Tauberian theorems for powers of bounded linear operators

Jarmo.Malinen@math.tkk.fi
Helsinki University of Technology

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Basic notions (1)

Let T be a bounded linear operator on a Banach space.

- T is power bounded if $\sup_{n\geq 0} \|T^n\| < \infty$.

- T satisfies the tauberian condition if

$$\sup_{n\geq 1} (n+1)\|(I-T)T^n\| := M < \infty.$$

- T satisfies the Ritt resolvent condition (by R. K. Ritt in 1953) if

$$\|(\lambda - 1)(\lambda - T)^{-1}\| \leq C < \infty \text{ for all } |\lambda| > 1.$$

Tauberian condition (1)

Esterle-Berkani' s conjecture was proposed in 1983:

Suppose $\sigma(T) = \{1\}$. Then either $T = I$ or

$$\liminf_{n\to\infty} (n+1)\| (I-T)\, T^n\| \geq 1/e.$$

Different proofs were independently given in 2000-2002 by N. Kalton, S. Montgomery-Smith, K. Oleszkiewicz, Y. Tomilov; and also by J. Malinen, O. Nevanlinna, V. Turunen, Z. Yuan.

Tauberian condition (2)

As a consequence of this result, we have:

Theorem 1. For any bounded linear T, either

(i) $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| \geq 1/e$ or

(ii) $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| = 0$ holds.

This was given by J. Malinen, O. Nevanlinna, V. Turunen, Z. Yuan in 2002.

Hence, the tauberian condition characterizes the fastest possible decay for $T^n - T^{n+1}$.

A fundamental result (1)

Theorem 2. The following are equivalent:

(i) T is power bounded, and it satisfies the tauberian condition; and

(ii) T satisfies the Ritt resolvent condition.

When the equivalent conditions hold, then there is $\delta > 0$ and $C_\delta < \infty$ such that

$$\|(\lambda - 1)(\lambda - T)^{-1}\| \leq C_\delta$$

for all $\lambda \in \mathbb{C}$ with $\arg(\lambda - 1) \in (-\pi/2 - \delta, \pi/2 + \delta)$.

A fundamental result (2)

This result has a long and interesting history.

The extension of the λ-domain in the Ritt condition outside a sector was observed independently by B. Nagy and J. Zemánek, Yu. Lyubich, and O. Nevanlinna in 1998. This observation led to the equivalence (i) $\Leftrightarrow$ (ii) as reported by B. Nagy and J. Zemánek.

It was pointed out earlier in 1993 by O. Nevanlinna that the (extended) Ritt condition implies power boundedness. Estimates for $\sup_{n\geq 0} \|T^n\|$ were given by N. Borovykh, D. Drissi and M. N. Spijker, and improved by O. El-Fallah and T. Ransford; both in 2000.

A fundamental result (3)

Assume $\|(\lambda - 1)(\lambda - T)^{-1}\| \le C$ for all $|\lambda| > 1$.

Then constants in estimates are as follows:

$$\sup_{n\ge 1} \|T^n\| \le C^2$$

by O. El-Fallah and T. Ransford in 2000; and

$$\sup_{n\ge 1} (n+1)\|(I-T)T^n\| \le 2C^2 + eC^3.$$

by J. Malinen, O. Nevanlinna, and Z. Yuan in 2002.

A Tauberian theorem (1)

Theorem 3. Assume that T satisfies the tauberian condition and the resolvent estimate

$$\|(\lambda - 1)(\lambda - T)^{-1}\| \leq C < \infty$$

for all $\lambda \in (1, 1 + \epsilon)$ for some $\epsilon > 0$.

Then

$$\sup_{n \geq 0} \|T^n\| \leq 2 + C\|T\| + 2M$$

and

$$\limsup_{n \to \infty} \|T^n\| \leq 2 + C\|T\| + (1 + 1/e)\, M.$$

A Tauberian theorem (2)

This result was given by J. Malinen, O. Nevanlinna, and Z. Yuan in 2002. The proof is a modification of the classical result by Tauber in 1897, taking formally $a_n = (I - T)T^n$:

Theorem 4. Define $f(r) = \sum_0^\infty a_n r^n$ for $0 < r < 1$. If $\lim_{n\to\infty}(n+1)a_n = 0$ and $\lim_{r\to 1_-} f(r) = s$, then $\lim_{n\to\infty} s_n = s$.

The possibility of this modification appears to be noted already in "Lectures on Summability" by A. Peyerimhoff, 1969.

A big equivalence theorem (1)

Many equivalent conditions can be given for T satisfying the tauberian condition

$$\sup_{n\geq 1}(n+1)\|(I-T)T^n\| < \infty.$$

Theorem 5. TFAE for T satisfying the tauberian condition:

(i) T is power bounded,

(ii) T satisfies the (extended) Ritt resolvent condition,

A big equivalence theorem (2)

(iii) there exists $C_K < \infty$ such that T satisfies the iterated Kreiss resolvent condition

$$\|(\lambda - T)^{-k}\| \leq \frac{C_K}{(|\lambda| - 1)^k} \quad \text{for all } |\lambda| > 1 \text{ and } k \in \mathbb{N},$$

(iv) there exists $C_{HY} < \infty$ such that $A = T - I$ satisfies the Hille – Yoshida resolvent condition

$$\|(\lambda - 1)^k (\lambda - T)^{-k}\| \leq C_{HY} \quad \text{for all } \lambda > 1 \text{ and } k \in \mathbb{N},$$

A big equivalence theorem (3)

(v) for some $k \in \mathbb{N}$ there exists $0 < \eta_k \leq 1 \leq C_k < \infty$ such that T satisfies the kth order resolvent condition

$$\|(\lambda - 1)^k (\lambda - T)^{-k}\| \leq C_k \quad \text{for all } \lambda \in (1, 1 + \eta_k),$$

(vi) $A = T - I$ generates an uniformly bounded, norm continuous, analytic semigroup $t \mapsto e^{At}$ of linear operators,

A big equivalence theorem (4)

(vii) the operators $M_n := \frac{1}{n+1}\sum_{j=0}^{n} T^j$ are uniformly bounded (i.e., T is uniformly Cesaro bounded), and

(viii) there exists $C_{UA} < \infty$ such that T is uniformly Abel bounded, i.e.,

$$\|(\lambda - 1)\sum_{k=0}^{n} \lambda^{-k-1} T^k\| < C_{UA} \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda > 1.$$

Comments on the equivalence (1)

Note that the equivalence (i) $\Leftrightarrow$ (v) is a slight improvement of Theorem 3 above.

It is well known and easy to see that

$$\text{(i)} \overset{p.b.}{\Rightarrow} \text{(iii)} \overset{Kreiss}{\Rightarrow} \text{(iv)} \overset{H-Y}{\Rightarrow} \text{(v)}$$

and

$$\text{(i)} \overset{p.b.}{\Rightarrow} \underset{}{\overset{Cesaro}{\text{(vii)}}} \;\&\; \overset{Abel}{\text{(viii)}}$$

hold even without the tauberian condition.

However, none of the converse implications hold without some additional assumption.

Comments on the equivalence (2)

Even

$$\text{(vii)} \overset{Cesaro \quad H-Y}{\Rightarrow} \text{(iv)} \qquad \text{and} \qquad \text{(vii)} \overset{Cesaro \quad Abel}{\Leftrightarrow} \text{(viii)}$$

hold without the tauberian condition.

The first claim was shown by O. Nevanlinna in 1997.

The direct implication $\text{(vii)} \overset{Cesaro \quad Abel}{\Rightarrow} \text{(viii)}$ was given by J. Grobler and C. B. Huijsmans in 1995. The converse part was given by A. Montes-Rodrígues, J. Sánchez-Álvarez, and J. Zemánek in 2005.

Comments on the equivalence (3)

It was pointed out by D. Tsedenbayar in 2002 in his dissertation that

$$\overset{p.b.}{\text{(i)}} \Leftrightarrow \overset{Cesaro}{\text{(vii)}}$$

if $\limsup_{n\to\infty}(n+1)\|M_n - M_{n-1}\| < \infty$.

The latter condition holds, if T satisfies the tauberian condition.

Comments on the equivalence (4)

Finally, if T satisfies the weaker form of tauberian condition

$$\sup_{n \geq 0} \sqrt{n+1} \|(I-T)T^n\| < \infty,$$

then

$$\overset{p.b.}{\text{(i)}} \Leftrightarrow \overset{H-Y}{\text{(iv)}}$$

follows from another classical tauberian theorem.

This fact was pointed out to us by an anonymous reviewer in 2006.

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That's all of it, folks!

Have a nice day.