MINIMAX CONTROL OF DISTRIBUTED DISCRETE TIME SYSTEMS THROUGH SPECTRAL FACTORIZATION

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Abstract

We introduce a Riccati equation theory for a class of well posed (I/O -stable) discrete time linear systems Φ as presented in [6].

We shall tie together three different problems: The first problem is the general question under which conditions a minimax control problem associated to Φ can be solved by a feedback law. The second problem is the existence of certain spectral factorization of the I/O -map of Φ. The third problem is about certain solution of a Riccati equation system associated to Φ.

We shall show that these three problems are in fact equivalent. This equivalence does not require any finite dimensional structure of the system Φ. The I/O-stability notion that we use throughout this paper is weaker than the conventional power stability ($\rho(A) < 1$). Finally, connections to the existing power stable and finite dimensional theories are presented.

1 Introduction

In this paper we study the nondefinite critical control problem in discrete time. Our basic object is a general H^∞ transfer function whose input, state and output spaces are not assumed to be finite dimensional.

Furthermore, we concentrate upon a fixed realization of the transfer function that is not assumed to be either input or output stable. In our setting, we regard the realization we are working with as given, no matter how (topologically) uncomfortable it is.

The realization that we are working with is called the *discrete time linear system* (DLS) Φ. It is given by the system of difference equations:

$$\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ y_j = Cx_j + Du_j, \quad j \geq 0, \end{cases} \tag{1}$$

where $u_j \in U$, $x_j \in H$, $y_j \in Y$, and A, B, C, D are bounded linear operators between appropriate Hilbert spaces. We call the ordered quadruple $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *a DLS in difference equation form*. The three Hilbert spaces are as follows: U is the input space, H is the state space and Y is the output space of ϕ. There is also another equivalent form for DLS, called *DLS in I/O form*. It consists of four linear operators in the ordered quadruple

$$\Phi := \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tag{2}$$

The operator $A \in \mathcal{L}(H)$ is called the *semi-group generator* of Φ, and it is the seme operator as in equation (1). $\mathcal{B} : \ell^2(Z_-; U) \supset \mathrm{dom}\,(\mathcal{B}) \to H$ is called the *controllability map* that maps the past input into the present state. $\mathcal{C} : H \supset \mathrm{dom}\,(\mathcal{C}) \to \ell^2(\mathbf{Z}_+; Y)$ is called the *observability map* that maps the present state into the future outputs. The last operator $\mathcal{D} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is called the *I/O map* that maps the input into output in a causal and shift invariant way.

The above notions are closely related to the concept of a continuous time *stable well-posed linear system* by O. Staffans in [10], [11] and G. Weiss in [13], [14]. For the basic properties of DLS's, see [6].

In the frequency plane, the action of $\mathcal{D}$ is the multiplication by the transfer function of the system. I/O -stability of Φ means that $\mathcal{D}$ is bounded; this is the H^∞-condition. I/O-stability implies that $\overline{\mathrm{range}\,(\mathcal{B})} \subset \mathrm{dom}\,(\mathcal{C})$. For definiteness, we assume that $\overline{\mathrm{dom}\,(\mathcal{C})} = H$.

As usual, the indefinite cost function is given by

$$J(x_0, \tilde{u}) = \langle \tilde{y}(x_0, \tilde{u}), J\tilde{y}(x_0, \tilde{u}) \rangle_{\ell^2(\mathbf{Z}_+; Y)}$$

where $\tilde{y}(x_0, \tilde{u}) := \mathcal{C}x_0 + \mathcal{D}\tilde{u}$ for a given input $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ and initial state $x_0 \in \mathrm{dom}\,(\mathcal{C})$. Here $J \in \mathcal{L}(Y)$ is self adjoint, not necessarily positive.

Given a cost function $J(x_0, \tilde{u})$, we define the control $\tilde{u}^{crit}(x_0)$ to be *critical* if the Frechet derivative of the cost $J(x_0, \tilde{u})$ with respect to $\tilde{u}$ vanishes at $\tilde{u}^{crit}(x_0)$. The study of certain feedback, factorization and Riccati equation properties of the critical control is the subject of this paper. Our main result is Theorem 6.

The theory of critical controls has connections to control theory (classical quadratic cost minimization theory when J is positive), function theory (via the representation of transfer functions and inner-outer factorizations; see [9]) and minimax game theory.

The lack of assumed finite-dimensionality means that $\sigma(A)$ could very well be a substantial (connected) subset of $\mathbf{C}$, or subset of the closed unit disk $\overline{\mathbf{D}}$ intersecting the boundary $\partial\mathbf{D}$. In general, this stops us from using tools such as matrix algebra or spectral projections. To fill this gap, we resort to certain spectral factorizations whose existence would sometimes be implied by, for example, finite dimensionality or power stability of the system. In this paper we shall concentrate upon equivalence results that are dimension independent, and not upon any particular existence results implied by finite-dimensionality.

2 Critical control

For each $x_0 \in \operatorname{dom}(\mathcal{C})$ the critical control $\tilde{u}^{crit}(x_0)$ is a saddle point of the cost functional $J(x_0, \tilde{u})$ as a mapping from $\ell^2(\mathbf{Z}_+; U)$ onto $\mathbf{R}$. We may have many critical controls, or none at all. If we assume that the Popov operator $\pi_+\mathcal{D}^* J \mathcal{D}\pi_+$ has a bounded inverse in $\ell^2(\mathbf{Z}_+; U)$ (i.e. $\mathcal{D}$ is J-coercive), we have:

Lemma 1. *Assume that the DLS $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ is I/O-stable and J-coercive. Then the following is true:*

(i) For each $x_0 \in \operatorname{dom}(\mathcal{C})$ there is a unique critical control $\tilde{u}^{crit}(x_0)$.

(ii) The critical control satisfies

$$\tilde{u}^{crit}(x_0) = \mathcal{K}^{crit} x_0, \tag{3}$$

where

$$\mathcal{K}^{crit} := -(\bar{\pi}_+\mathcal{D}^* J \mathcal{D}\bar{\pi}_+)^{-1}\bar{\pi}_+\mathcal{D}^* J \mathcal{C}$$

is a linear operator $\operatorname{dom}(\mathcal{C}) \to \ell^2(\mathbf{Z}_+; U)$, called the critical (closed loop) feedback operator.

The critical output is defined by $\tilde{y}^{crit}(x_0) = \mathcal{C}^{crit} x_0$, where $\mathcal{C}^{crit} := \mathcal{C} + \mathcal{D}\mathcal{K}^{crit}$. The conjugate symmetric sesquilinear form on $\operatorname{dom}(\mathcal{C}) \times \operatorname{dom}(\mathcal{C})$ defined by

$$P^{crit}(x_0, x_1) := \langle \mathcal{C}^{crit} x_0, J\mathcal{C}^{crit} x_1 \rangle_{\ell^2(\mathbf{Z}_+;Y)}$$

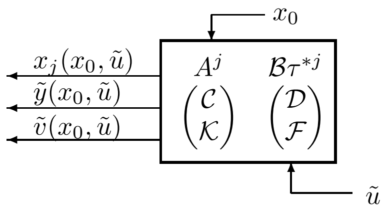
is called *the critical sesquilinear form.* Trivially $J(x_0, \tilde{u}^{crit}(x_0)) = P^{crit}(x_0, x_0)$.

3 Feedback connection

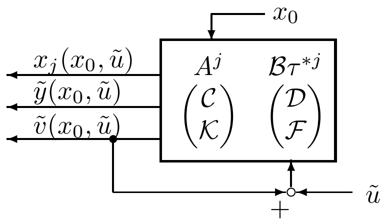
Sometimes we can write the critical control of Φ is a feedback form. Before we give the details we review what we mean by feedback.

We realize the state feedback by first adding a new equation $u_j = Kx_j$ to equations (1), where $K \in \mathcal{L}(U)$. This gives us an *extended DLS ϕ*. We get the *closed loop DLS $\phi_\diamond$* in difference equation form by simple manipulation. However, we need the same DLS in I/O -form.

In I/O -form, the new output signal given by K provides a new output $\tilde{v} \in \ell^2(\mathbf{Z}_+; U)$ to Φ, thus giving an *(open loop) extended DLS $\Phi^{ext} = [\Phi, [\mathcal{K}, \mathcal{F}]]$*. This is a cartesian product of two DLS's with the same input and semigroup structure, as presented in the following picture:



The ordered pair of operators $[\mathcal{K}, \mathcal{F}]$ is called a *feedback pair* of Φ. Here $\mathcal{K}$ is a valid observability map and $\mathcal{F}$ is a valid I/O-map for the system with semigroup generator A and controllability map $\mathcal{B}$. We require that $\operatorname{dom}(\mathcal{C}) \subset \operatorname{dom}(\mathcal{K})$, and $(\mathcal{I} - \mathcal{F})^{-1}$ is bounded and causal. $[\mathcal{K}, \mathcal{F}]$ is I/O-stable if, in addition, $\mathcal{F}$ is bounded. *The closed loop extended DLS $\Phi^{ext}_\diamond$* is the DLS that we obtain when we close the following state feedback connection:



For further results of discrete time feedback systems and their stability concepts, see [6].

The motivation for the following definition is clear.

Definition 2. *Assume that there is an I/O-stable feedback pair $[\mathcal{K}, \mathcal{F}]$ for Φ such that the extended system $\Phi^{ext} := \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \left(\begin{smallmatrix}\mathcal{C}\\ \mathcal{K}\end{smallmatrix}\right) & \left(\begin{smallmatrix}\mathcal{D}\\ \mathcal{F}\end{smallmatrix}\right) \end{smallmatrix}\right]$ has the following properties:*

(i) Both the extended DLS Φ^{ext} and the closed loop extended DLS $\Phi^{ext}_\diamond$ are I/O-stable.

(ii) With initial value $x_0 \in \operatorname{dom}(\mathcal{C})$ and zero input, $\Phi^{ext}_\diamond$ outputs the critical state sequence $\{x_j^{crit}(x_0)\}_{j\geq 0}$, critical output $\tilde{y}^{crit}(x_0)$ and critical control $\tilde{u}^{crit}(x_0)$ of the original system Φ.

Then we say that the critical control of Φ with cost functional $J(,)$ is of feedback form with the critical feedback pair $[\mathcal{K}, \mathcal{F}]$.

4 (J, S)-inner-outer factorization

We define a factorization of the I/O map that takes into account the interaction of the DLS Φ and the cost functional induced by J.

Definition 3. *The pair of operators* $(\mathcal{N}, \mathcal{M})$ *is a* (J, S) *-inner-outer factorization of* $\mathcal{D}$*, if the following conditions hold:*

(i) $\mathcal{N} \in \mathcal{L}(\ell^2(\mathbf{Z}; U), \ell^2(\mathbf{Z}; Y))$ *and* $\mathcal{M} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ *are causal shift invariant operators,*

(ii) $\mathcal{N}$ *is* (J, S) *-inner, i.e.* $\mathcal{N}^* J \mathcal{N} = S$*, where* $S = S^* \in \mathcal{L}(U)$*.*

(iii) $\mathcal{M}$ *is outer, i.e.* $\mathcal{M}^{-1} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$ *is causal shift invariant.*

(iv) $\mathcal{D} = \mathcal{N}\mathcal{M}^{-1}$*.*

The calculation of (J, S)-inner-outer factorization is a spectral factorization problem. $\mathcal{M}^{-1}$ is the S-spectral factor of $\mathcal{D}^* J \mathcal{D}$.

5 Riccati equation

Definition 4. *Let* $J \in \mathcal{L}(Y)$ *be self adjoint, and let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O-stable DLS. We say that the conjugate symmetric sesquilinear form* $P(\cdot, \cdot)$ *on* $\mathrm{dom}\,(\mathcal{C}) \times \mathrm{dom}\,(\mathcal{C})$ *satisfies the Riccati equation system (associated to* J *and* Φ*), if*

$$P(Ax_0, Ax_1) - P(x_0, x_1) + (C^* J C x_0, x_1)_H \tag{4}$$
$$= \left\langle Q_P^* \Lambda_P^{-1} Q_P x_0, x_1 \right\rangle_H,$$
$$\langle \Lambda_P u_0, u_1 \rangle = \langle D^* J D u_0, u_1 \rangle_U + P(Bu_0, Bu_1), \tag{5}$$
$$\langle Q_P x_1, u_2 \rangle = -\langle D^* J C x_1, u_2 \rangle_U - P(Ax_1, Bu_2) \tag{6}$$

for all $u_0, u_1, u_2 \in U$ *and* $x_0, x_1, x_2 \in \mathrm{dom}\,(\mathcal{C})$*. where the linear operators satisfy* $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ *and* $Q_P \in \mathcal{L}(H; U)$*.*

In particular the critical sesquilinear form $P^{crit}(\cdot, \cdot)$ satisfies the Riccati equation system. Note that if Φ is output stable ($\mathcal{C}$ bounded and $\mathrm{dom}\,(\mathcal{C}) = H$), then we could have stated the Riccati equation system as the familiar algebraic Riccati equation, and the critical sesquilinear form as a self adjoint $P^{crit} \in \mathcal{L}(H)$.

Also a converse is true: from certain solutions of the Riccati equation system we can recover a critical control feedback pair $[\mathcal{K}, \mathcal{F}]$ as stated in Theorem 6. For this end we have to define a concept of *indicator DLS* associated to a solution $P(\cdot, \cdot)$ of the Riccati equation system.

Definition 5. *Let* $J \in \mathcal{L}(Y)$ *be self adjoint, and let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O-stable DLS. Let* $P(\cdot, \cdot)$ *be a solution of the Riccati equation system (4)* - - (6). Then the DLS

$$\phi_P := \begin{pmatrix} A & B \\ -Q_P & \Lambda_P \end{pmatrix} \tag{7}$$

is called the indicator DLS ϕ_P (associated to J and Φ) of the sesquilinear form $P(\cdot, \cdot)$, where the bounded linear operators Q_P, Λ_P are a in Definition 4.

6 Equivalence theorem

Now we are ready to present the main results of this paper.

Theorem 6. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O-stable DLS, and* $J \in \mathcal{L}(Y)$ *be self adjoint. Then the following conditions (i), (ii) and (iii) are equivalent:*

(i) *a)* Φ *is J-coercive.*

b) There is an I/O-stable feedback pair $[\mathcal{K}, \mathcal{F}]$ *for* Φ *such that the critical control of* Φ *is of feedback form with the critical feedback pair* $[\mathcal{K}, \mathcal{F}]$*.*

(ii) *a) There is a boundedly invertible operator* $S \in \mathcal{L}(U)$ *such that* $\mathcal{D}$ *has a* (J, S)*-inner-outer factorization.*

b) $\pi_0 \mathcal{N}^* J \mathcal{C} \in \mathcal{L}(H; U)$*.*

(iii) There is a solution $P(\cdot, \cdot)$ *of the Riccati equation system (4)* - - (6) such that

a) the indicator DLS ϕ_P is both I/O-stable and outer,

b) $P(x_j, x_j) \to 0$ for all trajectories $\{x_j(x_0, \tilde{u})\}$ of Φ, with $x_0 \in \mathrm{dom}\,(\mathcal{C})$, $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$.

To say that the indicator DLS ϕ_P is outer means that the I/O-map of ϕ_P is outer. For proof, see [5]. Note that in this case the I/O-map $\mathcal{D}_{\phi_P}$ is the Λ_P^{-1} -spectral factor of $\mathcal{D}$, the I/O -map of the original system.

For finite dimensional, power stable partial analogues of equivalence (ii) $\Leftrightarrow$ (iii), see e.g. [4], [7] and [8]. For the equivalence (i) $\Leftrightarrow$ (ii), see also [2]. For infinite dimensional, power stable and time-variant analogue of Theorem 6, see [1, Theorems 3.2.8 and 3.2.10]. See also [3].

7 Conclusions

In this section we briefly look at the connections to the existing power stable and finite dimensional theories. Let us look at the implications of condition (iii) in Theorem 6 with this stronger stability assumption.

Lemma 7. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a power stable DLS, and* $J \in \mathcal{L}(Y)$ *be self adjoint. Assume that the Riccati equation system (4)* - - (6) has a solution $P(\cdot, \cdot)$ satisfying:

(i) $P(x_0, x_1,) = \langle x_0, P x_1 \rangle_H$ for a self adjoint $P \in \mathcal{L}(H)$,

(ii) $\Lambda_P^{-1} \in \mathcal{L}(U)$,

(iii) The closed loop semigroup generator $A + B\Lambda_P^{-1} Q_P$ is power stable, i.e. P is a (power) stabilizing solution of the Riccati equation system.

Then the conditions of Theorem 6 are satisfied.

So in the power stable case everything reduces to showing the existence of a *power* stabilizing solution for the Riccati equation. Note that now the Riccati equation can be written for bounded linear operators P rather than for conjugate symmetric sesquilinear forms $P(\cdot,\cdot)$. Assume now that the spaces U, H and Y are finite dimensional. The procedure of finding a power stabilizing solution is a central task in the matrix Riccati equation theory (see [8]). Under weak conditions, one can constructively prove the existence of a maximal self adjoint solution P_+ satisfying $P_+ \geq P$ for all other self adjoint solutions of the Riccati equation. P_+ is in fact an "almost power stabilizing" solution in the sense that $\rho(A + B\Lambda_P^{-1} Q_P) \leq 1$. For a positive definite cost functional J, the existence of the power stabilizing solution can be proved. For details, see [4, Theorems 13.1.1 and 13.5.2].

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