k\|_{L^2} \leq C_1 \sup_{k\in\mathbb{N}} \|\phi_k\|_{L^2}, \tag{4.127}$$

which is finite since $\phi_k \to f$. As $L^2(\mathbb{R}^n)$ is a reflexive and separable Banach space, the Banach-Alaoglu theorem (Theorem A.6) implies the existence of a weakly convergent subsequence $(D_j(R_1 * \phi_{k_l}))_{l\in\mathbb{N}}$. This means that there is a $v_j \in L^2(\mathbb{R}^n)$ such that for all test functions $\psi \in \mathcal{D}$,

$$\langle D_j(R_1 * \phi_{k_l}), \psi\rangle = (D_j(R_1 * \phi_{k_l}), \overline{\psi}) \xrightarrow{l\to\infty} (v_j, \overline{\psi}) = \langle v_j, \psi\rangle. \tag{4.128}$$

Corollary A.9 shows that for all compact sets $E \subset \mathbb{R}^n$ there are constants $q \geq 2$ and $C_2 > 0$ such that

$$\|\chi_E\, R_1 * g\|_{L^q} \leq C_2 \|g\|_{L^2} \tag{4.129}$$

for all $g \in L^2$, so that $R_1 * f$ and $R_1 * \phi_k$ are in $L^q_{\text{loc}} \subset L^1_{\text{loc}} \subset \mathcal{S}'$ and

$$\|\chi_E\, (R_1 * f - R_1 * \phi_k)\|_{L^q} \leq C_2 \|f - \phi_k\|_{L^2} \xrightarrow{k\to\infty} 0. \tag{4.130}$$

Thus for all test functions $\psi \in \mathcal{D}$,

$$\begin{aligned}\left|\left\langle R_1 * \phi_{k_l}, \frac{\partial\psi}{\partial x_j}\right\rangle - \left\langle R_1 * f, \frac{\partial\psi}{\partial x_j}\right\rangle\right| &\leq \left|\int_{\text{supp}\,\psi} (R_1 * \phi_{k_l} - R_1 * f)\, \frac{\partial\psi}{\partial x_j}\, dx\right| \\ &\leq \|\chi_{\text{supp}\,\psi}\, (R_1 * \phi_{k_l} - R_1 * f)\|_{L^q}\, \|\frac{\partial\psi}{\partial x_j}\|_{L^{q'}} \\ &\xrightarrow{l\to\infty} 0,\end{aligned} \tag{4.131}$$

which shows that

$$\langle D_j(R_1 * \phi_{k_l}), \psi\rangle = \left\langle R_1 * \phi_{k_l}, -\frac{\partial\psi}{\partial x_j}\right\rangle \xrightarrow{l\to\infty} \left\langle R_1 * f, -\frac{\partial\psi}{\partial x_j}\right\rangle = \langle D_j(R_1 * f), \psi\rangle. \tag{4.132}$$

When combined with (4.128), this implies that

$$\langle v_j, \psi\rangle = \langle D_j(R_1 * f), \psi\rangle \tag{4.133}$$

for all ψ in the dense set $\mathcal{D} \subset L^2$, which implies that $D_j(R_1 * f) \in L^2$ is the weak limit of the subsequence $(D_j(R_1 * \phi_{k_l}))_{l\in\mathbb{N}}$.

Again by Theorem 4.10,

$$\left\|\text{p.v.}\, \frac{\partial R_1}{\partial x_j} * \phi_{k_l} - \text{p.v.}\, \frac{\partial R_1}{\partial x_j} * f\right\|_{L^2} \leq C_1 \|\phi_{k_l} - f\|_{L^2} \xrightarrow{k\to\infty} 0, \tag{4.134}$$

so that

$$D_j(R_1 * \phi_{k_l}) = \text{p.v.}\, \frac{\partial R_1}{\partial x_j} * \phi_{k_l} \xrightarrow{l\to\infty} \text{p.v.}\, \frac{\partial R_1}{\partial x_j} * f \quad \text{strongly.} \tag{4.135}$$

Because strong convergence implies weak convergence, we conclude that $D_j(R_1 * f) = \text{p.v.}\, \frac{\partial R_1}{\partial x_j} * f$. □

Chapter 5

Derivation of the Reconstruction Formulae

We are now ready to derive rigorously the reconstruction formulae of Theorems 3.7 and 3.9, which will be repeated later on as Corollary 5.19 and Theorem 5.20, respectively. We shall follow the treatment in [SK].

5.1 The X-ray Domain $D_{\mathbf{xr}}$

The function to be reconstructed will be allowed to be in the space D_{xr}, defined as follows:

Definition 5.1. *For $n \geq 2$, the space*

$$D_{\mathrm{xr}}(\mathbb{R}^n) = \{g \in L^2(\mathbb{R}^n) \,|\, (1+|x|)^{1-n} g \in L^1(\mathbb{R}^n)\} \tag{5.1}$$

with norm

$$\|g\|_{\mathrm{xr}} = \|g\|_{L^2} + \|(1+|x|)^{1-n} g(x)\|_{L^1} \tag{5.2}$$

is called the x-ray domain *over $\mathbb{R}^n$.*

This is a useful domain because $R_1 * f$ behaves nicely for such f that $(1+|x|)^{1-n} f \in L^1$, as will be shown shortly in Theorem 5.3. The requirement that the function belong to L^2 is natural since the derivations of the properties of the Riesz transform $\frac{\partial R_1}{\partial x_j}*$ and the Calderón operator Λ were done in the previous chapter for L^2 spaces.

Note that when $n > 2$ and $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\|(1+|x|)^{1-n} f(x)\|_{L^1}^2 &\leq \|(1+|x|)^{1-n}\|_{L^2}^2 \|f\|_{L^2}^2 \\ &= |S^{n-1}| \int_0^\infty \frac{r^{n-1}}{(1+r)^{2n-2}}\, dr\, \|f\|_{L^2}^2 \\ &\leq |S^{n-1}| \left[\int_0^1 dr + \int_1^\infty \frac{r^{n-1}}{2^{2n-2} r^{2n-2}}\, dr\right] \|f\|_{L^2}^2 < \infty\end{aligned} \tag{5.3}$$

so that D_{xr} is simply L^2. In fact after proving Theorem 5.3 we will be able to show using Corollary A.9 that this is also true when $n = 2$. However, since Corollary A.9 uses Theorems A.7 and A.8 whose proofs are not presented in this work, we shall continue to speak of D_{xr} following the treatment in [SK], in attempt to minimize the number of places where proofs rely on outside references. This is also expedient for emphasizing that the difference between the norms $\|\cdot\|_{L^2}$ and $\|\cdot\|_{\mathrm{xr}}$.

In showing that $R_1 * f$ is well-behaved for $f \in D_{\mathrm{xr}}$, we shall need the following result:

Lemma 5.2. *If $n \geq 2$, $\phi \in L^1(\mathbb{R}^n)$ and $(1+|x|)^n\phi(x) \in L^\infty(\mathbb{R}^n)$, then*

$$|R_1 * \phi(x)| \leq C\,(1+|x|)^{1-n}\left[\|\phi\|_{L^1} + \|(1+|y|)^n\phi(y)\|_{L^\infty}\right] \tag{5.4}$$

for some constant $C > 0$, independent of ϕ.

Proof. It suffices to prove the claim for $\psi(x) := |\phi(x)|$ because then

$$\begin{aligned}|R_1 * \phi(x)| &\leq |R_1| * |\phi|(x) = R_1 * \psi(x)\\ &\leq C\,(1+|x|)^{1-n}\left(\|\psi\|_{L^1} + \|(1+|y|)^n\psi(y)\|_{L^\infty}\right)\\ &= C(1+|x|)^{1-n}\left(\|\phi\|_{L^1} + \|(1+|y|)^n\phi(y)\|_{L^\infty}\right).\end{aligned} \tag{5.5}$$

Observe that

$$\psi(x) = (1+|x|)^{-n}\,(1+|x|)^n\,\psi(x) \leq M\,(1+|x|)^{-n}, \tag{5.6}$$

where $M := \|(1+|y|)^n\psi(y)\|_{L^\infty}$ is finite by assumption. Distinguish two cases, $|x| < 1$ and $|x| \geq 1$.

When $|x| < 1$, divide the range of integration into two parts:

$$R_1 * \psi(x) = b_n \int_{|x-y|\leq 1} \frac{\psi(y)}{|x-y|^{n-1}}\,dy + b_n \int_{|x-y|>1} \frac{\psi(y)}{|x-y|^{n-1}}\,dy. \tag{5.7}$$

In the first integral, make the change of variable $z = y - x$, $dz = dy$ and note that $\psi(y) \leq M(1+|x|)^{-n} \leq M$. In the second integral, $1/|x-y|^{n-1} \leq 1$. Therefore,

$$\begin{aligned}R_1 * \psi(x) &\leq b_n M \int_{|z|\leq 1} \frac{dz}{|z|^{n-1}} + b_n \int_{|x-y|>1} \psi(y)\,dy\\ &\leq b_n M\,|S^{n-1}| \int_0^1 r^{1-n} r^{n-1}\,dr + b_n\|\psi\|_{L^1}\\ &\leq b_n\,|S^{n-1}|\left(M + \|\psi\|_{L^1}\right)\\ &\leq 2^{n-1} b_n\,|S^{n-1}|\,(1+|x|)^{1-n}\left(M + \|\psi\|_{L^1}\right),\end{aligned} \tag{5.8}$$

as $2^{n-1}\,(1+|x|)^{1-n} \geq 1$ for $|x| < 1$.

For $|x| \geq 1$, divide the domain of integration into three parts:

$$R_1 * \psi(x) = \int_{|y|\leq\frac{|x|}{2}} \frac{b_n\,\psi(y)\,dy}{|x-y|^{n-1}} + \int_{\frac{|x|}{2}<|y|<2|x|} \frac{b_n\,\psi(y)\,dy}{|x-y|^{n-1}} + \int_{|y|\geq 2|x|} \frac{b_n\,\psi(y)\,dy}{|x-y|^{n-1}}. \tag{5.9}$$

When $|y| \le |x|/2$,

$$|x-y| \ge ||x|-|y|| = |x|-|y| \ge |x| - \frac{|x|}{2} = \frac{|x|}{2} \tag{5.10}$$

and the first integral is majorized by

$$\frac{2^{n-1}\, b_n}{|x|^{n-1}} \int_{|y|\le \frac{|x|}{2}} \psi(y)\, dy \le \frac{2^{n-1}\, b_n}{|x|^{n-1}} \, \|\psi\|_{L^1}. \tag{5.11}$$

When $|x|/2 < |y| < 2\,|x|$, write $x = |x|\,\theta$, $\theta \in S^{n-1}$, and make the change of variable $y = |x|\, z$, $dy = |x|^n\, dz$, with $|z| = |y|/|x| \in (\frac{1}{2}, 2)$. Then $|x-y| = |x||\theta - z|$ and the integral becomes

$$\frac{b_n}{|x|^{n-1}} \int_{\frac{1}{2}<|z|<2} \frac{\psi(|x|\,z)\,|x|^n}{|\theta - z|^{n-1}}\, dz. \tag{5.12}$$

The estimates $|x|^n = (|x||z|)^n\, |z|^{-n} \le (1+|x||z|)^n\, 2^n$ and

$$\psi(|x|\,z) = (1+|x||z|)^{-n}\,(1+|x||z|)^n\,\psi(|x|\,z) \le M\,(1+|x||z|)^{-n} \tag{5.13}$$

show that the second integral is less than or equal to

$$\begin{aligned} \frac{2^n\, M\, b_n}{|x|^{n-1}} \int_{\frac{1}{2}<|z|<2} \frac{dz}{|\theta - z|^{n-1}} &\le \frac{2^n\, M\, b_n}{|x|^{n-1}} \int_{|w|<3} \frac{dw}{|w|^{n-1}} \\ &= \frac{2^n\, M\, b_n}{|x|^{n-1}}\, |S^{n-1}| \int_0^3 \frac{r^{n-1}\, dr}{r^{n-1}} \\ &= \frac{2^n\, 3\, M\, b_n\, |S^{n-1}|}{|x|^{n-1}}. \end{aligned} \tag{5.14}$$

Finally, when $|y| \ge 2\,|x|$, the situation is much like when $|y| \le |x|/2$: Now $|x-y| \ge ||x|-|y|| = |y|-|x| \ge 2\,|x| - |x| = |x|$, and the third integral has

$$\frac{b_n}{|x|^{n-1}} \int_{|y|\ge 2\,|x|} \psi(y)\, dy \le \frac{b_n}{|x|^{n-1}}\, \|\psi\|_{L^1} \tag{5.15}$$

as an upper bound.

The upper bounds (5.11), (5.14) and (5.15) for the three integrals in (5.9) combine to show that the claim holds:

$$\begin{aligned} R_1 * \psi(x) &\le \frac{(2^{n-1}+1)\, b_n}{|x|^{n-1}}\, \|\psi\|_{L^1} + \frac{2^n\, 3\, b_n\, |S^{n-1}|}{|x|^{n-1}}\, M \\ &\le 2^n\, 3\, b_n\, |S^{n-1}|\, (1+|x|)^{n-1}\, \big(\|\psi\|_{L^1} + M\big). \end{aligned} \tag{5.16}$$

□

This result helps to show that the condition that $(1+|x|)^{1-n} f \in L^1$ is both necessary and sufficient for $R_1 * f$ to converge absolutely almost everywhere:

Theorem 5.3. *If $n \geq 2$ and $(1+|x|)^{1-n} f(x) \in L^1(\mathbb{R}^n)$, then the convolution integral $R_1 * f(x)$ converges absolutely for almost almost all $x \in \mathbb{R}^n$, and $R_1 * f \in L^1_{\text{loc}}(\mathbb{R}^n)$. Furthermore, there is a constant $C > 0$ independent of f, such that*

$$\Big| \int_{\mathbb{R}^n} R_1 * f(x)\, \phi(x)\, dx \Big| \leq \big\|(1+|x|)^{1-n} f\big\|_{L^1} \left[\|\phi\|_{L^1} + \|(1+|x|)^n \phi\|_{L^\infty} \right] \tag{5.17}$$

for all measurable functions $\phi : \mathbb{R}^n \to \mathbb{C}$. Conversely, if the convolution integral $R_1 * f(x_0)$ converges absolutely for some $x_0 \in \mathbb{R}^n$, then $(1+|x|)^{1-n} f \in L^1(\mathbb{R}^n)$.

Proof. The triangle inequality and Fubini's theorem for non-negative functions show that

$$\begin{aligned} \Big| \int_{\mathbb{R}^n} R_1 * f(x)\, \phi(x)\, dx \Big| &= \Big| \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_1(x-y) f(y) \phi(x)\, dy\, dx \Big| \\ &\leq \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} R_1(x-y) |f(y)| |\phi(x)|\, dy\, dx \\ &= \int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} R_1(y-x)\, |\phi(x)|\, dx\, dy. \end{aligned} \tag{5.18}$$

By right of Lemma 5.2,

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(y)| \int_{\mathbb{R}^n} R_1(y-x)\, |\phi(x)|\, dx\, dy \\ &\qquad \leq C \int_{\mathbb{R}^n} |f(y)|\, (1+|y|)^{1-n}\, dy \left[\|\phi\|_{L^1} + \|(1+|y|)^n \phi(y)\|_{L^\infty} \right], \end{aligned} \tag{5.19}$$

which proves (5.17) and also shows that $R_1 * f \in L^1_{\text{loc}}$ indeed: For any compact $K \subset \mathbb{R}^n$,

$$\begin{aligned} \int_K |R_1 * f(x)|\, dx &\leq \int_{\mathbb{R}^n} R_1 * |f|(x)\, \chi_K(x)\, dx \\ &\leq C\, \big\|(1+|y|)^{1-n} f\big\|_{L^1} \left[\|\chi_K\|_{L^1} + \|(1+|y|)^n \chi_K(y)\|_{L^\infty} \right] < \infty. \end{aligned} \tag{5.20}$$

This also implies absolute convergence almost everywhere, for if there were a set $E \subset \mathbb{R}^n$ with $m(E) > 0$ such that $R_1 * |f|(x) = \infty$ for all $x \in E$, then the regularity of the Lebesgue measure would assure the existence of a compact set $K \subset E$ with $m(K) \geq m(E) - m(E)/2 > 0$, and consequently

$$\int_K R_1 * |f|(x) = \infty, \tag{5.21}$$

in contradiction with the local integrability just proved.

For the converse claim, note that if $|x_0| \leq 1$, $1 + |y| \geq |x_0| + |y| \geq |x_0 - y|$, and if $|x_0| > 1$,

$$1 + |y| \geq 1 + \frac{|y|}{|x_0|} = \frac{|x_0| + |y|}{|x_0|} \geq \frac{|x_0 - y|}{|x_0|}. \tag{5.22}$$

Therefore, $1 + |y| \geq \min\{1, |x_0|^{-1}\}\, |x_0 - y|$ and

$$\int_{\mathbb{R}^n} \big|(1+|y|)^{1-n} f(y)\big|\, dy \leq \max\{1, |x_0|^{n-1}\} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x_0 - y|^{n-1}}\, dy \tag{5.23}$$

is finite if $R_1 * f(x_0)$ converges absolutely. ∎

It now follows that the x-ray domain D_{xr} is actually just the space of square-integrable functions; the requirement that $(1+|x|)^{1-n} f(x) \in L^1$ is satisfied by all functions $f \in L^2$:

Corollary 5.4. *For all $n \geq 2$, $D_{\mathrm{xr}}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$.*

Proof. If $f \in L^2$, then also $|f| \in L^2$, so that Corollary A.9 asserts that $R_1 * |f| \in L^q_{\mathrm{loc}} \subset L^1_{\mathrm{loc}}$. Therefore the convolution integral $R_1 * f$ must converge absolutely almost everywhere. As this is equivalent to the condition that $(1+|x|)^{1-n} f(x) \in L^1$ by Theorem 5.3, we see that $f \in D_{\mathrm{xr}}$. The converse inclusion is clear from the definition of D_{xr}. □

As already stated after Definition 5.1 of the x-ray domain D_{xr}, we shall not use this result because its proof refers through Corollary A.9 to Theorems A.7 and A.8 whose proofs are not presented in this work.

Another property of the x-ray domain is its stability with respect to mollifying with certain kernels:

Theorem 5.5. *If $n \geq 2$, $g \in D_{\mathrm{xr}}(\mathbb{R}^n)$ and $(1+|x|)^{1-n} e(x) \in L^1(\mathbb{R}^n)$, then $e * g \in D_{\mathrm{xr}}(\mathbb{R}^n)$ and*

$$\|e * g\|_{\mathrm{xr}} \leq C \, \|(1+|x|)^{1-n} e(x)\|_{L^1} \|g\|_{\mathrm{xr}} \tag{5.24}$$

for some constant $C > 0$ independent of g and e.

Proof. Young's and Hölder's inequalities (Theorems A.5 and A.1) assert that

$$\begin{aligned} \|e * g\|_{L^2} &\leq \|e\|_{L^1} \|g\|_{L^2} \\ &= \|(1+|x|)^{1-n} (1+|x|)^{n-1} e\|_{L^1} \|g\|_{L^2} \\ &\leq \|(1+|x|)^{1-n}\|_{L^\infty} \|(1+|x|)^{n-1} e\|_{L^1} \|g\|_{L^2} \\ &\leq \|(1+|x|)^{n-1} e\|_{L^1} \|g\|_{\mathrm{xr}}. \end{aligned} \tag{5.25}$$

which settles the estimate for the L^2 part of $\|e*g\|_{\mathrm{xr}} = \|e*g\|_{L^2} + \|(1+|x|)^{1-n} \, e*g\|_{L^1}$.

For the other part, using the fact that

$$\begin{aligned} (1+a^2)(1+b)^2 &= 1 + 2b + b^2 + a^2 + 2ab + a^2 b^2 \\ &= 1 + (a+b)^2 + 2b(1-a+a^2) + a^2 b^2 \\ &\geq 1 + (a+b)^2 \end{aligned} \tag{5.26}$$

for $b \geq 0$, with $a = |x|$ and $b = |y|$, gives

$$1 + |x-y|^2 \leq 1 + (|x| + |y|)^2 \leq \left(1+|x|^2\right)(1+|y|)^2 \tag{5.27}$$

and consequently

$$\left(1+|x|^2\right)^{\frac{1-n}{2}} \leq \left(1+|x-y|^2\right)^{\frac{1-n}{2}} (1+|y|)^{n-1}. \tag{5.28}$$

As $1+|x| \le (1+|x|^2)^{1/2}$, estimate (5.28) and Young's inequality yield

$$\begin{aligned}\left\|(1+|x|)^{1-n}e*g\right\|_{L^1} &\le \left\|\left(1+|x|^2\right)^{\frac{1-n}{2}} e*g(x)\right\|_{L^1}\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |e(y)||g(x-y)| \left(1+|x|^2\right)^{\frac{1-n}{2}}\,dy\,dx\\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(1+|x-y|^2\right)^{\frac{1-n}{2}} |g(x-y)|\,(1+|y|)^{n-1}\,|e(y)|\,dy\,dx\\ &= \|e_1*g_1\|_{L^1} \le \|e_1\|_{L^1}\|g_1\|_{L^1},\end{aligned} \tag{5.29}$$

where

$$e_1(x) = (1+|x|)^{n-1}\,|e(x)| \qquad \text{and} \qquad g_1(x) = \left(1+|x|^2\right)^{\frac{1-n}{2}}\,|g(x)|. \tag{5.30}$$

Since

$$\|g_1\|_{L^1} = \left\|\left(\frac{1+2\,|x|+|x|^2}{1+|x|^2}\right)^{\frac{n-1}{2}}(1+|x|)^{1-n}g(x)\right\|_{L^1} \le 2^{\frac{n-1}{2}}\left\|(1+|x|)^{1-n}g(x)\right\|_{L^1}, \tag{5.31}$$

this shows that

$$\begin{aligned}\left\|(1+|x|)^{1-n}e*g\right\|_{L^1} &\le 2^{\frac{n-1}{2}}\left\|(1+|x|)^{n-1}e\right\|_{L^1}\left\|(1+|x|)^{1-n}g\right\|_{L^1}\\ &\le 2^{\frac{n-1}{2}}\left\|(1+|x|)^{n-1}e\right\|_{L^1}\|g\|_{\mathrm{xr}},\end{aligned} \tag{5.32}$$

which together with (5.25) implies the claim with $C = 2^{(n-1)/2}+1$. □

It is pleasant to know that elements of D_{xr} can be approximated using smooth functions with bounded support.

Lemma 5.6. *If $n \ge 2$ and $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then*

$$\|f - \chi_{B(0,R)}f\|_{\mathrm{xr}} \to 0 \tag{5.33}$$

as $R \to \infty$. Furthermore, $C_0^\infty(\mathbb{R}^n)$ is dense in $D_{\mathrm{xr}}(\mathbb{R}^n)$.

Proof. Fix $f \in D_{\mathrm{xr}}$ and $\varepsilon > 0$. As $(1+|x|)^{1-n}f \in L^1$,

$$\int_{|x|>r_1} (1+|x|)^{1-n}\,|f(x)|\,dx < \frac{\varepsilon}{4} \tag{5.34}$$

for some $r_1 > 0$. Analogously, as $f \in L^2$,

$$\int_{|x|>r_2} |f(x)|^2\,dx < \frac{\varepsilon^2}{16} \tag{5.35}$$

for some $r_2 > 0$. Thus for $R \ge \max\{r_1, r_2\} =: r_0$,

$$\|f - \chi_{B(0,R)}f\|_{\mathrm{xr}} = \int_{|x|>R}(1+|x|)^{1-n}\,|f(x)|\,dx + \left(\int_{|x|>R}|f(x)|^2\,dx\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}, \tag{5.36}$$

which proves the first claim.

Then set $f_0 = \chi_{B(0,r_0)} f$ and $\phi_\rho = e_\rho * f_0$ with e_ρ the approximate delta function of (A.48), $\rho > 0$. As described in Section A.1.1, $\phi_\rho \in C^\infty$ and $\|f_0 - \phi_\rho\|_{L^p} \to 0$ as $\rho \to 0$, for all $1 \le p < \infty$ for which $f_0 \in L^p$. Because $\|f_0\|_{L^1} \le \|\chi_{B(0,\rho)}\|_{L^2} \|f\|_{L^2} < \infty$,

$$\left\|(1+|x|)^{1-n}(f_0 - \phi_\rho)\right\|_{L^1} \le \|f_0 - \phi_\rho\|_{L^1} < \frac{\varepsilon}{4} \qquad \text{when } \rho \ge \rho_1 \tag{5.37}$$

for some $\rho_1 > 0$, and because $\|f_0\|_{L^2} \le \|f\|_{L^2} < \infty$,

$$\|f_0 - \phi_\rho\|_{L^2} < \frac{\varepsilon}{4} \qquad \text{when } \rho \ge \rho_2 \tag{5.38}$$

for some $\rho_2 > 0$. Choosing $\phi = \phi_{\max\{\rho_1,\rho_2\}}$ yields

$$\|f - \phi\|_{\mathrm{xr}} \le \|f - f_0\|_{\mathrm{xr}} + \|f_0 - \phi\|_{\mathrm{xr}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon. \tag{5.39}$$

Since

$$\operatorname{supp}\phi_\rho \subset \{x + y \in \mathbb{R}^n \mid x \in \operatorname{supp} e_\rho,\ y \in \operatorname{supp} f_0\} \subset B(0, R+\rho) \tag{5.40}$$

is compact, $\phi \in C_0^\infty$. □

Before showing that the relation $\mathcal{F}(R_1 * f) = |\xi|\,\hat{f}$, formally derived in Chapter 3, holds in D_{xr}, we still prove three auxiliary results.

The first one shows that the Riesz potential $R_1 * g$ of a non-negative x-ray attenuation coefficient g has the superharmonic-type property that its average over a ball is at most as large as a constant times its value at the centre of the ball. [Fro]

Lemma 5.7 (Frostman's mean value theorem). *Let $n \ge 2$, $h = \chi_{B(0,1)}$ and*

$$h_\rho(x) = \rho^{-n} h(\tfrac{x}{\rho}) = \begin{cases} \rho^{-n}, & |x| < \rho \\ 0, & |x| \ge \rho. \end{cases} \tag{5.41}$$

Then there is a constant $C > 0$, independent of ρ, such that for all $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$ and $\rho > 0$,

$$|\langle R_1 * f, h_\rho\rangle| \le C\, R_1 * |f|(0). \tag{5.42}$$

Proof. Fubini's theorem shows that since the integrand is non-negative,

$$\begin{aligned} |\langle R_1 * f, h_\rho\rangle| &\le b_n\, \rho^{-n} \int_{B(0,\rho)} \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-1}}\, dy\, dx \\ &= b_n \int_{\mathbb{R}^n} \frac{|f(y)|}{|y|^{n-1}}\, I(y,\rho)\, dy, \end{aligned} \tag{5.43}$$

where

$$I(y,\rho) = \frac{|y|^{n-1}}{\rho^n} \int_{B(0,\rho)} \frac{dx}{|x-y|^{n-1}}. \tag{5.44}$$

It suffices to show that $I(y,\rho)$ is bounded for all $y \in \mathbb{R}^n$, $\rho > 0$.

Distinguish two cases, according to whether $|y| < 2\rho$ or $|y| \geq 2\rho$. In the first case,

$$
\begin{aligned}
0 \leq I(y,\rho) &\leq \frac{|y|^{n-1}}{\rho^n} \int_{B(y,3\rho)} \frac{dx}{|x-y|^{n-1}} \\
&= \frac{|y|^{n-1}}{\rho^n} |S^{n-1}| \int_0^{3\rho} \frac{r^{n-1}\,dr}{r^{n-1}} \\
&= \frac{|y|^{n-1}\,3\rho}{\rho^n} |S^{n-1}| \\
&\leq 2^{n-1} 3\,|S^{n-1}|.
\end{aligned}
\tag{5.45}
$$

For the second case $|y| \geq 2\rho$, note that $x \in B(0,\rho)$ implies that $|x| \leq \rho \leq |y|/2$ and thus

$$
|x-y| \geq \big||y|-|x|\big| \geq |y| - \frac{|y|}{2} = \frac{|y|}{2}. \tag{5.46}
$$

Therefore,

$$
0 \leq I(y,\rho) \leq \frac{|y|^{n-1}}{\rho^n} \int_{B(0,\rho)} \frac{2^{n-1}}{|y|^{n-1}}\,dx = \frac{2^{n-1}\,m(B(0,\rho))}{\rho^n} = 2^{n-1}\,|B^n|. \tag{5.47}
$$

This shows that in both cases,

$$
I(y,\rho) \leq \max\left\{2^{n-1}3\,|S^{n-1}|,\, 2^{n-1}\,|B^n|\right\} =: C. \tag{5.48}
$$

$\square$

The second auxiliary result is that the (weighted) averages of f and $R_1 * f$ over a bounded set tend to zero as the diameter of the set grows. This indicates that in some respect, the mass of those functions is not concentrated at infinity.

Lemma 5.8. *For any $e \in L_0^\infty(\mathbb{R}^n)$, define $e_\rho(x) = \rho^{-n} e(\frac{x}{\rho})$, $\rho > 0$. Then*

(a) if $n \in \mathbb{Z}_+$ and $\hat{f} \in L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n)$, then $\langle f, e_\rho\rangle \to 0$ as $\rho \to \infty$

(b) if $n \geq 2$ and $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then $\langle R_1 * f, e_\rho\rangle \to 0$ as $\rho \to \infty$.

Proof. For claim (a), write $f = f_1 + f_2$ with $\widehat{f_1} \in L^1$ and $\widehat{f_2} \in L^2$. The claim for f_2 follows from the estimate

$$
\begin{aligned}
|\langle f_2, e_\rho\rangle| &\leq \|f_2 e_\rho\|_{L^1} \leq \|f_2\|_{L^2} \|e_\rho\|_{L^2} \\
&= \|f_2\|_{L^2} \sqrt{\int_{\mathbb{R}^n} \rho^{-2n}\,|e(\tfrac{x}{\rho})|^2\,dx} \\
&= \|\widehat{f_2}\|_{L^2}\,\rho^{-n/2} \sqrt{\int_{\mathbb{R}^n} |e(y)|^2\,dy} \\
&\leq \rho^{-n/2}\,\|\widehat{f_2}\|_{L^2}\,\|e\|_{L^\infty}\,\sqrt{m(\operatorname{supp} e)} \xrightarrow{\rho\to\infty} 0,
\end{aligned}
\tag{5.49}
$$

where the change of variable $y = x/\rho$, $dy = \rho^{-n}\,dx$ has been used.

For estimating $\langle f_1, e_\rho\rangle$, use Fubini's theorem to see that

$$\begin{aligned}\langle f_1, e_\rho\rangle &= \int (2\pi)^{-n/2} \int e^{ix\cdot\xi} \widehat{f}_1(\xi)\, d\xi\, e_\rho(x)\, dx \\ &= (2\pi)^{-n/2} \int\int e^{ix\cdot\xi} \rho^{-n} e(\tfrac{x}{\rho})\, dx\, \widehat{f}_1(\xi)\, d\xi \\ &= (2\pi)^{-n/2} \int\int e^{-iy\cdot(-\rho\xi)} e(y)\, dy\, \widehat{f}_1(\xi)\, d\xi \\ &= (2\pi)^{-n/2} \int \widehat{e}(-\rho\xi)\, \widehat{f}_1(\xi)\, d\xi \xrightarrow{\rho\to\infty} 0\end{aligned} \tag{5.50}$$

by Lebesgue's theorem of dominated convergence: The Riemann-Lebesgue lemma (A.64) tells that the constant function 0 is the pointwise limit of the sequence of functions $\widehat{e}(-\rho\xi)\, \widehat{f}_1(\xi)$, $\rho \in \mathbb{Z}_+$, and

$$|\widehat{e}(-\rho\xi)\, \widehat{f}_1(\xi)| \leq \|\widehat{e}\|_{L^\infty} |\widehat{f}_1(\xi)| \leq (2\pi)^{-n/2} \|e\|_{L^1} |\widehat{f}_1(\xi)| \in L^1. \tag{5.51}$$

This completes the proof of claim (a).

For claim (b), first choose $R > 0$ so large that $\operatorname{supp} e \subset B(0, R)$. Write $h = \chi_{B(0,1)}$ and $h_\lambda(x) = \lambda^{-n} h(x/\lambda)$ for $\lambda > 0$. Then with $y = x/\rho$, $dy = \rho^{-n}\, dx$,

$$\begin{aligned}|\langle R_1 * f, e_\rho\rangle| &\leq \int_{\mathbb{R}^n} |R_1 * f(x)|\, |e_\rho(x)|\, dx \\ &\leq \int_{\mathbb{R}^n} R_1 * |f|(x)\, |\rho^{-n} e(\tfrac{x}{\rho})|\, dx \\ &= \int_{\mathbb{R}^n} R_1 * |f|(\rho y)\, |e(y)|\, dy \\ &\leq \int_{B(0,R)} R_1 * |f|(\rho y)\, \|e\|_{L^\infty}\, dy \\ &= \|e\|_{L^\infty} \int_{\mathbb{R}^n} R_1 * |f|(\rho y)\, h(\tfrac{y}{R})\, dy \\ &= \|e\|_{L^\infty} \int_{\mathbb{R}^n} R_1 * |f|(x)\, \rho^{-n} h(\tfrac{x}{R\rho})\, dx \\ &= R^n\, \|e\|_{L^\infty} \int_{\mathbb{R}^n} R_1 * |f|(x)\, (R\rho)^{-n} h(\tfrac{x}{R\rho})\, dx \\ &= R^n\, \|e\|_{L^\infty} \langle R_1 * |f|, h_{R\rho}\rangle,\end{aligned} \tag{5.52}$$

so that it suffices to show that $\langle R_1 * |f|, h_{\tilde{\rho}}\rangle \to 0$ as $\tilde{\rho} = R\rho$ approaches infinity.

To this end, first consider $f_1 \in D_{\mathrm{xr}}$ with $f_1 \geq 0$ and $\operatorname{supp} f_1 \subset B(0, r_0)$. For $|x| \geq 2r_0$ and $y \in \operatorname{supp} f_1$,

$$|x - y| \geq \big||x| - |y|\big| \geq |x| - \frac{|x|}{2} = \frac{|x|}{2} \tag{5.53}$$

and $(1 + |y|)^{1-n} \geq (1 + r_0)^{1-n}$. Consequently,

$$\begin{aligned}R_1 * f_1(x) &= b_n \int_{\operatorname{supp} f_1} \frac{1}{|x - y|^{n-1}}\, |f_1(y)|\, dy \\ &\leq b_n \int_{\operatorname{supp} f_1} \left(\frac{2}{|x|}\right)^{n-1} (1 + r_0)^{n-1} (1 + |y|)^{1-n}\, |f_1(y)|\, dy \\ &= C_1\, \|f_1\|_{\mathrm{xr}}\, |x|^{1-n}\end{aligned} \tag{5.54}$$

whenever $|x| > 2r_0$, for some constant $C_1 > 0$ depending on the size of the support of f_1.

Now analyse

$$\langle R_1 * f_1, h_\rho \rangle = \rho^{-n} \int_{|x|<\rho} R_1 * f_1(x)\, dx \tag{5.55}$$

and split the range of integration into two parts, $|x| < 2r_0$ and $2r_0 \le |x| < \rho$, provided that $\rho > 2r_0$. In the first one, use Lemma 5.3 to see that

$$\begin{aligned} &\rho^{-n} \int_{|x|<2r_0} R_1 * f_1(x)\, dx \\ &= \rho^{-n} \langle R_1 * f_1, \chi_{B(0,2r_0)} \rangle \\ &\le \rho^{-n} C \, \|(1+|x|)^{1-n} f_1\|_{L^1} \left[\|\chi_{B(0,2r_0)}\|_{L^1} + \|(1+|x|)^n \chi_{B(0,2r_0)}\|_{L^\infty} \right] \\ &\le \rho^{1-n} C \, \|f_1\|_{\mathrm{xr}} \left[m(B(0,2r_0)) + (1+2r_0)^n \right], \end{aligned} \tag{5.56}$$

if we assume that $\rho \ge 1$. In the second part, (5.54) shows than

$$\begin{aligned} \rho^{-n} \int_{2r_0 \le |x| < \rho} R_1 * f_1(x)\, dx &\le \rho^{-n}\, C_1 \, \|f_1\|_{\mathrm{xr}} \int_{2r_0 \le |x| < \rho} |x|^{1-n}\, dx. \\ &= \rho^{-n}\, C_1 \, \|f_1\|_{\mathrm{xr}} \, |S^{n-1}| \int_{2r_0}^{\rho} r^{1-n} r^{n-1}\, dr \\ &\le \rho^{1-n}\, C_1 \, \|f_1\|_{\mathrm{xr}} \, |S^{n-1}|. \end{aligned} \tag{5.57}$$

Therefore

$$|\langle R_1 * f_1, h_\rho \rangle| \le \frac{C_2 \, \|f_1\|_{\mathrm{xr}}}{\rho^{n-1}} \tag{5.58}$$

when $\rho > 1$, for some constant $C_2 > 0$ depending on f_1.

On the other hand, for any $f_2 \in D_{\mathrm{xr}}$ with $f_2|_{B(0,2)} = 0$, Lemma 5.7 and the fact that $1 + |y| \le 2\,|y|$ whenever $|y| \ge 2$ show that

$$\begin{aligned} |\langle R_1 * f_2, h_\rho \rangle| &\le C_3 \, R_1 * f_2(0) \\ &= C_3 \, b_n \int_{|y|\ge 2} |y|^{1-n} |f_2(y)|\, dy \\ &\le C_3 \, b_n \int_{|y|\ge 2} 2^{n-1} \, (1+|y|)^{1-n} \, |f_2(y)|\, dy \\ &\le C_4 \, \|f_2\|_{\mathrm{xr}} \end{aligned} \tag{5.59}$$

for some constant $C_4 > 0$ independent of f_2.

Now given any $\varepsilon > 0$, Lemma 5.6 shows that with $R \ge 2$ sufficiently large, $f_1 := \chi_{B(0,R)}\, |f|$ satisfies $\||f| - f_1\|_{\mathrm{xr}} < \varepsilon / 2C_4$. This implies that with $f_2 = |f| - f_1$,

$$0 \le \langle R_1 * |f|, h_\rho \rangle = \langle R_1 * f_1, h_\rho \rangle + \langle R_1 * f_2, h_\rho \rangle \le \frac{C_2 \, \|f_1\|_{\mathrm{xr}}}{\rho^{n-1}} + C_4 \, \frac{\varepsilon}{2C_4} < \varepsilon \tag{5.60}$$

when ρ is sufficiently large. □

The third auxiliary result is that if two distributions agree on all test functions that vanish at the origin, then their difference is a constant times Dirac's delta distribution.

Lemma 5.9. *Let $n \in \mathbb{Z}_+$. If $f,\, g \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\langle f, \phi \rangle = \langle g, \phi \rangle \qquad \text{whenever } \phi(0) = 0, \tag{5.61}$$

then $f = g + c\delta$ for some constant $c \in \mathbb{C}$.

Proof. First show that $\operatorname{supp}(f - g) \subset \{0\}$. Let $y \neq 0$ and choose an open set $U \subset \mathbb{R}^n$ with $y \in U$, $0 \notin U$. Then $\phi(0) = 0$ for all $\phi \in \mathcal{D}$ with $\operatorname{supp}\phi \subset U$, so that by assumption $\langle f - g, \phi \rangle = 0$. In other words, every point $y \neq 0$ has a neighbourhood on which $f - g$ vanishes, and therefore $y \notin \operatorname{supp}(f - g)$.

Since $f - g$ has point support, it must be a finite linear combination of Dirac's delta distribution and its derivatives:

$$f - g = \sum_{|\alpha| \leq m} c_\alpha \, D^\alpha \delta. \tag{5.62}$$

It suffices to show that c_0 must be the only non-zero one of the constants c_α. Assume contrariwise that $c_\beta \neq 0$ and $\beta \neq (0, 0, \dots, 0)$. Choose a test function ψ such that $\partial^\beta \psi(0) \neq 0$ and $D^\alpha \psi(0) = 0$ for all $\alpha \neq \beta$; especially $\psi(0) = 0$. Such a function could be, for example, $x^\beta \tilde{\chi}_{1,2}$. This leads to the contradiction

$$0 = \langle f - g, \psi \rangle = \sum_{|\alpha| \leq m} c_\alpha \langle D^\alpha \delta, \psi \rangle = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \psi(0) = c_\beta \partial^\beta \psi(0) \neq 0. \tag{5.63}$$

□

We are now ready to extend to D_{xr} the property that $R_1 * g$ corresponds on the Fourier transformed side to $|\xi|^{-1}\,\widehat{g}$.

Theorem 5.10. *Assume that $n \geq 2$, $f \in D_{\text{xr}}(\mathbb{R}^n)$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\phi(0) = 0$. Then*

$$\langle \mathcal{F}(R_1 * f), \phi \rangle = \int |\xi|^{-1} \hat{f}(\xi)\, \phi(\xi)\, d\xi. \tag{5.64}$$

If, in addition, $|\xi|^{-1}\hat{f}(\xi) \in L^1_{\text{loc}}(\mathbb{R}^n)$, then

$$\mathcal{F}(R_1 * f)(\xi) = |\xi|^{-1}\hat{f}(\xi). \tag{5.65}$$

Proof. First construct functions $\phi_j \in \mathcal{S}'$ such that $\phi(x) = \sum_{j=1}^n x_j\, \phi_j(x)$, as follows.

Define the functions [FRS]

$$\begin{array}{ll} \phi^- = \tilde{\chi}_{2,3}\phi \in \mathcal{S} & \quad a^- = \tilde{\chi}_{3,4} \in C_0^\infty \\ \phi^+ = \phi - \phi^- \in \mathcal{S} & \quad a^+ = 1 - \tilde{\chi}_{1,2} \in C^\infty \end{array} \tag{5.66}$$

where $\tilde{\chi}_{\varepsilon_1,\varepsilon_2}$ is the approximate characteristic function of (A.34). They have the properties

$$\phi^-(x) = \begin{cases} \phi(x), & |x| \le 2 \\ 0, & |x| \ge 3 \end{cases} \qquad a^-(x) = \begin{cases} 1, & |x| \le 3 \\ 0, & |x| \ge 4 \end{cases}$$
$$\phi^+(x) = \begin{cases} 0, & |x| \le 2 \\ \phi(x), & |x| \ge 3 \end{cases} \qquad a^+(x) = \begin{cases} 0, & |x| \le 1 \\ 1, & |x| \ge 2. \end{cases} \tag{5.67}$$

Integrating along the straight line from 0 to x gives

$$\phi^-(x) = \int_0^1 \nabla\phi^-(tx)\cdot x\,dt = \sum_{j=1}^n x_j \int_0^1 \frac{\partial\phi^-}{\partial x_j}(tx)\,dt, \tag{5.68}$$

and along the ray starting from x and going away from the origin,

$$\phi^+(x) = -\int_1^\infty \nabla\phi^+(tx)\cdot x\,dt = -\sum_{j=1}^n x_j \int_1^\infty \frac{\partial\phi^+}{\partial x_j}(tx)\,dt. \tag{5.69}$$

Note that

$$\phi^\pm(x) = a^\pm(x)\,\phi^\pm(x) = \sum_{j=1}^n x_j\,\phi_j^\pm(x), \tag{5.70}$$

where

$$\phi_j^-(x) = a^-(x)\int_0^1 \frac{\partial\phi^-}{\partial x_j}(tx)\,dt \quad\text{and}\quad \phi_j^+(x) = -a^+(x)\int_1^\infty \frac{\partial\phi^+}{\partial x_j}(tx)\,dt \tag{5.71}$$

are rapidly decreasing functions: For ϕ_j^-, this follows from the estimate

$$\begin{aligned}
\|\phi_j^-\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n} \left| x^\alpha \sum_{\gamma\le\beta} c_{\beta\gamma}\partial^\gamma a^-(x)\int_0^1 \partial^{\beta-\gamma+e_j}\phi^-(tx)\,dt\right| \\
&\le \sup_{x\in\mathbb{R}^n} \sum_{\gamma\le\beta} c_{\beta\gamma}\left|x^\alpha\partial^\gamma a^-(x)\right| \int_0^1 \left|\partial^{\beta-\gamma+e_j}\phi^-(tx)\right|\,dt \\
&\le \sum_{\gamma\le\beta} c_{\beta\gamma}\|a^-\|_{\alpha,\gamma}\int_0^1 \|\phi^-\|_{0,\beta-\gamma+e_j}\,dt < \infty.
\end{aligned} \tag{5.72}$$

For ϕ_j^+,

$$\begin{aligned}
\|\phi_j^+\|_{\alpha,\beta} &= \sup_{x\in\mathbb{R}^n} \left| x^\alpha \sum_{\gamma\le\beta} c_{\beta\gamma}\partial^\gamma a^+(x)\int_1^\infty \partial^{\beta-\gamma+e_j}\phi^+(tx)\,dt\right| \\
&\le \sup_{x\in\mathbb{R}^n} \sum_{\gamma\le\beta} c_{\beta\gamma}\left|\partial^\gamma a^+(x)\right| \int_1^\infty |x^\alpha|\,|tx|^2\left|\partial^{\beta-\gamma+e_j}\phi^+(tx)\right|\,|x|^{-2}\,t^{-2}\,dt.
\end{aligned} \tag{5.73}$$

Note that for $\phi_j^+(x)$ to be non-zero, $|x|$ must be greater than 2, and that since $t \ge 1$ in the integral, $|x^\alpha| \le |(tx)^\alpha|$. As $|tx|^2 = \sum_{k=1}^n (tx)^{2e_k}$, this yields

$$\begin{aligned}
\|\phi_j^+\|_{\alpha,\beta} &\le \sup_{x\in\mathbb{R}^n} \sum_{\gamma\le\beta} c_{\beta\gamma}\left|\partial^\gamma a^+(x)\right| \int_1^\infty \left|(tx)^{\alpha+2e_k}\partial^{\beta-\gamma+e_j}\phi^+(tx)\right|\,t^{-2}\,dt \\
&\le \sum_{\gamma\le\beta}\sum_{k=1}^n c_{\beta\gamma}\|a^+\|_{0,\gamma}\,\|\phi^+\|_{\alpha+2e_k,\beta-\gamma+e_j}\int_1^\infty t^{-2}\,dt < \infty,
\end{aligned} \tag{5.74}$$

because although $a^+ \notin \mathcal{S}$, all of its derivatives are constant outside $B(0,2)$, so that $\|a^+\|_{0,\gamma} < \infty$ for all $\gamma \in \mathbb{N}^n$.

This gives us the desired functions ϕ_j:

$$\phi(x) = \phi^-(x) + \phi^+(x) = \sum_{j=1}^{n} x_j\,\phi_j(x), \qquad \text{where} \quad \phi_j = \phi_j^- + \phi_j^+ \in \mathcal{S}. \tag{5.75}$$

Now $R_1 * f \in L^1_{\mathrm{loc}} \subset \mathcal{S}'$ by Theorem 5.3, so that Theorem 4.12 shows that

$$\xi_j \mathcal{F}(R_1 * f) = -i\,\mathcal{F}\left(D_j(R_1 * f)\right) = -i\,\mathcal{F}\left(\mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * f\right) = \frac{\xi_j}{|\xi|}\,\hat{f}(\xi). \tag{5.76}$$

Therefore

$$\begin{aligned}
\langle \mathcal{F}(R_1 * f), \phi\rangle &= \sum_{j=1}^{n} \langle \mathcal{F}(R_1 * f), \xi_j \phi_j\rangle \\
&= \sum_{j=1}^{n} \langle \xi_j\,\mathcal{F}(R_1 * f), \phi_j\rangle \\
&= \sum_{j=1}^{n} \left\langle \frac{\xi_j}{|\xi|}\,\hat{f}, \phi_j \right\rangle \\
&= \int |\xi|^{-1}\,\hat{f}(\xi) \sum_{j=1}^{n} \xi_j\,\phi_j(\xi)\,d\xi \\
&= \int |\xi|^{-1}\,\hat{f}(\xi)\,\phi(\xi)\,d\xi,
\end{aligned} \tag{5.77}$$

which proves the first claim.

Then assume that $|\xi|^{-1}\hat{f} \in L^1_{\mathrm{loc}} \subset \mathcal{S}'$, so that

$$\langle \mathcal{F}(R_1 * f), \psi\rangle = \langle |\xi|^{-1}\hat{f}, \psi\rangle \tag{5.78}$$

for all $\psi \in \mathcal{S}$ with $\psi(0) = 0$. Lemma 5.9 then shows that $\mathcal{F}(R_1 * f) = |\xi|^{-1}\hat{f} + c\delta$ for some constant $c \in \mathbb{C}$. It remains to show that $c = 0$.

Write $u = \mathcal{F}^{-1}(|\xi|^{-1}\hat{f})$. Since $\widehat{u}$ was assumed to be locally integrable and

$$\|\chi_{\mathbb{R}^n \setminus B(0,1)} |\xi|^{-1}\hat{f}\|_{L^2} \le \|\hat{f}\|_{L^2} = \|f\|_{L^2} \le \|f\|_{\mathrm{xr}} < \infty, \tag{5.79}$$

we see that

$$\widehat{u} = \chi_{B(0,1)}\widehat{u} + \chi_{\mathbb{R}^n \setminus B(0,1)}\widehat{u} \in L^1 + L^2. \tag{5.80}$$

Thus for any $e \in C_0^\infty$ with $\int e(x)\,dx = 1$, and $e_\rho(x) = \rho^{-n} e(x/\rho)$,

$$c = \langle c, e_\rho\rangle = \langle \mathcal{F}^{-1}(c\delta), e_\rho\rangle = \langle R_1 * f, e_\rho\rangle - \langle u, e_\rho\rangle \xrightarrow{\rho\to\infty} 0 \tag{5.81}$$

by Lemma 5.8. This completes the proof. $\square$

Note that the condition that $|\xi|^{-1} f(\xi)$ be locally integrable in Theorem 5.10 is always true if $n > 2$, because then

$$\begin{aligned} \|\chi_{B(0,R)}(\xi)\,|\xi|^{-1}\hat{f}(\xi)\|_{L^1} &\leq \|\chi_{B(0,R)}(\xi)\,|\xi|^{-1}\|_{L^2}\|\hat{f}\|_{L^2} \\ &\leq \sqrt{|S^{n-1}|\int_0^R \frac{r^{n-1}\,dr}{r^2}}\,\|f\|_{L^2} \\ &\leq \sqrt{|S^{n-1}|\int_0^R r^{n-2}}\,\|f\|_{\mathrm{xr}} < \infty, \end{aligned} \tag{5.82}$$

or if $f \in L^p$, $p > 2$, because then $p' < 2$ and $n-1-p' \geq 1-p' > -1$, and consequently

$$\begin{aligned} \|\chi_{B(0,R)}(\xi)\,|\xi|^{-1}\hat{f}(\xi)\|_{L^1} &\leq \|\chi_{B(0,R)}(\xi)\,|\xi|^{-1}\|_{L^{p'}}\|\hat{f}\|_{L^p} \\ &= |S^{n-1}|\left(\int_0^R r^{n-1-p'}\,dr\right)^{\frac{1}{p'}}\|f\|_{L^p} < \infty. \end{aligned} \tag{5.83}$$

5.2 Properties of the Radiograph Operators

In this section, we shall investigate some properties of the divergent beam radiograph operator $\mathcal{D}$ and the parallel beam radiograph operator $\mathcal{P}$, introduced in Definitions 3.1 and 3.6.

The Riesz potential of an x-ray attenuation density can be calculated from the divergent or parallel beam radiographs:

Lemma 5.11. *If $n \geq 2$ and $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then for almost all $x \in \mathbb{R}^n$,*

$$\frac{1}{b_n} R_1 * f(x) = \int_{S^{n-1}} \mathcal{D}_x f(\theta)\,d\theta = \frac{1}{2}\int_{S^{n-1}} \mathcal{P}_\theta f(E_\theta x)\,d\theta \tag{5.84}$$

and the integrals converge absolutely.

Equation (5.84) also holds for all non-negative, measurable functions f, in the sense that when one of the members is infinite, so are the other two, and otherwise all three take the same finite value.

Proof. If $f \in D_{\mathrm{xr}}$, the changes of variable $t = |y|$, $\theta = y/|y|$, $dy = t^{n-1}\,dt\,d\theta$ and $y' = -y$ yield

$$\begin{aligned} \int_{S^{n-1}} \mathcal{D}_x f(\theta)\,d\theta &= \int_{S^{n-1}}\int_0^\infty f(x+t\theta)\,dt\,d\theta \\ &= \int_{\mathbb{R}^n} f(x+y)\,|y|^{1-n}\,dy \\ &= \int_{\mathbb{R}^n} f(x-y')\,|y'|^{1-n}\,dy' \\ &= \frac{1}{b_n} R_1 * g(x), \end{aligned} \tag{5.85}$$

for almost all $x \in \mathbb{R}^n$ by Fubini's theorem, since Theorem 5.3 asserts the absolute convergence of the integral. For the second equality in (5.84), observe that

$\mathcal{P}_\theta f(E_\theta x) = \mathcal{D}_x f(\theta) + \mathcal{D}_x f(-\theta)$, whence with $\theta' = -\theta$,

$$\begin{aligned} \frac{1}{2}\int_{S^{n-1}} \mathcal{P}_\theta f(E_\theta x)\, d\theta &= \frac{1}{2}\left[\int_{S^{n-1}} \mathcal{D}_x f(\theta)\, d\theta + \int_{S^{n-1}} \mathcal{D}_x f(-\theta)\, d\theta\right] \\ &= \frac{1}{2}\left[\int_{S^{n-1}} \mathcal{D}_x f(\theta)\, d\theta + \int_{S^{n-1}} \mathcal{D}_x f(\theta')\, d\theta'\right] \\ &= \frac{1}{b_n} R_1 * f(x). \end{aligned} \tag{5.86}$$

If then $f \geq 0$, (5.85) is justified by Fubini's theorem for non-negative functions, and (5.86) holds equally well. □

That D_{xr} is a reasonable domain for x-ray attenuation densities, as asserted by Theorem 5.3, can be formulated in terms of the divergent beam radiograph as follows:

Corollary 5.12. *If $n \geq 2$ and $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then for almost all $x \in \mathbb{R}^n$, $\mathcal{D}_x f$ is defined almost everywhere on S^{n-1} by an absolutely convergent integral and $\mathcal{D}_x f \in L^1(S^{n-1})$.*

If, on the contrary, $(1+|x|)^{1-n} f(x) \notin L^1(\mathbb{R}^n)$ and $f \geq 0$, then $\mathcal{D}_x f \notin L^1(S^{n-1})$ for any $x \in \mathbb{R}^n$.

Proof. By the triangle inequality and Lemma 5.11,

$$\int_{S^{n-1}} |\mathcal{D}_x f(\theta)|\, d\theta \leq \int_{S^{n-1}} \mathcal{D}_x |f|(\theta)\, d\theta = \frac{1}{b_n} R_1 * |f|(x), \tag{5.87}$$

which converges for almost all $x \in \mathbb{R}^n$ by Theorem 5.3. For such x, the absolute value integral

$$\mathcal{D}_x |f|(\theta) = \int_0^\infty |f(x + t\theta)|\, dt \tag{5.88}$$

must be finite for almost every $\theta \in S^{n-1}$.

If then $f \geq 0$ and $(1+|x|)^{1-n} f(x) \notin L^1(\mathbb{R}^n)$,

$$\int_{S^{n-1}} \mathcal{D}_x f(\theta)\, d\theta = \frac{1}{b_n} R_1 * f(x) = \infty \tag{5.89}$$

for all $x \in \mathbb{R}^n$, again by Theorem 5.3. □

For the parallel beam radiograph, we have the following result:

Lemma 5.13. *Let $n \geq 2$ and $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$. Then for almost all $\theta \in S^{n-1}$, $\mathcal{P}_\theta f$ is defined almost everywhere on $\theta^\perp$ by an absolutely convergent integral.*

If $\phi \in L^1(\mathbb{R}^n)$ and $(1+|x|)^n \phi \in L^\infty(\mathbb{R}^n)$, then

$$\int_{S^{n-1}} \langle \mathcal{P}_\theta f, \mathcal{P}_\theta \phi \rangle\, d\theta = \frac{2}{b_n} \langle R_1 * f, \phi \rangle \tag{5.90}$$

and

$$\left| \int_{S^{n-1}} \langle \mathcal{P}_\theta f, \mathcal{P}_\theta \phi \rangle\, d\theta \right| \leq C\, \|f\|_{\mathrm{xr}} \Big[\|\phi\|_{L^1} + \|(1+|x|)^n \phi\|_{L^\infty} \Big] \tag{5.91}$$

for some constant C.

Proof. By Lemma 5.11,

$$\begin{aligned}
\frac{2}{b_n}\langle R_1 * f, \phi\rangle &= \left\langle \int_{S^{n-1}} \mathcal{P}_\theta f(E_\theta x)\,d\theta, \phi \right\rangle \\
&= \int_{\mathbb{R}^n}\int_{S^{n-1}}\int_{-\infty}^{\infty} f(x+t\theta)\,dt\,d\theta\,\phi(x)\,dx \qquad (5.92)\\
&= \int_{S^{n-1}}\int_{\mathbb{R}^n}\int_{-\infty}^{\infty} f(x+t\theta)\,dt\,\phi(x)\,dx\,d\theta.
\end{aligned}$$

The changes of variable $x = x' + s\theta \in \theta^\perp \oplus \mathbb{R}\theta = \mathbb{R}^n$, $dx = dx'\,ds$ and $t' = s + t$, $dt' = dt$ yield

$$\begin{aligned}
\frac{2}{b_n}\langle R_1 * f, \phi\rangle &= \int_{S^{n-1}}\int_{\theta^\perp}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x'+s\theta+t\theta)\,dt\,\phi(x'+s\theta)\,ds\,dx'\,d\theta \\
&= \int_{S^{n-1}}\int_{\theta^\perp}\int_{-\infty}^{\infty} f(x'+t'\theta)\,dt'\int_{-\infty}^{\infty}\phi(x'+s\theta)\,ds\,dx'\,d\theta \qquad (5.93)\\
&= \int_{S^{n-1}}\langle \mathcal{P}_\theta f, \mathcal{P}_\theta\phi\rangle\,d\theta.
\end{aligned}$$

The order of integration can be changed above by Fubini's theorem, since the integral in (5.92) converges absolutely: By Lemma 5.11 and Theorem 5.3,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\int_{S^{n-1}}\int_{-\infty}^{\infty} |f(x+t\theta)|\,dt\,d\theta\,|\phi(x)|\,dx \\
&\qquad = \frac{2}{b_n}\langle R_1 * |f|, |\phi|\rangle \qquad (5.94)\\
&\qquad \le C\|(1+|x|)^{1-n} f\|_{L^1}\left[\|\phi\|_{L^1} + \|(1+|x|)^n\phi\|_{L^\infty}\right] \\
&\qquad \le C\|f\|_{\mathrm{xr}}\Big[\|\phi\|_{L^1} + \|(1+|x|)^n\phi\|_{L^\infty}\Big] < \infty
\end{aligned}$$

for some constant $C > 0$. This also proves (5.91) because

$$\begin{aligned}
\Big|\int_{S^{n-1}}\langle \mathcal{P}_\theta f, \mathcal{P}_\theta\phi\rangle\,d\theta\Big| &= \Big|\int_{\mathbb{R}^n}\int_{S^{n-1}}\int_{-\infty}^{\infty} f(x+t\theta)\,dt\,d\theta\,\phi(x)\,dx\Big| \qquad (5.95)\\
&\le \int_{\mathbb{R}^n}\int_{S^{n-1}}\int_{-\infty}^{\infty} |f(x+t\theta)|\,dt\,d\theta\,|\phi(x)|\,dx.
\end{aligned}$$

For showing the absolute convergence of $\mathcal{P}_\theta f(x)$ almost everywhere, fix $\phi(x) = e^{-|x|^2}$. As $|f|$ satisfies the conditions of the theorem as well as f,

$$\int_{S^{n-1}}\langle \mathcal{P}_\theta|f|, \mathcal{P}_\theta\phi\rangle < \infty \qquad (5.96)$$

by (5.91). The integrand $\langle \mathcal{P}_\theta|f|, \mathcal{P}_\theta\phi\rangle$ must therefore be finite for almost all $\theta \in S^{n-1}$.

For such θ, the integral $\mathcal{P}_\theta f(x)$ must converge absolutely almost everywhere, for if this were not the case, then there would be a set $E \subset \theta^\perp$ with $m(E) > 0$ and $\mathcal{P}_\theta|f|(x) = \infty$ for all $x \in E$. By the regularity of the Lebesgue measure, E could be chosen compact, so that $M := \inf_{x\in E}\mathcal{P}_\theta\phi(x) > 0$, and consequently

$$\langle \mathcal{P}_\theta|f|, \mathcal{P}_\theta\phi\rangle \ge \int_E \mathcal{P}_\theta|f|(x)\,\mathcal{P}_\theta\phi(x)\,dx \ge M\int_E \mathcal{P}_\theta|f|(x)\,dx = \infty. \qquad (5.97)$$

□

The following, so called projection slice theorem shows how the Fourier transform of the parallel beam radiograph can be obtained from the Fourier transform of the x-ray attenuation density:

Lemma 5.14. *If $f \in L^1(\mathbb{R}^n)$ and $\theta \in S^{n-1}$, then $\mathcal{P}_\theta f \in L^1(\theta^\perp)$ and*

$$\mathcal{F}\left(\mathcal{P}_\theta f\right)(\xi) = \sqrt{2\pi}\,\hat{f}(\xi) \tag{5.98}$$

for all $\xi \in \theta^\perp$.

Proof. Obviously,

$$\int_{\theta^\perp} |\mathcal{P}_\theta f(x)|\,dx \leq \int_{\theta^\perp}\int_{-\infty}^{\infty} |f(x+t\theta)|\,dt\,dx = \|f\|_{L^1} < \infty. \tag{5.99}$$

For $f \in L^1$, the change of variable $y = x + t\theta \in \theta^\perp \oplus \mathbb{R}\theta = \mathbb{R}^n$, $dy = dx\,dt$ yields, after noting that $\xi \in \theta^\perp$ implies $x \cdot \xi = x \cdot \xi + t\theta \cdot \xi = y \cdot \xi$,

$$\begin{aligned}\mathcal{F}\left(\mathcal{P}_\theta f\right)(\xi) &= (2\pi)^{-\frac{n-1}{2}} \int_{\theta^\perp}\int_{-\infty}^{\infty} f(x+t\theta) e^{-ix\cdot\xi}\,dt\,d\xi \\ &= \sqrt{2\pi}\,(2\pi)^{-n/2}\int_{\mathbb{R}^n} f(y)\,e^{-iy\cdot\xi}\,dx = \sqrt{2\pi}\,\hat{f}(\xi).\end{aligned} \tag{5.100}$$

□

The following result will also be needed:

Lemma 5.15. *If $h : \mathbb{R}^n \to [0,\infty)$ is measurable, then*

$$\int_{S^{n-1}}\int_{\zeta^\perp} |y|\,h(y)\,dy\,d\zeta = |S^{n-2}| \int_{\mathbb{R}^n} h(x)\,dx. \tag{5.101}$$

Proof. First observe that $S^{n-1} \cap \theta^\perp \cong S^{n-2}$, whence for $\tilde{h} : S^{n-1} \to [0,\infty)$,

$$\int_{S^{n-1}}\int_{S^{n-1}\cap\theta^\perp} \tilde{h}(\theta)\,d\zeta\,d\theta = |S^{n-2}| \int_{S^{n-1}} \tilde{h}(\theta)\,d\theta. \tag{5.102}$$

Consequently for $h : \mathbb{R}^n \to [0,\infty)$,

$$\begin{aligned}|S^{n-2}| \int_{\mathbb{R}^n} h(x)\,dx &= |S^{n-2}| \int_0^\infty \int_{S^{n-1}} h(r\theta)\,d\theta\,r^{n-1}\,dr \\ &= \int_0^\infty \int_{S^{n-1}}\int_{S^{n-1}\cap\theta^\perp} h(r\theta)\,d\zeta\,d\theta\,r^{n-1}\,dr.\end{aligned} \tag{5.103}$$

As

$$\begin{aligned}\theta \in S^{n-1} \;\wedge\; \zeta \in S^{n-1}\cap\theta^\perp \quad &\Leftrightarrow \quad \theta \in S^{n-1} \;\wedge\; \zeta \in S^{n-1} \;\wedge\; \theta\cdot\zeta = 0 \\ &\Leftrightarrow \quad \zeta \in S^{n-1} \;\wedge\; \theta \in S^{n-1}\cap\zeta^\perp,\end{aligned} \tag{5.104}$$

the change of variable $y = r\theta \in \zeta^\perp$, $dy = r^{n-2}\,dr\,d\theta$ gives

$$\begin{aligned}|S^{n-2}| \int_{\mathbb{R}^n} h(x)\,dx &= \int_0^\infty \int_{S^{n-1}}\int_{S^{n-1}\cap\zeta^\perp} h(r\theta)\,d\theta\,d\zeta\,r^{n-1}\,dr \\ &= \int_{S^{n-1}}\int_{S^{n-1}\cap\zeta^\perp}\int_0^\infty r\,h(r\theta)\,r^{n-2}\,dr\,d\theta\,d\zeta \\ &= \int_{S^{n-1}}\int_{\zeta^\perp} |y|\,h(y)\,dy\,d\zeta.\end{aligned} \tag{5.105}$$

□

The projection slice theorem (Lemma 5.14) can now be extended to cover certain functions in the x-ray domain:

Theorem 5.16. *Let $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$. Then for almost all $\theta \in S^{n-1}$,*

(a) $|\xi|^{1/2}\hat{f} \in L^2(\theta^\perp)$

(b) $\mathcal{P}_\theta f \in L^1_{\mathrm{loc}}(\theta^\perp) \subset \mathcal{S}'(\theta^\perp)$

(c) if $\phi \in \mathcal{S}(\theta^\perp)$ *and* $\phi(0) = 0$*, then*

$$\langle \mathcal{F}(\mathcal{P}_\theta f), \phi\rangle = \sqrt{2\pi}\int_{\theta^\perp} \hat{f}(\xi)\,\phi(\xi)\,d\xi \tag{5.106}$$

(d) if $|\xi|^{-1}\hat{f}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$*, then* $\hat{f} \in L^1_{\mathrm{loc}}(\theta^\perp)$ *and*

$$\mathcal{F}(\mathcal{P}_\theta f)(\xi) = \sqrt{2\pi}\,\hat{f}(\xi) \tag{5.107}$$

for almost all $\xi \in \theta^\perp$.

Proof. Claim (a) follows from Lemma 5.15:

$$\int_{S^{n-1}}\int_{\theta^\perp} |\xi|\,|\hat{f}(\xi)|^2\,d\xi\,d\theta = |S^{n-1}|\int_{\mathbb{R}^n} |\hat{f}(\xi)|^2\,d\xi = |S^{n-1}|\|f\|^2_{L^2} < \infty, \tag{5.108}$$

whence

$$\||\xi|^{1/2}\hat{f}\|^2_{L^2(\theta^\perp)} = \int_{\theta^\perp} |\xi|\,|\hat{f}(\xi)|^2\,d\xi < \infty \tag{5.109}$$

for almost all $x \in \theta^\perp$.

For claim (b), we show that for almost all $\theta \in S^{n-1}$,

$$\int_{B(0,R)\cap\theta^\perp} |\mathcal{P}_\theta f(x)|\,dx < \infty \tag{5.110}$$

for any $R > 0$. Note that

$$\begin{aligned}\int_{B(0,R)\cap\theta^\perp} |\mathcal{P}_\theta f(x)|\,dx &\le \int_{B(0,R)\cap\theta^\perp} |\mathcal{P}_\theta f(x)|\,\frac{(1+R)^n}{(1+|x|)^n}\,dx \\ &\le (1+R)^n\int_{\theta^\perp} |\mathcal{P}_\theta f(x)|\,(1+|x|)^{-n}\,dx\end{aligned} \tag{5.111}$$

and use the estimate

$$\begin{aligned}\mathcal{P}_\theta\left((1+|x|)^{-n-1}\right)(x) &= \int_{-\infty}^{\infty} (1+|x+t\theta|)^{-n-1}\,dt \\ &\ge \int_{-\infty}^{\infty} (1+|x|+|t|)^{-n-1}\,dt \\ &= 2\int_0^{\infty} (1+|x|+t)^{-n-1}\,dt \\ &= \frac{2}{n}(1+|x|)^{-n}\end{aligned} \tag{5.112}$$

to see that

$$\int_{B(0,R)\cap\theta^\perp} |\mathcal{P}_\theta f(x)|(1+|x|)^{-n}\,dx \le \frac{n}{2}(1+R)^n \left\langle |\mathcal{P}_\theta f(x)|, \mathcal{P}_\theta\left((1+|x|)^{-n-1}\right)\right\rangle. \tag{5.113}$$

Lemma 5.13 shows that for some constant $C_1 > 0$,

$$\begin{aligned}\int_{S^{n-1}}\int_{B(0,R)\cap\theta^\perp} &|\mathcal{P}_\theta f(x)|\,(1+|x|)^{-n}\,dx\,d\theta \\ &\le C_1\,\|f\|_{\mathrm{xr}}\left[\left\|(1+|x|)^{-n-1}\right\|_{L^1} + \left\|(1+|x|)^n(1+|x|)^{-n-1}\right\|_{L^\infty}\right],\end{aligned} \tag{5.114}$$

which is finite. This proves (b).

For claim (c), choose a sequence of functions $f_k \in L^1\cap D_{\mathrm{xr}}$ converging to f in the D_{xr} norm; this is possible since the subset C_0^∞ of $L^1\cap D_{\mathrm{xr}}$ is dense by Lemma 5.6. Then

$$\int_{S^{n-1}}\int_{\theta^\perp}|\mathcal{P}_\theta f(x)-\mathcal{P}_\theta f_k(x)|\,(1+|x|)^{-n}\,dx\,d\theta \le C_2\,\|f-f_k\|_{\mathrm{xr}} \xrightarrow{k\to\infty} 0 \tag{5.115}$$

by (5.114). This convergence in the $L^1(S^{n-1})$ norm implies that there is a subsequence $(f_{k_l})_{l\in\mathbb{N}}$ such that

$$\int_{\theta^\perp}|\mathcal{P}_\theta f(x)-\mathcal{P}_\theta f_{k_l}(x)|\,(1+|x|)^{-n}\,dx \xrightarrow{l\to\infty} 0 \tag{5.116}$$

for almost all $\theta\in S^{n-1}$. For such θ and any $\psi\in\mathcal{S}(\theta^\perp)$,

$$\begin{aligned}\langle \mathcal{P}_\theta f - \mathcal{P}_\theta f_{k_l},\psi\rangle &\le \int_{\theta^\perp}|\mathcal{P}_\theta f(x)-\mathcal{P}_\theta f_{k_l}|\;(1+|x|)^{-n}\;|(1+|x|)^n\psi(x)|\;dx \\ &\le C_3\int_{\theta^\perp}|\mathcal{P}_\theta f(x)-\mathcal{P}_\theta f_{k_l}|\;(1+|x|)^{-n}\,dx \xrightarrow{l\to\infty} 0,\end{aligned} \tag{5.117}$$

for some $C_3>0$ depending on the seminorms $\|\psi\|_{\alpha,0}$, $|\alpha|\le 2n$. Lemma 5.14 therefore tells us that

$$\begin{aligned}\sqrt{2\pi}\int_{\theta^\perp}\widehat{f_{k_l}}(\xi)\,\phi(\xi)\,d\xi &= \langle\mathcal{F}\left(\mathcal{P}_\theta f_{k_l}\right),\phi\rangle \\ &= \left\langle \mathcal{P}_\theta f_{k_l},\widehat{\phi}\right\rangle \xrightarrow{l\to\infty} \left\langle\mathcal{P}_\theta f,\widehat{\phi}\right\rangle = \langle\mathcal{F}\left(\mathcal{P}_\theta f\right),\phi\rangle.\end{aligned} \tag{5.118}$$

Before using this result, note that by (5.108),

$$\int_{S^{n-1}}\left\||\xi|^{1/2}\left(\hat{f}(\xi)-\widehat{f_{k_l}}(\xi)\right)\right\|^2_{L^2(\theta^\perp)}d\theta \le |S^{n-1}|\,\|f-f_{k_l}\|^2_{\mathrm{xr}} \xrightarrow{l\to\infty} 0, \tag{5.119}$$

so there is again a subsequence $\left(f_{k_{l_m}}\right)_{m\in\mathbb{N}}$ such that

$$\left\||\xi|^{1/2}\left(\hat{f}(\xi)-\widehat{f_{k_{l_m}}}(\xi)\right)\right\|_{L^2(\theta^\perp)} \xrightarrow{m\to\infty} 0 \tag{5.120}$$

for almost all $\theta\in S^{n-1}$. For such θ,

$$\begin{aligned}\int_{\theta^\perp\setminus B(0,1)}\left|\hat{f}(\xi)-\widehat{f_{k_{l_m}}}(\xi)\right|\,|\phi(\xi)|\,d\xi &\le \int_{\theta^\perp\setminus B(0,1)}|\xi|^{1/2}\left|\hat{f}(\xi)-\widehat{f_{k_{l_m}}}(\xi)\right|\,|\phi(\xi)|\,d\xi \\ &\le \int_{\theta^\perp}|\xi|^{1/2}\left|\hat{f}(\xi)-\widehat{f_{k_{l_m}}}(\xi)\right|\,|\phi(\xi)|\,d\xi \\ &\le \left\||\xi|^{1/2}\left(\hat{f}(\xi)-\widehat{f_{k_{l_m}}}(\xi)\right)\right\|_{L^2(\theta^\perp)}\|\phi\|_{L^2(\theta^\perp)} \\ &\xrightarrow{m\to\infty} 0.\end{aligned} \tag{5.121}$$

Since $\phi(0)$ was assumed to be 0,

$$|\phi(\xi)| = \left|\int_0^1 \nabla\phi(t\xi)\cdot\xi\,dt\right| \le \int_0^1 |\nabla\phi(t\xi)|\,|\xi|\,dt \le C_4\,|\xi|, \tag{5.122}$$

where $C_4 = \sqrt{\sum_{j=1}^n \|\phi\|_{0,e_j}^2}$. Consequently,

$$\begin{aligned}
\int_{\theta^\perp\cap B(0,1)} &\left|\hat{f}(\xi) - \widehat{f_{k_{l_m}}}(\xi)\right|\,|\phi(\xi)|\,d\xi \\
&\le C_4 \int_{\theta^\perp\cap B(0,1)} |\xi|^{1/2}\,|\xi|^{1/2}\left|\hat{f}(\xi) - \widehat{f_{k_{l_m}}}(\xi)\right|d\xi \\
&\le C_4 \left\||\xi|^{1/2}\right\|_{L^2(\theta^\perp\cap B(0,1))}\left\||\xi|^{1/2}\left(\hat{f}(\xi) - \widehat{f_{k_{l_m}}}(\xi)\right)\right\|_{L^2(\theta^\perp\cap B(0,1))} \\
&\le C_5 \left\||\xi|^{1/2}\left(\hat{f}(\xi) - \widehat{f_{k_{l_m}}}(\xi)\right)\right\|_{L^2(\theta^\perp)} \xrightarrow{m\to\infty} 0.
\end{aligned} \tag{5.123}$$

Combining estimates (5.121) and (5.123) yields

$$\left|\left\langle\widehat{f_{k_{l_m}}},\phi\right\rangle - \left\langle\hat{f},\phi\right\rangle\right| \le \int_{\theta^\perp}\left|\widehat{f_{k_{l_m}}}(\xi) - \hat{f}(\xi)\right|\,|\phi(\xi)|\,d\xi \xrightarrow{m\to\infty} 0. \tag{5.124}$$

Equations (5.118) and (5.124) together show that for almost all $\theta \in S^{n-1}$,

$$\langle\mathcal{F}(\mathcal{P}_\theta f),\phi\rangle = \sqrt{2\pi}\lim_{m\to\infty}\int_{\theta^\perp}\widehat{f_{k_{l_m}}}(\xi)\,\phi(\xi)\,d\xi = \sqrt{2\pi}\int_{\theta^\perp}\hat{f}(\xi)\,\phi(\xi)\,d\xi. \tag{5.125}$$

This proves claim (c).

In view of (d), Lemma 5.15 asserts that

$$\begin{aligned}
\int_{S^{n-1}}\int_{B(0,R)\cap\theta^\perp}|\hat{f}(\xi)|\,d\xi\,d\theta &= \int_{S^{n-1}}\int_{\theta^\perp}|\xi|\,|\xi|^{-1}\,|\hat{f}(\xi)|\,\chi_{B(0,R)}(\xi)\,d\xi\,d\theta \\
&= |S^{n-2}|\int_{\mathbb{R}^n}|\xi|^{-1}\,|\hat{f}(\xi)|\,\chi_{B(0,R)}(\xi)\,d\xi \\
&= |S^{n-2}|\int_{B(0,R)}|\xi|^{-1}\,|\hat{f}(\xi)|\,d\xi,
\end{aligned} \tag{5.126}$$

which is finite for any $R > 0$ by the assumption that $|\xi|^{-1}\hat{f}(\xi) \in L^1_{\text{loc}}(\mathbb{R}^n)$. Therefore $\hat{f} \in L^1_{\text{loc}}(\theta^\perp)$ for almost all $\theta \in S^{n-1}$.

Fix $\theta \in S^{n-1}$ such that $\hat{f} \in L^1_{\text{loc}}(\theta^\perp)$, $\mathcal{P}_\theta f \in \mathcal{S}'(\theta^\perp)$ and

$$\langle\mathcal{F}(\mathcal{P}_\theta f),\phi\rangle = \sqrt{2\pi}\int_{\theta^\perp}\hat{f}(\xi)\,\phi(\xi)\,d\xi \tag{5.127}$$

whenever $\phi \in \mathcal{S}(\theta^\perp)$ and $\phi(0) = 0$. We have already proved that almost all points of S^{n-1} satisfy these conditions.

Denote by $\delta_{\theta^\perp}$, $\mathcal{F}_{\theta^\perp}$ and $\mathcal{F}_{\theta^\perp}^{-1}$ the delta distribution, the Fourier transform and the inverse Fourier transform on $\theta^\perp$, respectively. By Lemma 5.9, $\mathcal{F}_{\theta^\perp}(\mathcal{P}_\theta f) = \sqrt{2\pi}\,\hat{f} + c\delta_{\theta^\perp}$ for some constant $c \in \mathbb{C}$, so that

$$\mathcal{P}_\theta f = u + c, \qquad \text{where} \qquad u = \sqrt{2\pi}\,\mathcal{F}_{\theta^\perp}^{-1}(\hat{f}). \tag{5.128}$$

Claim (d) will be proved once we succeed in showing that $c = 0$.

Set $e = \frac{1}{|B^n|}\chi_{B(0,1)}$ and $e_\rho(x) = \rho^{-n}e(x/\rho)$. Lemmata 5.13 and 5.8(b) show that

$$\int_{S^{n-1}} \langle \mathcal{P}_\theta f, \mathcal{P}_\theta e_\rho \rangle \, d\theta = \langle R_1 * f, e_\rho \rangle \xrightarrow{\rho \to \infty} 0. \tag{5.129}$$

Therefore the sequence $(e_\rho)_{\rho \in \mathbb{Z}_+}$ has a subsequence $(e_{\rho_k})_{k \in \mathbb{N}}$ such that

$$\langle \mathcal{P}_\theta f, \mathcal{P}_\theta e_{\rho_k} \rangle \xrightarrow{k \to \infty} 0 \qquad \text{for almost all } \theta \in S^{n-1}. \tag{5.130}$$

Since

$$\begin{aligned} c &= \int_{\mathbb{R}^n} c\, e_{\rho_k}(x) \, dx \\ &= \int_{\theta^\perp} c \int_{-\infty}^{\infty} e_{\rho_k}(x' + t\theta) \, dt \, dx' \\ &= \langle c, \mathcal{P}_\theta e_{\rho_k} \rangle \\ &= \langle \mathcal{P}_\theta f, \mathcal{P}_\theta e_{\rho_k} \rangle - \langle u, \mathcal{P}_\theta e_{\rho_k} \rangle, \end{aligned} \tag{5.131}$$

it will suffice to show that the second term on the right hand side approaches zero as $k \to \infty$. We shall do this with the help of Lemma 5.8(a) in $\theta^\perp \cong \mathbb{R}^{n-1}$, with $\mathcal{P}_\theta e$ in place of e.

The change of variable $s = t/\rho$, $ds = dt/\rho$ shows that

$$\mathcal{P}_\theta e_\rho(x) = \int_{-\infty}^{\infty} \rho^{-n} \, e\left(\frac{x + t\theta}{\rho}\right) dt = \rho^{1-n} \int_{-\infty}^{\infty} e\left(\frac{x}{\rho} + s\theta\right) ds = \left(\mathcal{P}_\theta e\right)_\rho (x) \tag{5.132}$$

since $\mathcal{P}_\theta e$ is a function on the $n-1$-dimensional space $\theta^\perp$. The function $\mathcal{P}_\theta e$ is in $L_0^\infty(\theta^\perp)$, since

$$\mathcal{P}_\theta e(x) = \frac{1}{|B^n|} \int_{-\infty}^{\infty} \chi_{B(0,1)}(x + t\theta) \, dt = \frac{2}{|B^n|} \max\left(0, \sqrt{1 - |x|^2}\right). \tag{5.133}$$

The function $\widehat{u}$ can be written as $\widehat{u} = \widehat{u}\chi_{B(0,1)} + \widehat{u}\chi_{\mathbb{R}^n \setminus B(0,1)}$, where

$$\widehat{u}\chi_{B(0,1)} = \sqrt{2\pi}\, \hat{f}\, \chi_{B(0,1)} \in L^1(\theta^\perp), \tag{5.134}$$

as $\hat{f} \in L^1_{\text{loc}}(\theta^\perp)$ by our choice of θ, and

$$\widehat{u}\chi_{\mathbb{R}^n \setminus B(0,1)} \in L^2(\theta^\perp) \qquad \text{for almost all } \theta \in S^{n-1}, \tag{5.135}$$

which can be seen as follows. Since

$$\begin{aligned} \int_{S^{n-1}} \int_{\theta^\perp \setminus B(0,1)} |\hat{f}(\xi)|^2 \, d\xi \, d\theta &\le \int_{S^{n-1}} \int_{\theta^\perp \setminus B(0,1)} |\xi| \, |\hat{f}(\xi)|^2 \, d\xi \, d\theta \\ &\le \int_{S^{n-1}} \int_{\theta^\perp} |\xi| \, |\hat{f}(\xi)|^2 \, d\xi \, d\theta \\ &= |S^{n-2}| \, \|\hat{f}\|_{L^2}^2 = |S^{n-2}| \, \|f\|_{L^2}^2 < \infty \end{aligned} \tag{5.136}$$

by Lemma 5.15,

$$\|\widehat{u}\chi_{\mathbb{R}^n \setminus B(0,1)}\|_{L^2(\theta^\perp)} = \sqrt{2\pi} \int_{\theta^\perp} |\hat{f}(\xi)|^2 \chi_{\mathbb{R}^n \setminus B(0,1)}(\xi) \, d\xi < \infty \tag{5.137}$$

for almost all $\theta \in S^{n-1}$.

All assumptions of Lemma 5.8(a) are therefore satisfied, whence we see that

$$\langle u, \mathcal{P}_\theta e_\rho \rangle = \langle u, (\mathcal{P}_\theta e)_\rho \rangle \xrightarrow{\rho \to \infty} 0. \tag{5.138}$$

This completes the proof. $\square$

The following theorem finally gives the approximate parallel beam reconstruction formula.

Theorem 5.17. *Let $n \geq 2$ and $e \in D_{\mathrm{xr}}(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n)$ be such that $|\xi|^{-1}\widehat{e}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Then for almost all $\theta \in S^{n-1}$,*

(a) $\mathcal{P}_\theta e \in L^1_{\mathrm{loc}}(\theta^\perp)$

(b) with any coordinates on $\theta^\perp$, $D_j(\mathcal{P}_\theta e) \in L^2(\theta^\perp)$ and

$$\mathcal{F}\big(D_j(\mathcal{P}_\theta e)\big)(\xi) = \sqrt{2\pi}\, i\, \xi_j\, \widehat{e}(\xi) \tag{5.139}$$

for almost all $\xi \in \theta^\perp$

(c) the function

$$\Lambda \mathcal{P}_\theta e = -\sum_{j=1}^{n-1} \mathrm{p.\,v.}\, \frac{\partial R_1}{\partial x_j} * D_j\left(\mathcal{P}_\theta e\right), \tag{5.140}$$

where $\mathrm{p.\,v.}\, \frac{\partial R_1}{\partial x_j} *$ is the Riesz transform on $\theta^\perp \cong \mathbb{R}^{n-1}$, is in $L^2(\theta^\perp)$ and

$$\mathcal{F}(\Lambda \mathcal{P}_\theta e)(\xi) = \sqrt{2\pi}\, |\xi|\, \widehat{e}(\xi) \tag{5.141}$$

almost everywhere

(d) if $f \in L^2_0(\mathbb{R}^n)$, then for all $x \in \mathbb{R}^n$

$$e * f(x) = \int_{S^{n-1}} k * \mathcal{P}_\theta f(E_\theta x)\, d\theta, \tag{5.142}$$

where $k = \frac{b_n}{2} \Lambda \mathcal{P}_\theta e$.

Proof. Claim (a) follows immediately from Theorem 5.16(b). In view of claim (b), Lemma 5.15 asserts that

$$\begin{aligned}
\int_{S^{n-1}} \int_{\theta^\perp} |\xi|^2\, |\widehat{e}(\xi)|^2\, d\xi\, d\theta &= |S^{n-2}| \int_{\mathbb{R}^n} |\xi|\, |\widehat{e}(\xi)|^2\, d\xi \\
&\leq |S^{n-2}| \int_{\mathbb{R}^n} (1 + |\xi|^2)^{1/2} |\widehat{e}(\xi)|^2\, d\xi \\
&= |S^{n-2}|\, \|e\|_{H^{1/2}(\mathbb{R}^n)} < \infty.
\end{aligned} \tag{5.143}$$

Therefore almost all $\theta \in S^{n-1}$ satisfy

$$\||\xi|\, \widehat{e}(\xi)\|^2_{L^2(\theta^\perp)} = \int_{\theta^\perp} |\xi|^2\, |\widehat{e}(\xi)|^2\, d\xi < \infty \tag{5.144}$$

and the claims of Theorem 5.16. Fix such a θ. By Theorem 5.16(d), $\mathcal{F}(\mathcal{P}_\theta e) = \sqrt{2\pi}\,\widehat{e}$ almost everywhere on $\theta^\perp$, and therefore $\mathcal{F}(D_j(\mathcal{P}_\theta e))(\xi) = \sqrt{2\pi}\, i\,\xi_j\,\widehat{e}(\xi)$, which is in $L^2(\theta^\perp)$ because

$$\|D_j(\mathcal{P}_\theta e)\|_{L^2(\theta^\perp)} = \sqrt{2\pi}\,\|\xi_j\,\widehat{e}(\xi)\|_{L^2(\theta^\perp)} \le \sqrt{2\pi}\,\||\xi|\,\widehat{e}(\xi)\|_{L^2(\theta^\perp)} < \infty. \tag{5.145}$$

This proves (b).

Since $D_j(\mathcal{P}_\theta e) \in L^2(\theta^\perp)$, Theorem 4.10 in $\theta^\perp \cong \mathbb{R}^{n-1}$ shows that

$$\Lambda\mathcal{P}_\theta e = -\sum_{j=1}^{n-1} \mathrm{p.\,v.}\,\frac{\partial R_1}{\partial x_j} * D_j\mathcal{P}_\theta e \in L^2(\theta^\perp), \tag{5.146}$$

and that

$$\mathcal{F}\left(\Lambda\mathcal{P}_\theta e\right)(\xi) = -\sum_{j=1}^{n-1} i\,\frac{\xi_j}{|\xi|}\,i\,\xi_j\,\mathcal{F}(\mathcal{P}_\theta e)(\xi) = \sqrt{2\pi}\,\widehat{e}(\xi)\,|\xi|^{-1}\sum_{j=1}^{n-1}\xi_j^2 = \sqrt{2\pi}\,|\xi|\,\widehat{e}(\xi), \tag{5.147}$$

as claimed in (c).

Claim (d) will be proved through a usual limit argument: we shall first derive the result for test functions, and then show that when more general functions are approximated by test functions, the result holds at the limit also for the more general functions.

Assume that $\phi \in L^2_0(\mathbb{R}^n)$ and $\operatorname{supp}\phi \subset B(0,R)$; we shall soon apply the results that we are going to derive, with ϕ a function closely related to f. Then

$$\|\mathcal{P}_\theta\phi\|^2_{L^2(\theta^\perp)} = \int_{\theta^\perp}\left|\int_{-\infty}^{\infty}\phi(y+t\theta)\,dt\right|^2 dy, \tag{5.148}$$

and applying the triangle inequality and Hölder's inequality to the inner integral gives

$$\begin{aligned}\|\mathcal{P}_\theta\phi\|^2_{L^2(\theta^\perp)} &\le \int_{\theta^\perp}\int_{-\infty}^{\infty}|\phi(y+t\theta)|^2\,\chi_{(-R,R)}(y+t\theta)^2\,dt\,dy\\ &\le \int_{\theta^\perp}2R\int_{-\infty}^{\infty}|\phi(y+t\theta)|^2\,dt\,dy\\ &= 2R\,\|\phi\|^2_{L^2(\mathbb{R}^n)} < \infty,\end{aligned} \tag{5.149}$$

so that $\mathcal{P}_\theta\phi \in L^2(\theta^\perp)$. Also, $\operatorname{supp}\mathcal{P}_\theta\phi \subset B(0,R)$, since whenever $y \in \theta^\perp \setminus B(0,R)$, $|y+t\theta| = \sqrt{|y|^2+t^2} \ge R$ for all t.

Choose a sequence of test functions $\phi_k \in \mathcal{D}(\mathbb{R}^n)$, such that $\operatorname{supp}\phi_k \subset B(0,R)$ and $\|\phi_k - \phi\|_{L^2(\mathbb{R}^n)} \to 0$. Then

$$\begin{aligned}\int_{\theta^\perp}\Lambda\mathcal{P}_\theta e(y)\,\mathcal{P}_\theta\phi_k(y)\,dy &= \langle\Lambda\mathcal{P}_\theta e, \mathcal{P}_\theta\phi_k\rangle\\ &= \left\langle\mathcal{F}\Lambda\mathcal{P}_\theta e, \mathcal{F}^{-1}\mathcal{F}^{-1}\mathcal{F}\mathcal{P}_\theta\phi_k\right\rangle.\end{aligned} \tag{5.150}$$

Now note that by Lemma 5.14, $\mathcal{F}\mathcal{P}_\theta\phi_k(\xi) = \sqrt{2\pi}\,\widehat{\phi_k}(\xi)$ and by (A.59), $\mathcal{F}^{-1}\mathcal{F}^{-1} = \check{\ }$. By claim (c) just proved above, $\mathcal{F}\Lambda\mathcal{P}_\theta e(\xi) = \sqrt{2\pi}\,|\xi|\,\widehat{e}(\xi)$. Therefore,

$$\int_{\theta^\perp}\Lambda\mathcal{P}_\theta e(y)\,\mathcal{P}_\theta\phi_k(y)\,dy = 2\pi\int_{\theta^\perp}|\xi|\,\widehat{e}(\xi)\,\widehat{\phi_k}(-\xi)\,d\xi, \tag{5.151}$$

and consequently

$$\int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(y)\, \mathcal{P}_\theta \phi_k(y)\, dy\, d\theta = 2\pi \int_{S^{n-1}} \int_{\theta^\perp} |\xi|\, \widehat{e}(\xi)\, \widehat{\phi_k}(-\xi)\, d\xi\, d\theta. \tag{5.152}$$

As $\||\xi|\, \widehat{e}(\xi)\, \widehat{\phi_k}(-\xi)\|_{L^1(\theta^\perp)} \leq \||\xi|\, \widehat{e}(\xi)\|_{L^2(\theta^\perp)} \|\widehat{\phi_k}(-\xi)\|_{L^2(\theta^\perp)} < \infty$ for almost all $\theta \in S^{n-1}$ by (5.144), Lemma 5.15 can be applied separately to the positive and negative parts of $\widehat{e}(\xi)\widehat{\phi_k}(-\xi)$, yielding

$$\begin{aligned} \int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(y)\, \mathcal{P}_\theta \phi_k(y)\, dy\, d\theta &= 2\pi\, |S^{n-2}| \int_{\mathbb{R}^n} \widehat{e}(\xi)\, \widehat{\phi_k}(-\xi)\, d\xi \\ &= 2\pi\, |S^{n-2}| \langle \mathcal{F} e, \mathcal{F}^{-1} \mathcal{F}^{-1} \mathcal{F} \phi_k \rangle \\ &= 2\pi\, |S^{n-2}| \langle e, \phi_k \rangle \\ &= 2\pi\, |S^{n-2}| \int_{\mathbb{R}^n} e(y)\, \phi_k(y)\, dy. \end{aligned} \tag{5.153}$$

The same holds at the limit for ϕ, which can be seen as follows. By Hölder's inequality and (5.149),

$$\begin{aligned} &\Big| \int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(y) \big(\mathcal{P}_\theta \phi_k(y) - \mathcal{P}_\theta \phi(y) \big)\, dy\, d\theta \Big| \\ &\qquad \leq \int_{S^{n-1}} \|\Lambda \mathcal{P}_\theta e\|_{L^2(\theta^\perp)} \|\mathcal{P}_\theta \phi_k - \mathcal{P}_\theta \phi\|_{L^2(\theta^\perp)}\, d\theta \\ &\qquad \leq |S^{n-1}| \|\Lambda \mathcal{P}_\theta e\|_{L^2(\theta^\perp)} \sqrt{2R}\, \|\phi_k - \phi\|_{L^2(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0. \end{aligned} \tag{5.154}$$

Since

$$\Big| \int_{\mathbb{R}^n} e(y) \big(\phi_k(y) - \phi(y) \big)\, dy \Big| \leq \|e\|_{L^2(\mathbb{R}^n)} \|\phi_k - \phi\|_{L^2(\mathbb{R}^n)} \xrightarrow{k \to \infty} 0, \tag{5.155}$$

this implies that

$$\begin{aligned} 2\pi\, |S^{n-2}| \int_{\mathbb{R}^n} e(y)\, \phi(y)\, dy &= \lim_{k \to \infty} 2\pi\, |S^{n-2}| \int_{\mathbb{R}^n} e(y)\, \phi_k(y)\, dy \\ &= \lim_{k \to \infty} \int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(y)\, \mathcal{P}_\theta \phi_k(y)\, dy\, d\theta \\ &= \int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(y)\, \mathcal{P}_\theta \phi(y)\, dy\, d\theta. \end{aligned} \tag{5.156}$$

If we now choose $\phi(y) = f(x - y)$ with $x \in \mathbb{R}^n$, we observe that using $\tau = t - x \cdot \theta$, $d\tau = -dt$,

$$\begin{aligned} \mathcal{P}_\theta \phi(y) &= \int_{-\infty}^{\infty} f(x - y - t\theta)\, dt \\ &= \int_{-\infty}^{\infty} f(E_\theta x + (x \cdot \theta)\, \theta - y - t\theta)\, dt \\ &= -\int_{\infty}^{-\infty} f(E_\theta x - y + \tau\theta)\, d\tau \\ &= \mathcal{P}_\theta f(E_\theta x - y). \end{aligned} \tag{5.157}$$

Therefore,

$$\begin{aligned} e * f(x) &= \int_{\mathbb{R}^n} e(y)\, f(x-y)\, dy \\ &= \frac{1}{2\pi\, |S^{n-2}|} \int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(y)\, \mathcal{P}_\theta f(E_\theta x - y)\, dy\, d\theta \\ &= \frac{b_n}{2} \int_{S^{n-1}} (\Lambda \mathcal{P}_\theta e * \mathcal{P}_\theta f)\, (E_\theta x)\, d\theta, \end{aligned} \tag{5.158}$$

as claimed. □

5.3 Reconstruction Formulae

The different reconstruction formulae can now be summarized as follows. The exact reconstruction formula is an immediate consequence of Theorem 5.10:

Theorem 5.18. *If $n \in \mathbb{Z}_+$, $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$ and $|\xi|^{-1} \hat{f}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for almost all x,*

$$\Lambda(R_1 * f)(x) = f(x). \tag{5.159}$$

Proof. By Theorem 5.10,

$$\mathcal{F}(\Lambda(R_1 * f))(\xi) = |\xi|\, |\xi|^{-1}\, \hat{f}(\xi) = \hat{f}(\xi). \tag{5.160}$$

Therefore $\Lambda(R_1 * f) = f$ as distributions. Since $f \in D_{\mathrm{xr}} \subset L^1_{\mathrm{loc}}$, this also implies equality almost everywhere. □

As Λ is a continuous operator from H^1 to L^2 by Theorem 4.11, the estimate for $\|\Lambda f\|_{L^2}$ depends on the L^2-norms of the derivatives of f through $\|f\|_{H^1}$. This is about as good as we can get, because

$$\Lambda = -\sum_{j=1}^{n} \mathrm{p.v.}\, \frac{\partial R_1}{\partial x_j} * D_j \tag{5.161}$$

explicitly involves the derivatives, and the Riesz transform $\mathrm{p.v.}\, \frac{\partial R_1}{\partial x_j} *$ is not a smoothing operator. The exact reconstruction is, therefore, sensitive to noise in the measurements.

The approximate reconstruction formula is obtained from Theorem 5.17:

Corollary 5.19. *Let $n \geq 2$ and $e \in D_{\mathrm{xr}}(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n)$ be such that $|\xi|^{-1} \widehat{e}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. If $f \in L^2_0(\mathbb{R}^n)$ and $\operatorname{supp} f \subset B(0,R)$, then for all $x \in \mathbb{R}^n$,*

$$\begin{aligned} e*f(x) &= \frac{b_n}{2} \int_{S^{n-1}} \int_{\theta^\perp} \Lambda \mathcal{P}_\theta e(E_\theta x - y)\, \mathcal{P}_\theta f(y)\, dy\, d\theta \\ &= \frac{b_n}{4R} \int_A \int_{S^{n-1}} \Lambda \mathcal{P}_\theta e\big(E_\theta (x-a)\big)\, \big(\mathcal{D}_a f(\theta) + \mathcal{D}_a f(-\theta)\big)\, |a \cdot \theta|\, d\theta\, da, \end{aligned} \tag{5.162}$$

where $A = S^{n-1}(0,R)$.

Proof. The first equality is Equation (5.142) of Theorem 5.17(d). To obtain the second, make the substitution $y = E_\theta a$, $dy = \frac{|a\cdot\theta|}{R}\,da$, where y runs twice over $B(0,R)\cap\theta^\perp$ (containing $\operatorname{supp}\mathcal{P}_\theta f$) as a runs over the two hemispheres of A (see Figure 3.4). This gives

$$\begin{aligned} e{*}f(x) &= \frac{b_n}{4R}\int_{S^{n-1}}\int_A \big(\Lambda\mathcal{P}_\theta e(E_\theta(x-a))\big)\,\mathcal{P}_\theta f(E_\theta a)\,|a\cdot\theta|\,da\,d\theta \\ &= \frac{b_n}{4R}\int_A\int_{S^{n-1}} \big(\Lambda\mathcal{P}_\theta e(E_\theta(x-a))\big)\big(\mathcal{D}_a f(\theta)+\mathcal{D}_a f(-\theta)\big)\,|a\cdot\theta|\,d\theta\,da. \end{aligned} \tag{5.163}$$

To justify changing the order of integration by Fubini's theorem, note that using Hölder's inequality twice, first on the outer integral and then on the inner one, yields

$$\begin{aligned} &\left(\int_{S^{n-1}}\int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(E_\theta x-y)|\,|\mathcal{P}_\theta f(y)|\,dy\,d\theta\right)^2 \\ &\qquad\le \int_{S^{n-1}}\Big(\int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(E_\theta x-y)||\mathcal{P}_\theta f(y)|\,dy\Big)^2 d\theta\int_{S^{n-1}}1^2\,d\theta \\ &\qquad\le |S^{n-1}|\int_{S^{n-1}}\int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(E_\theta x-y)|^2\,dy\int_{\theta^\perp}|\mathcal{P}_\theta f(y)|^2\,dy\,d\theta. \end{aligned} \tag{5.164}$$

This can be seen to be finite by analysing both inner integrals separately. Since

$$\begin{aligned} \int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(E_\theta x-y)|^2\,dy &= \int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(y)|^2\,dy \\ &= \|\Lambda\mathcal{P}_\theta e\|_{L^2(\theta^\perp)} = \|\mathcal{F}(\Lambda\mathcal{P}_\theta e)\|_{L^2(\theta^\perp)}, \end{aligned} \tag{5.165}$$

Theorem 5.17(c) and Lemma 5.15 assert that

$$\begin{aligned} \int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(E_\theta x-y)|^2\,dy &= \int_{\theta^\perp}|\xi|^2|\widehat{e}(\xi)|^2\,d\xi \\ &= |S^{n-2}|\int_{\mathbb{R}^n}|\xi||\widehat{e}(\xi)|^2\,d\xi \\ &\le |S^{n-2}|\,\|e\|^2_{H^{1/2}(\mathbb{R}^n)} =: C_1. \end{aligned} \tag{5.166}$$

If $\operatorname{supp} f\subset B(0,\tilde{R})$, then (5.149) tells that

$$\int_{\theta^\perp}|\mathcal{P}_\theta f(y)|^2\,dy = \|\mathcal{P}_\theta f\|^2_{L^2(\theta^\perp)} \le 2\tilde{R}\,\|f\|^2_{L^2(\mathbb{R}^n)} =: C_2. \tag{5.167}$$

Consequently,

$$\left(\int_{S^{n-1}}\int_{\theta^\perp}|\Lambda\mathcal{P}_\theta e(E_\theta x-y)|\,|\mathcal{P}_\theta f(y)|\,dy\,d\theta\right)^2 \le |S^{n-1}|^2\,C_1\,C_2<\infty. \tag{5.168}$$

□

The reconstruction formula for $\Lambda e * f$ is obtained by replacing e by Λe above:

Theorem 5.20. *Let $n\ge 2$ and let $e\in H^{3/2}(\mathbb{R}^n)\cap D_{\mathrm{xr}}(\mathbb{R}^n)$ be such that Λe, $D_1 e$, $\dots$, $D_n e\in D_{\mathrm{xr}}(\mathbb{R}^n)$. If $f\in L^2_0(\mathbb{R}^n)$, then*

$$\begin{aligned} \Lambda(e{*}f)(x) &= (\Lambda e)*f(x) \\ &= -\frac{b_n}{2}\int_{S^{n-1}}\big(\triangle\mathcal{P}_\theta e\big)*\big(\mathcal{P}_\theta f\big)(E_\theta x)\,d\theta \\ &= -\frac{b_n}{4R}\int_A\int_{S^{n-1}}\triangle\mathcal{P}_\theta e(E_\theta(x-a))\,(\mathcal{D}_a f(\theta)+\mathcal{D}_a f(-\theta))\,|a\cdot\theta|\,d\theta\,da \end{aligned} \tag{5.169}$$

for all $x \in \mathbb{R}^n$. If, in addition, e has compact support, then

$$\Lambda(e * f) = e * \Lambda f. \tag{5.170}$$

Proof. First observe that

$$\|\Lambda e\|_{H^{1/2}} = \|(1+|\xi|^2)^{1/4}|\xi|\,\widehat{e}(\xi)\|_{L^2} \le \|(1+|\xi|^2)^{3/4}\widehat{e}(\xi)\|_{L^2} = \|e\|_{H^{3/2}} < \infty, \tag{5.171}$$

and that

$$\begin{aligned}\int_K |\xi|^{-1}\,|\mathcal{F}\Lambda e(\xi)|\,d\xi &= \int_K |\widehat{e}(\xi)|\,d\xi \\ &\le \|\widehat{e}\|_{L^2}\|\chi_K\|_{L^2} \\ &\le \|e\|_{H^{3/2}}\sqrt{m(K)} < \infty\end{aligned} \tag{5.172}$$

whenever K is compact. Therefore Corollary 5.19 can be applied to Λe:

$$\begin{aligned}(\Lambda e) * f(x) &= \frac{b_n}{2}\int_{S^{n-1}} \big(\Lambda\mathcal{P}_\theta\Lambda e\big) * \big(\mathcal{P}_\theta f\big)(E_\theta x)\,d\theta \qquad (5.173)\\ &= \frac{b_n}{2R}\int_A\int_{S^{n-1}} \big(\Lambda\mathcal{P}_\theta\Lambda e(E_\theta(x-a))\big)\big(\mathcal{D}_a f(\theta)+\mathcal{D}_a f(-\theta)\big)\,|a\cdot\theta|\,d\theta\,da.\end{aligned}$$

Now Theorem 5.17(c), applied to Λe, asserts that for almost all $\theta \in S^{n-1}$,

$$\mathcal{F}(\Lambda\mathcal{P}_\theta\Lambda e) = \sqrt{2\pi}\,|\xi|\,\mathcal{F}(\Lambda e) = \sqrt{2\pi}\,|\xi|^2\,\widehat{e}(\xi) = -\sum_{j=1}^n \sqrt{2\pi}\,i\xi_j\,\mathcal{F}(D_j e). \tag{5.174}$$

Since

$$|\mathcal{F}D_j e| = |\xi_j|\,\widehat{e}(\xi) \le |\xi|\,\widehat{e}(\xi) = |\mathcal{F}\Lambda e|, \tag{5.175}$$

Equations (5.171) and (5.172) show that $D_j e \in H^{1/2}$ and $|\xi|^{-1}\mathcal{F}(D_j e) \in L^1_{\text{loc}}$. Therefore Theorem 5.17(b) can be applied to $D_j e$, yielding that

$$\sqrt{2\pi}\,i\xi_j\,\mathcal{F}(D_j e) = \mathcal{F}(D_j\mathcal{P}_\theta D_j e) \tag{5.176}$$

for almost all $\theta \in S^{n-1}$. Now $\mathcal{P}_\theta D_j e = D_j\mathcal{P}_\theta e$ as distributions for almost all $\theta \in S^{n-1}$, which can be seen as follows. Choose any $\phi \in \mathcal{D}(\theta^\perp)$. By Theorem 5.16(b), $\mathcal{P}_\theta e \in L^1_{\text{loc}}(\theta^\perp)$ for almost all $\theta \in S^{n-1}$, so that

$$\langle D_j\mathcal{P}_\theta e, \phi\rangle = -\left\langle \mathcal{P}_\theta e, \frac{\partial\phi}{\partial x_j}\right\rangle = -\int_{\operatorname{supp}\phi}\int_{-\infty}^{\infty} e(x+t\theta)\,dt\,\frac{\partial\phi}{\partial x_j}(x)\,dx. \tag{5.177}$$

Using Fubini's theorem, partial integration and the fact that $D_j e \in H^{1/2} \subset L^1_{\text{loc}}$, gives

$$\begin{aligned}\langle D_j\mathcal{P}_\theta e, \phi\rangle &= -\int_{-\infty}^{\infty}\int_{\operatorname{supp}\phi} e(x+t\theta)\,\frac{\partial\phi}{\partial x_j}(x)\,dx\,dt \\ &= \int_{-\infty}^{\infty}\int_{\operatorname{supp}\phi} D_j e(x+t\theta)\,\phi(x)\,dx\,dt \\ &= \int_{\operatorname{supp}\phi}\int_{-\infty}^{\infty} D_j e(x+t\theta)\,dt\,\phi(x)\,dx \\ &= \langle \mathcal{P}_\theta D_j e, \phi\rangle\end{aligned} \tag{5.178}$$

and shows that $\mathcal{P}_\theta D_j e = D_j \mathcal{P}_\theta e$ indeed, and consequently

$$\mathcal{F}(\Lambda \mathcal{P}_\theta \Lambda e) = -\sum_{j=1}^{n} \mathcal{F}(D_j D_j \mathcal{P}_\theta e) = \mathcal{F}(-\triangle \mathcal{P}_\theta e). \tag{5.179}$$

Therefore $\Lambda \mathcal{P}_\theta \Lambda e = -\triangle \mathcal{P}_\theta e$, and substituting this into (5.173) proves the first part of the claim. Since e, $\Lambda e \in L^2 \subset \mathcal{S}'$ and $f \in L^2_0 \subset \mathcal{D}'_0$,

$$\mathcal{F}\big(\Lambda(e * f)\big)(\xi) = |\xi|\,(2\pi)^{n/2}\,\widehat{e}(\xi)\hat{f}(\xi) = \mathcal{F}\big((\Lambda e) * f)\big)(\xi). \tag{5.180}$$

If e has bounded support, then $e \in \mathcal{D}'_0$ and we can write

$$\mathcal{F}\big(\Lambda(e * f)\big)(\xi) = \mathcal{F}(e * \Lambda f)(\xi). \tag{5.181}$$

□

5.4 Stability Results

In contrast with reconstructing $f = \Lambda R_1 * f$, which is sensitive to noise in the measurements, reconstructing $e * f$ is a stable operation. The following theorem tells that the error in $e * f$ is uniformly bounded by the L^2 norm of the error in the measurements $\mathcal{D}f$:

Theorem 5.21. *If $n \geq 2$ and $e \in D_{\mathrm{xr}}(\mathbb{R}^n) \cap H^{1/2}(\mathbb{R}^n)$ is such that $|\xi|^{-1}\widehat{e}(\xi) \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$|e * f(x)| \leq C\,\|\mathcal{D}f\|_{L^2(S^{n-1}(0,R)\times S^{n-1})} \tag{5.182}$$

for all $f \in L^2_0(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B(0,R)$ and all $x \in \mathbb{R}^n$.

Proof. By Corollary 5.19, Young's inequality on $\theta^\perp$ and Hölder's inequality on S^{n-1},

$$\begin{aligned} |e * f(x)|^2 &\leq \frac{b_n^2}{4}\left[\int_{S^{n-1}} \big|\Lambda \mathcal{P}_\theta e * \mathcal{P}_\theta f(E_\theta x)\big|\,d\theta\right]^2 \\ &\leq \frac{b_n^2}{4}\left[\int_{S^{n-1}} \|\Lambda \mathcal{P}_\theta e\|_{L^2(\theta^\perp)} \|\mathcal{P}_\theta f\|_{L^2(\theta^\perp)}\,d\theta\right]^2 \\ &\leq \frac{b_n^2}{4}\int_{S^{n-1}} \|\Lambda \mathcal{P}_\theta e\|^2_{L^2(\theta^\perp)}\,d\theta \int_{S^{n-1}} \|\mathcal{P}_\theta f\|^2_{L^2(\theta^\perp)}\,d\theta. \end{aligned} \tag{5.183}$$

Theorem 5.17(c) asserts that

$$\|\Lambda \mathcal{P}_\theta e\|^2_{L^2(\theta^\perp)} = 2\pi\,\||\xi|\,\widehat{e}(\xi)\|^2_{L^2(\theta^\perp)} = 2\pi \int_{\theta^\perp} |\xi|^2\,|\widehat{e}(\xi)|^2\,d\xi, \tag{5.184}$$

whence

$$\begin{aligned} \int_{S^{n-1}} \|\Lambda \mathcal{P}_\theta e\|^2_{L^2(\theta^\perp)}\,d\theta &= 2\pi \int_{S^{n-1}} \int_{\theta^\perp} |\xi|^2\,|\widehat{e}(\xi)|^2\,d\xi\,d\theta \\ &= 2\pi\,|S^{n-2}| \int_{\mathbb{R}^n} |\xi|\,|\widehat{e}(\xi)|^2\,d\xi = 2\pi\,|S^{n-2}|\,\|e\|^2_{H^{1/2}} \end{aligned} \tag{5.185}$$

by Lemma 5.15. Performing the already familiar change of coordinates $y = E_\theta a$, $dy = \frac{|a\cdot\theta|}{R}\,da$, where y runs twice over $B(0,R)\cap\theta^\perp$ (containing $\operatorname{supp}\mathcal{P}_\theta f$) as a runs over the two hemispheres of $A := S^{n-1}(0,R)$ (see Figure 3.4), shows that the latter integral in (5.183) is equal to

$$\begin{aligned}\int_{S^{n-1}}\int_{\theta^\perp}|\mathcal{P}_\theta f(y)|^2\,dy\,d\theta &= \frac{1}{2}\int_{S^{n-1}}\int_A|\mathcal{P}_\theta f(E_\theta a)|^2\,\frac{|a\cdot\theta|}{R}\,da\,d\theta \\ &\le \int_{S^{n-1}}\int_A|\mathcal{D}_a f(\theta)|^2\,da\,d\theta = \|\mathcal{D}f\|^2_{L^2(A\times S^{n-1})}.\end{aligned}\tag{5.186}$$

This shows that the claim holds with $C = b_n\sqrt{\frac{\pi}{2}\,|S^{n-2}|}\,\|e\|_{H^{1/2}}$. ☐

Almost the same derivation yields an analogous result for $\Lambda e * f$:

Theorem 5.22. *If $n \ge 2$ and $e \in H^{3/2}(\mathbb{R}^n)$ is such that $\Lambda e \in D_{\mathrm{xr}}(\mathbb{R}^n)$, then there exists a constant $C > 0$ such that*

$$|\Lambda e * f(x)| \le C\,\|\mathcal{D}f\|_{L^2(S^{n-1}(0,R)\times S^{n-1})}\tag{5.187}$$

for all $f \in L^2_0(\mathbb{R}^n)$ with $\operatorname{supp} f \subset B(0,R)$ and all $x \in \mathbb{R}^n$.

Proof. As in the proof of Theorem 5.21,

$$|e * f(x)|^2 \le \frac{b_n^2}{4}\int_{S^{n-1}}\int_{\theta^\perp}|\triangle\mathcal{P}_\theta e(y)|^2\,dy\,d\theta\int_{S^{n-1}}\int_{\theta^\perp}|\mathcal{P}_\theta f(y)|^2\,dy\,d\theta.\tag{5.188}$$

by Theorem 5.20, Young's inequality on $\theta^\perp$ and Hölder's inequality on S^{n-1}. The first part of Theorem 5.20 and Lemma 5.15 show that

$$\begin{aligned}\int_{S^{n-1}}\int_{\theta^\perp}|\triangle\mathcal{P}_\theta e(y)|^2\,dy\,d\theta &= 2\pi\int_{S^{n-1}}\int_{\theta^\perp}\left||\xi|^4\,|\widehat{e}(y)|^2\,dy\,d\theta\right. \\ &= 2\pi\,|S^{n-2}|\int_{\mathbb{R}^n}|\xi|^3|\widehat{e}(y)|^2\,dy = 2\pi\,|S^{n-2}|\|e\|_{H^{3/2}},\end{aligned}\tag{5.189}$$

so that the same change of variable as in the proof of Theorem 5.21 yields

$$|e * f(x)| \le b_n\sqrt{\frac{\pi\,S^{n-2}}{2}}\,\|e\|_{H^{3/2}}\|\mathcal{D}f\|_{L^2(S^{n-1}(0,R)\times S^{n-1})},\tag{5.190}$$

as claimed. ☐

Chapter 6

Equivalence of Wave Fronts

6.1 Wave Front of a Distribution

If Λf is to be reconstructed instead of the real x-ray attenuation density f, it is important to know that these two functions yield similar information. Especially the discontinuities in the density structure, indicating boundaries between different materials or different types of tissues, are often of interest, and it fortunately turns out that Λf and f have exactly the same discontinuities. More precisely, they have the same *wave fronts*, which are defined as follows: [Hör]

Definition 6.1. *Let $X \subset \mathbb{R}^n$ be an open set and let $f \in \mathcal{S}'(X)$ be a tempered distribution. The point $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})$ does not belong to the* wave front *of f, if there exist an $\varepsilon > 0$ and a function $\phi \in C_0^\infty(\mathbb{R}^n)$ with $\phi(x_0) \neq 0$, such that for each $N \in \mathbb{Z}_+$ there is a constant c_N for which*

$$|(\mathcal{F}(\phi f))(\xi)| < c_N (1 + |\xi|)^{-N} \tag{6.1}$$

whenever ξ is in the cone

$$K_\varepsilon(\xi_0) := \left\{ \xi \in \mathbb{R}^n \setminus \{0\} \,\middle|\, \left|\tfrac{\xi}{|\xi|} - \tfrac{\xi_0}{|\xi_0|}\right| < \varepsilon \right\}. \tag{6.2}$$

All other points $(x_0, \xi_0) \in X \times (\mathbb{R}^n \setminus \{0\})$ are said to belong to the wave front *of f, which is denoted by $WF(f)$.*

The projection $\{x_0 \in X \mid (x_0, \xi_0) \in WF(f)\}$ is called the singular support *of f and denoted by* $\operatorname{singsupp} f$.

The wave front set describes the singularities of f, because if f is smooth in some neighbourhood U of a point $x_0 \in X$, then for any $\phi \in C_0^\infty$ supported in U, ϕf is in $C_0^\infty \subset \mathcal{S}$ and therefore also $\mathcal{F}(\phi f) \in \mathcal{S}$. Consequently, $(1 + |\xi|)^N \mathcal{F}(\phi f)(\xi)$ is bounded by some constant c_N depending on the seminorms $\|\mathcal{F}(\phi f)\|_{\alpha,0}$, $|\alpha| \leq N$.

Also conversely, if (6.1) holds, then ϕf is smooth. Thus, the singular support of f consists of the points $x_0 \in X$ at which f is singular. For a fixed $x_0 \in \operatorname{singsupp} f$, the set of points $\{\xi_0 \in \mathbb{R}^n \setminus \{0\} \mid (x_0, \xi_0) \in WF(f)\}$ tells in which directions f is not smooth at x_0. For details and proofs, see [Hör].

We shall now show that under certain assumptions on f, the distributions Λf and f have the same wave fronts. This result holds under more general conditions, too, but to avoid using results from the theory of pseudodifferential operators, we shall confine our study to square-integrable functions for which Λf is in the x-ray domain D_{xr} [RK]. Such functions are, of course, sufficient in practical tomography applications.

6.2 Auxiliary Results

A few lemmata will be needed for the proof.

First of all, the cutoff function ϕ above can be chosen to be identically 1 in a neighbourhood of x_0:

Lemma 6.2. *If $n \geq 2$, $X \subset \mathbb{R}^n$, $f \in \mathcal{S}'(X)$ and $(x_0, \xi_0) \notin WF(f)$, then there exist an open set $U \subset X$, a function $\phi \in C_0^\infty$ and constants $\varepsilon > 0$, $c_1, c_2, c_3, \ldots$ such that $\phi|_U = 1$, $x_0 \in U$ and*

$$|\mathcal{F}(\phi f)(\xi)| < c_N \, (1 + |\xi|)^{-N} \quad \text{whenever } \xi \in K_\varepsilon(\xi_0) \tag{6.3}$$

for all $N \in \mathbb{Z}_+$.

Proof. Since $(x_0, \xi_0) \in WF(f)$, there is a function $\tilde{\psi} \in C_0^\infty$ with $\tilde{\psi}(x_0) \neq 0$ and constants $\varepsilon > 0$ and $\tilde{c}_N$, $N \in \mathbb{Z}_+$, such that

$$\left|\mathcal{F}(\tilde{\psi} f)(\xi)\right| < \tilde{c}_N \, (1 + |\xi|)^{-N} \tag{6.4}$$

whenever $\xi \in K_\varepsilon(\xi_0)$. Set

$$\psi(x) = \frac{|\tilde{\psi}(x_0)|}{\tilde{\psi}(x_0)} \, \tilde{\psi}(x) \tag{6.5}$$

so that $a := \psi(x_0) = |\tilde{\psi}(x_0)| > 0$. Since ψ is continuous, an open neighbourhood V of x_0 can be chosen such that $\operatorname{Re}\psi(x) > \frac{a}{2}$ whenever $x \in V$. Choose $\varepsilon_1, \varepsilon_2 > 0$ such that $B(x_0, \varepsilon_1) \subset B(x_0, \varepsilon_2) \subset V$ and a function $\eta \in C_0^\infty$ with properties $0 \leq \eta(x) \leq 1$ for all $x \in X$, $\eta|_{B(x_0,\varepsilon_1)} = 1$ and $\eta(x) = 0$ when $x \notin B(x_0, \varepsilon_2)$; for instance the function $\tilde{\chi}_{\varepsilon_1,\varepsilon_2}$ in (A.34) will do.

Now set $g(x) = \eta(x)\psi(x) + (1 - \eta(x))a/4$. This function is clearly smooth and it does not vanish anywhere: When $|x - x_0| < \varepsilon_1$, $|g(x)| \geq \operatorname{Re}\psi(x) > \frac{a}{2}$; when $|x - x_0| > \varepsilon_2$, $|g(x)| = \frac{a}{4} > 0$, and when $\varepsilon_1 \leq |x - x_0| \leq \varepsilon_2$,

$$|g(x)| \geq \operatorname{Re} g(x) = \eta(x) \operatorname{Re}\psi(x) + (1 - \eta(x)) \, \frac{a}{4} \geq \frac{a}{4} > 0. \tag{6.6}$$

We can therefore define the function $\phi(x) = \psi(x)/g(x) \in C_0^\infty(\mathbb{R}^n)$ with the property $\phi(x) = 1$ whenever $x \in U := B(x_0, \varepsilon_1)$.

For showing that ϕ satisfies (6.3), first fix $h \in C_0^\infty$ such that $h|_{\operatorname{supp}\psi} = 1$; once again, $\tilde{\chi}_{\varepsilon_1',\varepsilon_2'}$ can be used if ε_1' is chosen so large that $\operatorname{supp}\psi \subset B(0, \varepsilon_1')$ and $\varepsilon_2' > \varepsilon_1'$. Then $h\psi = \psi$ and

$$\mathcal{F}(\phi f) = \mathcal{F}\left(\frac{\psi f}{g}\right) = \mathcal{F}\left(\frac{h\psi f}{g}\right) = \mathcal{F}\left(\frac{h}{g}\right) * \mathcal{F}(\psi f). \tag{6.7}$$

Therefore,

$$|\mathcal{F}(\phi f)(\xi)| \leq \int_{\mathbb{R}^n} \left|\mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right| |\mathcal{F}(\psi f)(\xi-\zeta)| \, d\zeta. \tag{6.8}$$

Using (6.4) and the fact that the constant $|\tilde{\psi}(x_0)|/\tilde{\psi}(x_0)$ in (6.5) has modulus 1, yields

$$|\mathcal{F}(\phi f)(\xi)| \leq \tilde{c}_N \int_{\mathbb{R}^n} \left|\mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right| (1+|\xi-\zeta|)^{-N} \, d\zeta. \tag{6.9}$$

We then split the domain of integration into two parts, $|\zeta| \leq \frac{1}{2}|\xi|$ and $|\zeta| > \frac{1}{2}|\xi|$, and call the integrals over these two parts I_1 and I_2, respectively. We estimate them separately.

When $|\zeta| \leq \frac{1}{2}|\xi|$, $|\xi-\zeta| \geq ||\xi|-|\zeta|| \geq \frac{1}{2}|\xi|$, and therefore

$$(1+|\xi-\zeta|)^{-N} \leq \left(1+\frac{1}{2}|\xi|\right)^{-N} \leq \left(\frac{1}{2}+\frac{1}{2}|\xi|\right)^{-N} = 2^N (1+|\xi|)^{-N}, \tag{6.10}$$

which shows that

$$\begin{aligned} I_1 &\leq \tilde{c}_N \int_{|\zeta|\leq\frac{1}{2}|\xi|} \left|\mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right| 2^N (1+|\xi|)^{-N} \, d\zeta \\ &\leq 2^N \tilde{c}_N \left\|\mathcal{F}\left(\tfrac{h}{g}\right)\right\|_{L^1} (1+|\xi|)^{-N}. \end{aligned} \tag{6.11}$$

For estimating I_2, note that $(1+|\xi-\zeta|)^{-N} < 1$, whence

$$I_2 \leq \int_{|\zeta|>\frac{1}{2}|\xi|} (1+|\zeta|)^N (1+|\zeta|)^{-N} \left|\mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right| d\zeta. \tag{6.12}$$

Since in the domain of integration,

$$(1+|\zeta|)^{-N} < \left(1+\frac{1}{2}|\xi|\right)^{-N} < \left(\frac{1}{2}+\frac{1}{2}|\xi|\right)^{-N} = 2^N (1+|\xi|)^{-N}, \tag{6.13}$$

we see that

$$\begin{aligned} I_2 &\leq 2^N \tilde{c}_N \int_{|\zeta|>\frac{1}{2}|\xi|} (1+|\zeta|)^N \left|\mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right| d\zeta \, (1+|\xi|)^{-N} \\ &\leq 2^N \tilde{c}_N \left\|(1+|\zeta|)^N \mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right\|_{L^1} (1+|\xi|)^{-N}. \end{aligned} \tag{6.14}$$

Combining estimates (6.11) and (6.14) we see that (6.3) holds, if we choose

$$c_N = 2^N \tilde{c}_N \left(\left\|\mathcal{F}\left(\tfrac{h}{g}\right)\right\|_{L^1} + \left\|(1+|\zeta|)^N \mathcal{F}\left(\tfrac{h}{g}\right)(\zeta)\right\|_{L^1} \right), \tag{6.15}$$

which is certainly finite, as $\mathcal{F}(h/g) \in \mathcal{S}(\mathbb{R}^n)$. ☐

The geometry of two coaxial cones is described by

Lemma 6.3. *If $n \geq 2$, $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $\varepsilon \in (0, 1/\sqrt{2})$, then there is a constant $C > 0$ such that*

$$|\xi-\zeta| \geq C\,|\xi| \qquad \textit{and} \qquad |\xi-\zeta| \geq C\,|\zeta| \tag{6.16}$$

whenever $\xi \in K_{\varepsilon/2}(\xi_0)$ and $\zeta \in \mathbb{R}^n \setminus K_\varepsilon(\xi_0)$.

Proof. Because $K_\delta(\xi_0) = K_\delta(\lambda\xi_0)$ for all $\lambda > 0$, we can assume that $|\xi_0| = 1$. Write $\tilde{K} := K_\varepsilon(\xi_0)$ and $\tilde{K}' := K_{\varepsilon/2}(\xi_0)$. Consider the plane $T \subset \mathbb{R}^n$ spanned by the origin, ξ and ζ and call ξ_0' the orthogonal projection of ξ_0 onto T. The intersection of a cone with a plane through its vertex is a sector of a circle, so the intersection of T with the hull of $\tilde{K}$ consists of two rays from the origin. Let κ be the point located at distance 1 from the origin on the closer one to ζ of these two rays, and let κ' be the point with the same properties with respect to $\tilde{K}'$. (See Figure 6.1.)

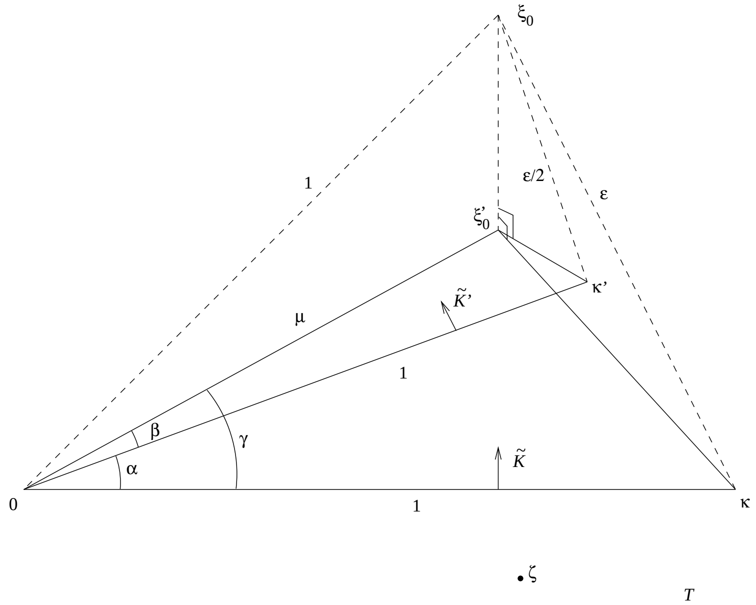


Figure 6.1: Geometry for constructing the lower bound α_0 in the proof of Lemma 6.3. Lines not in the plane T are drawn as dashed.

Write $\mu = |\xi_0'|$, $\alpha = \measuredangle \kappa' 0 \kappa$, $\beta = \measuredangle \xi_0' 0 \kappa'$ and $\gamma = \measuredangle \xi_0' 0 \kappa = \alpha + \beta$, where

$$\measuredangle x_1 x_2 x_3 = \arccos \frac{(x_1 - x_2) \cdot (x_3 - x_2)}{|x_1 - x_2|\,|x_3 - x_2|} \in [0, \pi] \tag{6.17}$$

denotes the angle between the line segments from x_2 to x_1 and x_3. As ζ moves in $\mathbb{R}^n \setminus \tilde{K}$, μ varies in $(\sqrt{1 - \varepsilon^2/4}, 1]$.

Applying the Pythagorean and cosine theorems for the triangles $\xi_0\xi_0'\kappa'$ and $\xi_0'0\kappa'$, respectively, yields

$$\frac{\varepsilon^2}{4} - (1^2 - \mu^2) = \mu^2 + 1^2 - 2\mu\cos\beta, \tag{6.18}$$

and for $\xi_0\xi_0'\kappa$ and $\xi_0'0\kappa$, yields

$$\varepsilon^2 - (1^2 - \mu^2) = \mu^2 + 1^2 - 2\mu\cos\gamma, \tag{6.19}$$

and consequently

$$\alpha = \gamma - \beta = \arccos\frac{2 - \varepsilon^2}{2\mu} - \arccos\frac{8 - \varepsilon^2}{8\mu}. \tag{6.20}$$

A few calculations show that given $\varepsilon \in (0, 1/\sqrt{2})$, this is a decreasing function of $\mu \in (\sqrt{1-\varepsilon^2/4}, 1]$ and has a lower bound $\alpha_0 > 0$ such that

$$\alpha \geq \alpha_0 \qquad \text{for all } \mu \in (\sqrt{1-\varepsilon^2/4}, 1]. \tag{6.21}$$

Denote by ζ' the point which is further away from ξ_0 of the two points of T with the properties $|\xi - \zeta'| = |\xi - \zeta|$ and $\zeta' - \xi \perp \xi$.

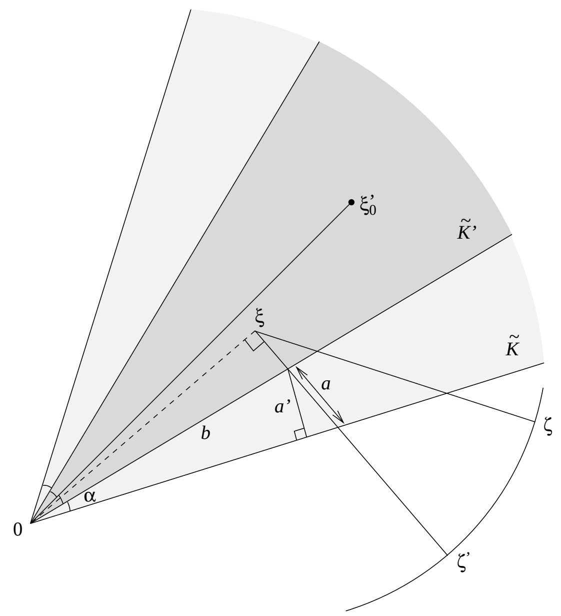


Figure 6.2: Geometry in the proof of the first part of Lemma 6.3.

Figure 6.2 depicts this situation and demonstrates that since α is clearly at most as great as the aperture of $K_\varepsilon(\xi_0)$, which is less that $\pi/2$,

$$\frac{|\xi-\zeta|}{|\xi|} = \frac{|\xi-\zeta'|}{|\xi|} \geq \frac{|\xi-\zeta'|}{b} \geq \frac{a}{b} \geq \frac{a'}{b} = \sin\alpha \geq \sin\alpha_0 =: C \tag{6.22}$$

for all $\xi \in \tilde{K}'$, $\zeta \in \mathbb{R}^n \setminus \tilde{K}$, proving the first claim.

For the second one, write $\delta = \measuredangle \zeta 0 \xi$ and distinguish two cases according to whether δ is greater or smaller than $\pi/2$.

If $\delta < \pi/2$, the situation is as in Figure 6.3, and we see that

$$\frac{|\xi-\zeta|}{|\zeta|} \geq \frac{\tilde{a}}{|\zeta|} \geq \frac{\tilde{b}}{|\zeta|} = \sin(\alpha+\beta_1) \geq \sin\alpha \geq \sin\alpha_0. \tag{6.23}$$

If then $\delta \geq \pi/2$, simply denote by ξ' the point on the line joining ξ and ζ that is perpendicular to ζ. Figure 6.4 demonstrates that

$$|\zeta - \xi| = |\zeta - \xi'| + |\xi' - \xi| \geq |\zeta - \xi'| = \sqrt{|\xi'|^2 + |\zeta|^2} \geq |\zeta| \geq \sin\alpha_0\,|\zeta|. \tag{6.24}$$

Combining estimates (6.23) and (6.24) shows that the second claim holds, too. $\square$

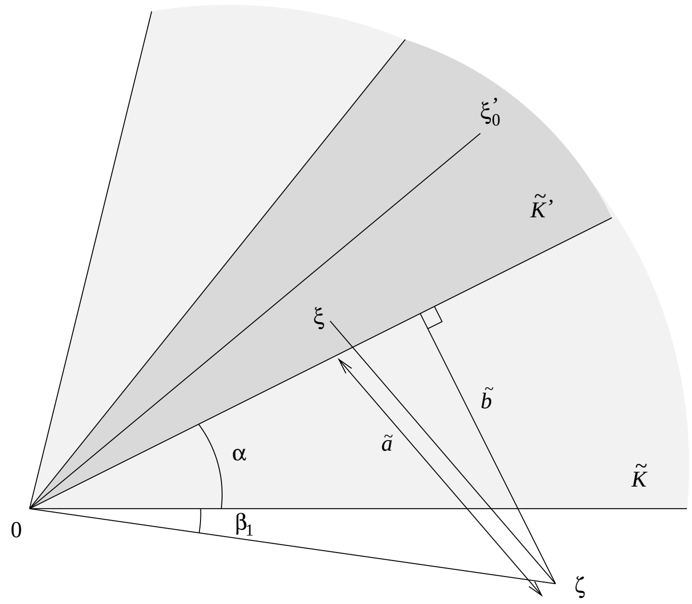


Figure 6.3: Geometry in the proof of the second part of Lemma 6.3.

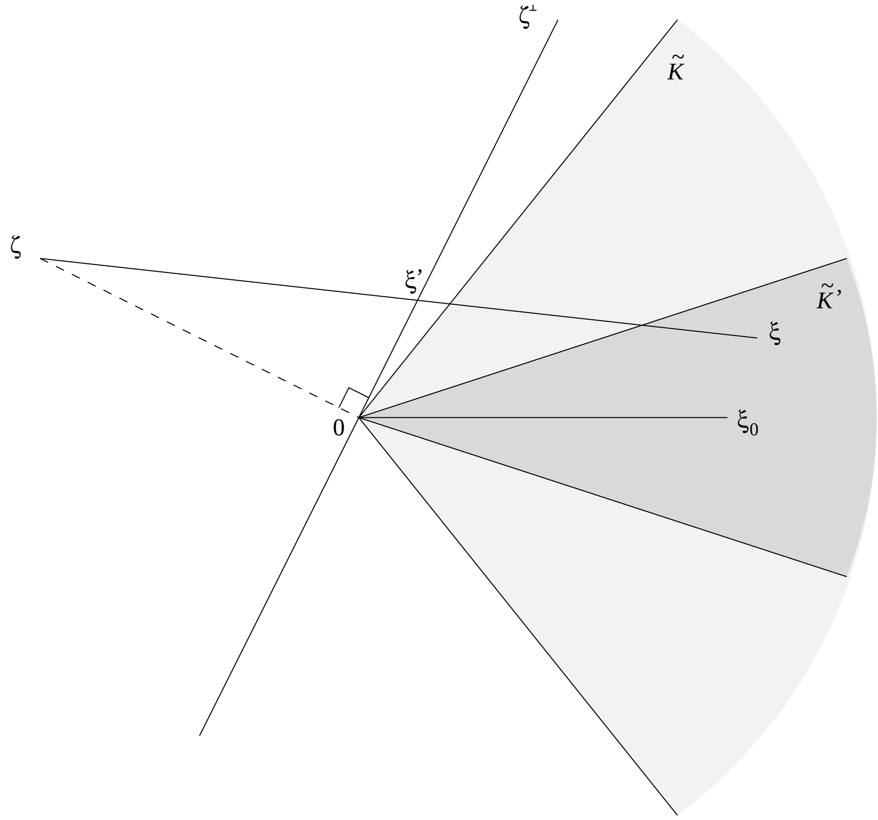


Figure 6.4: Geometry in the proof of the second part of Lemma 6.3.

Before proceeding to show that Λf and f have the same wave fronts, we shall derive the following result. It will be used for seeing that when f is decomposed as $(1-\phi)f + \phi f$, the former term does not produce singularities in Λf or $R_1 * f$:

Lemma 6.4. *Let $n \geq 2$, let $U \subset \mathbb{R}^n$ be open and let $\eta, \phi \in C_0^\infty(\mathbb{R}^n)$ be such that* $\operatorname{supp}\eta \subset U$ *and* $\phi|_U = 1$. *If $f \in D_{\mathrm{xr}}(\mathbb{R}^n)$ and $R_1 * f \in L^2(\mathbb{R}^n)$, then*

$$\eta\, R_1 * \big[(1-\phi)f\big] \in C_0^\infty(\mathbb{R}^n), \tag{6.25}$$

and if $f \in L^2(\mathbb{R}^n)$ and $\Lambda f \in L^2(\mathbb{R}^n)$, then

$$\eta\, \Lambda\big[(1-\phi)f\big] \in C_0^\infty(\mathbb{R}^n). \tag{6.26}$$

Proof. Since η has bounded support, so do the two functions in the claims. For proving that

$$\eta(x) R_1 * \big[(1-\phi)f\big](x) = b_n \eta(x) \int_{\mathbb{R}^n} \frac{1-\phi(y)}{|x-y|^{n-1}}\, f(y)\, dy \tag{6.27}$$

is smooth, note that in order for both $1-\phi(y)$ and $\eta(x)$ to be non-zero,

$$|x-y| \geq \operatorname{dist}\left(\operatorname{supp}\eta,\, \operatorname{supp}(1-\phi)\right) =: d_0 \tag{6.28}$$

which is positive because $\operatorname{supp}(1-\phi) \subset \mathbb{R}^n \setminus U$ and $\operatorname{supp}\eta \subset U$ are disjoint closed sets (see Figure 6.5).

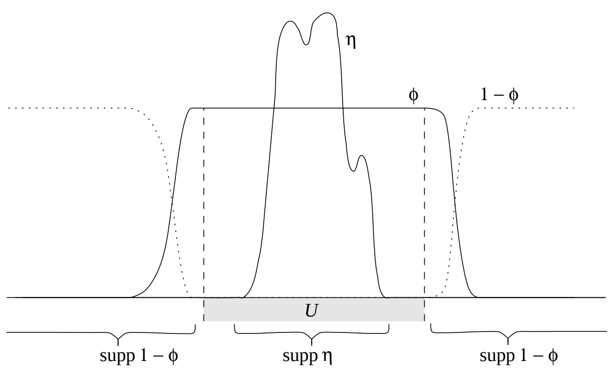


Figure 6.5: The distance between the supports of η and $1-\phi$ in the proof of Lemma 6.4 is positive.

The integrand in (6.27) is therefore nonsingular, and the Leibniz rule gives

$$\begin{aligned} &\partial^\alpha\big(\eta(x) R_1 * \big[(1-\phi)f\big](x)\big) \\ &\qquad = b_n \sum_{\beta \leq \alpha} c_{\alpha\beta}\, \partial^{\alpha-\beta}\eta(x) \int_{\mathbb{R}^n \setminus B(x_0,d_0)} \partial_x^\beta \frac{1-\phi(y)}{|x-y|^{n-1}}\, f(y)\, dy. \end{aligned} \tag{6.29}$$

Differentiation under the integral sign is permitted because, with $h := (1-\phi)\, f \in L^2$,

$$\begin{aligned} &\partial_k \int_{\mathbb{R}^n \setminus B(x_0,d_0)} \frac{h(y)}{|x-y|^{n-1}}\, dy \\ &\qquad = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x_0,d_0)} \frac{h(y)}{\varepsilon} \left[\frac{1}{|x+\varepsilon e_k - y|^{n-1}} - \frac{1}{|x-y|^{n-1}} \right] dy \quad (6.30) \\ &\qquad = \int_{\mathbb{R}^n \setminus B(x_0,d_0)} \partial_k \frac{h(y)}{|x-y|^{n-1}}\, dy \end{aligned}$$

by Lebesgue's theorem of dominated convergence: When $|\varepsilon| \leq d_0/2$,

$$|x + \varepsilon e_k - y| \geq \big||x-y| - |\varepsilon|\big| \geq d_0 - \frac{d_0}{2} = \frac{d_0}{2} > 0, \tag{6.31}$$

and therefore $|x-y|^{1-n}$ is smooth, so that the mean value theorem yields

$$\begin{aligned}\left[\frac{1}{|x+\varepsilon e_k - y|^{n-1}} - \frac{1}{|x-y|^{n-1}}\right] &= \frac{\partial}{\partial \tilde{x}_k} \frac{1}{|\tilde{x} - y|^{n-1}}\Big|_{\tilde{x}=x+\xi e_k} \varepsilon \\ &= \frac{x_k + \xi - y_k}{|x + \xi e_k - y|^{n+1}}\, \varepsilon\end{aligned} \tag{6.32}$$

for some $\xi \in \mathbb{R}$ with $|\xi| < |\varepsilon|$. The fact that $|\xi| < |\varepsilon| \leq d_0/2 \leq |x-y|/2$ also implies that

$$|x + \xi e_k - y| \geq \big||x-y| - |\xi|\big| \geq |x-y| - \frac{|x-y|}{2} = \frac{|x-y|}{2}, \tag{6.33}$$

which yields

$$\begin{aligned}\left|\frac{h(y)}{\varepsilon}\left[\frac{1}{|x+\varepsilon e_k - y|^{n-1}} - \frac{1}{|x-y|^{n-1}}\right]\right| &= \left|\frac{h(y)}{\varepsilon} \frac{x_k + \xi - y_k}{|x+\xi e_k - y|^{n+1}}\, \varepsilon\right| \\ &\leq \left|\frac{h(y)}{|x + \xi e_k - y|^n}\right| \\ &\leq \frac{2^n\, |h(y)|}{|x-y|^n} =: g(y).\end{aligned} \tag{6.34}$$

The dominant function g is integrable because

$$\|g\|_{L^1}^2 \leq 2^{2n}\, |S^{n-1}| \int_{d_0}^{\infty} r^{-2n}\, r^{n-1}\, dr\, \|h\|_{L^2}^2 < \infty. \tag{6.35}$$

The same procedure also works inductively for higher-order derivatives, since

$$\begin{aligned}&\partial^{\gamma+e_k} \int_{\mathbb{R}^n \setminus B(x_0,d_0)} \frac{h(y)}{|x-y|^{n-1}}\, dy \\ &\qquad = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B(x_0,d_0)} [\partial^\gamma R_1(x+\varepsilon e_k - y) - \partial^\gamma R_1(x-y)]\, dy\end{aligned} \tag{6.36}$$

and here the integral is dominated in absolute value by

$$\left|\frac{h(y)}{\varepsilon}\, \partial^{\gamma+\varepsilon_k} R_1(x - \xi e_k - y)\, \varepsilon\right| \leq \frac{2^{n+|\gamma|}\, |h(y)|}{|x-y|^{n+|\gamma|}} \in L^1, \qquad |\xi| < |\varepsilon| \leq \frac{d_0}{2}. \tag{6.37}$$

This shows that $\eta\, R_1 * \big[(1-\phi) f\big] \in C^\infty(\mathbb{R}^n)$.

Analogously for the second part of the claim,

$$\begin{aligned}&\partial^\alpha \Big(\eta(x)\, \Lambda\big[(1-\phi)f\big](x)\Big) \\ &\quad = \partial^\alpha \Big(-\eta(x) \sum_{j=1}^n \mathrm{p.v.}\, \frac{\partial R_1}{\partial x_j} * D_j\big[(1-\phi)f\big](x)\Big) \\ &\quad = -\sum_{\beta \leq \alpha} c_{\alpha\beta}\, \partial^{\alpha-\beta}\eta(x) \sum_{j=1}^n \int_{\mathbb{R}^n \setminus B(x_0,d_0)} \partial_x^{e_j+\beta} R_1(x-y)\, D_j\big[(1-\phi)f\big](y)\, dy.\end{aligned} \tag{6.38}$$

This time the induction step is

$$\begin{aligned}
&\partial^{\gamma+e_k}\int_{\mathbb{R}^n\setminus B(x_0,d_0)}\frac{\partial R_1}{\partial x_j}(x-y)\,D_jh(y)\,dy\\
&=\lim_{\varepsilon\to 0}\int_{\mathbb{R}^n\setminus B(x_0,d_0)}\frac{D_jh(y)}{\varepsilon}\left[\partial^{e_j+\beta}R_1(x+\varepsilon e_k-y)-\partial^{e_j+\beta}R_1(x-y)\right]dy. \qquad (6.39)
\end{aligned}$$

Now the integrand is dominated in absolute value by

$$\begin{aligned}
\left|\frac{D_jh(y)}{\varepsilon}\,\partial^{e_j+\beta+e_k}R_1(x+\xi e_k-y)\,\varepsilon\right| &\leq \frac{|D_jh(y)|}{|x+\xi e_k-y|^{n-1+1+|\beta|+1}}\\
&\leq \frac{2^{n+|\beta|+1}\,|D_jh(y)|}{|x-y|^{n+|\beta|+1}} =: g(y),
\end{aligned} \qquad (6.40)$$

which is integrable since

$$\|g\|_{L^1}^2 \leq 2^{2n+2|\beta|+2}\,|S^{n-1}|\int_{d_0}^{\infty} r^{-2n-2|\beta|-2}\,r^{n-1}\,dr\,\|D_jh\|_{L^2}^2 \qquad (6.41)$$

and

$$\|D_jh\|_{L^2} = \|(D_j\phi)f+(1-\phi)D_jf\|_{L^2} \leq \|D_j\phi\|_{L^\infty}\|f\|_{L^2}+\|1-\phi\|_{L^\infty}\|D_jf\|_{L^2} \qquad (6.42)$$

is finite because $f\in L^2$ and $\|D_jf\|_{L^2}=\|i\xi_j\hat{f}\|_{L^2}\leq \||\xi|\,\hat{f}\|_{L^2}=\|\Lambda f\|_{L^2}<\infty$. □

6.3 Λf and f Have Equal Wave Fronts

We are now ready to present the main result of this chapter:

Theorem 6.5. *If $n\geq 2$, $f\in L^2(\mathbb{R}^n)$ and $\Lambda f\in D_{\mathrm{xr}}(\mathbb{R}^n)$, then $WF(f)=WF(\Lambda f)$.*

Proof. First for showing the inclusion $WF(\Lambda f)\subset WF(f)$, choose $(x_0,\,\xi_0)\notin WF(f)$ and show that $(x_0,\,\xi_0)\notin WF(\Lambda f)$. The converse inclusion will be shown later. Fix $U,\,\phi,\,\varepsilon,\,c_1,\,c_2,\,c_3,\,\ldots$ according to Lemma 6.2. Assume that $\varepsilon<1/\sqrt{2}$, because if the condition (6.3) holds for an $\varepsilon\geq 1/\sqrt{2}$, it holds for a smaller one as well.

Select a function $\eta\in C_0^\infty(\mathbb{R}^n)$ such that $\operatorname{supp}\eta\subset U$. We shall prove that there are constants $\varepsilon'>0,\,c_1',\,c_2',\,c_3',\,\ldots$ such that

$$|\mathcal{F}(\eta\Lambda f)(\xi)| < c_N'\,(1+|\xi|)^{-N} \qquad (6.43)$$

for all $N\in\mathbb{Z}_+$, whenever $\xi\in K_{\varepsilon'}(\xi_0)$. To this end, write $\Lambda f=\Lambda(\phi f)+\Lambda\left((1-\phi)f\right)$. By Lemma 6.4, $\eta\Lambda\left((1-\phi)f\right)\in C_0^\infty$, which implies that $\mathcal{F}\left(\eta\Lambda\left((1-\phi)f\right)\right)\in\mathcal{S}$. Therefore, $(1+|\xi|)^N\mathcal{F}(\eta\Lambda((1-\phi)f))$ is bounded for all $N\in\mathbb{Z}_+$.

It is therefore sufficient to find constants $\varepsilon'>0,\,c_1',\,c_2',\,c_3'\ldots$ such that

$$|\mathcal{F}(\eta\Lambda(\phi f))(\xi)| < c_N'(1+|\xi|)^{-N} \qquad (6.44)$$

for all $N\in\mathbb{Z}_+$ and $\xi\in K_{\varepsilon'}(\xi_0)$. In fact, because $1+|\xi|^N>\frac{1}{2}\,(1+|\xi|)^N$ for large $|\xi|$, say for $|\xi|>R_0$, and because

$$d_N=\max_{|\xi|\leq R_0}\frac{(1+|\xi|)^N}{1+|\xi|^N}<\infty, \qquad (6.45)$$

we see that always

$$\left(1+|\xi|^N\right)^{-1} \le \max\{d_N,\, 2\}\,(1+|\xi|)^{-N}. \tag{6.46}$$

Hence it suffices to find constants c'_N such that

$$|\mathcal{F}(\eta\Lambda(\phi f))(\xi)| < c'_N\left(1+|\xi|^N\right)^{-1} \tag{6.47}$$

whenever $\xi \in K_{\varepsilon'}(\xi_0)$ for some $\varepsilon' > 0$, or equivalently to show that for each N,

$$\left(1+|\xi|^N\right)|\mathcal{F}(\eta\Lambda(\phi f))(\xi)| \tag{6.48}$$

is bounded as ξ moves in $K_{\varepsilon'}(\xi_0)$. This is what will be done next, with $\varepsilon' = \varepsilon/2$.

Choose $N \in \mathbb{Z}_+$ arbitrarily and examine the expression

$$\mathcal{F}(\eta\Lambda(\phi f))(\xi) = \mathcal{F}\eta * \mathcal{F}(\Lambda(\phi f))(\xi) = \int_{\mathbb{R}^n} \widehat{\eta}(\xi-\zeta)\,|\zeta|\,\mathcal{F}(\phi f)(\zeta)\,d\zeta. \tag{6.49}$$

Split the domain of integration into two parts, the cone $\tilde{K} := K_\varepsilon(\xi_0)$, and its complement. Using the triangle inequality, we obtain

$$\left(1+|\xi|^N\right)|\mathcal{F}(\eta\Lambda(\phi f))(\xi)| \le I_1(\xi) + I_2(\xi), \tag{6.50}$$

where

$$I_1(\xi) = \left(1+|\xi|^N\right)\int_{\tilde{K}} |\widehat{\eta}(\xi-\zeta)|\,|\zeta|\,|\mathcal{F}(\phi f)(\zeta)|\,d\zeta \tag{6.51}$$

and

$$I_2(\xi) = \left(1+|\xi|^N\right)\int_{\mathbb{R}^n\setminus\tilde{K}} |\widehat{\eta}(\xi-\zeta)|\,|\zeta|\,|\mathcal{F}(\phi f)(\zeta)|\,d\zeta. \tag{6.52}$$

We shall show that both $I_1(\xi)$ and $I_2(\xi)$ are bounded.

For estimating I_1, note that since $\widehat{\eta} \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{\eta}(\xi-\zeta) \le C_1(1+|\xi-\zeta|^N)^{-1} \tag{6.53}$$

for some $C_1 > 0$, and that by the choice of ϕ,

$$|\mathcal{F}(\phi f)(\zeta)| \le c_{N+n+2}(1+|\zeta|)^{-N-n-2} \le c_{N+n+2}(1+|\zeta|^{N+n+2})^{-1}. \tag{6.54}$$

Therefore

$$I_1(\xi) \le C_1\,c_{N+n+2}\int_{\tilde{K}} \frac{1+|\xi|^N}{1+|\xi-\zeta|^N}\,\frac{|\zeta|}{1+|\zeta|^{N+n+2}}\,d\zeta. \tag{6.55}$$

To see that this is bounded, observe that

$$\begin{aligned}
\frac{1+|\xi|^N}{1+|\xi-\zeta|^N} &= \frac{1+|(\xi-\zeta)+\zeta|^N}{1+|\xi-\zeta|^N} \le \frac{1+(|\xi-\zeta|+|\zeta|)^N}{1+|\xi-\zeta|^N} \\
&= \frac{1+|\xi-\zeta|^N}{1+|\xi-\zeta|^N} + \sum_{j=1}^{N}\binom{n}{j}\frac{|\xi-\zeta|^{N-j}|\zeta|^j}{1+|\xi-\zeta|^N} \\
&= 1 + \sum_{j=1}^{N}\binom{n}{j}|\zeta|^j\,\frac{1}{|\xi-\zeta|^{j-N}+|\xi-\zeta|^j} \\
&\le 1 + \sum_{j=1}^{N}\binom{n}{j}|\zeta|^j = (1+|\zeta|)^N,
\end{aligned} \tag{6.56}$$

which shows that

$$\begin{aligned} I_1(\xi) &\le C_1\, c_{N+n+2} \int_{\mathbb{R}^n} \frac{(1+|\zeta|)^N\,|\zeta|}{1+|\zeta|^{N+n+2}}\, d\zeta \\ &= C_1\, c_{N+n+2}\, |S^{n-1}| \int_0^\infty \frac{(1+r)^N\, r\, r^{n-1}}{1+r^{N+n+2}}\, dr. \end{aligned} \tag{6.57}$$

The integrand is majorized on $[0,1]$ by $2^N 1^n/(1+0^{N+n+2}) = 2^N$ and on $[1,\infty)$ by $(2r)^N r^n/r^{N+n+2} = 2^N r^{-2}$, both of which are integrable over the respective intervals. Therefore $I_1(\xi)$ is bounded.

For I_2, use the fact that as $\widehat{\eta} \in \mathcal{S}$,

$$\widehat{\eta}(\xi-\zeta) \le C_2(1+|\xi-\zeta|)^{-N-n-2} \tag{6.58}$$

for some $C_2 > 0$. Also,

$$\begin{aligned} |\mathcal{F}(\phi f)(\xi)| &\le (2\pi)^{-n/2} \int_{\mathbb{R}^n} |e^{-ix\cdot\xi}|\,|\phi(x)|\,|f(x))|\, dx \\ &= (2\pi)^{-n/2}\|\phi f\|_{L^1} \le (2\pi)^{-n/2}\|f\|_{L^2}\|\phi\|_{L^2} =: C_3, \end{aligned} \tag{6.59}$$

so that

$$I_2(\xi) \le \left(1+|\xi|^N\right) \int_{\mathbb{R}^n\setminus\tilde{K}} C_2\,(1+|\xi-\zeta|)^{-N-n-2}\,|\zeta|\, C_3\, d\zeta. \tag{6.60}$$

Analyse each factor of

$$(1+|\xi-\zeta|)^{-N-n-2} = (1+|\xi-\zeta|)^{-1}(1+|\xi-\zeta|)^{-N}(1+|\xi-\zeta|)^{-n-1} \tag{6.61}$$

separately. First of all, the second claim of Lemma 6.3 asserts that

$$(1+|\xi-\zeta|)^{-1} \le |\xi-\zeta|^{-1} \le C^{-1}\,|\zeta|^{-1}, \tag{6.62}$$

which cancels out the factor $|\zeta|$ in (6.60). By the first claim of Lemma 6.3, the second factor

$$(1+|\xi-\zeta|)^{-N} \le \max\{1, C^{-N}\}\,(1+|\xi|)^{-N} \le \max\{1, C^{-N}\}\,(1+|\xi|^N)^{-1} \tag{6.63}$$

cancels out the factor $1+|\xi|^N$ showing the required speed of decrease. The third factor is dominated by something that leaves the integral finite:

$$\begin{aligned} (1+|\xi-\zeta|)^{-n-1} &\le \max\{1, C^{-n-1}\}\,(1+|\zeta|)^{-n-1} \\ &\le \max\{1, C^{-n-1}\}\,(1+|\zeta|^{n+1})^{-1}. \end{aligned} \tag{6.64}$$

Therefore,

$$\begin{aligned} I_2(\xi) &\le C_4\left(1+|\xi|^N\right) \int_{\mathbb{R}^n\setminus\tilde{K}} \frac{|\zeta|}{|\zeta|\,(1+|\xi|^N)\,(1+|\zeta|^{n+1})}\, d\zeta \\ &\le C_4\,|S^{n-1}| \int_0^\infty \frac{r^{n-1}\,dr}{1+r^{n+1}} \\ &\le C_4\,|S^{n-1}| \left(\int_0^1 \frac{dr}{1+0^{n+1}} + \int_1^\infty \frac{dr}{r^2}\right) < \infty. \end{aligned} \tag{6.65}$$

This completes the proof that $WF(\Lambda f) \subset WF(f)$.

For the converse inclusion, choose $(x_0, \xi_0) \notin WF(\Lambda f)$ and show that $(x_0, \xi_0) \notin WF(f)$. Write $\psi = \Lambda f \in D_{\mathrm{xr}}$ so that

$$|\xi|^{-1}\widehat{\psi}(\xi) = |\xi|^{-1}\,|\xi|\,\hat{f}(\xi) = \hat{f}(\xi) \in L^2 \subset L^1_{\mathrm{loc}}, \tag{6.66}$$

and consequently

$$f = \mathcal{F}^{-1}\big(|\xi|^{-1}\,|\xi|\hat{f}(\xi)\big) = \mathcal{F}^{-1}\big(|\xi|^{-1}\,\widehat{\psi}(\xi)\big) = R_1 * \psi \tag{6.67}$$

by Theorem 5.10. Choose $U \subset \mathbb{R}^n$, $\phi \in C_0^\infty$, $\varepsilon \in (0, 1/\sqrt{2})$ and constants c_1, c_2, ... , according to Lemma 6.2, as before, but now with f replaced by ψ. Then choose any $\eta \in C_0^\infty$ with $\operatorname{supp}\eta \subset U$ and proceed as follows to find constants c'_1, c'_2, ... , such that

$$|\mathcal{F}(\eta R_1 * \psi)(\xi)| < c'_N(1 + |\xi|)^{-N}, \tag{6.68}$$

or equivalently

$$|\mathcal{F}(\eta R_1 * \psi)(\xi)| < c'_N(1 + |\xi|^N)^{-1}, \tag{6.69}$$

for all $N \in \mathbb{Z}_+$ and $\xi \in K_{\varepsilon/2}(\xi_0)$. As before, it suffices to show this for $\eta R_1 * (\phi\psi)$ instead of $\eta R_1 * \psi$, because

$$\eta R_1 * \psi = \eta R_1 * \big[(1 - \phi)\,\psi\big] + \eta R_1 * (\phi\psi) \tag{6.70}$$

and $\eta R_1 * [(1 - \phi)\,\psi] \in C_0^\infty(\mathbb{R}^n)$ by Lemma 6.4, so that $\mathcal{F}(\eta R_1 * [(1 - \phi)\,\psi]) \in \mathcal{S}(\mathbb{R}^n)$ and the estimate (6.69) holds with $\eta R_1 * \psi$ replaced by $\eta R_1 * [(1 - \phi)\psi]$.

Once again, use the fact that

$$\begin{aligned} |\mathcal{F}\,(\eta R_1 * (\phi\psi))\,(\xi)| &= \big|\widehat{\eta} * \big(|\xi|^{-1}\,\mathcal{F}(\phi\psi)\big)\,(\xi)\big| \\ &\le \int_{\mathbb{R}^n} |\widehat{\eta}(\xi - \zeta)|\,|\zeta|^{-1}\,|\mathcal{F}(\phi\psi)(\zeta)|\;d\zeta \end{aligned} \tag{6.71}$$

and split the range of integration into two parts, $\tilde{K} = K_\varepsilon(\xi_0)$ and $\mathbb{R}^n \setminus \tilde{K}$. For estimating the integral over $\tilde{K}$, note that since $\widehat{\eta} \in \mathcal{S}$, $|\widehat{\eta}(\xi - \zeta)| \le C_5(1 + |\xi - \zeta|^N)^{-1}$ for some $C_5 > 0$, so that (6.56) yields

$$\begin{aligned} &\big(1 + |\xi|^N\big) \int_{\tilde{K}} |\widehat{\eta}(\xi - \zeta)|\,|\zeta|^{-1}\,|\mathcal{F}(\phi\psi)(\zeta)|\;d\zeta \\ &\le \big(1 + |\xi|^N\big) \int_{\tilde{K}} C_5\,\big(1 + |\xi - \zeta|^N\big)^{-1}\;|\zeta|^{-1} c_{N+n}\,(1 + |\zeta|)^{-N-n}\;d\zeta \\ &\le C_5\,c_{N+n} \int_{\tilde{K}} \frac{(1 + |\zeta|)^N}{|\zeta|\,(1 + |\zeta|)^{N+n}}\,d\zeta \\ &\le C_5\,c_{N+n} \int_{\mathbb{R}^n} \frac{1}{|\zeta|\,(1 + |\zeta|)^n}\,d\zeta \\ &= C_5\,c_{N+n}\,|S^{n-1}| \int_0^\infty \frac{r^{n-2}}{(1 + r)^n}\,dr \\ &\le C_5\,c_{N+n}\,|S^{n-1}| \left(\int_0^1 dr + \int_1^\infty \frac{r^{n-2}}{r^n}\,dr\right) < \infty. \end{aligned} \tag{6.72}$$

In the integral over $\mathbb{R}^n \setminus \tilde{K}$, use the fact that $\widehat{\eta} \in \mathcal{S}$ together with Lemma 6.3 to see that

$$\begin{aligned} |\widehat{\eta}(\xi - \zeta)| &\le C_6(1 + |\xi - \zeta|)^{-N}(1 + |\xi - \zeta|)^{-n} \\ &\le C_7(1 + |\xi|)^{-N}(1 + |\zeta|)^{-n} \\ &\le C_7\left(1 + |\xi|^N\right)^{-1}(1 + |\zeta|)^{-n}. \end{aligned} \tag{6.73}$$

whenever $\xi \in K_{\varepsilon/2}(\xi_0)$. This estimate and the analogue of (6.59) yield

$$\begin{aligned} &\left(1 + |\xi|^N\right) \int_{\mathbb{R}^n \setminus \tilde{K}} |\widehat{\eta}(\xi - \zeta)||\zeta|^{-1} |\mathcal{F}(\phi\psi)(\zeta)| \, d\zeta \\ &\qquad \le C_7\left(1 + |\xi|^N\right) \int_{\mathbb{R}^n} \frac{(2\pi)^{-n/2}\|\phi\|_{L^2}\|\psi\|_{L^2}}{\left(1 + |\xi|^N\right)(1 + |\zeta|)^n\,|\zeta|}\, d\zeta \\ &\qquad \le C_7\,(2\pi)^{-n/2}\,\|\phi\|_{L^2}\,\|\psi\|_{L^2}\,|S^{n-1}| \int_0^\infty \frac{r^{n-2}}{(1 + r)^n}\, dr < \infty. \end{aligned} \tag{6.74}$$

Combining estimates (6.72) and (6.74) with (6.71) shows that $(x_0, \xi_0) \notin WF(f)$. $\square$

Chapter 7

Further Properties of Λf and Conclusions

Even though $WF(f) = WF(\Lambda f)$, jumps of Λf might of course be much smaller than those of f. In this case, discontinuities of f would perhaps not be visible in $e * \Lambda f$. Fortunately, discontinuities are clearly visible, as the following results show.

Because at least for $f \in \mathcal{S}$

$$\mathcal{F}(\Lambda^2 f) = |\xi|^2 \hat{f} = -\sum_{j=1}^{n} (i\,\xi_j)^2 \hat{f} = \mathcal{F}(-\triangle f), \tag{7.1}$$

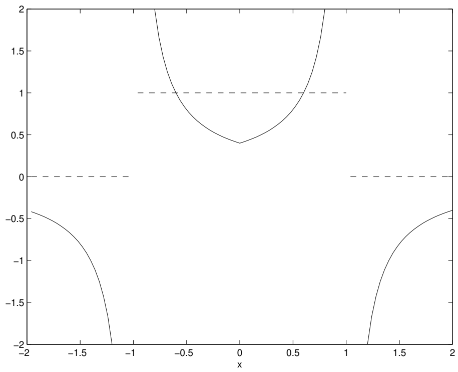
the operator Λ is sometimes called the square root of the negative Laplacian, "$\Lambda = \sqrt{-\triangle}$". By emphasizing edges, it indeed behaves like a first order differential operator. Tomography objects are often made up of areas of constant density, which makes detecting their edges an important objective in many applications.

Because a typical x-ray attenuation coefficient function f can be thought of as a linear combination of characteristic functions,

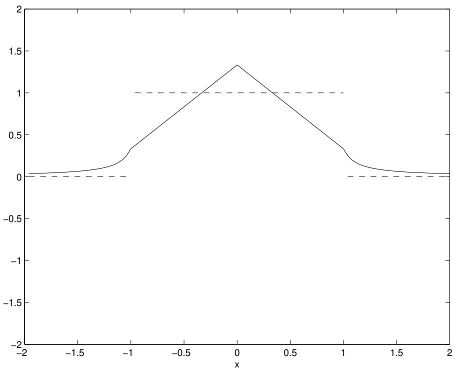
$$f = \sum_{j=1}^{n} c_j\, \chi_{X_j}, \tag{7.2}$$

and because only linear operations on f are considered, studying $\Lambda\chi_X$ gives some insight into the behaviour of Λ. It turns out that if X is sufficiently regular, then $\Lambda\chi_X$ behaves like $\pm 1/\mathrm{dist}(x, \partial X)$ near the edges of X, being positive inside X and negative outside, and is cupped inside X; see Figure 7.1(a). [FRS]

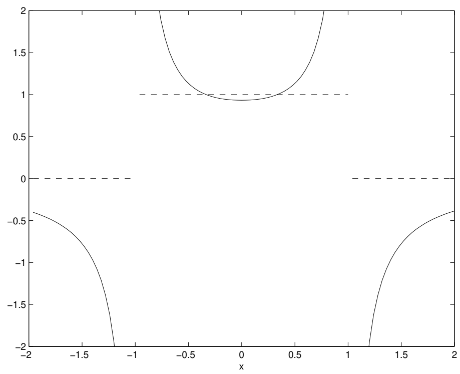
These features of Λf cause small details of low contrast to be highlighted. This is often useful to some extent, but it would normally be expedient to be able to alleviate the cup effect. A heuristic method has been proposed for this purpose: adding to Λf a multiple of $R_1 * \chi_X$, which can also be computed locally as is evident from (3.8). It turns out that $R_1 * \chi_X$ is continuous everywhere and analytic in $\mathbb{R}^n \setminus \partial X$, and that it behaves like $C + \mathrm{dist}(x, \partial X)$ inside X and like $(C + \mathrm{dist}(x, \partial X))^{1-n}$ outside X; see Figure 7.1(b). This linear combination $Lf := \alpha(\Lambda f + \mu R_1 * f)$, where α and μ are constants, is not an approximation of f, but its qualitative behaviour is quite similar, and this reconstruction has proved to be useful in practice. See Figure 7.1(c). [FRS, FRSex, FFRS]



(a) $\Lambda\chi_X$



(b) $R_1 * \chi_X$



(c) $L\chi_X$

Figure 7.1: Graph of χ_X (dotted line) and approximate graphs of (a) $\Lambda\chi_X$, (b) $R_1 * \chi_X$ and (c) $L\chi_X = \alpha(\Lambda\chi_X + \mu R_1 * \chi_X)$ (solid line) for $X = B^2 \subset \mathbb{R}^2$; section along a straight line through the origin.

The value of μ that yields the most informative images must be determined empirically for each picture. The graph of $\Lambda\chi_X$ is on average steeper for smaller sets X, which implies a stronger need for cup correction. Therefore μ should be large if the principal interest is in small details, and small if Lf is to be nearly flat in large areas of constant density. A rule of thumb of $\mu = cr_0^{-2}$, where $c \approx 6$ is an experimental constant, has been devised for making $L\chi_{B(0,r_0)}$ nearly constant in most of $B(0, r_0)$. [FFRS]

Although the difference in sign of $\Lambda\chi_X$ inside and outside X permits qualitative inference of discontinuities in the x-ray attenuation density, quantitative estimates for the sizes of the jumps in f cannot be directly made from the jumps in Λf. Procedures for calculating the jumps in f have, however, been developed. They first require dividing the object into subsets X_j in which f is assumed to be constant. [FFRS, KR1]

Observe that the reconstruction of Λf (or $\Lambda e * f$) is called *local* because the convolution kernel $\triangle\mathcal{P}_\theta e$ in the reconstruction formula (5.169) is compactly supported, so that the computation of $\Lambda f(x)$ only requires measurements of the attenuation along lines passing close to x. On the other hand, the result is *global* in the sense that the value of f at a point x affects Λf everywhere since Λ is a non-local operator, as mentioned in Section 3.6. In the reconstruction of f (or $e * f$) the situation is reversed. The kernel $\Lambda\mathcal{P}_\theta e$ is not compactly supported, whence the reconstruction of f is *global*, using attenuation measurements along all lines going through the object, but the result is *local*: changing the value of f at a point x does not change the reconstructed value at other points, except through the convolution with the point spread function e.

This local-global duality is reflected in the fact that $\Lambda\chi_X$ does not vanish completely outside X. In particular, the high contrast details show in other parts of the locally reconstructed image. Methods for reducing this effect have been developed. [FFRS]

Several extensions to the concept of local tomography have also been proposed. *Pseudolocal tomography*, for instance, expresses the reconstructed function as $f = f_d + f_d^c$, where f_d^c is smooth. The points of discontinuity and jumps in f_d and its derivatives are exactly the same as those in f and its derivatives. The reconstruction of f_d can be done using measurements from only a small neighbourhood of the region of interest. [KR2, RK]

Other generalisations include considering different pseudodifferential operators $\mathcal{B}$ of f than just Λf. Under certain assumptions, $\mathcal{B}$ preserves the singularities of f as was shown in Section 6.3 for Λ. [KLM, RK]

The theory can also be extended to the attenuated Radon transform, which appears in the context of nuclear emission tomography. Incomplete data problems, where measurements can only be made from a limited angle, are also of both theoretical and practical interest. [KLM, Nat, RK]

Any of these questions would make interesting subjects for further study.

Appendix A

Mathematical Tools

A.1 Spaces of Functions and Distributions

A.1.1 L^p Spaces

We state below some of the most often used features of L^p spaces. For an introduction to the theory of L^p spaces, see [Ru2]. We shall give proofs or references to them only for those claims that cannot be found in it.

In what follows, we shall denote the space $L^p(X)$, where $1 \leq p \leq \infty$ and X is a measure space, simply by L^p. In this work, X is always $\mathbb{R}^n$ or a measurable subset of it, equipped with the Lebesgue measure.

For all $1 \leq p \leq \infty$, the spaces L^p are complete normed spaces, *i.e. Banach spaces*, with norm $\| \cdot \|_{L^p}$.

Convergence in L^p ($\|f_k - f\|_{L^p} \to 0$) and pointwise convergence ($f_k(x) \to f(x)$) do not imply each other, but the following is true: If a sequence of functions (f_k) converges to f in L^p, it has a subsequence that converges to f almost everywhere.

If $1 \leq p, p' \leq \infty$ are such that

$$\frac{1}{p} + \frac{1}{p'} = 1, \tag{A.1}$$

then p and p' are called *conjugate exponents*; here $1/\infty = 0$. The notation p' is often used without separate mention for the conjugate exponent of p. The spaces L^p and $L^{p'}$ are closely related, as the following results show.

Theorem A.1 (Hölder's inequality). *Let p and p' be conjugate exponents. Then*

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^{p'}}. \tag{A.2}$$

Corollary A.2. *Let $0 \leq p_1, p_2, \ldots, p_n \leq \infty$ be such that*

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1. \tag{A.3}$$

Then

$$\|f_1 f_2 \cdots f_n\|_{L^1} \leq \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \cdots \|f_n\|_{L^{p_n}}. \tag{A.4}$$

Proof. We proceed by induction. For $n = 1$, the claim is trivial. Assume that the claim holds for $n = k$. Writing $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_k}$ so that $\frac{1}{p} + \frac{1}{p_{k+1}} = 1$, Hölder's inequality gives us

$$\begin{aligned} \|f_1 \cdots f_k f_{k+1}\|_{L^1} &\le \left(\int |f_1 \cdots f_k|^p \, dx \right)^{1/p} \|f_{k+1}\|_{L^{p_{k+1}}} \\ &= \left(\|f_1^p \cdots f_k^p\|_{L^1} \right)^{1/p} \|f_{k+1}\|_{L^{p_{k+1}}} . \end{aligned} \tag{A.5}$$

Now, since $\frac{1}{p_1/p} + \frac{1}{p_2/p} + \cdots + \frac{1}{p_k/p} = 1$, we can use our assumption:

$$\begin{aligned} \|f_1^p \cdots f_k^p\|_{L^1} &\le \left(\int |f_1^p|^{p_1/p} \, dx \right)^{p/p_1} \cdots \left(\int |f_k^p|^{p_k/p} \, dx \right)^{p/p_k} \\ &= \|f_1\|_{L^{p_1}}^p \cdots \|f_k\|_{L^{p_k}}^p , \end{aligned} \tag{A.6}$$

and hence

$$\|f_1 \cdots f_k f_{k+1}\|_{L^1} \le \|f_1\|_{L^{p_1}} \cdots \|f_k\|_{L^{p_k}} \|f_{k+1}\|_{L^{p_{k+1}}} . \tag{A.7}$$

Thus, the claim holds equally well for $n = k + 1$. □

If $1 \le p < \infty$ and p and q are conjugate exponents, then L^q is isometrically isomorphic to the *dual space* of L^p. This means that each bounded linear functional $\Phi : L^p \to \mathbb{C}$ can be represented by a unique $g \in L^q$ in the sense that for all $f \in L^p$,

$$\Phi(f) = \int f(x) \, \overline{g(x)} \, dx. \tag{A.8}$$

This also implies

Theorem A.3 (Converse of Hölder's inequality). *Let p and p' be conjugate exponents, $1 \le p < \infty$. Then*

$$\|f\|_{L^p} = \sup_{\|g\|_{L^{p'}} \le 1} \left| \int f(x) \, \overline{g(x)} \, dx \right| . \tag{A.9}$$

We shall often need to change orders of integration, which is justified by

Theorem A.4 (Fubini). *Let either*

(a) $f : X_1 \times X_2 \to [0, \infty]$, *or*

(b) $f : X_1 \times X_2 \to \mathbb{C}$ *and* $f \in L^1(X_1 \times X_2)$.

Then

$$\int_{X_1} \left(\int_{X_2} f(x,y) \, dy \right) dx = \int_{X_2} \left(\int_{X_1} f(x,y) \, dx \right) dy. \tag{A.10}$$

The *convolution* of two sufficiently well-behaved functions f and g is the function

$$(f * g)(x) = \int f(x - y) g(y) \, dy. \tag{A.11}$$

Convolution has many useful properties, being for instance commutative ($f * g = g * f$), associative ($f * (g * h) = (f * g) * h$), and linear ($f * (g + h) = f * g + f * h$). Also,

$$\frac{\partial}{\partial x_j}(f * g) = \frac{\partial f}{\partial x_j} * g = f * \frac{\partial g}{\partial x_j}, \tag{A.12}$$

when f and g are sufficiently smooth (see Section A.1.3). Continuity is asserted by

Theorem A.5 (Young's inequality). *Let p, q and r be such that $1 \le p, q, r \le \infty$ and $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Then*

$$\|f * g\|_{L^r} \le \|f\|_{L^p} \|g\|_{L^q}. \tag{A.13}$$

Proof. Since

$$|f * g(x)| = \left| \int f(x-y) g(y)\, dy \right| \le \int |f(x-y)||g(y)|\, dy = |f| * |g|(x), \tag{A.14}$$

assume that f and g are real-valued and nonnegative; if they are not, set $\tilde{f} = |f|$, $\tilde{g} = |g|$ and drop the tildes. Thus also $(f * g)(x) \ge 0$ and $|f * g| = f * g$. Let $1 \le \mu, \nu \le \infty$ be such that

$$\frac{1}{\mu} = \frac{1}{p} - \frac{1}{r}, \qquad \frac{1}{\nu} = \frac{1}{q} - \frac{1}{r}, \tag{A.15}$$

so that $\frac{1}{r} + \frac{1}{\mu} + \frac{1}{\nu} = \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. Thus, using the Corollary A.2 of Hölder's inequality, we have

$$\begin{aligned}
(f * g)(x) &= \int f(y)^{p(\frac{1}{p} - \frac{1}{\mu})} g(x-y)^{q(\frac{1}{q} - \frac{1}{\nu})} f(y)^{p/\mu} g(x-y)^{q/\nu}\, dy \\
&\le \left(\int f(y)^{rp(\frac{1}{p} - \frac{1}{\mu})} g(x-y)^{rq(\frac{1}{q} - \frac{1}{\nu})}\, dy \right)^{\frac{1}{r}} \times \\
&\qquad \times \left(\int f(y)^p\, dy \right)^{\frac{1}{\mu}} \left(\int g(x-y)^q\, dy \right)^{\frac{1}{\nu}}.
\end{aligned} \tag{A.16}$$

Since

$$\frac{1}{p} - \frac{1}{\mu} = \frac{1}{q} - \frac{1}{\nu} = \frac{1}{r}, \tag{A.17}$$

this simplifies to

$$(f * g)(x) \le \|f\|_{L^p}^{p/\mu} \|g\|_{L^q}^{q/\nu} \left(\int f(y)^p g(x-y)^q\, dy \right)^{\frac{1}{r}} \tag{A.18}$$

yielding

$$\begin{aligned}
\|f * g\|_{L^r}^r &= \int |(f * g)(x)|^r\, dx \\
&\le \|f\|_{L^p}^{pr/\mu} \|g\|_{L^q}^{qr/\nu} \int \int f(y)^p g(x-y)^q\, dy\, dx \\
&= \|f\|_{L^p}^{pr/\mu} \|g\|_{L^q}^{qr/\nu} \int f(y)^p \int g(x-y)^q\, dx\, dy \\
&= \|f\|_{L^p}^{pr/\mu} \|g\|_{L^q}^{qr/\nu} \|f\|_{L^p}^p \|g\|_{L^q}^q
\end{aligned} \tag{A.19}$$

where reversing the order of integration is justified by Fubini's theorem (Theorem A.4) since f^p and g^q are nonnegative. Thus, noting that

$$\frac{1}{\mu} + \frac{1}{r} = \frac{1}{p} \qquad \text{and} \qquad \frac{1}{\nu} + \frac{1}{r} = \frac{1}{q}, \tag{A.20}$$

we get

$$\|f * g\|_{L^r} \leq \|f\|_{L^p}^{\frac{p}{\mu}+\frac{p}{r}} \|g\|_{L^q}^{\frac{q}{\nu}+\frac{q}{r}} = \|f\|_{L^p}\|g\|_{L^q}. \tag{A.21}$$

□

In general, L^p functions are of course not continuous. Since $\|f - g\|_{L^p} = 0$ if f and g disagree on a set of measure zero, one cannot even speak of point values of an L^p function. Instead, elements of L^p must be interpreted as equivalence classes of functions agreeing almost everywhere.

Continuous, and what is more, smooth functions are, however, dense in L^p for $1 \leq p < \infty$: If the non-negative functions $\phi_j \in C^k(\mathbb{R}^n)$, $j \in \mathbb{N} \cup \{\infty\}$, are such that $\|\phi_j\|_{L^1} = 1$ for all j and

$$\int_{|x|\geq\delta} \phi_j(x)\, dx \to 0 \qquad \text{as } j \to \infty, \tag{A.22}$$

then the sequence of functions $\phi_j * f$ converges to f in the L^p norm;[1] the fact that they are in L^p is clear from Young's inequality above. The functions $\phi_j * f$ are in C^k because $\partial^\alpha(\phi_j * f) = (\partial^\alpha \phi_j) * f$.

The kernels ϕ_j, called mollifiers, can be chosen to be, for instance, the functions $e_{1/j}$ introduced below in (A.48) as approximations of the delta distribution. In that case, $e_{1/j} * f \in C^\infty$.

Of the L^2 spaces, L^2 plays a special role. As $p = 2$ is its own conjugate exponent, L^2 can be thought of as its own dual space; this is expressed by saying that L^2 is *reflexive*. Since L^2 is also separable, which is to say that it has a dense countable subset, the Banach-Alaoglu theorem can be applied:

Theorem A.6 (Banach-Alaoglu). *The closed unit ball in a separable reflexive Banach space $\mathcal{B}$ is weakly sequentially compact. That is, every bounded sequence $(x_j)_{j\in\mathbb{N}}$ in $\mathcal{B}$ has a weakly convergent subsequence $(x_{j_k})_{k\in\mathbb{N}}$ whose weak limit x satisfies*

$$\|x\| \leq \sup_{j\in\mathbb{N}} \|x_j\|. \tag{A.24}$$

Proof. See [HP, Theorem 6.3.7] or [Ru3, Theorem 3.15]. □

[1] Reference [Ru2, Theorem 9.10] covers this with the particular choice of

$$\phi_j(x) = \sqrt{\frac{2}{\pi}}\, \frac{j}{1 + j^2x^2}. \tag{A.23}$$

The more general form can be found in [HS, Theorem 21.37].

Another remarkable property of L^2 is that for $p = 2$, the duality in (A.8) defines an inner product through

$$(f, g) = \int f(x)\, \overline{g(x)}\, dx, \tag{A.25}$$

This makes L^2 into a *Hilbert space*, *i.e.*, a Banach space whose norm is induced from the inner product:

$$\|f\|_{L^2} = \sqrt{(f, f)}. \tag{A.26}$$

A.1.2 Principal Value Integrals

If $f = u^+ - u^- + i(v^+ - v^-)$, where u^+, u^-, v^+ and v^- are nonnegative, measurable functions, the Lebesgue integral of f is defined to be

$$\int_E f\, dx = \int_E u^+\, dx - \int_E u^-\, dx + i \int_E v^+\, dx - i \int_E v^-\, dx, \tag{A.27}$$

whenever all terms on the right hand side are finite. If they are not, and this is caused by a singularity at one point, say the origin, it may happen that the positive and negative parts of the function cancel out if the singularity is approached at an equal speed from all directions, *i.e.* the limit

$$\mathrm{p.\,v.} \int f(x)\, dx = \lim_{\varepsilon \to 0^+} \int_{E \setminus B(0,\varepsilon)} f(x)\, dx \tag{A.28}$$

may be finite. In this case, we call this limit the *Cauchy principal value* of the integral.

Convolution being an integral, too, we define the *Cauchy principal value convolution* of f and g as

$$\mathrm{p.\,v.}\, f * g(x) = \mathrm{p.\,v.} \int f(y) g(x - y)\, dy. \tag{A.29}$$

Note that the definition is not symmetric with respect to f and g.

The letters p. v. are also sometimes used as part of the name of a distribution; this is the case when the principal value integral defines a distribution through

$$\langle \mathrm{p.\,v.}\, f, \phi \rangle = \mathrm{p.\,v.} \int f(x)\, \phi(x)\, dx. \tag{A.30}$$

One such situation is given by Theorem 4.8. Distributions are introduced in the following section.

A.1.3 Distributions

For an open set $X \subset \mathbb{R}^n$, we shall call $\mathcal{D}(X)$ the space $C_0^\infty(X)$ with a topology such that a sequence (ϕ_k) converges *to*ϕ in $\mathcal{D}(X)$, if

1. there exists a fixed compact set $K \subset X$ such that $\operatorname{supp} \phi_k \subset K$ for all $k \in \mathbb{N}$
2. for all $m \in \mathbb{N}$, $\sup \left\{ |\partial^\alpha \phi_k(x) - \partial^\alpha \phi(x)| \,\middle|\, |\alpha| \le m,\, x \in K \right\} \xrightarrow{k \to \infty} 0.$

Distributions are defined as generalisations of functions in the following way. For $\phi, \psi \in \mathcal{D}(X)$, the bilinear pairing

$$\langle \psi, \phi \rangle = \int_X \psi(x)\, \phi(x)\, dx \tag{A.31}$$

makes $\langle \psi, \cdot \rangle \colon \mathcal{D}(X) \to \mathbb{C}$ a continuous linear functional, as the integral is absolutely convergent. More generally, we shall call *distributions* on X the continuous linear functionals on $\mathcal{D}(X)$, and denote by $\mathcal{D}'(X)$ their complex vector space. The value of a distribution f at $\phi \in \mathcal{D}(X)$ can be written

$$f(\phi) = \langle f, \phi \rangle. \tag{A.32}$$

The latter form is normally used. Convergence in $\mathcal{D}'(X)$ is defined as follows: $f_k \to f$ if $\langle f_k, \phi \rangle \to \langle f, \phi \rangle$ for all $\phi \in \mathcal{D}(X)$. For these reasons, the elements of $\mathcal{D}(X)$ are called *test functions*.

According to Equation (A.31), every test function $\psi \in \mathcal{D}(X)$ can also be considered a distribution. In the sequel, we shall often make no distinction between a test function and the corresponding distribution, and we shall even write[2]

$$\mathcal{D}(X) \subset \mathcal{D}'(X). \tag{A.33}$$

Consequently, when we extend operations on functions to operations on distributions, we shall have to make sure that when restricted to $\mathcal{D}(X)$, they agree with the familiar definitions.

Figure A.1 shows some inclusion relations between spaces of functions and distributions used in this work.

A particularly useful class of test functions are the approximate characteristic functions of balls $B(0, \varepsilon_1)$, that is, functions that assume the value 1 in $B(0, \varepsilon_1)$ and the value 0 outside larger balls $B(0, \varepsilon_2)$. Such functions are given, for instance, by the formula

$$\tilde{\chi}_{\varepsilon_1,\varepsilon_2}(x) = \begin{cases} 1, & |x| \le \varepsilon_1 \\ \exp\left(\frac{1}{(\varepsilon_2-\varepsilon_1)^2} + \frac{1}{(|x|-\varepsilon_1)^2-(\varepsilon_2-\varepsilon_1)^2} \right), & \varepsilon_1 \le |x| \le \varepsilon_2 \\ 0, & |x| \ge \varepsilon_2. \end{cases} \tag{A.34}$$

(See Figure A.2.)

The *Schwartz space of rapidly decreasing functions* is the set

$$\mathcal{S}(X) = \left\{ \phi \in C^\infty(X) \,\middle|\, \|\phi\|_{\alpha,\beta} < \infty \ \forall\, \alpha, \beta \in \mathbb{N}^n \right\}, \tag{A.35}$$

where the numbers

$$\|\phi\|_{\alpha,\beta} = \sup_{x \in X} |x^\alpha\, \partial^\beta \phi(x)| \tag{A.36}$$

are seminorms defining the topology of $\mathcal{S}$. The space $\mathcal{S}(X)$ is metrisable, but for our purposes, it suffices to know that $\phi_k \to \phi$ in $\mathcal{S}$ if and only if $\|\phi_k - \phi\|_{\alpha,\beta} \to 0$ for all $\alpha, \beta \in \mathbb{N}^n$.

[2] In fact, this should be interpreted as $\iota : \mathcal{D}(X) \to \mathcal{D}'(X)$, where $\langle \iota\psi, \phi \rangle = \int_X \psi(x)\, \phi(x)\, dx$. The inclusions in Figure A.1 should be understood similarly.

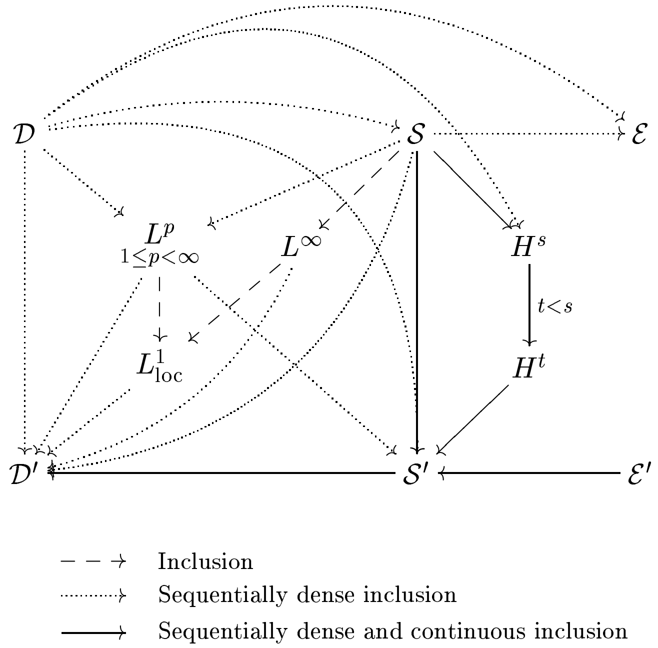


Figure A.1: Inclusion relations between various function spaces.

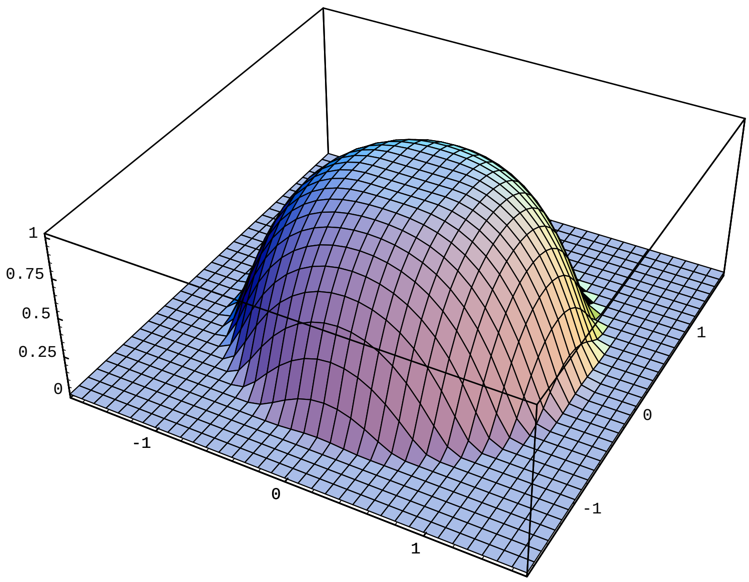


Figure A.2: The approximate characteristic function $\tilde{\chi}_{\varepsilon_1,\varepsilon_2}$ in (A.34) with $n = 2$, $\varepsilon_1 = 0.3$, $\varepsilon_2 = 1.4$.

A simple consequence of the finiteness of the seminorms $\|\phi\|_{\alpha,\beta}$ is that $p(x,|x|)\phi(x)$ is bounded and integrable for all polynomials p of the $n+1$ variables $x_1, \ldots, x_n, |x|$.

Analogously to $\mathcal{D}'(X)$, we define $\mathcal{S}'(X)$, the space of *tempered distributions* on X, to be the set of continuous linear functionals on $\mathcal{S}(X)$. As before in (A.32), the notation $\langle f, \phi\rangle$ will be used for the value that a tempered distribution f assumes at $\phi \in \mathcal{S}(X)$. An equivalent definition for $\mathcal{S}'$ is

$$\mathcal{S}'(X) = \big\{ f : \mathcal{S}(X) \to \mathbb{C} \,\big|\, \exists\, C \geq 0,\, N \in \mathbb{N} : \, |\langle f, \phi\rangle| \leq C \sum_{\substack{|\alpha|\leq N \\ |\beta| \leq N}} \|\phi\|_{\alpha,\beta} \;\; \forall\, \phi \in \mathcal{S} \big\}. \tag{A.37}$$

A sequence (f_k) in $\mathcal{S}'(X)$ converges to f if $\langle f_k, \phi\rangle \to \langle f, \phi\rangle$ for all $\phi \in \mathcal{S}(X)$. This weak topology of $\mathcal{S}'(X)$ is not metrisable. As arguments of tempered distributions, rapidly decreasing functions are also called test functions.

Again, $\mathcal{S}(X)$ can be considered a subset of $\mathcal{S}'(X)$ if we are careful with extending operations on rapidly decreasing functions to operations on tempered distributions. The fact that this inclusion is dense is remarkable, because it allows us to extend many operators $L : \mathcal{S}(X) \to \mathcal{S}(X)$ to $\tilde{L} : \mathcal{S}'(X) \to \mathcal{S}'(X)$ in a sequentially continuous way, *i.e.*, in such a way that if $\psi_k \to f \in \mathcal{S}'(X)$ with $\psi_k \in \mathcal{S}(X)$, then

$$\langle L\psi_k, \phi\rangle \xrightarrow{k\to\infty} \langle \tilde{L} f, \phi\rangle \tag{A.38}$$

for all $\phi \in \mathcal{S}(X)$. Definitions of sequentially continuous extensions include multiplication by a function $M \in C^\infty$ whose all derivatives are dominated in absolute value by a polynomial,

$$\langle M f, \phi\rangle = \langle f, M\phi\rangle, \tag{A.39}$$

and differentiation: The *distribution derivative* or *weak derivative* D^α is defined by

$$\langle D^\alpha f, \phi\rangle = \langle f, (-\partial)^\alpha \phi\rangle, \tag{A.40}$$

whose consistency can be seen by partial integration. For first order distribution derivatives, the notation $D_j := D^{e_j}$ is also used.

The *dilation operator* σ_λ, $\lambda > 0$, is defined for a function $\phi \in \mathcal{S}$ by $\sigma_\lambda \phi(x) = \phi(\lambda x)$. For $f \in \mathcal{S}'$, we set

$$\langle \sigma_\lambda f, \phi\rangle = \lambda^{-n} \langle f, \sigma_{1/\lambda}\phi\rangle. \tag{A.41}$$

This, too, is a natural extension to the definition above, because for $\psi \in \mathcal{S}$, the change of variable $y = x/\lambda$, $dy = \lambda^{-n} dx$ yields

$$\lambda^{-n}\langle \psi, \sigma_{1/\lambda}\phi\rangle = \int \psi(x)\, \phi(\tfrac{x}{\lambda})\, \lambda^{-n}\, dx = \int \psi(\lambda y)\, \phi(y)\, dy = \langle \sigma_\lambda \psi, y\rangle. \tag{A.42}$$

The *support* of a function f defined on X is the set

$$\operatorname{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}; \tag{A.43}$$

$\overline{A}$ denotes the *closure* of a set A, *i.e.*, the smallest closed set containing A.

A distribution $f \in \mathcal{D}'(X)$ is said to vanish on a set $U \subset X$, if $\langle f, \phi \rangle = 0$ for all ϕ with $\operatorname{supp} \phi \subset U$. The *support of a distribution* $f \in \mathcal{D}'(X)$ is defined as the set of points that do not have a neighbourhood on which f vanishes.

The space of distributions with compact support is denoted by $\mathcal{E}'(X) = \mathcal{D}'_0(X)$. These distributions can be extended to linear functionals on $C^\infty(X) =: \mathcal{E}$, since for $f \in \mathcal{D}'_0$ with $\operatorname{supp} f \subset B(0, R)$ and $\phi \in C^\infty$,

$$\langle f, \phi \rangle = \langle f, \tilde{\chi}_{R,R+1}\phi \rangle \tag{A.44}$$

is finite as $\tilde{\chi}_{R,R+1}\phi \in C_0^\infty$. The same topology is used as for $\mathcal{D}'(X)$.

Convolutions $f * g$ can be defined for $f \in \mathcal{D}'(X)$ and $g \in \mathcal{D}(X)$ by

$$\langle f * g, \phi \rangle = \langle f, \check{g} * \phi \rangle, \tag{A.45}$$

where $\check{g}$ denotes the *reflection* of g, $\check{g}(x) = g(-x)$. This is the natural extension from the case where $f \in L^1_{\text{loc}}$. The same formula can then be used to extend the definition to cases where $g \in \mathcal{D}'_0(X)$. Most properties of the convolution, including commutativity, linearity, the Convolution theorem (A.60) and Formula (A.12) for the derivatives, remain valid after this extension.

Inclusion relations between these function and distribution spaces are shown in Figure A.1. An inclusion $A \subset B$ is denoted by $A \to B$. All inclusions are proper.

An example of a distribution which is not a conventional function is *Dirac's delta distribution* δ, defined by

$$\langle \delta, \phi \rangle = \phi(0). \tag{A.46}$$

Clearly, there is no function $e : X \to \mathbb{C}$ such that

$$\int_X e(y)\phi(y)\, dy = \phi(0), \tag{A.47}$$

for all $\phi \in \mathcal{D}$, but as $\mathcal{D}$ is dense in $\mathcal{S}'$, δ can be seen as the limit of a sequence of functions having higher and higher, narrower and narrower peaks around the origin. For instance, if we define

$$e_\varepsilon(x) = \varepsilon^{-n} e_1(x/\varepsilon), \quad \text{where} \quad e_1(x) = \tilde{\chi}_{0,1}(x) = \begin{cases} Ce^{\frac{1}{|x|^2-1}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \tag{A.48}$$

and choose C in such a way that $\int e_1(x)\, dx = 1$, we have $e_\varepsilon \in \mathcal{S}$ for all $\varepsilon \geq 0$ and

$$\delta = \lim_{\varepsilon \to 0} e_\varepsilon. \tag{A.49}$$

Such a function e_ε is often called an *approximate delta function*, *blurring kernel* or *point spread function*.

The derivatives of the delta distribution give the values of the corresponding derivatives of the test function at the origin,

$$\langle D^\alpha \delta, \phi \rangle = (-1)^{|\alpha|} \langle \delta, \partial^\alpha \phi \rangle = (-1)^{|\alpha|}\, \partial^\alpha \phi(0). \tag{A.50}$$

The support of the delta distribution and its derivatives consists of the single point 0. In fact, finite linear combinations of δ and its derivatives are the only functions with point support: if $\operatorname{supp} f = \{0\}$, then $f = \sum_{|\alpha|\leq m} c_\alpha D^\alpha \delta$ for some constants m and c_α.

For an introduction to the theory distributions, see references [Bre, Fri, Hör, Ner, Rau].[3]

A.1.4 Fourier Transform

Various different definitions are commonly used for the *Fourier transform* $\mathcal{F}f = \hat{f}$ of a function $f : \mathbb{R}^n \to \mathbb{C}$ in an appropriate class, for instance, $f \in \mathcal{S}$. We shall use the following one:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx. \tag{A.51}$$

The *inverse Fourier transform* is then given by

$$f(x) = \mathcal{F}^{-1}\hat{f}(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi. \tag{A.52}$$

The Fourier transform can be considered as a linear operator between various function spaces. Linearity is of major significance, as many operations considered in this work are linear.

The Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ is a continuous linear bijection. Also,

$$|\hat{f}(\xi)| \leq (2\pi)^{-n/2} \int |f(x)|\, |e^{-ix\cdot\xi}|\, dx = (2\pi)^{-n/2} \|f\|_{L^1}, \tag{A.53}$$

whence $\|\hat{f}\|_{L^\infty} \leq (2\pi)^{-n/2}\, \|f\|_{L^1}$, so that the Fourier transform can be extended to a continuous linear injection from L^1 to L^∞, using the same Formula (A.51).

The Fourier transform $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ can also be extended to a bijection $\mathcal{F} : L^2 \to L^2$ as the limit of the truncated Fourier transform:

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = L^2\text{-}\lim_{R\to\infty}\, (2\pi)^{-n/2} \int_{|x|\leq R} f(x)\, e^{-ix\cdot\xi}\, dx. \tag{A.54}$$

The inverse transform is then

$$f(x) = L^2\text{-}\lim_{R\to\infty} (2\pi)^{-n/2} \int_{|x|\leq R} \hat{f}(\xi)\, e^{ix\cdot\xi}\, d\xi. \tag{A.55}$$

[3] [Bre] includes a gentle, intuitive prologue. [Fri] provides a good treatment of the basic theory. An exhaustive account of the theory can be found in [Hör].

Reference [Ner] also gives a short introduction to the theory of distributions. Like some other texts, it defines distributions as the continuous *conjugate* linear functionals on $\mathcal{D}$, which has the advantage that $\langle f, \phi\rangle = (f, \phi)$ for $f \in L^2$, $\phi \in \mathcal{S}$, but this changes somewhat most formulae. Qualitatively, however, the results remain obviously the same, because each test function is, in fact, only exchanged with its complex conjugate.

[Rau] deals with $\mathcal{D}'$ only briefly in an appendix, but considers more thoroughly tempered distributions, their Fourier transforms and their relations with L^p spaces.

The Fourier transform and its inverse are extended to members of $\mathcal{S}'$ by

$$\langle \mathcal{F}f, \phi\rangle = \langle f, \mathcal{F}\phi\rangle \qquad \langle \mathcal{F}^{-1}f, \phi\rangle = \langle f, \mathcal{F}^{-1}\phi\rangle \qquad \text{for } \phi \in \mathcal{S}. \tag{A.56}$$

This operator $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$ is sequentially continuous, and its restrictions to $\mathcal{S}$, L^1 and L^2 agree with the Fourier transforms defined in (A.51) and (A.54).

The letter ξ will commonly be used without separate mention for the variable on the Fourier transform side, whereas x will be used for the variable on the untransformed side.

Some additional properties of the Fourier transform include the following:

$$\mathcal{F}(\partial^\alpha f)(\xi) = (i\xi)^\alpha \hat{f}(\xi) \qquad \text{for } f \in \mathcal{S}' \tag{A.57}$$

$$\mathcal{F}(\sigma_\lambda f) = \lambda^{-n}\sigma_{1/\lambda}\hat{f} \qquad \text{for } f \in \mathcal{S}', \tag{A.58}$$

$$\mathcal{F}\mathcal{F}f(x) = \mathcal{F}^{-1}\mathcal{F}^{-1}f(x) = \check{f}(x) := f(-x) \qquad \text{for } f \in \mathcal{S}', \tag{A.59}$$

the *Convolution theorem*

$$\mathcal{F}(f * g)(\xi) = (2\pi)^{n/2}\,\hat{f}(\xi)\,\widehat{g}(\xi) \tag{A.60}$$

for $f \in L^1$, $g \in L^1 \cup L^2$ or $f \in \mathcal{S}'$, $g \in \mathcal{D}'_0$, the *Parseval formula*

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} \qquad \text{for } f \in L^2, \tag{A.61}$$

the *Plancherel formula*[4]

$$(f, g) = (\hat{f}, \widehat{g}) \qquad \text{for } f, g \in L^2 \tag{A.62}$$

$$\langle f, \overline{\phi}\rangle = \langle \hat{f}, \overline{\hat{\phi}}\rangle \qquad \text{for } f \in \mathcal{S}',\ \phi \in \mathcal{S} \tag{A.63}$$

and the Riemann-Lebesgue lemma

$$\lim_{|x|\to\infty} \hat{f}(x) = 0 \qquad \text{for } f \in L^1. \tag{A.64}$$

As (A.58) suggests,

$$\mathcal{F}(\delta) = \lim_{\varepsilon\to 0} \mathcal{F}(\varepsilon^{-n}\sigma_{1/\varepsilon}e) = \lim_{\varepsilon\to 0} \sigma_\varepsilon \widehat{e} = 1, \tag{A.65}$$

when e is an approximate delta function.

References [Hör], [Ner], [Rau, Chapter 2], and [Ru2, Chapter 9] provide an introduction to the Fourier transform, together with proofs of these facts.

[4]The resemblance between the names Parseval and Plancherel causes some inconsistency as to which name refers to which one of these two equivalent formulae. We choose to call the former one (A.61) the Parseval formula, and the latter one (A.62), (A.63) the Plancherel formula.

A.1.5 H^s Spaces

For the Sobolev spaces $H^k(\mathbb{R}^n)$ with $k \in \mathbb{N}$, defined in Chapter 2, we note that an equivalent definition is

$$H^k(\mathbb{R}^n) = \left\{ f \in L^2(\mathbb{R}^n) \mid D^\alpha f \in L^2(\mathbb{R}^n) \text{ for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq k \right\}. \tag{A.66}$$

This is obvious because if $|\alpha| \leq k$,

$$|\xi^\alpha| = \prod_{j=1}^{n} |\xi_j|^{\alpha_j} \leq \prod_{j=1}^{n} |\xi|^{\alpha_j} = |\xi|^k \leq (1 + |\xi|^2)^{k/2} \tag{A.67}$$

implies that

$$\|D^\alpha f\|_{L^2} = \|\mathcal{F}(D^\alpha f)\|_{L^2} = \|\xi^\alpha \hat{f}\|_{L^2} \leq \|(1 + |\xi|^2)^{k/2}\, \hat{f}\|_{L^2}, \tag{A.68}$$

and on the other hand,

$$(1 + |\xi|^2)^{k/2} \leq \Big(1 + \sum_{j=1}^{n} |\xi_j|\Big)^k = \sum_{|\alpha| \leq k} c_\alpha \, |\xi^\alpha| \tag{A.69}$$

implies that

$$\begin{aligned} \|(1 + |\xi|^2)^{k/2}\, \hat{f}\|_{L^2} &\leq \sum_{|\alpha| \leq k} |c_\alpha| \, \||\xi^\alpha|\, \hat{f}\|_{L^2} \\ &= \sum_{|\alpha| \leq k} |c_\alpha| \, \|\mathcal{F}(D^\alpha f)\|_{L^2} \\ &= \sum_{|\alpha| \leq k} |c_\alpha| \, \|D^\alpha f\|_{L^2} \end{aligned} \tag{A.70}$$

so that $\|(1 + |\xi|^2)^{k/2}\, \hat{f}\|_{L^2} < \infty$ if and only if $\|D^\alpha f\|_{L^2} < \infty$ whenever $|\alpha| < k$.

A.2 Continuity of the Riesz Potentials

The definition of the Riesz potential, given in Definition 3.2, can be generalised: The *Riesz potential of order* $\alpha \in \mathbb{R}$ of the function $f : \mathbb{R}^n \to \mathbb{C}$ is the function $R_\alpha * f$, where

$$R_\alpha = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right)} \, |x|^{\alpha - n}. \tag{A.71}$$

Texts that use the notation I_1 instead of $R_1 *$, also use I_α instead of $R_\alpha *$.

The mappings $f \mapsto R_\alpha * f$ have the following continuity properties, which are needed in the proof of Theorem 4.12. Instead of giving the proofs, we refer to [Ste] and [Zie].

Theorem A.7. *If $\alpha > 0$, $1 < p < \infty$ and $\alpha p < n$, then there is a constant C, depending on n and p, such that*

$$\| |x|^{\alpha-n} * f(x) \|_{L^q} \le C \, \|f\|_{L^p}, \tag{A.72}$$

where

$$q = \frac{np}{n - \alpha p} \tag{A.73}$$

for all $f \in L^p(\mathbb{R}^n)$.

Proof. See [Zie, Theorem 2.8.4, page 86] or [Ste, Theorem V.1(b), page 119]. □

Theorem A.8. *If $E \subset \mathbb{R}^n$ is compact and $p > 1$, then there are constants C_1 and C_2, depending only on p, n and E, such that*

$$\frac{1}{m(E)} \int_E \exp\left[\left(\frac{|x|^{n/p-n} * f(x)}{C_1 \, \|f\|_{L^p}} \right)^{p'} \right] dx \le C_2 \tag{A.74}$$

for all $f \in L^p(E)$.

Proof. See [Zie, Theorem 2.9.1, page 89]. □

These two theorems imply that for all $n \ge 2$, the Riesz potential operator is locally bounded from $L^2(\mathbb{R}^n)$ to some $L^q(\mathbb{R}^n)$.

Corollary A.9. *If $n \ge 2$, and $E \subset \mathbb{R}^n$ is compact, then there are constants $q \ge 2$ and $C > 0$ such that*

$$\|\chi_E \, R_1 * f\|_{L^q} \le C \, \|f\|_{L^2} \tag{A.75}$$

for all $f \in L^2(\mathbb{R}^n)$.

Proof. For $n \ge 3$, the result follows directly from the global estimate of Theorem A.7 with $\alpha = 1$ and $p = 2$. If $n = 2$, note that the expression te^{-t} is bounded by some constant C_3 for all $t > 0$. Theorem A.8 with $p = 2$ therefore implies that for any compact set $E \subset \mathbb{R}^n$,

$$\begin{aligned} \int_E \left| \frac{R_1 * f(x)}{C_1 \, \|\chi_E f\|_{L^2}} \right|^2 dx &\le \int_E \left(\frac{R_1 * |f|(x)}{C_1 \, \|\chi_E f\|_{L^2}} \right)^2 dx \\ &\le C_3 \int_E \exp\left[\left(\frac{R_1 * |f|(x)}{C_1 \, \|\chi_E f\|_{L^2}} \right)^2 \right] dx \\ &\le C_3 \, C_2 \, m(E) \end{aligned} \tag{A.76}$$

and therefore

$$\|\chi_E \, R_1 * f\|_{L^2} = \sqrt{\int_E |R_1 * f(x)|^2 \, dx} \le C \, \|\chi_E \, f\|_{L^2} \le C \, \|f\|_{L^2}, \tag{A.77}$$

where $C = C_1 \sqrt{C_3 \, C_2 \, m(E)}$. □

A.3 Polar and Spherical Coordinates in $\mathbb{R}^n$

In $\mathbb{R}^n$, $n \geq 2$, an integral $\int_D f(x)\,dx$ can often be easily evaluated using *polar coordinates* if the integrand f and domain of integration $D \subset \mathbb{R}^n$ possess appropriate symmetry.

If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, the simplest form of polar coordinates is the well known

$$r = |x| \in [0, \infty), \qquad \theta = \frac{x}{r} \in S^{n-1}, \qquad x = r\theta \tag{A.78}$$

so that

$$\int_D f(x)\,dx = \int_{D_\theta} \int_{D_r} f(r, \theta)\, r^{n-1}\, dr\, d\theta \tag{A.79}$$

when

$$D = \left\{ r\theta \in \mathbb{R}^n \,\middle|\, r \in D_r \subset [0, \infty),\, \theta \in D_\theta \subset S^{n-1} \right\}. \tag{A.80}$$

When $n = 1$, this holds equally well, if the counting measure is used on $D_\theta \subset S^0 = \{-1, +1\}$:

$$\int_{D_\theta} f(\theta)\, d\theta = \sum_{\theta \in D_\theta} f(\theta). \tag{A.81}$$

In this case, (A.79) simply becomes

$$\int_D f(x)\,dx = \int_{(-\infty,0]\cap D} f(x)\,dx + \int_{[0,\infty)\cap D} f(x)\,dx. \tag{A.82}$$

The same formalism can thus be used regardless of the dimension. We shall focus on the cases $n \geq 2$ for the rest of this section.

If $f = f(r)$ is spherically symmetric,

$$\int_D f(x)\,dx = m(D_\theta) \int_{D_r} f(r)\, r^{n-1}\, dr. \tag{A.83}$$

A simple application of this is the calculation of the measures of the unit balls and spheres in different dimensions:

Lemma A.10. *For all $n \in \mathbb{Z}_+$,*

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} = n\,|B^n|, \qquad |B^n| = \frac{2\pi^{n/2}}{n\,\Gamma\left(\frac{n}{2}\right)}. \tag{A.84}$$

Proof. For calculating $|S^{n-1}|$, we shall evaluate the integral

$$I = \int_{\mathbb{R}^n} e^{-|x|^2} \tag{A.85}$$

in two ways. Firstly, since

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2}\,dx &= \sqrt{\int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy} = \sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy} \\ &= \sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-r^2}\, r\,dr\,d\theta} = \sqrt{2\pi \Big/_{0}^{\infty} -\frac{e^{-r^2}}{2}} = \sqrt{\pi}, \end{aligned} \tag{A.86}$$

we get

$$I = \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx \right)^n = \pi^{n/2}. \tag{A.87}$$

Secondly,

$$I = \int_{S^{n-1}} \int_0^{\infty} e^{-r^2} r^{n-1}\, dr\, d\theta, \tag{A.88}$$

whence, by making the substitution $u = r^2$, $du = 2r\, dr$, we get

$$I = |S^{n-1}| \frac{1}{2} \int_0^{\infty} e^{-u} u^{\frac{n}{2}-1}\, du = \frac{1}{2} |S^{n-1}|\, \Gamma\left(\frac{n}{2}\right). \tag{A.89}$$

Therefore,

$$|S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}, \tag{A.90}$$

proving the first part of the claim. The second part follows immediately by using polar coordinates:

$$|B^n| = \int_{B^n} dx = \int_{S^{n-1}} \int_0^1 r^{n-1}\, dr\, d\theta = \frac{|S^{n-1}|}{n}. \tag{A.91}$$

□

Even when f is not spherically symmetric, the evaluation of (A.79) is rather straightforward in the two-dimensional case, in the sense that both $D_r \subset [0, \infty)$ and $D_\theta \subset S^1$ are one-dimensional.

For $n \geq 3$, D_θ must be parametrised using $n - 1 \geq 2$ real numbers. In some cases, this can be done by introducing the *spherical coordinates* $r \in D_r \subset [0, \infty)$, $\varphi \in D_\varphi \subset [0, \pi]$ and $\omega = (\omega_1, \omega_2, \dots, \omega_{n-2}) \in D_\omega \subset B^{n-2}$:

$$\begin{aligned} x_1 &= r \cos\varphi \\ x_2 &= r\omega_1 \sin\varphi \\ x_3 &= r\omega_2 \sin\varphi \\ &\vdots \\ x_{n-1} &= r\omega_{n-2} \sin\varphi \\ x_n &= \pm r \sin\varphi \sqrt{1 - \sum_{k=1}^{n-2} \omega_k^2}. \end{aligned} \tag{A.92}$$

The expression for x_n can be simplified to $x_n = r\omega_{n-1} \sin\varphi$ by considering

$$\tilde{\omega} := (\omega_1, \omega_2, \dots, \omega_{n-2}, \omega_{n-1}) \in S^{n-2}, \qquad \omega_{n-1} = \pm\sqrt{1 - \sum_{k=1}^{n-1} \omega_k^2}. \tag{A.93}$$

When r and φ are fixed, $\tilde{\omega}$ runs over S^{n-2} and ω runs twice over B^{n-2}, so both cases of the $\pm$ sign must be covered. (See Figure A.3.)

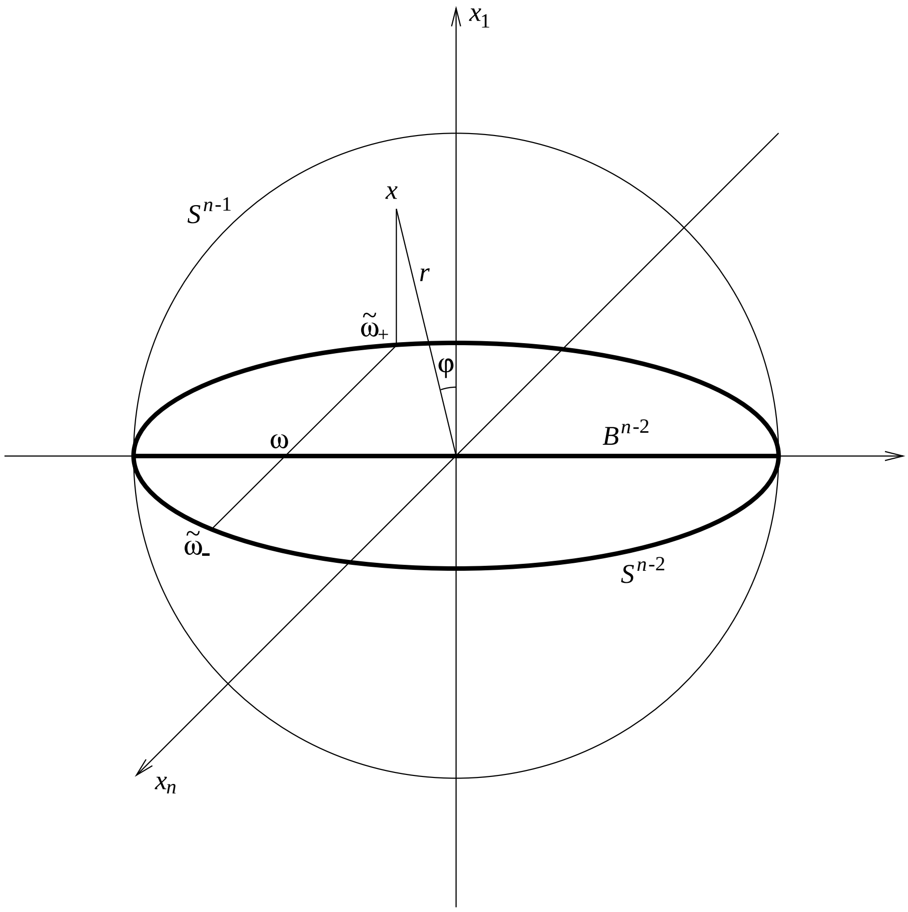


Figure A.3: Spherical coordinates in $\mathbb{R}^n$, $n \geq 3$.

The Jacobian determinant is obtained as follows:

$$\det J = \begin{vmatrix} \frac{\partial x_1}{\partial r} & \frac{\partial x_1}{\partial \varphi} & \frac{\partial x_1}{\partial \omega_1} & \frac{\partial x_1}{\partial \omega_2} & \cdots & \frac{\partial x_1}{\partial \omega_{n-2}} \\ \frac{\partial x_2}{\partial r} & \frac{\partial x_2}{\partial \varphi} & \frac{\partial x_2}{\partial \omega_1} & \frac{\partial x_2}{\partial \omega_2} & \cdots & \frac{\partial x_2}{\partial \omega_{n-2}} \\ \frac{\partial x_3}{\partial r} & \frac{\partial x_3}{\partial \varphi} & \frac{\partial x_3}{\partial \omega_1} & \frac{\partial x_3}{\partial \omega_2} & \cdots & \frac{\partial x_3}{\partial \omega_{n-2}} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_{n-1}}{\partial r} & \frac{\partial x_{n-1}}{\partial \varphi} & \frac{\partial x_{n-1}}{\partial \omega_1} & \frac{\partial x_{n-1}}{\partial \omega_2} & \cdots & \frac{\partial x_{n-1}}{\partial \omega_{n-2}} \\ \frac{\partial x_n}{\partial r} & \frac{\partial x_n}{\partial \varphi} & \frac{\partial x_n}{\partial \omega_1} & \frac{\partial x_n}{\partial \omega_2} & \cdots & \frac{\partial x_n}{\partial \omega_{n-2}} \end{vmatrix} \tag{A.94}$$

$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 & 0 & \cdots & 0 \\ \omega_1 \sin\varphi & r\omega_1\cos\varphi & r\sin\varphi & 0 & \cdots & 0 \\ \omega_2\sin\varphi & r\omega_2\cos\varphi & 0 & r\sin\varphi & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-2}\sin\varphi & r\omega_{n-2}\cos\varphi & 0 & 0 & \cdots & r\sin\varphi \\ \omega_{n-1}\sin\varphi & r\omega_{n-1}\cos\varphi & -r\frac{\omega_1}{\omega_{n-1}}\sin\varphi & -r\frac{\omega_2}{\omega_{n-1}}\sin\varphi & \cdots & -r\frac{\omega_{n-2}}{\omega_{n-1}}\sin\varphi \end{vmatrix}$$

Subtracting the second column, multiplied by $\frac{\sin\varphi}{\cos\varphi}$, from the first column and then developing the determinant with respect to the first column, we get

$$\det J = r^{n-1} \begin{vmatrix} \frac{1}{\cos\varphi} & -\frac{\sin\varphi\cos\varphi}{\cos\varphi} & 0 & 0 & \cdots & 0 \\ 0 & \omega_1\cos\varphi & \sin\varphi & 0 & \cdots & 0 \\ 0 & \omega_2\cos\varphi & 0 & \sin\varphi & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \omega_{n-2}\cos\varphi & 0 & 0 & \cdots & \sin\varphi \\ 0 & \omega_{n-1}\cos\varphi & -\frac{\omega_1}{\omega_{n-1}}\sin\varphi & -\frac{\omega_2}{\omega_{n-1}}\sin\varphi & \cdots & -\frac{\omega_{n-2}}{\omega_{n-1}}\sin\varphi \end{vmatrix}$$

$$= r^{n-1}\sin^{n-2}\varphi \begin{vmatrix} \omega_1 & 1 & 0 & \cdots & 0 \\ \omega_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{n-2} & 0 & 0 & \cdots & 1 \\ \omega_{n-1} & -\frac{\omega_1}{\omega_{n-1}} & -\frac{\omega_2}{\omega_{n-1}} & \cdots & -\frac{\omega_{n-2}}{\omega_{n-1}} \end{vmatrix} \tag{A.95}$$

Now, subtracting the second through last column from the first, multiplied by ω_1, ω_2, … , ω_{n-2}, respectively, yields, since $\omega_1^2 + \omega_2^2 + \cdots + \omega_{n-2}^2 + \omega_{n-1}^2 = 1$,

$$\det J = r^{n-1}\sin^{n-2}\varphi \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \frac{1}{\omega_{n-1}} & -\frac{\omega_1}{\omega_{n-1}} & -\frac{\omega_2}{\omega_{n-1}} & \cdots & -\frac{\omega_{n-2}}{\omega_{n-1}} \end{vmatrix}$$
$$= (-1)^n \frac{r^{n-1}\sin^{n-2}\varphi}{\sqrt{1-\sum_{k=1}^{n-2}\omega_k^2}}. \tag{A.96}$$

Thus,

$$\int_D f(x)\,dx = \int_{D_r}\int_{D_\varphi}\int_{D_\omega} \big(f_+(r,\varphi,\omega) + f_-(r,\varphi,\omega)\big) \frac{r^{n-1}\sin^{n-2}\varphi}{\sqrt{1-\sum_{k=1}^{n-2}\omega_k^2}}\,d\omega\,d\varphi\,dr, \tag{A.97}$$

where f_+ and f_- correspond to the two signs in (A.92). Analogously for $D \subset S^{n-1}$, we have

$$\int_D f(\theta)\, d\theta = \int_{D_\varphi} \int_{D_\omega} \big(f_+(\varphi,\omega) + f_-(\varphi,\omega)\big) \frac{\sin^{n-2}\varphi}{\sqrt{1-\sum_{k=1}^{n-2}\omega_k^2}}\, d\omega\, d\varphi. \tag{A.98}$$

In cases where the domain of integration D_ω is all of B^{n-2} and f does not depend on ω or the sign of ω_{n-1}, the integral with respect to ω in (A.97) and (A.98) evaluates to $|S^{n-2}|$, as expected:

Lemma A.11.

$$2\int_{B^{n-2}} \frac{d\omega}{\sqrt{1-|\omega|^2}} = |S^{n-2}|. \tag{A.99}$$

Proof. Using the polar coordinates of $\mathbb{R}^{n-2} \supset B^{n-2}$,

$$2\int_{B^{n-2}} \frac{d\omega}{\sqrt{1-|\omega|^2}} = 2\,|S^{n-3}| \int_0^1 \frac{r^{n-3}}{\sqrt{1-r^2}}\, dr. \tag{A.100}$$

On the other hand, remembering Formula (A.84) for $|S^{n-1}|$, we have

$$2\,|S^{n-3}|\, \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2\,\Gamma\left(\frac{n-1}{2}\right)} = 2\,\frac{2\pi^{(n-2)/2}}{\Gamma\left(\frac{n-2}{2}\right)}\, \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2\,\Gamma\left(\frac{n-1}{2}\right)} = \frac{2\pi^{(n-1)/2}}{\Gamma\left(\frac{n-1}{2}\right)} = |S^{n-2}|, \tag{A.101}$$

whence it suffices to show that

$$\int_0^1 \frac{r^{n-3}}{\sqrt{1-r^2}}\, dr = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2\,\Gamma\left(\frac{n-1}{2}\right)}. \tag{A.102}$$

To do this, we first note that since

$$\begin{aligned}
\int_0^1 \frac{r^k}{\sqrt{1-r^2}}\, dr &= \int_0^1 r^{k-1} \frac{r}{\sqrt{1-r^2}}\, dr \\
&= -\Big/_0^1 \left(r^{k-1}\sqrt{1-r^2}\right) + (k-1)\int_0^1 r^{k-2}\sqrt{1-r^2}\, dr \\
&= (k-1)\int_0^1 \frac{r^{k-2}}{\sqrt{1-r^2}}\,(1-r^2)\, dr \\
&= (k-1)\int_0^1 \frac{r^{k-2}}{\sqrt{1-r^2}}\, dr - (k-1)\int_0^1 \frac{r^k}{\sqrt{1-r^2}}\, dr,
\end{aligned} \tag{A.103}$$

we have the recursion formula

$$\int_0^1 \frac{r^k}{\sqrt{1-r^2}}\, dr = \frac{k-1}{k} \int_0^1 \frac{r^{k-2}}{\sqrt{1-r^2}}\, dr \tag{A.104}$$

for all $k \geq 2$. Now if n is even,

$$\begin{aligned}
\Gamma\left(\frac{n-2}{2}\right) &= \frac{n-4}{2}\,\Gamma\left(\frac{n-4}{2}\right) = \cdots \\
&= \frac{n-4}{2}\frac{n-6}{2}\cdots\frac{4}{2}\frac{2}{2}\,\Gamma(1) = \frac{n-4}{2}\frac{n-6}{2}\cdots\frac{4}{2}\frac{2}{2}
\end{aligned} \tag{A.105}$$

and analogously

$$\Gamma\left(\frac{n-1}{2}\right) = \frac{n-3}{2}\frac{n-5}{2}\cdots\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}, \tag{A.106}$$

so (A.104) gives us

$$\begin{aligned}\int_0^1 \frac{r^{n-3}}{\sqrt{1-r^2}}\,dr &= \frac{n-4}{n-3}\frac{n-6}{n-5}\cdots\frac{4}{5}\frac{2}{3}\int_0^1 \frac{r\,dr}{\sqrt{1-r^2}} \\ &= \frac{\frac{n-4}{2}\frac{n-6}{2}\cdots\frac{4}{2}\frac{2}{2}}{\frac{n-3}{2}\frac{n-5}{2}\cdots\frac{5}{2}\frac{3}{2}}\frac{\frac{\sqrt{\pi}}{2}}{\frac{1}{2}\sqrt{\pi}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2\,\Gamma\left(\frac{n-1}{2}\right)},\end{aligned} \tag{A.107}$$

as claimed. If n is odd, we see similarly that

$$\begin{aligned}\int_0^1 \frac{r^{n-3}}{\sqrt{1-r^2}}\,dr &= \frac{n-4}{n-3}\frac{n-6}{n-5}\cdots\frac{3}{4}\frac{1}{2}\int_0^1 \frac{dr}{\sqrt{1-r^2}} \\ &= \frac{\frac{n-4}{2}\frac{n-6}{2}\cdots\frac{3}{2}\frac{1}{2}}{\frac{n-3}{2}\frac{n-5}{2}\cdots\frac{4}{2}\frac{2}{2}}\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n-2}{2}\right)}{2\,\Gamma\left(\frac{n-1}{2}\right)},\end{aligned} \tag{A.108}$$

which completes the proof. □

We have thus derived the following rule for evaluating integrals using spherical coordinates:

Lemma A.12. *If $n \geq 2$ and*

$$D = \left\{(x_1, \ldots, x_{n-1}, \pm x_n) \,\middle|\, r \in D_r,\ \varphi \in D_\varphi,\ \omega \in D_\omega\right\} \subset \mathbb{R}^n \tag{A.109}$$

where

$$\begin{aligned} x_1(r,\varphi,\omega) &= r\cos\varphi \\ x_2(r,\varphi,\omega) &= r\omega_1\sin\varphi \\ x_3(r,\varphi,\omega) &= r\omega_2\sin\varphi \\ &\vdots \\ x_{n-1}(r,\varphi,\omega) &= r\omega_{n-2}\sin\varphi \\ x_n(r,\varphi,\omega) &= r\sin\varphi\sqrt{1-\sum_{k=1}^{n-2}\omega_k^2}, \end{aligned} \tag{A.110}$$

$D_r \subset [0,\infty)$, $D_\varphi \subset [0,\pi]$, $D_\omega \subset B^{n-2}$, $f \in L^1(D)$ and

$$f_\pm(r,\varphi,\omega) = f\big(x_1(r,\varphi,\omega), \ldots, x_{n-1}(r,\varphi,\omega), \pm x_n(r,\varphi,\omega)\big), \tag{A.111}$$

then

$$\int_D f(x)\,dx = \int_{D_r}\int_{D_\varphi}\int_{D_\omega}\big(f_+(r,\varphi,\omega)+f_-(r,\varphi,\omega)\big)\,\frac{r^{n-1}\sin^{n-2}\varphi}{\sqrt{1-\sum_{k=1}^{n-2}\omega_k^2}}\,d\omega\,d\varphi\,dr. \tag{A.112}$$

If furthermore $D_\omega = B^{n-2}$ and $g(r,\varphi) = f_+(r,\varphi,\omega) = f_-(r,\varphi,\omega)$ for all r, φ and ω, then

$$\int_D f(x)\,dx = |S^{n-2}|\int_{D_r}\int_{D_\varphi} g(r,\varphi)\,r^{n-1}\,\sin^{n-2}\varphi\,d\varphi\,dr. \tag{A.113}$$

Note that in the special case $n = 2$, we have $D_\omega = B^0 = \{0\}$,

$$\int_{D_\omega} \psi(r,\varphi,\omega)\, d\omega = \psi(r,\varphi,0) \tag{A.114}$$

using the counting measure, and

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases} \tag{A.115}$$

is just the familiar polar coordinate representation in the plane, with the exception that here $\varphi \in [0,\pi]$ and the two semicircles $S^1_+(\pm e_2)$ are covered using $f_\pm$.

When integrating over the unit sphere or part of it, we have the following analogous result:

Lemma A.13. *If $n \geq 2$ and*

$$D = \big\{(\theta_1,\dots,\theta_{n-1},\pm\theta_n) \,\big|\, \varphi \in D_\varphi,\ \omega \in D_\omega\big\} \subset S^{n-1}, \tag{A.116}$$

where $\theta_i(\varphi,\omega) = x_i(1,\varphi,\omega)$ with x_i as in (A.110) for all $i \in \{1,\dots,n\}$, $f \in L^1(D)$ and

$$f_\pm(\varphi,\omega) = f\big(\theta_1(\varphi,\omega),\dots,\theta_{n-1}(\varphi,\omega),\pm\theta_n(\varphi,\omega)\big), \tag{A.117}$$

then

$$\int_D f(\theta)\, d\theta = \int_{D_\varphi}\int_{D_\omega} \big(f_+(\varphi,\omega) + f_-(\varphi,\omega)\big) \frac{\sin^{n-2}\varphi}{\sqrt{1-\sum_{k=1}^{n-2}\omega_k^2}}\, d\omega\, d\varphi. \tag{A.118}$$

If furthermore $D_\omega = B^{n-2}$ and $g(\varphi) = f_+(\varphi,\omega) = f_-(\varphi,\omega)$ for all φ and ω, then

$$\int_D f(\theta)\, d\theta = |S^{n-2}| \int_{D_\varphi} g(\varphi)\, \sin^{n-2}\varphi\, d\varphi. \tag{A.119}$$

The domains D_r, D_φ and D_ω are here, for some sample domains D, the following:

D	D_r	D_φ	D_ω
$\mathbb{R}^n$	$[0,\infty)$	$[0,\pi]$	B^{n-2}
$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_1 \geq 0\}$	$[0,\infty)$	$[0,\frac{\pi}{2}]$	B^{n-2}
S^{n-1}	$-$	$[0,\pi]$	B^{n-2}
$S^{n-1}_+(e_1) \subset S^{n-1}$	$-$	$[0,\frac{\pi}{2}]$	B^{n-2}

Appendix B

Table of Notation

Notation	**Meaning**	**See page**
$\measuredangle$	Angle	71
$:=, =:$	Definition, assignment of value	5
$*$	Convolution	85, 92
$\widehat{\ }$	Fourier transform	93
$^\vee$	Reflection	92
$\theta^\perp$	Orthogonal complement of θ	13
$\overline{A}$	Closure of set A	92
$\overline{z}$	Complex conjugate of complex number z	
B^k	Unit ball of $\mathbb{R}^k$	5
$B(x_0, R)$	Open ball with radius R and centre at x_0	5
C^k	Space of k times continuously differentiable functions	7
C_0^k	Space of C^k functions with compact support	7
D^α, D_j	Distribution derivative	91
$\mathcal{D}$	Divergent beam radiograph	8
$\mathcal{D}$	Space of smooth functions with bounded support	88
$\mathcal{D}'$	Space of distributions	89
$\mathcal{D}'_0$	Space of distributions with compact support	92
dist	Distance	5
D_{xr}	X-ray domain	39
e_j	j^{th} unit basis vector	4
$\mathcal{E}$	Space of smooth functions	92
$\mathcal{E}'$	Space of distributions with compact support	92
E_θ	Orthogonal projection onto $\theta^\perp$	13
$\mathcal{F}$	Fourier transform	93
$\mathcal{F}^{-1}$	Inverse Fourier transform	93
$H^s(\mathbb{R}^n)$	Sobolev space of order s	7
$K_\varepsilon(\xi_0)$	Cone with aperture ε and axis ξ_0	68
L^p	Space of functions whose p^{th} power is integrable	7
L^p_{loc}	Space of functions whose p^{th} power is locally integrable	7
L^∞	Space of essentially bounded functions	7
$\mathbb{N}$	The set of natural numbers $\{0, 1, 2, \dots\}$	4
$\mathcal{P}$	Parallel beam radiograph	13
p. v.	Principal value	22, 88

$\mathcal{R}$	Radon transform	9
R_1	Riesz kernel	10
$\mathbb{R}$	Real field	4
$\mathbb{R}^n$	n-dimensional space	4
$\mathcal{S}$	Schwartz space of rapidly decreasing functions	89
$\mathcal{S}'$	Space of tempered distributions	91
S^{n-1}	Unit sphere in $\mathbb{R}^n$	5
$S^{n-1}(x_0, R)$	Sphere with radius R and centre x_0	5
$S_+^{n-1}(y)$	Hemisphere closest to y	5
singsupp	Singular support	68
supp	Support	91
WF	Wave front	68
$\mathbb{Z}$	Set of integers $\{\dots, -2, -1, 0, 1, 2, \dots\}$	4
$\mathbb{Z}_+$	Set of positive integers $\{1, 2, 3, \dots\}$	4
$\partial^\alpha, \partial_j$	Classical partial derivative	6
∂A	Boundary of the set A	5
χ_X	Characteristic function of the set X	5
$\tilde{\chi}_{\varepsilon_1,\varepsilon_2}$	Approximate characteristic function of $B(0, \varepsilon_1)$	89
Λ	Calderón operator, inverse of R_1*	11
σ_λ	Dilation	91

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