Tustin's method for final state approximation of conservative dynamical systems

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Abstract: We study a Crank–Nicolson type time discretisation (known as Tustin's method in engineering literature) for a conservative, infinite-dimensional linear dynamical system whose transfer function is scalar and inner. We show that this discretisation approximates the state trajectory at any given time. We first prove the result for canonical Hankel range realisations, and the general case is then obtained using the state space isomorphism.

Keywords: Distributed parameter systems, Mathematical systems theory

1. INTRODUCTION

Let us introduce the purpose of this paper in the finite-dimensional case where $X = \mathbb{C}^n$ for $n < \infty$, and the state-space system S is defined by the block matrix $S := \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ \mathbb{C} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ \mathbb{C} \end{smallmatrix}\right]$. The corresponding dynamical equations are

$$\begin{cases} z'(t) &= Az(t) + Bu(t), \\ y(t) &= Cz(t) + Du(t), \quad t \geq -T, \\ z(-T) &= z_{-T}. \end{cases} \tag{1}$$

That such a system S is *(scattering) conservative* means the following: for all initial times $-T < 0$, input signals $u, u^d \in C^2([-T,\infty))$, and initial states $z_{-T}, z^d_{-T} \in X$, the energy balance equations

$$\begin{aligned} &\frac{d}{dt}\|z(t)\|_X^2 = |u(t)|^2 - |y(t)|^2 \quad \text{and} \\ &\frac{d}{dt}\|z^d(t)\|_X^2 = |u^d(t)|^2 - |y^d(t)|^2 \end{aligned} \tag{2}$$

hold for all $t > -T$ where z, y are given by (1), and z^d, y^d are given by

$$\begin{cases} \frac{d}{dt}z^d(t) &= A^* z^d(t) + C^* u^d(t), \\ y^d(t) &= B^* z^d(t) + D^* u^d(t), \quad t \geq -T, \\ z^d(-T) &= z^d_{-T}. \end{cases} \tag{3}$$

The *Cayley–Tustin discretisation* (or *transform*) of (1) is defined for any *time step* $h > 0$ by

$$\begin{cases} z_j^{(h)} &= A_\sigma z_{j-1}^{(h)} + B_\sigma u_j^{(h)}, \\ y_j^{(h)} &= C_\sigma z_{j-1}^{(h)} + D_\sigma u_j^{(h)}, \quad j > -J_h, \\ z_{-J_h}^{(h)} &= 0 \end{cases} \tag{4}$$

where we define[1] $A_\sigma := (\sigma + A)(\sigma - A)^{-1}$, $B_\sigma := \sqrt{2\sigma}(\sigma - A)^{-1}B$, $C_\sigma := \sqrt{2\sigma}C(\sigma - A)^{-1}$, $D_\sigma := \hat{\mathbf{G}}(\sigma)$,

$$J_h := \lceil T/h \rceil \in \{1, 2, \dots\} \quad \text{and} \quad \sigma := 2/h > 0. \tag{5}$$

[1] Thus, J_h is the unique integer satisfying $hJ_h \in (T, T + 1/h]$.

Suppose now that $z_{-T} = 0$ in (1), and that equations (1) and (4) are connected by $\{u_j^{(h)}\}_{j\in\mathbb{Z}} = T_\sigma u$ where the *discretising (or sampling) operator* T_σ is given by[2]

$$u_j^{(h)} = \frac{1}{\sqrt{h}} \int_{(j-1)h}^{jh} u(t)\, dt \quad \text{for all} \quad j \in \mathbb{Z}. \tag{6}$$

In the finite-dimensional case, the main result – Theorem 12 – of this paper takes the following form:

Theorem 1. Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a finite-dimensional, conservative state space system with scalar signals, such that the contraction semigroup $\mathbb{T}(t) := e^{At}$ is exponentially stable. For any $T > 0$ and $u \in C^2([-T,\infty))$ with $\mathrm{supp}(u) \subset (-T, 0]$, define the continuous trajectory z by (1) with $z_{-T} = 0$. Define the discrete trajectory $\{z_j^{(h)}\}_{j=-J_h\ldots 0}$ by (4) – (5) for all $h > 0$. Then $\lim_{h\to 0+} \|z_0^{(h)} - z(0)\|_X = 0$.

The proof of this theorem is given at the end of the paper.

Remark 2. Instead of using $t = 0$ as the final state to be approximated in Theorem 1, this time point can be chosen arbitrarily by translation invariance. That also the input-output mapping of system S is approximated by Tustin's method, has already been treated in Havu and Malinen (2005, 2007).

In this paper, Theorem 1 is proved in a considerably more general form. Indeed, the assumption $\dim X < \infty$ is not required at all, allowing infinite-dimensional *conservative system nodes* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ on a separable Hilbert (state) space X. A thorough introduction to such system nodes and their Cayley–Tustin transforms can be found in (Havu and Malinen, 2007, Section 1), and we assume that reader has access to this text. Moreover, the exponential stability assumption in Theorem 1 can be weakened to mere strong stability of both the contraction semigroups $\mathbb{T}(\cdot)$ and $\mathbb{T}^d(t) := e^{A^* t}$; this is equivalent with the property that the (scalar) transfer function $\hat{\mathbf{G}}(\cdot)$ of system node S is inner. All these generalisations are presented in Theorem 12.

[2] We extend $u \in C^2([-T,\infty))$ by zero for $t < -T$ in (6).

It is possible to further extend Theorem 12 for systems whose external signals u and y are not scalar but live in a separable Hilbert space. After this final generalisation (not presented here to reduce technicality), applications cover many practical linear systems in applied mathematics and physics; e.g., the scattering conservative, boundary controlled wave equation treated in Malinen and Staffans (2006), wave propagation in transmission graphs as treated in Aalto and Malinen (2011) using Webster's equation.

System nodes have been introduced under different names (such as *operator colligations* or *Livšic – Brodskiĭ nodes*) in, e.g., Helton (1976); Brodskiĭ (1971a,b, 1978); Malinen et al. (2006); Malinen and Staffans (2006, 2007); Livšic and Yantsevich (1977); Smuljan (1986); Staffans (2004); Sz.-Nagy and Foias (1970). References to Cayley transform in numerical analysis include, e.g., Arov and Gavrilyuk (1993); Gavrilyuk and Makarov (1994, 1998).

Outline of this paper is as follows: In Section 2 we introduce canonical Hankel range realisations of $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$, express them as system nodes $S^{\mathbf{G}}$, and compute their Cayley–Tustin transforms $\phi_\sigma^{\mathbf{G}}$. The main result of this paper is given in Section 3 for the special case of $S^{\mathbf{G}}$; see Theorem 11. Using the state space isomorphism, this result is translated to general systems $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ (satisfying the conditions of Theorem 12) in Section 4.

Notation

The real axis and the complex plane are $\mathbb{R}$ and $\mathbb{C}$, and we write $\mathbb{R}_+ = (0,\infty)$, $i\mathbb{R} = \{z : \operatorname{Re} z = 0\}$, $\mathbb{C}_+ = \{z : \operatorname{Re} z > 0\}$, and $\mathbb{D} = \{z : |z| < 1\}$. The set $\mathbb{Z}$ denotes integers, and $\mathbb{Z}_+ = \{0, 1, \ldots\}$, $\mathbb{Z}_- = \{\ldots, -1, 0\}$, and $\mathbb{N} = \{1, \ldots\}$. Square summable sequences are denoted by $\ell^2(\mathbb{Z})$, $\ell^2(\mathbb{N})$, etc., with the norm $\|\{u_j\}\|^2_{\ell^2(\mathbb{Z})} = \sum_{j\in\mathbb{Z}} |u_j|^2$.

Bounded linear operators between Hilbert spaces X, Z are denoted by $\mathcal{L}(X; Z)$ and $\mathcal{L}(X)$ if $Z = X$. The spectrum of $A \in \mathcal{L}(X)$ are denoted by $\sigma(A)$. By $C(I)$ we denote the continuous functions on $I = [a, b] \subset \mathbb{R}$. For $n = 1, 2, \ldots$, the space $C^n(\mathbb{R})$ denotes the n times continuously differentiable functions, and $C^n(I)$ denotes the restrictions of $C^n(\mathbb{R})$ to I. By $C_0^n(I)$ denote those $f \in C^n(\mathbb{R})$ for which $\operatorname{supp}(f) \subset I$.

The Laplace transform is defined by $\hat{f}(s) = (\mathcal{L}f)(s) := \int_0^\infty e^{-st} f(t)\, dt$ for $s \in \mathbb{C}_+$. By Plancherel's theorem, $\mathcal{L} : L^2(\mathbb{R}_+) \to H^2(\mathbb{C}_+)$ is a unitary operator. The Fourier transform is defined by $(\mathcal{F}f)(i\omega) := \int_{-\infty}^\infty e^{-i\omega t} f(t)\, dt$ for $f \in L^1(\mathbb{R})$ and $i\omega \in i\mathbb{R}$. The operator $\mathcal{F}$ has an extension from $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ to a unitary operator $\mathcal{F} : L^2(\mathbb{R}) \to L^2(i\mathbb{R})$ if $\|f\|^2_{L^2(i\mathbb{R})} := \frac{1}{2\pi} \int_{-\infty}^\infty |f(i\omega)|^2\, d\omega$.

2. TUSTIN TRANSFORM OF HANKEL RANGE REALISATIONS

2.1 Generating operators and conservativity

For any $z \in L^2(\mathbb{R})$ and $t' \in \mathbb{R}$, define the translation group by $(\tau^t z)(t') := z(t + t')$. For a closed interval $I \subset \mathbb{R}$, define the orthogonal projection π_I on $L^2(\mathbb{R})$ by

$$(\pi_I z)(t') := \begin{cases} z(t') & \text{for } t' \in I, \\ 0 & \text{for } t' \notin I; \end{cases}$$

$\pi_+ := \pi_{\mathbb{R}_+}$, and $\pi_- := \pi_{\mathbb{R}_-}$. We identify $L^2(\mathbb{R}_+) = \operatorname{range}(\pi_+)$ and $L^2(\mathbb{R}_-) = \operatorname{range}(\pi_-)$.

Let $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$ be arbitrary, denote by $\mathbf{G} \in \mathcal{L}(L^2(\mathbb{R}))$ the corresponding I/O-map, and define the Hankel state space

$$H := \overline{\operatorname{range}(\pi_+ \mathbf{G} \pi_-)} \subset L^2(\mathbb{R}_+) \tag{7}$$

where the closure is taken in $L^2(\mathbb{R}_+)$. The space H is given the norm of $L^2(\mathbb{R})$, its inner product is denoted by $\langle \cdot, \cdot \rangle_H$, and it is invariant under the unilateral backward translation; i.e., $\pi_+ \tau^t H \subset H$ holds for all $t \geq 0$.

Consider now the well-posed linear system $\Sigma_{\mathbf{G}}$ defined by

$$\Sigma_{\mathbf{G}} : \begin{cases} \gamma(t) & = \pi_+ \tau^t \gamma_0 + \pi_+ \mathbf{G} \pi_- \tau^t u \quad \text{for} \quad t \geq 0, \\ y & = \pi_+ \gamma_0 + \pi_+ \mathbf{G} \pi_+ u \end{cases} \tag{8}$$

for initial state $\gamma_0 \in H$ and input signal $u \in L^2(\mathbb{R}_+)$ where the functions $\gamma(\cdot)$ and y are the state trajectory and output signal of $\Sigma_{\mathbf{G}}$, respectively. As a well-posed linear system, $\Sigma_{\mathbf{G}}$ is associated to a unique system node; see (Havu and Malinen, 2007, Definition 1.1) and (Staffans, 2004, Theorem 4.6.5).

Definition 3. Let $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$. Denote by $S^{\mathbf{G}} = \left[\begin{smallmatrix} [A\&B]_{\mathbf{G}} \\ [C\&D]_{\mathbf{G}} \end{smallmatrix}\right]$ the system node on Hilbert spaces $(\mathbb{C}, H, \mathbb{C})$ with the domain $\operatorname{dom}(S^{\mathbf{G}})$, associated to the well-posed linear system $\Sigma_{\mathbf{G}}$ in (8). Denote for $\sigma > 0$ by $\phi_\sigma^{\mathbf{G}} = \left[\begin{smallmatrix} A_\sigma^{\mathbf{G}} & B_\sigma^{\mathbf{G}} \\ C_\sigma^{\mathbf{G}} & D_\sigma^{\mathbf{G}} \end{smallmatrix}\right]$ the Cayley–Tustin transform of $S^{\mathbf{G}}$ as defined in (Havu and Malinen, 2007, Section 1.3).

The transfer function of $\phi_\sigma^{\mathbf{G}}$ is given by

$$\hat{\mathbf{D}}_\sigma(z) = \hat{\mathbf{G}}\left(\frac{1-z}{1+z}\sigma\right) \text{ for } z \in \mathbb{D}. \tag{9}$$

Since we need an explicit expression for $\phi_\sigma^{\mathbf{G}}$, we must compute the generating operators of $S^{\mathbf{G}}$:

Proposition 4. Let $S^{\mathbf{G}}$ be as in Definition 3, and define H by (7). Then the main operator $A^{\mathbf{G}}$, input operator $B^{\mathbf{G}}$, and the output operator $C^{\mathbf{G}}$ of $S^{\mathbf{G}}$ are given by the equations

$$A^{\mathbf{G}} = \frac{d}{dt}\Big|_{\operatorname{dom}(A^{\mathbf{G}})} \text{ where } \operatorname{dom}\left(A^{\mathbf{G}}\right) = H^1(\mathbb{R}_+) \cap H, \tag{10}$$

$$(\sigma - A_{-1}^{\mathbf{G}})^{-1} B^{\mathbf{G}} u = \pi_+ \mathbf{G}(e_\sigma u) \quad \text{for all } u \in U \text{ and } \sigma > 0, \text{ and} \tag{11}$$

$$C^{\mathbf{G}} z = z(0) \text{ for all } z \in \operatorname{dom}\left(A^{\mathbf{G}}\right) \tag{12}$$

where $A_{-1}^{\mathbf{G}} \in \mathcal{L}(H; H_{-1})$ is the Yosida extension of $A_{-1}^{\mathbf{G}}$ (see (Havu and Malinen, 2007, Section 1.3)), and

$$e_\sigma(t) := \begin{cases} 0 & \text{for} \quad t > 0, \\ e^{\sigma t} & \text{for} \quad t \leq 0. \end{cases} \tag{13}$$

The domain of $S^{\mathbf{G}}$ is given by[3] $\operatorname{dom}\left(S^{\mathbf{G}}\right) = \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} H \\ U \end{smallmatrix}\right] : x - \pi_+ \mathbf{G}(e_\sigma u) \in H^1(\mathbb{R}_+) \text{ for some } \sigma > 0 \right\}$, and the transfer function of $S^{\mathbf{G}}$ is $\hat{\mathbf{G}}$.

Proof. The generator of the backward translation semigroup $\mathbb{S}(t) := \pi_+ \tau^t$ on $L^2(\mathbb{R}_+)$ is

$$A = \tfrac{d}{dt} \quad \text{with domain} \quad \operatorname{dom}(A) = H^1(\mathbb{R}_+).$$

[3] $H^1(\mathbb{R}_+) := \left\{ h \in L^2(\mathbb{R}_+) : \exists h'(t) \text{ a.e. } t \in \mathbb{R}_+ \text{ and } h' \in L^2(\mathbb{R}_+) \right\}$ is the Sobolev space.

Because $\mathbb{S}(t)H \subset H$, it follows that the generator $A^{\mathbf{G}}$ of $\mathbb{S}(t)|_H$ is given by (10). That (11) holds follows from (Staffans, 2004, p. 214). The characterisation of $\operatorname{dom}\left(S^{\mathbf{G}}\right)$ follows from what we have already computed, together with the definition $\operatorname{dom}\left(S^{\mathbf{G}}\right) := \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} H \\ U \end{smallmatrix}\right] : A^{\mathbf{G}}_{-1}x + B^{\mathbf{G}}u \in H\right\}$. Remaining claims follow from (8) and the correspondence of $S^{\mathbf{G}}$ and $\Sigma_{\mathbf{G}}$ by (Staffans, 2004, Theorem 4.6.5).

Using the operators given by Proposition 4 and assuming that $u \in C^2(\mathbb{R}_+)$ and $\left[\begin{smallmatrix} \gamma_0 \\ u(0) \end{smallmatrix}\right] \in \operatorname{dom}\left(S^{\mathbf{G}}\right)$, the dynamical equations (8) for the *classical solution* can be written in the differential form

$$\begin{cases} \gamma'(t) &= A^{\mathbf{G}}_{-1}\gamma(t) + B^{\mathbf{G}}u(t), \\ y(t) &= [C\&D]_{\mathbf{G}} \left[\begin{smallmatrix} \gamma(t) \\ u(t) \end{smallmatrix}\right] \end{cases} \tag{14}$$

for $t \geq 0$ where $[C\&D]_{\mathbf{G}}$ is given for $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}\left(S^{\mathbf{G}}\right)$ by

$$[C\&D]_{\mathbf{G}} \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \left[x - \pi_+ \mathbf{G}(e_\sigma u)\right](0) + \hat{\mathbf{G}}(\sigma)u.$$

Recall that $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$ is *inner* if $|\hat{\mathbf{G}}(iy)| = 1$ for almost all $y \in \mathbb{R}$. By Parseval's identity, the corresponding I/O-mapping of such a transfer function satisfies

$$\mathbf{G}^*\mathbf{G} = \mathbf{G}\mathbf{G}^* = \mathcal{I} \quad \text{on} \quad L^2(\mathbb{R}). \tag{15}$$

If $\mathbf{G}$ is the I/O-mapping of a *conservative* system S, then (15) means that S (and also its dual system S^d as characterised in (Malinen et al., 2006, Proposition 2.4)) cannot permanently trap a strictly positive amount of energy inside its state space.

Proposition 5. Suppose that $\hat{\mathbf{G}}$ is inner. Then its Hankel range realisation $S^{\mathbf{G}}$ is a linear system that is exactly controllable (in infinite time), exactly observable (in infinite time), and (scattering) conservative.

This is well-known in the model theory for Hilbert space contractions (see, e.g., (Staffans, 2004, Corollary 11.7.4)).

2.2 Cayley–Tustin transform of $S^{\mathbf{G}}$

We next compute the Cayley–Tustin transform $\phi^{\mathbf{G}}_\sigma = \left[\begin{smallmatrix} A^{\mathbf{G}}_\sigma & B^{\mathbf{G}}_\sigma \\ C^{\mathbf{G}}_\sigma & D^{\mathbf{G}}_\sigma \end{smallmatrix}\right]$ of $S^{\mathbf{G}}$. The feed-through operator $D^{\mathbf{G}}_\sigma$ of $\phi^{\mathbf{G}}_\sigma$ can be directly read from (9):

$$D^{\mathbf{G}}_\sigma = \hat{\mathbf{D}}_\sigma(0) = \hat{\mathbf{G}}(\sigma). \tag{16}$$

The input operator $B^{\mathbf{G}}_\sigma = \sqrt{2\sigma}(\sigma - A^{\mathbf{G}})^{-1}B^{\mathbf{G}}$ of $\phi^{\mathbf{G}}_\sigma$ is easy to obtain by (Staffans, 2004, p. 214). We get

$$B^{\mathbf{G}}_\sigma u = \sqrt{2\sigma}\pi_+\mathbf{G}(e_\sigma u) \quad \text{for all } u \in U \tag{17}$$

where $e_\sigma(t)$ is given by (13) with the Fourier transform

$$\hat{e}_\sigma(i\omega) = (\sigma - i\omega)^{-1} \quad \text{for all} \quad \omega \in \mathbb{R}. \tag{18}$$

Let us proceed to the *cogenerator* $A^{\mathbf{G}}_\sigma := (\sigma + A^{\mathbf{G}})(\sigma - A^{\mathbf{G}})^{-1} \in \mathcal{L}(H)$. We need an auxiliary result:

Lemma 6. For all $\sigma > 0$, the operator

$$\left(\mathcal{D}_\sigma \hat{g}\right)(s) := \begin{cases} \dfrac{\hat{g}(s) - \hat{g}(\sigma)}{s - \sigma} & \text{for } s \neq \sigma, \\ g'(\sigma) & \text{for } s = \sigma \end{cases} \tag{19}$$

for $\hat{g} \in H^2(\mathbb{C}_+)$ satisfies $\|\sigma\mathcal{D}_\sigma\|_{\mathcal{L}(H^2(\mathbb{C}_+))} \leq 1 + \frac{1}{\sqrt{8\pi}}$, and its adjoint is given by $(\mathcal{D}^*_\sigma \hat{f})(s) = -\frac{\hat{f}(s)}{s+\sigma}$ for $\hat{f} \in H^2(\mathbb{C}_+)$.

All this follows from a direct computation.

Proposition 7. Take $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$ and $\sigma > 0$, and let $\phi^{\mathbf{G}}_\sigma$ be as in Definition 3 with the state space H given by (7).

(i) Then the cogenerator in $\phi^{\mathbf{G}}_\sigma$ satisfies $A^{\mathbf{G}}_\sigma = (2V_\sigma - I)|_H$ where

$$(\sigma^{-1}V_\sigma z)(t) := \int_t^\infty e^{\sigma(t-v)}z(v)\,dv = \left(\mathcal{L}\pi_+\tau^t z\right)(\sigma); \tag{20}$$

for $z \in L^2(\mathbb{R}_+)$. Moreover, $\mathcal{L}(A^{\mathbf{G}}_\sigma)^j z = (-I - 2\sigma\mathcal{D}_\sigma)^j\mathcal{L}z$ for all $z \in H$.

(ii) For all $u \in U$ and $j \in \mathbb{Z}_+$, we have

$$\mathcal{L}\left(A^{\mathbf{G}}_\sigma\right)^j B^{\mathbf{G}}_\sigma u = -\sqrt{2\sigma}(-I - 2\sigma\mathcal{D}_\sigma)^j\mathcal{D}_\sigma\hat{\mathbf{G}}u. \tag{21}$$

Proof. First part of claim (i): Since $A^{\mathbf{G}}_\sigma = 2\sigma\left(\sigma - A^{\mathbf{G}}\right)^{-1} - I$, it is enough to show that $\sigma\left(\sigma - A^{\mathbf{G}}\right)^{-1} = V_\sigma|_H$ where $A^{\mathbf{G}}$ is given by (10), and V_σ is given by (20). For any $u \in H^1(\mathbb{R}_+)$, we have by the Schwartz inequality

$$|u(t)| \leq |u(0)| + \sqrt{t}\|u\|_{H^1(\mathbb{R}_+)} \text{ for } t \geq 0. \tag{22}$$

If $u = (\sigma - A^{\mathbf{G}})^{-1}z$ for $z \in H \subset L^2(\mathbb{R}_+)$, then $u \in \operatorname{dom}\left(A^{\mathbf{G}}\right) \subset H^1(\mathbb{R}_+)$ and $u'(t) = \sigma u(t) - z(t)$ a.e. $t \in \mathbb{R}_+$. By the variation of constants formula we get $u(t) = e^{\sigma t}\left(u(0) - \int_0^t e^{-\sigma v}z(v)\,dv\right)$ a.e. $t \in \mathbb{R}_+$. Since u cannot grow exponentially by (22) and $\sigma > 0$, it follows that $u(0) = \int_0^\infty e^{-\sigma v}z(v)\,dv$ and hence $u(t) = e^{\sigma t}\int_t^\infty e^{-\sigma v}z(v)\,dv = \left(\sigma^{-1}V_\sigma z\right)(t)$. We conclude that $(\sigma - A^{\mathbf{G}})^{-1} = \sigma^{-1}V_\sigma|_H$.

The second part of claim (i): We obtain by changing the order of integration

$$\left(\mathcal{L}V_\sigma z\right)(s) = -\left(\sigma\mathcal{D}_\sigma z\right)(s).$$

for $s \in \mathbb{C}_+$ and $z \in L^2(\mathbb{R}_+)$. Thus $\mathcal{L}V_\sigma = -\sigma\mathcal{D}_\sigma\mathcal{L}$ on $L^2(\mathbb{R}_+)$. Therefore $\mathcal{L}A^{\mathbf{G}}_\sigma z = (-I - 2\sigma\mathcal{D}_\sigma)\mathcal{L}z$ holds for all $z \in H$, and the claim follows.

Claim (ii): By (13), (18), and the fact that $\mathbf{G}$ operates as multiplication by $\hat{\mathbf{G}}(\cdot)$ in the frequency domain, we get for all $\omega \in \mathbb{R}$

$$\begin{aligned} \mathcal{F}\left[\mathbf{G}(e_\sigma u)\right](i\omega) &= \frac{\hat{\mathbf{G}}(i\omega)u}{\sigma - i\omega} \\ &= -(\mathcal{D}_\sigma\hat{\mathbf{G}})(i\omega)u - (i\omega - \sigma)^{-1}\hat{\mathbf{G}}(\sigma)u. \end{aligned} \tag{23}$$

Now, $\mathcal{D}_\sigma\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$ since $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$, and also the nontangential boundary limit function $i\omega \mapsto (\mathcal{D}_\sigma\hat{\mathbf{G}})(i\omega)u$ is in $L^2(i\mathbb{R})$ for all $u \in U$. Hence, $\mathcal{D}_\sigma\hat{\mathbf{G}}u \in H^2(\mathbb{C}_+)$, and it is easy to see that $(\cdot - \sigma)^{-1}\hat{\mathbf{G}}(\sigma)u \in H^2(\mathbb{C}_-)$. Thus the splitting on the right hand side of (23) is an orthogonal direct sum, and we get for a.e. $\omega \in \mathbb{R}$ the identity $\mathcal{L}\left[\pi_+\mathbf{G}(e_\sigma u)\right] = -\mathcal{D}_\sigma\hat{\mathbf{G}}u$. Using claim (ii) together with (17), we get (21) for all $j \geq 0$ and $u \in U$.

Remark 8. The adjoint of $-\sqrt{2\sigma}(-I - 2\sigma\mathcal{D}_\sigma)^j\mathcal{D}_\sigma$ in (21) satisfies for $\hat{f} \in H^2(\mathbb{C}_+)$ and $s \in \mathbb{C}_+$ (see Lemma 6)

$$-\sqrt{2\sigma}\left(\mathcal{D}^*_\sigma(-I - 2\sigma\mathcal{D}^*_\sigma)^j\hat{f}\right)(s) = \frac{\sqrt{2\sigma}}{\sigma + s}\left(\frac{\sigma - s}{\sigma + s}\right)^j\hat{f}(s);$$

Note that the *Laguerre basis* of $H^2(\mathbb{C}_+)$ appears on the right hand side.

3. CONVERGENCE RESULTS

Now that we have explicit descriptions for $S^{\mathbf{G}}$ and $\phi_\sigma^{\mathbf{G}}$ of Definition 3 in familiar terms, it is possible to prove the main result – Theorem 12 – in a special case.

The discretised Hankel state trajectory $\{\gamma_j^{(h)}\}_{j\in\mathbb{Z}} \subset H$ for input $u \in C_0^2([-T,0])$ is given by

$$\begin{cases} \gamma_j^{(h)} &= A_\sigma^{\mathbf{G}}\gamma_{j-1}^{(h)} + B_\sigma^{\mathbf{G}} u_j^{(h)}, \\ y_j^{(h)} &= C_\sigma^{\mathbf{G}}\gamma_{j-1}^{(h)} + D_\sigma^{\mathbf{G}} u_j^{(h)}, \quad j > -J_h, \\ \gamma_{-J_h}^{(h)} &= 0 \end{cases} \tag{24}$$

where J_h, h are given (5), and $\{u_j^{(h)}\} = T_\sigma u$ by (6). The main task in this section is to prove the weak convergence

$$\lim_{h\to 0+} \left\langle \gamma, \gamma(0) - \gamma_0^{(h)} \right\rangle_H = 0 \quad \text{for all} \quad \gamma \in H \tag{25}$$

which is easiest carried out in frequency domain. Theorem 11 follows from (25) and the following:

Proposition 9. Let $\hat{\mathbf{G}}(\cdot)$, u, $S^{\mathbf{G}}$, and $\phi_\sigma^{\mathbf{G}}$ be as above. Assume, in addition, that $\hat{\mathbf{G}}(\cdot)$ is inner from both sides. Define $\gamma(0)$ and $\gamma_0^{(h)}$ for $h>0$ by (14) and (24), respectively. Define $u^{(h)} \in L^2(\mathbb{R}_-)$ as the down-sampled input given by (see (6))

$$u^{(h)} := T_\sigma^* T_\sigma u \quad \text{with } \sigma = 2/h. \tag{26}$$

Then

(i) $\lim_{h\to 0+} \left\langle \gamma, \gamma_0^{(h)} - \pi_+ \mathbf{G} u^{(h)} \right\rangle_H = 0$ for all $\gamma \in H$ if and only if (25) holds; and

(ii) if (25) holds, then $\lim_{h\to 0+} \|\gamma_0^{(h)} - \gamma(0)\|_H = 0$.

Proof. Claim (i): Because

$$\lim_{h\to 0+} \|u^{(h)} - u\|_{L^2(\mathbb{R})} = 0 \text{ for all } u \in L^2(\mathbb{R}) \tag{27}$$

and $\|\mathbf{G}\|_{\mathcal{L}(L^2(\mathbb{R}))} = 1$, we get

$$\|\gamma(0) - \pi_+ \mathbf{G} u^{(h)}\|_H \le \|u - u^{(h)}\|_{L^2(\mathbb{R}_-)} \to 0 \text{ as } h \to 0+.$$

Claim (ii) follows from (Havu and Malinen, 2007, Proposition 7) and the fact that $\lim_{h\to 0+} \|\gamma_0^{(h)}\|_H = \|\gamma(0)\|_H$ which is a consequence of the conservativity of both $S^{\mathbf{G}}$ and $\phi_\sigma^{\mathbf{G}}$; see Proposition 5 and (Havu and Malinen, 2007, Proposition 1.4). The details are as follows: The conservativity of $S^{\mathbf{G}}$ implies the energy balance $\|\gamma(0)\|_H^2 = \|u\|_{L^2(\mathbb{R}_-)}^2 - \|y\,|_{\mathbb{R}_-}\|_{L^2(\mathbb{R}_-)}^2$ for the solution of (14), and the conservativity of $\phi_\sigma^{\mathbf{G}}$ implies the energy balance $\|\gamma_0^{(h)}\|_H^2 = \|\{u_j^{(h)}\}\|_{\ell^2(\mathbb{Z}_-)}^2 - \|\{y_j^{(h)}\}_{j\le 0}\|_{\ell^2(\mathbb{Z}_-)}^2$ for the solution of (24) where $\{u_j^{(h)}\} := T_\sigma u \in \ell^2(\mathbb{Z}_-)$ since $u \in C_0^2([-T,0])$.

Because the hold operator $T_\sigma^* : \ell^2(\mathbb{Z}_-) \to L^2(\mathbb{R}_-)$ is an isometry, we have $\|u^{(h)}\|_{L^2(\mathbb{R}_-)} = \|\{u_j^{(h)}\}\|_{\ell^2(\mathbb{Z}_-)}^2$ for $u^{(h)} = T_\sigma^*\{u_j^{(h)}\} = T_\sigma^* T_\sigma u \in L^2(\mathbb{R}_-)$, and similarly $\|y^{(h)}\,|_{\mathbb{R}_-}\|_{L^2(\mathbb{R}_-)} = \|\{y_j^{(h)}\}_{j\le 0}\|_{\ell^2(\mathbb{Z}_-)}^2$ where we define $y^{(h)} := T_\sigma^*\{y_j^{(h)}\}_{j\in\mathbb{Z}} \in L^2(\mathbb{R})$. The claim now follows by taking the limit as $h \to 0+$ (with $\sigma = 2/h$) of these energy balances because clearly $\|u^{(h)}\|_{L^2(\mathbb{R}_-)} \to \|u\|_{L^2(\mathbb{R}_-)}$, and $\|(y^{(h)} - y)\,|_{\mathbb{R}_-}\|_{L^2(\mathbb{R}_-)} \to 0$ by time translation from (Havu and Malinen, 2007, Theorem 4.3).

In frequency domain, we have by Proposition 7 the following formula for the controllability map of $\phi_\sigma^{\mathbf{G}}$:

$$\hat{\gamma}_0^{(h)} = -\sqrt{2\sigma}\mathcal{D}_\sigma \sum_{j=0}^{J_h} (-I - 2\sigma\mathcal{D}_\sigma)^j (\hat{\mathbf{G}} u_{-j}^{(h)}) \tag{28}$$

where $\sigma = 2/h > 0$ and $\{u_j^{(h)}\} = T_\sigma u$ for $u \in C_0^2([-T,0])$. The Fourier transform of $u^{(h)} \in L^2(\mathbb{R}_-)$ (see (26)) is

$$\hat{u}^{(h)}(s) = \frac{e^{sh} - 1}{s\sqrt{h}} \sum_{j\ge 0} e^{sjh} u_{-j}^{(h)} \text{ for } s \in i\mathbb{R}. \tag{29}$$

By Proposition 9 together with (28) – (29), equation (25) in frequency domain takes the form[4]

$$\begin{aligned} \lim_{h\to 0+} \Big\langle \hat{\gamma}, -\sqrt{2\sigma}\mathcal{D}_\sigma \sum_{j=0}^{J_h} (-I - 2\sigma\mathcal{D}_\sigma)^j (\hat{\mathbf{G}}(\cdot) u_{-j}^{(h)}) \\ -\hat{\mathbf{G}}\hat{u}^{(h)} \Big\rangle_{L^2(i\mathbb{R})} = 0 \text{ for all } \hat{\gamma} \in H^2(\mathbb{C}_+). \end{aligned} \tag{30}$$

By Remark 8, the first part of (30) takes the form

$$\begin{aligned} &\left\langle \hat{\gamma}, -\sqrt{2\sigma}\mathcal{D}_\sigma \sum_{j=0}^{J_h} (-I - 2\sigma\mathcal{D}_\sigma)^j (\hat{\mathbf{G}}(\cdot) u_{-j}^{(h)}) \right\rangle_{L^2(i\mathbb{R})} \\ &= \sum_{j=0}^{J_h} \left\langle f_j^{(h)}, F(\cdot) u_{-j}^{(h)} \right\rangle_{L^2(i\mathbb{R})} \end{aligned}$$

where $f_j^{(h)}(s) := \frac{(1-s)\sqrt{2\sigma}}{s+\sigma}\left(1 - \frac{2s}{s+\sigma}\right)^j \hat{\gamma}(s)$, and

$$F(s) := \frac{\hat{\mathbf{G}}(s)}{s+1} \quad \text{satisfies} \quad \|F\|_{H^2(\mathbb{C}_+)} \le \sqrt{\pi}. \tag{31}$$

Similarly, the latter part in (30) takes the form

$$\left\langle \hat{\gamma}, \hat{\mathbf{G}}\hat{u}^{(h)} \right\rangle_{L^2(i\mathbb{R})} = \sum_{j=0}^{J_h} \left\langle g_j^{(h)}, F(\cdot) u_{-j}^{(h)} \right\rangle_{L^2(i\mathbb{R})}$$

where $g_j^{(h)}(s) := \frac{1}{s\sqrt{h}}(1-s)\left(1 - e^{-sh}\right) e^{-sjh}\hat{\gamma}(s)$. Note that $f_j^{(h)}, g_j^{(h)} \in H^2(\mathbb{C}_+)$, and we estimate their difference using the multiplication operators on $L^2(i\mathbb{R})$, defined by

$$(M_{h,j}\hat{\gamma})(s) := (1-s) r_j^{(h)}(s)\hat{\gamma}(s) \text{ for a.e. } s \in i\mathbb{R} \tag{32}$$

where

$$r_j^{(h)}(s) := \frac{\sqrt{2\sigma}}{s+\sigma}\left(1 - \frac{2s}{s+\sigma}\right)^j - \frac{\left(1 - e^{-sh}\right)}{s\sqrt{h}} e^{-sjh} \tag{33}$$

for $s \in \mathbb{C} \setminus \{0, -\sigma\}$ and $h = 2/\sigma$. Clearly $r_j^{(h)} \in H^\infty(\mathbb{C}_+)$, and we show that $r_j^{(h)} \to 0$ sufficiently fast on compact sets as $h \to 0+$:

Proposition 10. For any $\omega > 0$ define

$$M_\omega := \max_{|z| \le \frac{2}{1+1/\omega}} \left| \sum_{k\ge 0} \left(2^{-k-2} - \frac{1}{(k+3)!}\right) z^k \right| < \infty. \tag{34}$$

(i) For all $h < 1/(\omega+1)$, $s \in [-i\omega, i\omega]$, and $j \in \mathbb{N}$ we have

$$|r_j^{(h)}(s)| \le M_\omega h^{5/2}|s|^2 + h^{3/2} j^{1/2}|s|.$$

[4] We regard $H^2(\mathbb{C}_\pm)$ as subspaces of $L^2(i\mathbb{R})$ using nontangential limits.

(ii) For any $\hat\gamma \in L^2(i\mathbb{R})$ with $\mathrm{supp}(\hat\gamma) \subset [-i\omega, i\omega]$ we have

$$\sum_{j=0}^{J_h} \|M_{h,j}\hat\gamma\|^2_{L^2(i\mathbb{R})} \leq (M_\omega + \tfrac{1}{2})(1+\omega^2)^2 \cdot h^3(J_h+1)(J_h+M_\omega)\,\|\hat\gamma\|^2_{L^2(i\mathbb{R})}.$$

Proof. Claim (i): Because $|1 - \frac{2s}{s+\sigma}|^j = 1$ and $|\frac{1-e^{-sh}}{s\sqrt{h}}| \leq h^{1/2}$ for all imaginary s and $h = 2/\sigma > 0$, we get

$$h^{-1/2}|r_j^{(h)}(s)| \leq \left|\frac{1}{1+\frac{sh}{2}} - \frac{1-e^{-sh}}{sh}\right| + \left|\left(1 - \frac{2s}{s+\sigma}\right)^j - e^{-sjh}\right|. \tag{35}$$

For the first term on the right hand side of (35), we write

$$\frac{1}{1+\frac{z}{2}} - \frac{1-e^{-z}}{z} = z^2\sum_{k\geq 0}\left(2^{-k-2} - \frac{1}{(k+3)!}\right)(-z)^k$$

that clearly converges for all $|z| < 2$. For any $s \in \mathbb{C}$ with $|s| \leq \omega$ and $h < 2/(\omega+1)$ we get $M_\omega < \infty$ in (34) and hence $|\frac{1}{1+(sh)/2} - \frac{1-e^{-sh}}{sh}| \leq M_\omega h^2|s|^2$.

It remains to estimate the absolute value $m_j(sh) := \left|\left(\frac{1-\frac{sh}{2}}{1+\frac{sh}{2}}\right)^j - \left(e^{-sh}\right)^j\right|$ for $s = yi \in i\mathbb{R}$. Writing $(1 - \frac{sh}{2})(1+\frac{sh}{2})^{-1} = e^{-i\theta_1}$ for $\theta_1 \in (-\pi/2, \pi/2)$ and $\theta_2 = yh$, elementary trigonometry yields $m_j(sh)^2 := |e^{-ij\theta_1} - e^{-ij\theta_2}|^2 \leq 2j|\theta_1 - \theta_2|$. Because $\tan\frac{\theta_1}{2} = \frac{yh}{2} = \frac{\theta_2}{2}$, we get from Taylor's theorem

$$\theta_2 - \theta_1 = 2(\tan\tfrac{\theta_1}{2} - \tfrac{\theta_1}{2}) = 2\cdot\frac{2\sin\frac{\theta'}{2}}{\cos^3\frac{\theta'}{2}}\left(\frac{\theta_1}{2}\right)^2 = \frac{\sin\frac{\theta'}{2}}{\cos^3\frac{\theta'}{2}}\theta_1^2$$

where either $0 \leq \theta' \leq \theta_1$ or $\theta_1 \leq \theta' \leq 0$, depending on the sign of θ_1. We also have $|\theta_1| < |\theta_2|$. Since $|\theta_1| < \pi/2$, we have $|\theta_1 - \theta_2| \leq \frac{\sin\frac{\pi}{4}}{\cos^3\frac{\pi}{4}}\theta_1^2 = \frac{\theta_1^2}{2}$ and so $m_j(sh)^2 \leq j\theta_1^2 \leq j\theta_2^2 = jh^2|s|^2$.

Claim (ii): By the first claim of this proposition, we have

$$\begin{aligned}\|M_{h,j}\hat\gamma\|^2_{L^2(i\mathbb{R})} &= \int_{-\omega}^{\omega}(1+y^2)|\hat r_j^{(h)}(iy)|^2|\hat\gamma(iy)|^2\,dy \\ &\leq h^3\omega^2(1+\omega^2)\left(M_\omega + j^{1/2}\right)^2\|\hat\gamma\|^2_{L^2(i\mathbb{R})} \\ &\leq h^3(1+\omega^2)^2\left(M_\omega^2 + (2M_\omega+1)j\right)\|\hat\gamma\|^2_{L^2(i\mathbb{R})}\end{aligned}$$

for all $j \in \mathbb{N}$ and $h > 1/(\omega+1)$. Claim (ii) follows from this since $\sum_{j=0}^{J_h}\left(M_\omega^2 + (2M_\omega+1)j\right) = (J_h+1)((M_\omega+\frac{1}{2})J_h + M_\omega^2) \leq (M_\omega+\frac{1}{2})(J_h+1)(J_h+M_\omega)$.

Putting together all these ingredients, we obtain the main result of this paper for Hankel range realisations:

Theorem 11. Suppose that $\hat{\mathbf{G}} \in H^\infty(\mathbb{C}_+)$ is an inner function whose associated I/O-map is denoted by $\mathbf{G}$. Let the system node $S^{\mathbf{G}}$, its Cayley–Tustin transform $\phi_\sigma^{\mathbf{G}}$ for $\sigma > 0$, and the state space H be as in Section 2.1.

Fix $T > 0$ and take $u \in C^2(\mathbb{R})$ with $\mathrm{supp}(u) \subset [-T, 0]$. Define $\gamma(0) \in H$ by (14) and $\gamma_0^{(h)} \in H$ by (24) for all $h = 2/\sigma > 0$. Then $\lim_{h\to 0+}\|\gamma(0) - \gamma_0^{(h)}\|_H = 0$.

Proof. We need to verify (30) for $\hat\gamma \in H^2(\mathbb{C}_+)$. Defining F and $M_{h,j}$ by (31) – (32), this takes the form

$$\begin{aligned}&\left\langle \hat\gamma, -\sqrt{2\sigma}\sum_{j=0}^{J_h}\mathcal{D}_\sigma(-I-2\sigma\mathcal{D}_\sigma)^j(\hat{\mathbf{G}}u_{-j}^{(h)}) - \hat{\mathbf{G}}\hat u^{(h)}\right\rangle_{L^2(i\mathbb{R})} \\ &\quad= \sum_{j=0}^{J_h}\left\langle M_{h,j}\hat\gamma, F(\cdot)u_{-j}^{(h)}\right\rangle_{H^2(\mathbb{C}_+)} \to 0\end{aligned} \tag{36}$$

as $h \to 0+$ where σ and J_h satisfy (5), and the equality holds for all $\hat\gamma \in H^2(\mathbb{C}_+)$. We first show the convergence (on the right hand side of (36)) for a dense set of $\hat\gamma \in L^2(i\mathbb{R})$ that are compactly supported as in Proposition 10. By (31) and the contractivity of $T_\sigma \in \mathcal{L}(L^2(\mathbb{R}); \ell^2(\mathbb{Z}))$, we have

$$\sum_{j=0}^{J_h}\|F(\cdot)u_{-j}^{(h)}\|^2_{H^2(\mathbb{C}_+)} \leq \pi\sum_{j\in\mathbb{Z}_+}|u_{-j}^{(h)}|^2 \leq \pi\|u\|^2_{L^2(\mathbb{R}_-)} \tag{37}$$

since $\{u_j^{(h)}\}_{j\in\mathbb{Z}} = T_\sigma u$. We can now estimate using Proposition 10 and Schwartz inequality

$$\begin{aligned}&\left|\sum_{j=0}^{J_h}\left\langle M_{h,j}\hat\gamma, F(\cdot)u_{-j}^{(h)}\right\rangle_{H^2(\mathbb{C}_+)}\right| \\ &\leq \left(\sum_{j=0}^{J_h}\|M_{h,j}\hat\gamma\|^2_{L^2(i\mathbb{R})}\right)^{1/2}\cdot\left(\sum_{j=0}^{J_h}\|F(\cdot)u_{-j}^{(h)}\|^2_{L^2(i\mathbb{R})}\right)^{1/2} \\ &\leq h^{1/2}\cdot\sqrt{\pi}(M_\omega+\tfrac{1}{2})^{1/2}(1+\omega^2)\|\hat\gamma\|_cL^2(i\mathbb{R})\|u\|_{L^2(\mathbb{R})}\cdot \\ &\qquad\cdot\left(h^2(J_h+1)(J_h+M_\omega)\right)^{1/2} \to 0\end{aligned}$$

since $T < hJ_h \leq T + h$ by (5), and hence $h^2(J_h+1)(J_h+M_\omega) \to T^2$ as $h \to 0+$. From (35) we get $\sum_{j=0}^{J_h}\|M_{h,j}\hat\gamma\|^2_{L^2(i\mathbb{R})} \leq M\|(1+y^2)\hat\gamma(iy)\|^2_{L^2(i\mathbb{R})}$, and thus the convergence in (36) holds for all $\hat\gamma$ in this weighted $L^2(i\mathbb{R})$-space that has a dense intersection with $H^2(\mathbb{C}_+)$.

Because $S^{\mathbf{G}}$ and each $\phi_\sigma^{\mathbf{G}}$ for $\sigma > 0$ are (continuous, resp. discrete time) conservative, their controllability maps are contractions (see, e.g., (Havu and Malinen, 2007, Proposition 1.4)), and the same holds for all the discretising operators that contribute to the left hand side of (36). Thus, convergence in (36) holds for all $\hat\gamma \in H^2(\mathbb{C}_+)$, and the proof is now complete.

4. FINAL STATE APPROXIMATION

Theorem 12 below is reduced to Theorem 11 using the state space isomorphism, and we next remind the reader of the basic facts of it. For a full treatment, see, e.g., (Staffans, 2004, Theorem 11.4.13), and the references therein.

A contraction semigroup $\{\mathbb{T}(t)\}_{t\geq 0}$ on Hilbert space X is called *completely nonunitary* (shortly, *c.n.u.*) if there is no reducing (closed) subspace $X' \subset X$ such that $\{\mathbb{T}(t)|_{X'}\}_{t\geq 0}$ is a unitary group on X'. Suppose now that we are given two conservative system nodes $S_1$and S_2 that

(i) have the same transfer functions, [5] and
(ii) their semigroups $\mathbb{T}_1(t)$, $\mathbb{T}_2(t)$ are c.n.u. in respective state spaces X_1, X_2.

[5] Also known as *characteristic functions* in operator literature such as, e.g., Brodskiĭ (1971a); Sz.-Nagy and Foias (1970)

If conditions (i) – (ii) hold, the systems $S_1$and S_2 may only differ by an unitary change of coordinates $V : X_1 \to X_2$ between the two state spaces. In particular, given same input signal for both such $S_1$and S_2, the state trajectories are mapped to each other by the same unitary operator V, too. For $\sigma > 0$, the extension to Cayley–Tustin transforms $\phi_{1,\sigma}$, $\phi_{2,\sigma}$ of S_1, S_2 behave as expected: S_1, S_2 are state space isomorphic if and only if $\phi_{1,\sigma}$, $\phi_{2,\sigma}$ are isomorphic in the discrete time sense with the same operator V.

Theorem 12. Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a conservative system node on Hilbert spaces $(\mathbb{C}, X, \mathbb{C})$ whose semigroup is c.n.u. and transfer function $\hat{\mathbf{G}}$ is inner. For $\sigma > 0$, denote the Cayley–Tustin transform of S by $\phi_\sigma = \left[\begin{smallmatrix} A_\sigma & B_\sigma \\ C_\sigma & D_\sigma \end{smallmatrix}\right]$.

For any $T > 0$ and $u \in C^2([-T, 0])$, define the continuous trajectory $z(\cdot)$ by

$$\begin{bmatrix} z'(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}, \quad t \geq -T, \tag{38}$$

with the initial condition $z(-T) = 0$. For $h = 2/\sigma$, define the discrete trajectory $\{z_j^{(h)}\}_{j=-J_h \ldots 0}$ by (4) – (5). Then $\lim_{h\to 0+} \|z_0^{(h)} - z(0)\|_X = 0$.

Remark 13. The unique solvability of (38) is treated in, e.g., (Malinen et al., 2006, Section 2) for system nodes.

Proof of Theorem 12. The system node $S^{\mathbf{G}}$ is conservative and exactly observable by Proposition 5. For contradiction, suppose that the semigroup $\mathbb{T}^{\mathbf{G}}(\cdot)$ of $S^{\mathbf{G}}$, generated by $A^{\mathbf{G}}$ in (10), is not c.n.u.. Because $\mathrm{dom}\left(A^{\mathbf{G}}\right)$ is dense in H, there exists $x_0 \in \mathrm{dom}\left(A^{\mathbf{G}}\right)$ satisfying $\|\mathbb{T}^{\mathbf{G}}(t)x_0\|_H = 1$ for all $t \in \mathbb{R}$. This would imply $\|C^{\mathbf{G}}\mathbb{T}^{\mathbf{G}}(t)x_0\|_Y^2 = \left\langle (C^{\mathbf{G}})^* C^{\mathbf{G}}\mathbb{T}^{\mathbf{G}}(t)x_0, \mathbb{T}^{\mathbf{G}}(t)x_0\right\rangle_{H_{-1}^d, \mathrm{dom}(A^{\mathbf{G}})}$ $= -\left\langle (A^{\mathbf{G}} + (A^{\mathbf{G}})_{-1}^*)\mathbb{T}^{\mathbf{G}}(t)x_0, \mathbb{T}^{\mathbf{G}}(t)x_0\right\rangle_{H_{-1}^d, \mathrm{dom}(A^{\mathbf{G}})}$ $= -\frac{d}{dt}\|\mathbb{T}^{\mathbf{G}}(t)x_0\|_H^2 = 0$ since $A^{\mathbf{G}} + (A^{\mathbf{G}})_{-1}^* = -(C^{\mathbf{G}})^*C^{\mathbf{G}}$ on $\mathrm{dom}\left(A^{\mathbf{G}}\right)$ by (Malinen et al., 2006, Theorems 4.4 and 4.5) where the extrapolation space H_{-1}^d is the dual of $\mathrm{dom}\left(A^{\mathbf{G}}\right)$ using H as the pivot space. Thus, such x_0 would be unobservable, and we conclude that $A^{\mathbf{G}}$, indeed, generates a c.n.u. contraction semigroup. By the state space isomorphism, the node $S^{\mathbf{G}}$ is isomorphic to S, and $\phi_\sigma^{\mathbf{G}}$ is isomorphic to ϕ_σ by some unitary operator $V \in \mathcal{L}(X; H)$.

Assume that $z(\cdot)$ satisfies (38) and $\gamma(\cdot)$ safisfies (14) for $t \geq -T$ with the initial conditions connected by $\gamma(-T) = Vz_{-T}$. The state space isomorphism gives now $z(t) = V^*\gamma(t)$ for all $t \geq -T$. Because $\phi_\sigma^{\mathbf{G}}$ is isomorphic to ϕ_σ by the same operator V, we conclude that the discrete trajectories $\{z_j^{(h)}\}$ in (4) and $\{\gamma_j^{(h)}\}$ in (24) satisfy $z_j^{(h)} = V^*\gamma_j^{(h)}$ for all $j \geq -J_h$. Now by Theorem 11, we have $\|z_0^{(h)} - z(0)\|_X = \|V^*(\gamma_0^{(h)} - \gamma(0))\|_X = \|\gamma_0^{(h)} - \gamma(0)\|_H \to 0$ as $h \to 0$, and the proof is complete.

Proof of Theorem 1. If $\lambda \in i\mathbb{R} \cap \sigma(A)$ and $A\bar{x} = \lambda\bar{x}$ for $\bar{x} \neq 0$, then $\|e^{At}\bar{x}\|_X = \|e^{\lambda t}\bar{x}\|_X = \|\bar{x}\|_X$ for all $t \in \mathbb{R}$. This is impossible since the semigroup is assumed to be exponentially stable; hence, c.n.u.. We conclude that $i\mathbb{R} \cap \sigma(A) = \emptyset$, and thus $\hat{\mathbf{G}}(s) = D + C(s - A)^{-1}B$ for $s \notin \sigma(A)$ exists for all $s \in i\mathbb{R}$. By (Malinen et al., 2006, Proposition 1.4) and finite-dimensionality, we thus have $A + A^* = -C^*C, \quad C = -DB^*$ and $D^*D = I$. Using $A + A^* = -C^*C$ gives the identity $B^*(\bar{s} - A^*)^{-1}C^*C(s - A)^{-1}B = B^*(\bar{s} - A^*)^{-1}B + B^*(s - A)^{-1}$ for any $s \in i\mathbb{R}$. Using $C = -DB^*$ gives $\hat{\mathbf{G}}(s)^*\hat{\mathbf{G}}(s) = D^*D = 1$, and thus $\hat{\mathbf{G}}$ is inner. Now Theorem 1 follows from Theorem 12.

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