Cauchy problems from networks of passive boundary control systems

Atte Aalto and Jarmo Malinen

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Abstract

We show under mild assumptions that a composition of internally well-posed, impedance passive (or conservative) boundary control systems through Kirchhoff type connections is also an internally well-posed, impedance passive (resp., conservative) boundary control system. The proof is based on results of [20]. We also present two examples of such compositions involving Webster's equation and the Timoshenko beam equation.

Keywords: *Boundary control, passive system, distributed parameter system, well-posedness, composition, Cauchy problem*

1 Introduction

We treat the solvability (forward in time) of dynamical boundary control systems that are composed by interconnecting a finite number of more simple boundary control *subsystems* that are already known to be solvable forward in time. The interconnections are given in terms of algebraic equations satisfied by the boundary control/observation operators of the subsystems. The aggregate formed by the subsystems and their interconnections is called *a transmission graph* (see Definition 3.1), and it can be seen as a generalisation of mathematical transmission lines and networks. We assume throughout this work that all the subsystems are passive or conservative as described in, *e.g.*, [8], [16], [19], [20], [23], [24], and [25], and they are represented by equations of the form (5) below involving *strong boundary nodes*. Moreover, the interconnections respect passivity in the sense that they do not create energy. In Theorem 3.4 — the main result of this paper — we give conditions for checking the solvability (*i.e.*, internal well-posedness) and passivity of the transmission graph in terms of simple conditions on the subsystems and interconnections.

To illuminate the purpose of this paper, let us consider the following example from acoustic wave propagation. Given the interconnection graph in Fig. 1, the longitudinal wave propagation on its edges (*i.e.*, wave guides) is governed by

$$\frac{\partial^2 \psi^{(j)}}{\partial t^2}(x,t) = c^2 \frac{\partial^2 \psi^{(j)}}{\partial x^2}(x,t), \qquad x \in [0, L_j], \text{ and } t \in \mathbb{R}^+. \tag{1}$$

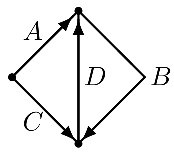


Figure 1: The example graph

Here the index $j = A, ..., D$ refers to the index of the edge, and the arrows in Fig. 1 show the positive direction of the parametrisation $x \in [0, L_j]$. To the vertices ABD and BCD we impose Kirchhoff law type coupling (boundary) conditions (take vertex ABD for example):

$$\begin{cases} \frac{\partial \psi^{(A)}}{\partial t}(L_A, t) = \frac{\partial \psi^{(B)}}{\partial t}(0, t) = \frac{\partial \psi^{(D)}}{\partial t}(L_D, t), \\ A_A \frac{\partial \psi^{(A)}}{\partial x}(L_A, t) - A_B \frac{\partial \psi^{(B)}}{\partial x}(0, t) + A_D \frac{\partial \psi^{(D)}}{\partial x}(L_D, t) = 0. \end{cases} \tag{2}$$

We remark that in acoustics applications the state $\psi^{(j)}$ is chosen to be a velocity potential; then $p^{(j)} = \rho \frac{\partial \psi^{(j)}}{\partial t}$ gives the perturbation pressure and $v^{(j)} = -\frac{\partial \psi^{(j)}}{\partial x}$ gives the perturbation velocity for each edge. Thus, the first equation in (2) says that the pressure is continuous, and the second equation is a flow conservation law (the weights A_j can be understood as the cross-sectional areas of the wave guides).

We want to control the pressure at the vertex AC and observe the perturbation flux to the wave guides A and C. Defining the input and output

$$\begin{cases} u(t) := & \frac{\partial \psi^{(A)}}{\partial t}(0, t) = \frac{\partial \psi^{(C)}}{\partial t}(0, t), \\ y(t) := & -A_A \frac{\partial \psi^{(A)}}{\partial x}(0, t) - A_C \frac{\partial \psi^{(C)}}{\partial x}(0, t), \end{cases} \tag{3}$$

respectively, then equations (1) for $j = A, ..., D$ and (2) define a dynamical system whose solvability and energy conservation we wish to verify using Theorem 3.4 below.

We must consider first the solvability of the subsystems, that is, equations (1) on the edges with boundary conditions

$$\begin{bmatrix} \frac{\partial \psi^{(j)}}{\partial t}(0, t) \\ \frac{\partial \psi^{(j)}}{\partial t}(L_j, t) \end{bmatrix} = \begin{bmatrix} u_1^{(j)}(t) \\ u_2^{(j)}(t) \end{bmatrix} =: u^{(j)}(t). \tag{4}$$

After reducing (1) to a first order equation of form $\dot{z} = Lz$ with $z = \left[\begin{smallmatrix} \psi^{(j)} \\ p^{(j)} \end{smallmatrix} \right]$, defining operator G by (4), that is, by $Gz(t) = u^{(j)}(t)$, and K in a similar manner, we obtain *an internally well-posed boundary node* $\Xi^{(j)} = (G, L, K)$ that is *impedance conservative*, see Definitions 2.2 and 2.3. As explained after

Definition 2.2, the initial value problem

$$\begin{array}{rcl} u(t) & = & Gz(t), \\ \dot{z}(t) & = & Lz(t), \\ y(t) & = & Kz(t), \qquad t \in \mathbb{R}^+, \\ z(0) & = & z_0 \end{array} \tag{5}$$

has a solution such that $\psi^{(j)}$ in equation (1) satisfies $\psi^{(j)} \in C^1(\mathbb{R}^+, L^2(0, L_j)) \cap C(\mathbb{R}^+, H^1(0, L_j))$ for all inputs $u^{(j)} \in C^2(\mathbb{R}^+, \mathbb{C}^2)$ and for all initial states $\psi^{(j)}(0)$ that satisfy the boundary condition (4), too. For technical details, see the (more general) example of Webster's equation presented in Section 5.1.

Now we have boundary nodes $\Xi^{(j)}$, $j = A, ..., D$ and coupling conditions of the form (2) for all vertices except the one that defines the external input and output through (3). They form a transmission graph as defined in Definition 3.1. Since the components $\Xi^{(j)}$ are solvable and conservative, then by Theorem 3.4, also the resulting composed system is solvable forward in time and conservative in a similar way as any of its components.

Let us review the most relevant literature on compositions of (boundary control) systems. The feedback theory for (regular) well-posed linear systems is treated in [25: Chapter 7] and [28] whose concept of admissibility of the feedback loops is related to the (internal) well-posedness of the composed system, but the theory can be used only when well-posedness of the components is verified by other means.

Transport equation on graphs is studied in [5] by using semigroup techniques. Compositions of PDEs on 1D spatial domains are treated in [27] in terms of port-Hamiltonian framework. Compositions of more general systems are studied in, *e.g.*, [2] and [14] who treat systems that give raise to Dirac structures on their state spaces (see also [4]). These contain impedance conservative, strong boundary control systems (as characterised in Definitions 2.2 and 2.3) as a special case. However, our approach is based on results of [19; 20] that are reviewed in Section 2, and we are able to treat couplings of both passive and conservative systems at once.

Further practical examples of compositions of PDEs with 1D spatial domains include semiconductor strips and lattice structures constructed of Timoshenko beams. Such systems have also been studied from the spectral point of view: asymptotic spectral properties of the Laplacian are studied in [12] and [22] when its "graph-like" 3D spatial domain collapses to a graph with 1D edges. See also [15] for the spectral properties of the Sturm-Liouville equation on a Y-shaped graph.

We present two concrete examples of transmission graphs in Section 5, namely the human vocal tract (modelled by Webster's equation on a Y-shaped graph) and a lattice structure composed of homogeneous Timoshenko beams (see [18] for more examples of passive boundary control systems). In the latter example we also consider partial connections where some degrees of freedom are not transmitted over the boundary coupling. This happens when the beams are connected by a hinge which cannot convey moment to the direction of the axle of the hinge.

2 Background

In this work we treat linear *boundary control systems* described by operator differential equations of the form (5) involving linear mappings G, L, and K:

Definition 2.1. *Let $\Xi := (G, L, K)$ be a triple of linear mappings.*

(i) Ξ is a colligation *on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ if G, L, and K have the same domain $\mathcal{Z} = \mathrm{dom}(\Xi) \subset \mathcal{X}$ and values in $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$, respectively;*

(ii) A colligation Ξ is strong *if $\begin{bmatrix} G \\ L \\ K \end{bmatrix}$ is closed as an operator $\mathcal{X} \to \begin{bmatrix} \mathcal{U} \\ \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$ with domain $\mathcal{Z}$, and L is closed with $\mathrm{dom}(L) = \mathcal{Z}$.*

We call L the interior operator*, G the* input (boundary) operator*, and K the* output (boundary) operator*. The space $\mathcal{Z}$ we call the* solution space*, $\mathcal{X}$ the* state space*, and $\mathcal{U}$ and $\mathcal{Y}$ the* input *and* output spaces*, respectively. In $\mathcal{Z}$ we use the graph norm of $\begin{bmatrix} G \\ L \\ K \end{bmatrix}$ in $\mathrm{dom}(\Xi)$.*

In this paper we use the notations $\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$ and $\bigoplus$ to represent orthogonal direct sum of (sub)spaces. See also Remark 6.4 for a discussion on the terms input and output.

Many dynamical systems defined by boundary controlled partial differential equations naturally adopt the form (5) associated with some colligation (G, L, K) on properly chosen spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$, see examples in Section 5. Equations (5) are solvable forward in time (at least) if Ξ satisfies somewhat stronger assumptions:

Definition 2.2. *A strong colligation $\Xi = (G, L, K)$ is a* boundary node *on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ if the following conditions are satisfied:*

(i) G is surjective and $\mathcal{N}(G)$ is dense in $\mathcal{X}$;

(ii) The operator $L|_{\mathcal{N}(G)}$ (interpreted as an operator in $\mathcal{X}$ with domain $\mathcal{N}(G)$) has a nonempty resolvent set.

This boundary node is internally well-posed *(in the forward time direction) if, in addition,*

(iii) $L|_{\mathcal{N}(G)}$ generates a C_0 semigroup.

This definition coincides with [19: Definition 1.1] for strong colligations. There are, in fact, well-posed boundary nodes that are not strong (see [20: Proposition 6.3]) but we do not consider such nodes in this paper[1]. We remark that also [6], [8], and [14] treat strong colligations (with different names), see [20: Theorem 5.2] and [14: Remark 4.4].

If $\Xi = (G, L, K)$ is an internally well-posed boundary node, then (5) has a unique solution for sufficiently smooth input functions u and initial states z_0

[1]To avoid confusion, we shall use the term strong boundary node below.

compatible with $u(0)$. More precisely, as shown in [19: Lemma 2.6], for all $z_0 \in \mathcal{Z}$ and $u \in C^2(\mathbb{R}^+;\mathcal{U})$ with $Gz_0 = u(0)$ the first, second, and fourth of the equations in (5) have a unique solution $z \in C^1(\mathbb{R}^+;\mathcal{X}) \cap C(\mathbb{R}^+;\mathcal{Z})$, and hence we can define $y \in C(\mathbb{R}^+;\mathcal{Y})$ by the third equation in (5). In the rest of this article, when we say "a smooth solution of (5) on $\mathbb{R}^+$" we mean a solution with the above properties.

In a practical application, checking the solvability of (5), that is, verifying the conditions of Definition 2.2 may be difficult. However, in many cases this is not necessary because the system satisfies energy (in)equalities that can be verified using the Green–Lagrange inequality without an *a priori* knowledge of the well-posedness. Such energy laws make it easier to check the solvability, see Proposition 2.4 below. First we shall define impedance passivity/conservativity. To keep the notation simple, we assume that $\mathcal{U} = \mathcal{Y}$ even though it would be enough to assume that $\mathcal{U}$ and $\mathcal{Y}$ are a dual pair of Hilbert spaces with duality pairing $\langle \cdot, \cdot \rangle_{(\mathcal{Y},\mathcal{U})}$; see [20: Definition 3.6] and the discussion preceding it.

Definition 2.3. *Let $\Xi = (G, L, K)$ be a colligation on Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$.*

(i) Ξ is impedance passive *if the following conditions hold:*

(a) $\begin{bmatrix} \beta G + K \\ \alpha - L \end{bmatrix}$ *is surjective for some $\alpha, \beta \in \mathbb{C}^+$;*

(b) For all $z \in \mathrm{dom}(\Xi)$ we have the Green–Lagrange inequality

$$\mathrm{Re}\langle z, Lz \rangle_{\mathcal{X}} \le \langle Kz, Gz \rangle_{\mathcal{U}}. \tag{6}$$

(ii) Impedance passive Ξ is impedance conservative *if* (6) *holds as an equality, and (a) holds also for some $\alpha, \beta \in \mathbb{C}^-$.*

Impedance passivity/conservativity is defined in [20: Definition 3.2] using the external Cayley transform of scattering passivity/conservativity (see also the discussion there). These definitions are equivalent by [20: Theorem 3.4]. We further remark that [20: Theorem 3.4] also states that for an impedance passive Ξ, condition (a) holds for all $\alpha, \beta \in \mathbb{C}^+$, and for an impedance conservative Ξ, condition (a) holds also for all $\alpha, \beta \in \mathbb{C}^-$.

Suppose now that Ξ is an internally well-posed, impedance passive boundary node and z a smooth solution of (5). Then (6) means plainly the energy inequality

$$\frac{d}{dt}\|z(t)\|^2_{\mathcal{X}} \le \langle y(t), u(t) \rangle_{\mathcal{U}} \quad \text{for all } t \in \mathbb{R}^+$$

where the right hand side stands for the instantaneous power inflicting the system, and the norm of $\mathcal{X}$ measures the energy stored in the state.

The following proposition utilising the energy balance laws is needed for checking internal well-posedness and impedance passivity/conservativity.

Proposition 2.4. *Let $\Xi = (G, L, K)$ be a strong colligation on Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{U})$.*

(i) Suppose that (6) *holds for all* $z \in \operatorname{dom}(\Xi)$*, and that* $\left[\begin{smallmatrix} G \\ \alpha - L \end{smallmatrix}\right]$ *is surjective for some* $\alpha \in \mathbb{C}$ *with* $\operatorname{Re}(\alpha) \geq 0$*. Then* Ξ *is an internally well-posed, impedance passive boundary node. If, in addition,* (6) *holds as an equality and* $\left[\begin{smallmatrix} G \\ \alpha - L \end{smallmatrix}\right]$ *is surjective also for some* $\operatorname{Re}(\alpha) \leq 0$*, then the internally well-posed boundary node* Ξ *is impedance conservative.*

(ii) If Ξ *is impedance passive, then it is an internally well-posed boundary node if and only if its input operator* G *is surjective.*

For a proof, see [20: Theorem 4.3 and Remark 4.6] for part (i) and [20: Theorem 4.7] for part (ii).

Internally well-posed boundary nodes can always be written in terms of more general and complicated *system nodes* (see [19], [21], and [25]) but they are excluded from *state linear systems* studied in [3]. A functional analytic setting of boundary control systems, that is independent of the system node setting, was formulated in [6] and significant progress was made in [23] and [24]. See also [9] for a similar presentation.

3 Transmission graphs as colligations

Assume that we have colligations $\Xi^{(j)} = \left(G^{(j)}, L^{(j)}, K^{(j)}\right)$ on Hilbert spaces $\left(\mathcal{U}^{(j)}, \mathcal{X}^{(j)}, \mathcal{Y}^{(j)}\right)$ with solution spaces $\mathcal{Z}^{(j)}$, $j = 1, ..., m$, where

$$
\begin{aligned}
G^{(j)} &= \begin{bmatrix} G_1^{(j)} \\ \vdots \\ G_{k_j}^{(j)} \end{bmatrix} : \operatorname{dom}(\Xi^{(j)}) \to \mathcal{U}^{(j)} = \begin{bmatrix} \mathcal{U}_1^{(j)} \\ \vdots \\ \mathcal{U}_{k_j}^{(j)} \end{bmatrix} \quad \text{and} \\
K^{(j)} &= \begin{bmatrix} K_1^{(j)} \\ \vdots \\ K_{k_j}^{(j)} \end{bmatrix} : \operatorname{dom}(\Xi^{(j)}) \to \mathcal{Y}^{(j)} = \begin{bmatrix} \mathcal{Y}_1^{(j)} \\ \vdots \\ \mathcal{Y}_{k_j}^{(j)} \end{bmatrix}.
\end{aligned}
$$

That is, the Hilbert spaces $\mathcal{U}^{(j)}$ and $\mathcal{Y}^{(j)}$ are represented by an orthogonal direct sum of k_j subspaces each, and the corresponding input and output operators are split accordingly. Following this splitting, we define the index set

$$
\mathcal{I}nd := \left\{(j,i) \in \mathbb{N} \times \mathbb{N} \;\middle|\; j = 1, ..., m;\ i = 1, ..., k_j\right\} = \bigcup_{k=1}^{N} \mathcal{I}^k \ \cup \ \bigcup_{l=1}^{M} \mathcal{J}^l
$$

where the sets $\mathcal{I}^1, ..., \mathcal{I}^N$ and $\mathcal{J}^1, ..., \mathcal{J}^M$ are pairwise disjoint. The sets $\mathcal{I}^k$ and $\mathcal{J}^l$ define a graph structure where inputs and outputs of nodes $\Xi^{(j)}$ are coupled by algebraic equations (8) and (9) below. In order to make the couplings possible, we require that the compatibility conditions

$$
\mathcal{U}_i^{(j)} = \mathcal{U}_q^{(p)} \quad \text{and} \quad \mathcal{Y}_i^{(j)} = \mathcal{Y}_q^{(p)} \tag{7}
$$

hold for all $(j,i),(p,q) \in \mathcal{I}^k,\ k = 1,...,N$ and for all $(j,i),(p,q) \in \mathcal{J}^l,\ l = 1,...,M$. Each of the sets $\mathcal{I}^k$ or $\mathcal{J}^l$ describes individual couplings of signals, and we name the sets *control* and *closed* vertices, respectively.

Definition 3.1. *Assume that $\Xi^{(j)}$ are colligations with splittings as described above. Suppose that sets $\mathcal{I}^1,...,\mathcal{I}^N$ and $\mathcal{J}^1,...,\mathcal{J}^M$ are defined consistently with this splitting so that the compatibility conditions* (7) *hold.*

The ordered triple

$$\Gamma := \left(\left\{ \Xi^{(j)} \right\}_{j=1}^{m}, \left\{ \mathcal{I}^k \right\}_{k=1}^{N}, \left\{ \mathcal{J}^l \right\}_{l=1}^{M} \right)$$

is a transmission graph *with (Kirchhoff) couplings*

(i) for all control and closed vertices, the continuity equations

$$G_i^{(j)} z^{(j)} = G_q^{(p)} z^{(p)} \qquad \text{for } z^{(j)} \in \mathcal{Z}^{(j)} \text{ and } z^{(p)} \in \mathcal{Z}^{(p)} \tag{8}$$

hold, i.e., (8) *holds for all $(j,i),(p,q) \in \mathcal{I}^k,\ k = 1,...,N$ and for all $(j,i),(p,q) \in \mathcal{J}^l,\ l = 1,...,M$; and*

(ii) for closed vertices, also the flow conservation equations

$$\sum_{(j,i)\in\mathcal{J}^l} K_i^{(j)} z^{(j)} = 0 \qquad \text{for } z^{(j)} \in \mathcal{Z}^{(j)} \text{ and } l = 1,...,M \tag{9}$$

hold.

For a general transmission graph, the set of control vertices is nonempty. Control vertices are exactly those couplings where external signals are applied. If the transfer function (see [19: Section 2]) of each $\Xi^{(j)}$ represents electrical admittance, then the physical dimensions of $\mathcal{U}^{(j)}$ and $\mathcal{Y}^{(j)}$ are the voltage and current, respectively. Equations (8) and (9) are the classical Kirchhoff laws, namely, the continuity of voltage and the conservation of charge.

Definition 3.2. *Let Γ be a transmission graph as in Definition 3.1. Using the same notation, we define the* colligation of the transmission graph *as the triple $\Xi_\Gamma = (G,L,K)$ on the Hilbert spaces $(\mathcal{U},\mathcal{X},\mathcal{Y})$ where*[2]

$$\mathcal{X} := \bigoplus_{j=1}^{m} \mathcal{X}^{(j)}, \qquad \mathcal{U} := \bigoplus_{\substack{(j,i)\in\mathcal{I}^k \\ k=1,...,M}} \mathcal{U}_i^{(j)}, \qquad \mathcal{Y} := \bigoplus_{\substack{(j,i)\in\mathcal{I}^k \\ k=1,...,M}} \mathcal{Y}_i^{(j)},$$

$$\mathrm{dom}(\Xi_\Gamma) := \left\{ \bigoplus_{j=1}^{m} \mathcal{Z}^{(j)} \;\middle|\; (8) \text{ and } (9) \text{ hold} \right\},$$

[2]In sums of $\mathcal{U}$ and $\mathcal{Y}$, pick *one* pair $(j,i) \in \mathcal{I}^k$ for each k.

$$G := [G_{k,j}]_{\substack{k=1,\dots,N \\ j=1,\dots,m}}, \quad L := \begin{bmatrix} L^{(1)} & & \\ & \ddots & \\ & & L^{(m)} \end{bmatrix}, \text{ and } K := [K_{k,j}]_{\substack{k=1,\dots,N \\ j=1,\dots,m}}$$

where

$$G_{k,j} := \begin{cases} G_k^{(j)}/|\mathcal{I}^k|, & \text{if } (j,k) \in \mathcal{I}^k, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad K_{k,j} := \begin{cases} K_k^{(j)}, & \text{if } (j,k) \in \mathcal{I}^k, \\ 0, & \text{otherwise.} \end{cases}$$

In order to make the preceding definitions more intuitive, let us return to the example on the wave equation on the graph of Fig. 1, presented in the introduction.

Example 3.3. We have four boundary nodes $\Xi^{(j)}$, $j = A, ..., D$ whose input and output spaces are split into two parts, see equation (4). Thus, our index set is

$$\mathcal{I}nd = \left\{(j,i) \;\middle|\; j = A, ..., D,\ i = 1, 2\right\}.$$

We have one control vertex $\mathcal{I}^1 = \{(A,1),(C,1)\}$ and two closed vertices $\mathcal{J}^1 = \{(A,2),(B,1),(D,2)\}$ and $\mathcal{J}^2 = \{(B,2),(C,2),(D,1)\}$.

The dynamical system given by (1), (2), and (3) corresponds to the colligation of the transmission graph $\Gamma := \left(\left\{\Xi^{(j)}\right\}_{j=A}^{D}, \left\{\mathcal{I}^1\right\}, \left\{\mathcal{J}^1, \mathcal{J}^2\right\}\right)$. More precisely, equations in (2) are equivalent with (8) and (9) and the input and output operators given in Definition 3.2 yield the input/output of equation (3).

The main result of this paper is the following:

Theorem 3.4. *Assume that the transmission graph Γ is composed of internally well-posed, impedance passive (conservative), strong boundary nodes $\Xi^{(j)} = \left(G^{(j)}, L^{(j)}, K^{(j)}\right)$ with the following property:*

$$\text{all of the operators } \begin{bmatrix} G^{(j)} \\ K^{(j)} \end{bmatrix} \text{ are surjective.} \tag{10}$$

Then the colligation of Γ is an impedance passive (resp., conservative), internally well-posed, strong boundary node.

This is proved in three steps (Lemmas 4.1, 4.2, and 4.3) presented in the following section. The assumption (10) can be relaxed (see Remark 6.2) but this condition appears to hold in many applications (as in both of our examples in Section 5).

4 Proof of Theorem 3.4

Suppose we are given a transmission graph Γ. We reconstruct this graph by a finite number of three different kinds of steps, starting from its components $\Xi^{(j)}$. In step 1, we form a partial parallel connection between two compatible colligations to obtain a new colligation, see Fig. 2a. We remark that such parallel connections are treated in [25: Examples 2.3.13 and 5.1.17] for system nodes.

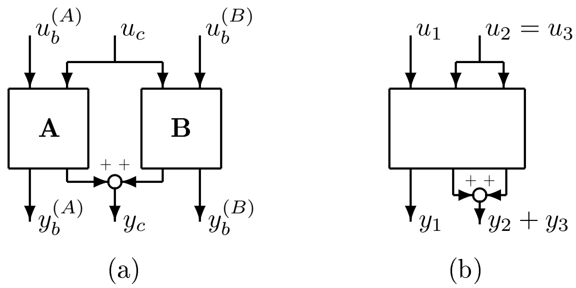


Figure 2: (a) The partial parallel coupling; (b) The loop coupling

In step 2, we form loops by joining two signals of a single colligation to obtain a new colligation, see Fig. 2b. Both the control vertices and the closed vertices are treated similarly at this stage: all the vertices are left "open" so that (8) is satisfied but (9) is not. After constructing the full coupling graph structure by taking a finite number of steps 1 and 2 in some order, the final step 3 is taken to close those vertices that are not used for control/observation; then condition (9) is satisfied, too. The transmission graph Γ and its colligation have now been reconstructed, and the remaining (open) vertices are exactly the control vertices of Γ.

By this procedure, it is possible to synthesise any transmission graph. In Lemmas 4.1, 4.2, and 4.3, we show that if we start from internally well-posed, impedance passive/conservative strong boundary nodes, then the resulting colligations after steps 1, 2, and 3 (respectively) are internally well-posed, impedance passive/conservative, strong boundary nodes as well. This is required for iterated application of these steps in order to prove Theorem 3.4. The reconstruction procedure is demonstrated in Section 4.4 by using the graph of Fig. 1.

4.1 Step 1: partial parallel coupling

Assume that we have two colligations $\Xi^{(A)} = \left(\begin{bmatrix} G_b^{(A)} \\ G_c^{(A)} \end{bmatrix}, L^{(A)}, \begin{bmatrix} K_b^{(A)} \\ K_c^{(A)} \end{bmatrix} \right)$ and $\Xi^{(B)} = \left(\begin{bmatrix} G_b^{(B)} \\ G_c^{(B)} \end{bmatrix}, L^{(B)}, \begin{bmatrix} K_b^{(B)} \\ K_c^{(B)} \end{bmatrix} \right)$ on Hilbert spaces $\left(\begin{bmatrix} \mathcal{U}_b^{(A)} \\ \mathcal{U}_c \end{bmatrix}, \mathcal{X}^{(A)}, \begin{bmatrix} \mathcal{Y}_b^{(A)} \\ \mathcal{Y}_c \end{bmatrix} \right)$ and $\left(\begin{bmatrix} \mathcal{U}_b^{(B)} \\ \mathcal{U}_c \end{bmatrix}, \mathcal{X}^{(B)}, \begin{bmatrix} \mathcal{Y}_b^{(B)} \\ \mathcal{Y}_c \end{bmatrix} \right)$ with solution spaces $\mathcal{Z}^{(A)}$ and $\mathcal{Z}^{(B)}$, respectively.

Now define the composed colligation $\Xi^{(AB)} := \left(G^{(AB)}, L^{(AB)}, K^{(AB)} \right)$ on the Hilbert spaces

$$\mathcal{X}^{(AB)} := \begin{bmatrix} \mathcal{X}^{(A)} \\ \mathcal{X}^{(B)} \end{bmatrix}, \quad \mathcal{U}^{(AB)} := \begin{bmatrix} \mathcal{U}_b^{(A)} \\ \mathcal{U}_c \\ \mathcal{U}_b^{(B)} \end{bmatrix}, \quad \text{and} \quad \mathcal{Y}^{(AB)} := \begin{bmatrix} \mathcal{Y}_b^{(A)} \\ \mathcal{Y}_c \\ \mathcal{Y}_b^{(B)} \end{bmatrix}$$

by
$$L^{(AB)} := \begin{bmatrix} L^{(A)} & 0 \\ 0 & L^{(B)} \end{bmatrix},$$

$$G^{(AB)} := \begin{bmatrix} G_b^{(A)} & 0 \\ G_c^{(A)} & 0 \\ 0 & G_b^{(B)} \end{bmatrix}, \quad \text{and} \quad K^{(AB)} := \begin{bmatrix} K_b^{(A)} & 0 \\ K_c^{(A)} & K_c^{(B)} \\ 0 & K_b^{(B)} \end{bmatrix}.$$

The domain of the colligation is

$$\operatorname{dom}(\Xi^{(AB)}) := \left\{ \begin{bmatrix} z^{(A)} \\ z^{(B)} \end{bmatrix} \in \begin{bmatrix} \operatorname{dom}(\Xi^{(A)}) \\ \operatorname{dom}(\Xi^{(B)}) \end{bmatrix} \;\middle|\; G_c^{(A)} z^{(A)} = G_c^{(B)} z^{(B)} \right\}.$$

Such partial parallel coupling is illustrated in Fig. 2a. We now show that such coupling of two boundary nodes is also a boundary node and the coupling preserves internal well-posedness and passivity/conservativity.

Lemma 4.1. *Let $\Xi^{(A)}$, $\Xi^{(B)}$, and $\Xi^{(AB)}$ be as defined above. If the colligations $\Xi^{(A)}$ and $\Xi^{(B)}$ are internally well-posed, impedance passive (conservative), strong boundary nodes such that both $\left[\begin{smallmatrix} G^{(A)} \\ K^{(A)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} G^{(B)} \\ K^{(B)} \end{smallmatrix}\right]$ are surjective, then the composed colligation $\Xi^{(AB)}$ is an internally well-posed, impedance passive (resp., conservative), strong boundary node with the property that $\left[\begin{smallmatrix} G^{(AB)} \\ K^{(AB)} \end{smallmatrix}\right]$ is surjective.*

Proof. We start by showing that $\Xi^{(AB)}$ is a strong colligation. First, we show that $\Xi^{(AB)}$ is closed. Assume that $\operatorname{dom}(\Xi^{(AB)}) \ni \left[\begin{smallmatrix} z_n^{(A)} \\ z_n^{(B)} \end{smallmatrix}\right] \to \left[\begin{smallmatrix} z^{(A)} \\ z^{(B)} \end{smallmatrix}\right]$ and

$$\begin{bmatrix} G_b^{(A)} & 0 \\ G_c^{(A)} & 0 \\ 0 & G_b^{(B)} \end{bmatrix} \begin{bmatrix} z_n^{(A)} \\ z_n^{(B)} \end{bmatrix} \to \begin{bmatrix} u_b^{(A)} \\ u_c \\ u_b^{(B)} \end{bmatrix}, \quad \begin{bmatrix} L^{(A)} & 0 \\ 0 & L^{(B)} \end{bmatrix} \begin{bmatrix} z_n^{(A)} \\ z_n^{(B)} \end{bmatrix} \to \begin{bmatrix} x^{(A)} \\ x^{(B)} \end{bmatrix},$$

$$\text{and} \quad \begin{bmatrix} K_b^{(A)} & 0 \\ K_c^{(A)} & K_c^{(B)} \\ 0 & K_b^{(B)} \end{bmatrix} \begin{bmatrix} z_n^{(A)} \\ z_n^{(B)} \end{bmatrix} \to \begin{bmatrix} y_b^{(A)} \\ y_c \\ y_b^{(B)} \end{bmatrix}.$$

Since colligations $\Xi^{(A)}$ and $\Xi^{(B)}$ are strong, the operators $L^{(A)}$ and $L^{(B)}$ are closed, $\left[\begin{smallmatrix} z^{(A)} \\ z^{(B)} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \operatorname{dom}(\Xi^{(A)}) \\ \operatorname{dom}(\Xi^{(B)}) \end{smallmatrix}\right]$, and also $L^{(A)} z^{(A)} = x^{(A)}$ and $L^{(B)} z^{(B)} = x^{(B)}$. To show that $\left[\begin{smallmatrix} z^{(A)} \\ z^{(B)} \end{smallmatrix}\right] \in \operatorname{dom}(\Xi^{(AB)})$, we need to use the strongness of $\Xi^{(A)}$ and $\Xi^{(B)}$ which implies that $G_c^{(A)}$ and $G_c^{(B)}$ are continuous with respect to the graph norms of $L^{(A)}$ and $L^{(B)}$, respectively, by [20: Lemma 4.5]. Hence

$$\begin{aligned} &\|G_c^{(A)} z^{(A)} - G_c^{(B)} z^{(B)}\|_{\mathcal{U}_c} \le \|G_c^{(A)}(z^{(A)} - z_n^{(A)})\|_{\mathcal{U}_c} + \|G_c^{(B)}(z^{(B)} - z_n^{(B)})\|_{\mathcal{U}_c} \\ &\le M_A \left(\|z^{(A)} - z_n^{(A)}\|_{\mathcal{X}^{(A)}} + \|L^{(A)}(z^{(A)} - z_n^{(A)})\|_{\mathcal{X}^{(A)}} \right) + \\ &\quad + M_B \left(\|z^{(B)} - z_n^{(B)}\|_{\mathcal{X}^{(B)}} + \|L^{(B)}(z^{(B)} - z_n^{(B)})\|_{\mathcal{X}^{(B)}} \right) \to 0 \text{ when } n \to \infty \end{aligned}$$

where we have used the fact $G_c^{(A)} z_n^{(A)} = G_c^{(B)} z_n^{(B)}$. This implies $G_c^{(A)} z^{(A)} = G_c^{(B)} z^{(B)}$ meaning that $\left[\begin{smallmatrix} z^{(A)} \\ z^{(B)} \end{smallmatrix}\right] \in \operatorname{dom}(\Xi^{(AB)})$. By a similar computation we can verify

$$\begin{bmatrix} G_b^{(A)} & 0 \\ G_c^{(A)} & 0 \\ 0 & G_b^{(B)} \end{bmatrix} \begin{bmatrix} z^{(A)} \\ z^{(B)} \end{bmatrix} = \begin{bmatrix} u_b^{(A)} \\ u_c \\ u_b^{(B)} \end{bmatrix} \text{ and } \begin{bmatrix} K_b^{(A)} & 0 \\ K_c^{(A)} & K_c^{(B)} \\ 0 & K_b^{(B)} \end{bmatrix} \begin{bmatrix} z^{(A)} \\ z^{(B)} \end{bmatrix} = \begin{bmatrix} y_b^{(A)} \\ y_c \\ y_b^{(B)} \end{bmatrix}.$$

Closedness of $L^{(AB)}$ with domain $\mathrm{dom}(L^{(AB)}) = \mathrm{dom}(\Xi^{(AB)})$ is shown similarly. Thus, $\Xi^{(AB)}$ is strong colligation. Note that in the preceding computation, we did not need $G_c^{(A)} z_n^{(A)} \to u_c$ to show $\left[\begin{smallmatrix} z^{(A)} \\ z^{(B)} \end{smallmatrix}\right] \in \mathrm{dom}(\Xi^{(AB)})$, *i.e.*, $G_c^{(A)} z^{(A)} = G_c^{(B)} z^{(B)}$.

We proceed to show that $\Xi^{(AB)}$ is an internally well-posed, impedance passive boundary node with the help of Proposition 2.4. Surjectivity of $\left[\begin{smallmatrix} G^{(AB)} \\ \alpha - L^{(AB)} \end{smallmatrix}\right]$ (with domain $\mathrm{dom}(\Xi^{(AB)})$) for some $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha \geq 0$ follows from the fact that $\left[\begin{smallmatrix} G^{(A)} \\ \alpha - L^{(A)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} G^{(B)} \\ \alpha - L^{(B)} \end{smallmatrix}\right]$ are surjective for the same α. All that is left is to show that the Green–Lagrange identity (6) holds:

$$\begin{aligned}
& \mathrm{Re}\big\langle z, L^{(AB)} z \big\rangle_{\mathcal{X}^{(AB)}} \\
= \; & \mathrm{Re}\big\langle z^{(A)}, L^{(A)} z^{(A)} \big\rangle_{\mathcal{X}^{(A)}} + \mathrm{Re}\big\langle z^{(B)}, L^{(B)} z^{(B)} \big\rangle_{\mathcal{X}^{(B)}} \\
\leq \; & \mathrm{Re}\big\langle K_b^{(A)} z^{(A)}, G_b^{(A)} z^{(A)} \big\rangle_{\mathcal{U}_b^{(A)}} + \mathrm{Re}\big\langle K_c^{(A)} z^{(A)}, G_c^{(A)} z^{(A)} \big\rangle_{\mathcal{U}_c} + \\
& + \mathrm{Re}\big\langle K_b^{(B)} z^{(B)}, G_b^{(B)} z^{(B)} \big\rangle_{\mathcal{U}_b^{(B)}} + \mathrm{Re}\big\langle K_c^{(B)} z^{(B)}, G_c^{(B)} z^{(B)} \big\rangle_{\mathcal{U}_c} \\
= \; & \mathrm{Re}\big\langle K^{(AB)} z, G^{(AB)} z \big\rangle_{\mathcal{U}^{(AB)}}
\end{aligned}$$

where the last equation follows from $G_c^{(A)} z^{(A)} = G_c^{(B)} z^{(B)}$ and definitions of $G^{(AB)}$ and $K^{(AB)}$. Surjectivity of $\left[\begin{smallmatrix} G^{(AB)} \\ K^{(AB)} \end{smallmatrix}\right]$ follows from surjectivity of $\left[\begin{smallmatrix} G^{(A)} \\ K^{(A)} \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} G^{(B)} \\ K^{(B)} \end{smallmatrix}\right]$.

The conservativity is verified by repeating the latter part of the proof with $-\alpha$ in place of α and replacing the inequality in Green–Lagrange identity by equality. $\square$

4.2 Step 2: loop coupling

Now assume that we have a colligation $\Xi = (G, L, K)$ on the Hilbert spaces $\left(\left[\begin{smallmatrix} \mathcal{U}_1 \\ \mathcal{U}_c \\ \mathcal{U}_c \end{smallmatrix}\right], \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_c \\ \mathcal{Y}_c \end{smallmatrix}\right] \right)$ where $G = \left[\begin{smallmatrix} G_1 \\ G_2 \\ G_3 \end{smallmatrix}\right]$ and $K = \left[\begin{smallmatrix} K_1 \\ K_2 \\ K_3 \end{smallmatrix}\right]$, *i.e.*, the input and output operators and spaces can be split into (at least) three parts. We "glue" two of these parts together to form another colligation $\widehat{\Xi} := \left(\widehat{G}, \widehat{L}, \widehat{K}\right)$ on the Hilbert spaces $\left(\left[\begin{smallmatrix} \mathcal{U}_1 \\ \mathcal{U}_c \end{smallmatrix}\right], \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_c \end{smallmatrix}\right] \right)$ with $\mathrm{dom}(\widehat{\Xi}) := \left\{ z \in \mathrm{dom}(\Xi) \;\middle|\; G_2 z = G_3 z \right\}$, $\widehat{L} := L|_{\mathrm{dom}(\widehat{\Xi})}$, $\widehat{G} := \left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right]$, and $\widehat{K} := \left[\begin{smallmatrix} K_1 \\ K_2 + K_3 \end{smallmatrix}\right]$.

The block diagram of such coupling is shown in Fig. 2b. As in step 1, we show that if the original colligation Ξ is an internally well-posed, impedance passive (conservative), strong boundary node, then $\widehat{\Xi}$ is one as well.

Lemma 4.2. *Let Ξ and $\widehat{\Xi}$ be as defined above. If the colligation Ξ is an internally well-posed, impedance passive (conservative), strong boundary node such that $\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right]$ is surjective, then also $\widehat{\Xi}$ is an internally well-posed, impedance passive (resp., conservative), strong boundary node with the property that $\left[\begin{smallmatrix} \widehat{G} \\ \widehat{K} \end{smallmatrix}\right]$ is surjective.*

Proof. Strongness of $\widehat{\Xi}$ is shown as before in Lemma 4.1.

Surjectivity of $\left[\begin{smallmatrix} \widehat{G} \\ \alpha-\widehat{L} \end{smallmatrix}\right]$ for some $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha \geq 0$ is easy to see, and also Green–Lagrange identity holds in $\operatorname{dom}(\widehat{\Xi})$:

$$\begin{aligned} \operatorname{Re}\langle z, \widehat{L}z\rangle_{\widehat{\mathcal{X}}} &\leq \operatorname{Re}\langle K_1 z, G_1 z\rangle_{\mathcal{U}_1} + \operatorname{Re}\langle K_2 z, G_2 z\rangle_{\mathcal{U}_c} + \operatorname{Re}\langle K_3 z, G_3 z\rangle_{\mathcal{U}_c} \\ &= \operatorname{Re}\langle K_1 z, G_1 z\rangle_{\mathcal{U}_1} + \operatorname{Re}\langle (K_2+K_3) z, G_2 z\rangle_{\mathcal{U}_c} \\ &= \operatorname{Re}\langle \widehat{K} z, \widehat{G} z\rangle_{\widehat{\mathcal{U}}} \end{aligned}$$

where the second equality follows from $G_2 z = G_3 z$ and the last from the definitions of $\widehat{G}$ and $\widehat{K}$. Surjectivity of $\left[\begin{smallmatrix} \widehat{G} \\ \widehat{K} \end{smallmatrix}\right]$ follows from surjectivity of $\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right]$.

If Ξ is conservative, then to show conservativity of $\widehat{\Xi}$, just repeat the proof with $-\alpha$ in place of α and replace the inequality in the Green–Lagrange identity with equality. $\square$

4.3 Step 3: closing the vertices

In this step, we single out some vertices as control/observation vertices and permanently "close" all others with respect to additional external signals. Note that after steps 1 and 2, under the assumptions of Lemmas 4.1 and 4.2, the resulting colligation is an internally well-posed boundary node, such that condition (i) of Definition 3.1 is satisfied. This closing means that we require also the condition (ii) of Definition 3.1 to be satisfied, and we now show that this can be done without sacrificing the internal well-posedness or passivity/conservativity.

So let $\Xi = (G, L, K)$ be a colligation on the Hilbert spaces $\left(\left[\begin{smallmatrix} \mathcal{U}_1 \\ \mathcal{U}_2 \end{smallmatrix}\right], \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{smallmatrix}\right]\right)$ with splittings $G = \left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right]$ and $K = \left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right]$ where G_2 and K_2 correspond to vertices we want to close. Define the new colligation by $\widehat{\Xi} := \left(G_1, \widehat{L}, K_1\right)$ on the Hilbert spaces $(\mathcal{U}_1, \mathcal{X}, \mathcal{Y}_1)$ with $\operatorname{dom}(\widehat{\Xi}) := \operatorname{dom}(\Xi) \cap \mathcal{N}(K_2)$ and $\widehat{L} := L|_{\operatorname{dom}(\widehat{\Xi})}$.

Lemma 4.3. *Let Ξ and $\widehat{\Xi}$ be as defined above. If Ξ is an internally well-posed, impedance passive (conservative), strong boundary node with the property that $\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right]$ is surjective, then also $\widehat{\Xi}$ is an internally well-posed, impedance passive (resp., conservative), strong boundary node.*

Proof. We carry out a partial flow inversion and interchange the roles of G_2 and K_2. More precisely, we shall prove that $\widetilde{\Xi} := \left(\widetilde{G}, L, \widetilde{K}\right)$ on Hilbert spaces $\left(\left[\begin{smallmatrix} \mathcal{U}_1 \\ \mathcal{Y}_2 \end{smallmatrix}\right], \mathcal{X}, \left[\begin{smallmatrix} \mathcal{Y}_1 \\ \mathcal{U}_2 \end{smallmatrix}\right]\right)$ where $\widetilde{G} := \left[\begin{smallmatrix} G_1 \\ K_2 \end{smallmatrix}\right]$, $\widetilde{K} := \left[\begin{smallmatrix} K_1 \\ G_2 \end{smallmatrix}\right]$, and $\operatorname{dom}(\widetilde{\Xi}) := \operatorname{dom}(\Xi)$, is an internally well-posed, impedance passive (conservative), strong boundary node. Colligation $\widehat{\widetilde{\Xi}}$ is then obtained from $\widetilde{\Xi}$ by restricting the solution space to $\mathcal{N}(K_2)$, and it clearly has all the properties as claimed, see Definition 2.2 and [20: Lemma 4.5] concerning the strongness of $\widehat{\widetilde{\Xi}}$.

It is trivial that $\widetilde{\widetilde{\Xi}}$ is a strong colligation. One way to see the interchangeability of G_2 and K_2 is directly by Definition 2.3 with $\beta = 1$:

$$\begin{bmatrix} \widetilde{G} + \widetilde{K} \\ \alpha - L \end{bmatrix} = \begin{bmatrix} \left[\begin{smallmatrix} G_1 \\ K_2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} K_1 \\ G_2 \end{smallmatrix}\right] \\ \alpha - L \end{bmatrix} = \begin{bmatrix} \left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right] \\ \alpha - L \end{bmatrix} = \begin{bmatrix} G + K \\ \alpha - L \end{bmatrix}.$$

The surjectivity of this operator follows from impedance passivity of Ξ. Similarly for the conservative system we also need the operator

$$\begin{bmatrix} \widetilde{G} - \widetilde{K} \\ \alpha - L \end{bmatrix} = \left[\begin{array}{cc|c} I & 0 & 0 \\ 0 & -I & 0 \\ \hline 0 & 0 & I \end{array}\right] \begin{bmatrix} G - K \\ \alpha - L \end{bmatrix}$$

to be surjective which holds by the conservativity of Ξ, see Definition 2.3 with $\beta = -1$. The Green–Lagrange (in)equality is also trivial, and it follows that $\widetilde{\widetilde{\Xi}}$ is an impedance passive (conservative), strong colligation.

Finally, by Proposition 2.4, the surjectivity of $\left[\begin{smallmatrix} G_1 \\ K_2 \end{smallmatrix}\right]$ implies that $\widetilde{\widetilde{\Xi}}$ is an internally well-posed boundary node. $\square$

4.4 Example on constructing the composition

Let us once more return to the example of the introduction. We reconstruct the interconnection graph in four phases which are illustrated in Fig. 3. We start with four boundary nodes labelled with A, B, C, and D. The input and output operators and spaces of each system are split into two parts, *i.e.*, $k_j = 2$. The vertices are labelled with 1 and 2 and the arrows in Fig. 3 point from 1 to 2.

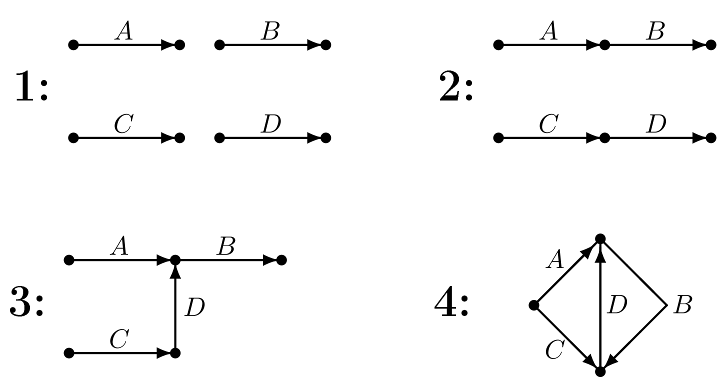


Figure 3: Composing a transmission graph

• **Phase 1**

We start with colligations $\Xi^{(j)} = \left(\begin{bmatrix} G_1^{(j)} \\ G_2^{(j)} \end{bmatrix}, L^{(j)}, \begin{bmatrix} K_1^{(j)} \\ K_2^{(j)} \end{bmatrix}\right)$ on the Hilbert spaces $\left(\begin{bmatrix} \mathcal{U}_1^{(j)} \\ \mathcal{U}_2^{(j)} \end{bmatrix}, \mathcal{X}^{(j)}, \begin{bmatrix} \mathcal{Y}_1^{(j)} \\ \mathcal{Y}_2^{(j)} \end{bmatrix}\right)$, $j = A, B, C, D$.

• **Phase 2**

The system A is connected to B, and C to D, by a partial parallel coupling so that we obtain two colligations $\Xi^{(AB)}$ and $\Xi^{(CD)}$ with

$$G^{(AB)} = \begin{bmatrix} G_1^{(A)} & 0 \\ G_2^{(A)} & 0 \\ 0 & G_2^{(B)} \end{bmatrix}, \quad K^{(AB)} = \begin{bmatrix} K_1^{(A)} & 0 \\ K_2^{(A)} & K_1^{(B)} \\ 0 & K_2^{(B)} \end{bmatrix},$$

$$\text{and} \quad \mathrm{dom}(\Xi^{(AB)}) = \left\{ \begin{bmatrix} z^{(A)} \\ z^{(B)} \end{bmatrix} \in \begin{bmatrix} \mathrm{dom}(\Xi^{(A)}) \\ \mathrm{dom}(\Xi^{(B)}) \end{bmatrix} \;\middle|\; G_2^{(A)} z^{(A)} = G_1^{(B)} z^{(B)} \right\}$$

and similarly $G^{(CD)}$, $K^{(CD)}$, and $\mathrm{dom}(\Xi^{(CD)})$.

Note that these colligations are induced by transmission graphs; for example the colligation of $\Gamma^{(AB)} := \left(\left\{\Xi^{(A)}, \Xi^{(B)}\right\}, \{\{(A,1)\}, \{(A,2),(B,1)\}, \{(B,2)\}\}, \emptyset\right)$ is exactly $\Xi^{(AB)}$.

• **Phase 3**

Now $\Xi^{(AB)}$ is connected to $\Xi^{(CD)}$ by a partial parallel coupling. The part of the operator $G^{(AB)}$ which is not involved in the connection is $G_b^{(AB)} = \begin{bmatrix} G_1^{(A)} & 0 \\ 0 & G_2^{(B)} \end{bmatrix}$ and the part that is, is $G_c^{(AB)} = \begin{bmatrix} G_2^{(A)} & 0 \end{bmatrix}$. Correspondingly $K_b^{(AB)} = \begin{bmatrix} K_1^{(A)} & 0 \\ 0 & K_2^{(B)} \end{bmatrix}$ and $K_c^{(AB)} = \begin{bmatrix} K_2^{(A)} & K_1^{(B)} \end{bmatrix}$. The system $\Xi^{(CD)}$ is connected by its free vertex $\{(D,2)\}$ to the common vertex $\{(A,2),(B,1)\}$ of $\Xi^{(AB)}$ so the CD-splitting is done differently, namely $G_b^{(CD)} = \begin{bmatrix} G_1^{(C)} & 0 \\ 0 & G_1^{(D)} \end{bmatrix}$, $G_c^{(CD)} = \begin{bmatrix} 0 & G_2^{(D)} \end{bmatrix}$, $K_b^{(CD)} = \begin{bmatrix} K_1^{(C)} & 0 \\ K_2^{(C)} & K_1^{(D)} \end{bmatrix}$, and $K_c^{(CD)} = \begin{bmatrix} 0 & K_2^{(D)} \end{bmatrix}$.

Thus, as described in Section 4.1, we obtain a system with

$$G = \left[\begin{array}{cc|cc} G_1^{(A)} & 0 & 0 & 0 \\ 0 & G_2^{(B)} & 0 & 0 \\ \hline G_2^{(A)} & 0 & 0 & 0 \\ \hline 0 & 0 & G_1^{(C)} & 0 \\ 0 & 0 & 0 & G_1^{(D)} \end{array}\right], \quad K = \left[\begin{array}{cc|cc} K_1^{(A)} & 0 & 0 & 0 \\ 0 & K_2^{(B)} & 0 & 0 \\ \hline K_2^{(A)} & K_1^{(B)} & 0 & K_2^{(D)} \\ \hline 0 & 0 & K_1^{(C)} & 0 \\ 0 & 0 & K_2^{(C)} & K_1^{(D)} \end{array}\right],$$

and $$\mathrm{dom}(\Xi) = \Bigg\{ z^{(j)} \in \mathrm{dom}(\Xi^{(j)}),\ j = A, B, C, D \ \Bigg| \ G_2^{(A)} z^{(A)} = G_1^{(B)} z^{(B)} = G_2^{(D)} z^{(D)},\ G_2^{(C)} z^{(C)} = G_1^{(D)} z^{(D)} \Bigg\}.$$

Again, the colligation Ξ is induced by a transmission graph $\Gamma := \left(\{\Xi^{(j)}\}_{j=A}^{D}, \{\mathcal{I}_l\}_{l=1}^{5}, \emptyset \right)$ where $\mathcal{I}_1 = \{(A,1)\}$, $\mathcal{I}_2 = \{(A,2),(B,1),(D,2)\}$, $\mathcal{I}_3 = \{(B,2)\}$, $\mathcal{I}_4 = \{(C,1)\}$, and $\mathcal{I}_5 = \{(C,2),(D,1)\}$.

- **Phase 4**

In the last phase, the vertex $\{(B,2)\}$ is connected to $\{(C,2),(D,1)\}$, and $\{(A,1)\}$ to $\{(C,1)\}$, by a loop coupling. The parts of input and output that are not involved in the connection are $G_1 = [G_2^{(A)}\ 0\ 0\ 0]$ and $K_1 = [K_2^{(A)}\ K_1^{(B)}\ 0\ K_2^{(D)}]$. The operators that are involved are $G_2 = \begin{bmatrix} G_1^{(A)} & 0 & 0 & 0 \\ 0 & G_2^{(B)} & 0 & 0 \end{bmatrix}$, $K_2 = \begin{bmatrix} K_1^{(A)} & 0 & 0 & 0 \\ 0 & K_2^{(B)} & 0 & 0 \end{bmatrix}$, $G_3 = \begin{bmatrix} 0 & 0 & G_1^{(C)} & 0 \\ 0 & 0 & 0 & G_1^{(D)} \end{bmatrix}$, and $K_3 = \begin{bmatrix} 0 & 0 & K_1^{(C)} & 0 \\ 0 & 0 & K_2^{(C)} & K_1^{(D)} \end{bmatrix}$. As described in Section 4.2, the new input and output operators are $G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$ and $K = \begin{bmatrix} K_1 \\ K_2 + K_3 \end{bmatrix}$. To $\mathrm{dom}(\Xi)$ we impose the additional condition $G_2 z_2 = G_3 z_3$. In terms of the original blocks, this means $G_1^{(A)} z^{(A)} = G_1^{(C)} z^{(C)}$ and $G_2^{(B)} z^{(B)} = G_1^{(D)} z^{(D)}$.

In block operators G and K, before closing any vertices, each column corresponds to one system (an edge of the graph) and each row corresponds to a coupling (a vertex of the graph). Thus, in phase 2, the block operators $G^{(AB)}$, $K^{(AB)}$, $G^{(CD)}$, and $K^{(CD)}$ have three rows and two columns. In phase 3, G and K have five rows and four columns. And finally, when connecting vertex $\{(B,2)\}$ to $\{(C,2),(D,1)\}$ and $\{(A,1)\}$ to $\{(C,1)\}$, two rows are lost.

5 Examples

5.1 Vocal tract: Webster's equation with dissipation

The human vocal tract can be considered as a Y-shaped graph whose three free vertices are at the vocal folds, mouth, and nose. The closed vertex with three outgoing edges is located at the velum. Wave propagation in such domain can be computed by Webster's equation up to frequencies of about 4 kHz where the effect of the transversal resonances become significant, see [10: Section 5 and Fig. 1]. As explained in [17], the solution of 1D Webster's equation approximates the solution of the corresponding 3D wave equation in a tubular domain, averaged over the cross-sections of the tube. Hence, Webster's equation cannot capture transversal dynamics of the wave propagation.

The generalised Webster's equation is derived in [17], and it is given by

$$\psi_{tt}(x,t) + \frac{2\pi\theta W(x)c(x)^2}{A(x)}\psi_t(x,t) - \frac{c(x)^2}{A(x)}\frac{\partial}{\partial x}\left(A(x)\frac{\partial \psi}{\partial x}(x,t) \right) = 0. \quad (11)$$

The solution ψ is *Webster's velocity potential* that approximates the wave equation velocity potential when averaged over a transversal cross-section at distance $x \in [0, l]$ from the tube end. Thus, $\psi_x = -v$ and $\rho\psi_t = p$ where v is the perturbation velocity, ρ is the density, and p is the perturbation pressure. The function $A(x) = \pi R(x)^2$ in (11) is the cross-sectional area of the tube, $R(\cdot)$ being the radius of the cross-section. Define the curvature ratio by $\eta(x) = R(x)\kappa(x)$ where $\kappa(\cdot)$ is the curvature of the tube centreline. Note that it is assumed that $\eta(x) < 1$ meaning that the tube does not fold onto itself. The surface area factor $W(x) = R(x)\sqrt{R'(x)^2 + (\eta(x) - 1)^2}$ and the corrected sound velocity $c(x) = \frac{c}{\sqrt{1+\frac{1}{4}\eta(x)^2}}$ in (11) depend on the curvature. The coefficient $\theta \geq 0$ regulates the dissipation at the tube walls.

Equation (11) contains a distributed dissipation term due to a dissipative boundary condition at the tube walls[3] and inhomogeneous sound velocity, which occurs when the tube is curved. The classical Webster's equation (that is, without curvature or dissipation) is obtained by setting $\theta = 0$ and $\kappa(x) = 0$. We say that the boundary control of (11) is of impedance type, when the control signals are chosen as $\rho\psi_t(0, t)$ and $\rho\psi_t(l, t)$ and observation signals as $\psi_x(0, t)$ and $-\psi_x(l, t)$. Note that the velocity potential given by (11) alone is not unique since $\psi + C$ for arbitrary C gives the same pressure and velocity fields as ψ. This affects to the choice of the state and solution spaces later.

As explained above, the model for the vocal tract is divided into three parts. In each of these parts we have a velocity potential $\psi^{(j)} : (0, l_j) \times \mathbb{C} \to \mathbb{C},\ j = A, B, C$ that satisfies (11) with respective functions $A_j \in C^1[0, l]$ such that $A_j(x) > \epsilon > 0$ and c_j such that $\infty > c_j(x) > \epsilon > 0$ and $c_j^{-2}(x) \in L^2(0, l)$. The potentials are connected through Kirchhoff conditions

$$\begin{cases} \frac{\partial\psi^{(A)}}{\partial t}(0,t) = \frac{\partial\psi^{(B)}}{\partial t}(0,t) = \frac{\partial\psi^{(C)}}{\partial t}(0,t), \\ A_A(0)\frac{\partial\psi^{(A)}}{\partial x}(0,t) + A_B(0)\frac{\partial\psi^{(B)}}{\partial x}(0,t) + A_C(0)\frac{\partial\psi^{(C)}}{\partial t}(0,t) = 0. \end{cases} \tag{12}$$

The system is controlled by the flow u through the vocal folds, and there is an acoustic resistance at the mouth and nose openings:

$$\begin{cases} \frac{\partial\psi^{(A)}}{\partial x}(l_A,t) = u(t) & \text{at vocal folds,} \\ \frac{\partial\psi^{(B)}}{\partial t}(l_B,t) + \theta_B c\frac{\partial\psi^{(B)}}{\partial x}(l_B,t) = 0 & \text{at mouth, and} \\ \frac{\partial\psi^{(C)}}{\partial t}(l_C,t) + \theta_C c\frac{\partial\psi^{(C)}}{\partial x}(l_C,t) = 0 & \text{at nose} \end{cases} \tag{13}$$

where θ_B and θ_C are the dimensionless normalised acoustic resistances.

We proceed to formulate this model as a boundary control system. First, we write Webster's equation as a first order system by choosing the state vector as $z = \begin{bmatrix} \psi \\ \psi_t \end{bmatrix}$. The state and solution spaces are

$$\mathcal{X}^{(j)} := h^1[0, l_j] \times L^2(0, l_j) \qquad \text{and} \qquad \mathcal{Z}^{(j)} := h^2[0, l_j] \times H^1[0, l_j]$$

[3]The classical Webster's equation is derived using Neumann boundary conditions at the walls of the tube.

respectively, where $h^1[0,l_j] = H^1[0,l_j]\big/\!\sim$ and $h^2[0,l_j] = H^2[0,l_j]\big/\!\sim$ where the equivalence relation $x \sim y$ holds if $x-y$ is constant Lebesgue almost everywhere in $(0,l_j)$. We equip $h^1[0,l_j]$ with the norm $\|z\|_{h^1[0,l_j]} := \left\| \frac{\partial z}{\partial x} \right\|_{L^2(0,l_j)}$, and the state spaces with inner products

$$\langle z, y \rangle_{\mathcal{X}^{(j)}} := \frac{\rho}{2} \left(\int_0^{l_j} \frac{\partial z_1}{\partial x} \overline{\frac{\partial y_1}{\partial x}} \, A_j(x) dx + \int_0^{l_j} z_2(x) \overline{y_2(x)} \, \frac{A_j(x) dx}{c_j(x)^2} \right).$$

The norms induced by $\langle \cdot, \cdot \rangle_{\mathcal{X}^{(j)}}$ correspond to the physical energy. In the solution spaces we use norms

$$\|z\|^2_{\mathcal{Z}^{(j)}} := \|z_1\|^2_{h^1[0,l_j]} + \left\| \frac{\partial^2 z_1}{\partial x^2}(x) \right\|^2_{L^2(0,l_j)} + \|z_2\|^2_{H^1[0,l_j]}.$$

The input and output spaces are $\mathcal{U}^{(j)} = \mathcal{Y}^{(j)} = \mathbb{C}^2$ with the Euclidian norm. The interior operators are defined by

$$L^{(j)} := W^{(j)} + D^{(j)} : \ \mathcal{Z}^{(j)} \to \mathcal{X}^{(j)}$$

where

$$W^{(j)} := \begin{bmatrix} 0 & 1 \\ \frac{c_j(x)^2}{A_j(x)} \frac{\partial}{\partial x} \left(A_j(x) \frac{\partial}{\partial x} \right) & 0 \end{bmatrix} \quad \text{and} \quad D^{(j)} := \begin{bmatrix} 0 & 0 \\ 0 & -\frac{2\pi\theta W_j(x) c_j(x)^2}{A_j(x)} \end{bmatrix};$$

the dissipative part $D^{(j)}$ acts as a bounded perturbation (in $\mathcal{X}^{(j)}$) to the classical Webster-related part $W^{(j)}$. The input and output operators are defined by

$$G^{(j)} z^{(j)} := \begin{bmatrix} \rho z_2^{(j)}(0,t) \\ \rho z_2^{(j)}(l_j,t) \end{bmatrix} \quad \text{and} \quad K^{(j)} z^{(j)} := \begin{bmatrix} A_j(0) \frac{\partial z_1^{(A)}}{\partial x}(0,t) \\ -A_j(l_j) \frac{\partial z_1^{(j)}}{\partial x}(l_j,t) \end{bmatrix}.$$

The pressure controlled, velocity observed Webster's equation can finally be written in the form

$$\begin{cases} u^{(j)}(t) &= G^{(j)} z^{(j)}(t), \\ \dot{z}^{(j)}(t) &= L^{(j)} z^{(j)}(t), \\ y^{(j)}(t) &= K^{(j)} z^{(j)}(t), \quad t \in \mathbb{R}^+, \end{cases}$$

and it remains to show that each $\Xi^{(j)} = (G^{(j)}, L^{(j)}, K^{(j)})$ satisfies the conditions of Definitions 2.2 and 2.3.

Theorem 5.1. *Each colligation $\Xi^{(j)} = (G^{(j)}, L^{(j)}, K^{(j)})$ on spaces $\left(\mathbb{C}^2, \mathcal{X}^{(j)}, \mathbb{C}^2\right)$ defined above is an impedance passive (even conservative if $\theta = 0$), internally well-posed, strong boundary node.*

Proof. Here we drop the index j, and begin by showing the claim in the special impedance conservative case $\widehat{\Xi} = (G, W, K)$ on $\left(\mathbb{C}^2, \mathcal{X}, \mathbb{C}^2\right)$.

It is easy to see that $\widehat{\Xi}$ is a strong colligation, and that G is surjective. Thus, to show surjectivity of $\left[\begin{smallmatrix} G \\ \alpha - W \end{smallmatrix}\right]$ it is sufficient to show $(\alpha - W)|_{\mathcal{N}(G)}$ to be bijective.

Fix $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{X}$ (in the following we treat $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ as a representative from the equivalence class) and $\alpha \neq 0$. We wish to find $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in \mathcal{N}(G)$, s.t.

$$\begin{bmatrix} \alpha & -1 \\ -\frac{c(x)^2}{A(x)} \frac{\partial}{\partial x}\left(A(x)\frac{\partial}{\partial x}\right) & \alpha \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}. \tag{14}$$

The first row implies $\alpha z_1 - z_2 = f$ (in $H^1[0,l]$). The condition $\left[\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right] \in \mathcal{N}(G)$ is equivalent to $z_2(0) = z_2(l) = 0$ so that $z_1(0) = \frac{f(0)}{\alpha}$ and $z_1(l) = \frac{f(l)}{\alpha}$. Multiplying the first row in (14) with α and adding it to the second row gives

$$\alpha^2 z_1(x) - \frac{c(x)^2}{A(x)} \frac{\partial}{\partial x}\left(A(x)\frac{\partial z_1}{\partial x}(x)\right) = \alpha f(x) + g(x) \quad \left(\in L^2(0,l)\right).$$

This equation with the aforementioned boundary conditions has a unique variational solution $z_1 \in H^2[0,l]$ that satisfies $\left[\begin{smallmatrix} z_1 \\ \alpha z_1 - f \end{smallmatrix}\right] \in \mathcal{N}(G)$. If we solve (14) for a different representative of the same equivalence class, that is, with right hand side $\left[\begin{smallmatrix} f+C \\ g \end{smallmatrix}\right]$ where $C \in \mathbb{C}$, then we get for (14) the respective solution $\left[\begin{smallmatrix} z_1 + C/\alpha \\ \alpha(z_1 + C/\alpha) - f - C \end{smallmatrix}\right] = \left[\begin{smallmatrix} z_1 + C/\alpha \\ \alpha z_1 - f \end{smallmatrix}\right]$ which is in the same equivalence class with $\left[\begin{smallmatrix} z_1 \\ \alpha z_1 - f \end{smallmatrix}\right]$. Hence, equation (14) has a unique solution in $\mathcal{Z}$ for all $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \in \mathcal{X}$. The Green–Lagrange identity (6) for $\widehat{\Xi}$ as an equality can be shown by partial integration. The claim is now proved for $\widehat{\Xi}$ by Proposition 2.4.

Since $D : \mathcal{X} \to \mathcal{X}$ is bounded, also $L|_{\mathcal{N}(G)} = (W + D)|_{\mathcal{N}(G)}$ generates a C_0-semigroup by [1: Corollary 3.5.6]. Because $W(x) \geq 0$ and $\theta > 0$, it follows

$$\langle z, Dz \rangle_{\mathcal{X}} = -\pi\theta\rho \int_0^l W(x) z_2(x)^2 \, dx \leq 0$$

which means that Green–Lagrange identity for Ξ holds as an inequality. Because bounded perturbations of closed operators are closed, nodes Ξ and $\widehat{\Xi}$ are simultaneously strong. □

The boundary conditions (12) at velum correspond to conditions (8) and (9). Thus, after noting that operators $\left[\begin{smallmatrix} G^{(j)} \\ K^{(j)} \end{smallmatrix}\right]$ are surjective (try polynomial functions in $\mathcal{Z}$), Theorem 3.4 yields:

Theorem 5.2. *Define a colligation* $\Xi = (G, L, K)$ *on spaces* $\left(\mathbb{C}^2, \mathcal{X}, \mathbb{C}^2\right)$ *where*

$$G \begin{bmatrix} z^{(A)} \\ z^{(B)} \\ z^{(C)} \end{bmatrix} := \begin{bmatrix} \rho z_2^{(A)}(l_A, t) \\ \rho z_2^{(B)}(l_B, t) \\ \rho z_2^{(C)}(l_C, t) \end{bmatrix}, \qquad L := \begin{bmatrix} L^{(A)} & 0 & 0 \\ 0 & L^{(B)} & 0 \\ 0 & 0 & L^{(C)} \end{bmatrix},$$

$$\text{and} \qquad K \begin{bmatrix} z^{(A)} \\ z^{(B)} \\ z^{(C)} \end{bmatrix} := \begin{bmatrix} -A_A(l_A) \frac{\partial z_1^{(A)}}{\partial x}(l_A, t) \\ -A_B(l_B) \frac{\partial z_1^{(B)}}{\partial x}(l_B, t) \\ -A_C(l_C) \frac{\partial z_1^{(C)}}{\partial x}(l_C, t) \end{bmatrix}, \quad \text{with}$$

$$\operatorname{dom}(\Xi) := \left\{ \begin{bmatrix} z^{(A)} \\ z^{(B)} \\ z^{(C)} \end{bmatrix} \in \begin{bmatrix} \mathcal{Z}^{(A)} \\ \mathcal{Z}^{(B)} \\ \mathcal{Z}^{(C)} \end{bmatrix} \;\middle|\; z_1^{(A)}(0,t) = z_2^{(B)}(0,t) = z_2^{(C)}(0,t), \right.$$
$$\left. A_A(0)\frac{\partial z_1^{(A)}}{\partial x}(0,t) + A_B(0)\frac{\partial z_1^{(B)}}{\partial x}(0,t) + A_C(0)\frac{\partial z_1^{(C)}}{\partial x}(0,t) = 0 \right\}.$$

Then Ξ is an impedance passive, internally well-posed, strong boundary node. The node Ξ is conservative if and only if $\theta = 0$.

Here also the vertices corresponding to the mouth and nose are also chosen to be control vertices which does not correspond to boundary conditions (13). It can be shown that an impedance passive internally well-posed system remains as one with such resistive termination but we do not do it here.

5.2 Homogeneous Timoshenko beam

In this example we consider the Timoshenko beam (see, *e.g.*, [26: Section 9.6]) with six degrees of freedom. Assume we have a beam of length l with a unit vector $q_1 \in \mathbb{R}^3$ showing its orientation, and vectors q_2 and q_3 such that $\{q_1, q_2, q_3\}$ forms a right-handed orthonormal system. Denote the unitary change of basis matrix by $Q := [q_1|q_2|q_3] \in \mathbb{R}^{3\times 3}$. Define the functions w_j to be the displacements of the points of the beam in the direction of q_j. The rotations of the beam cross-sections with respect to axis q_1 is given by ϕ_1, with respect[4] to axis $-q_2$ by function ϕ_3, and with respect to axis q_3 by function ϕ_2.

The *longitudinal* vibrations (that is, in the direction of q_1) and the rotational vibrations with respect to axis q_1 are governed by equations

$$\rho A \frac{\partial^2 w_1}{\partial t^2} = AE \frac{\partial^2 w_1}{\partial x^2} \qquad \text{and} \qquad \rho I_r \frac{\partial^2 \phi_1}{\partial t^2} = g J_{tor} \frac{\partial^2 \phi_1}{\partial x^2}$$

where ρ is the density of the material, A is the cross-sectional area of the beam, and E is the elastic modulus. In the second equation I_r is the inertial moment of the cross-section with respect to q_1-axis, gJ_{tor} is torsional rigidity (and g itself is the shear modulus and it is usually denoted by G). The dynamics of the other four degrees of freedom — corresponding to *transversal* vibrations — is governed by the homogeneous Timoshenko beam equations (without external load)

$$\left\{ \begin{array}{lcl} \rho A \frac{\partial^2 w_j}{\partial t^2} & = & \kappa_j A g \left(\frac{\partial^2 w_j}{\partial x^2} - \frac{\partial \phi_j}{\partial x} \right), \\ \rho I_j \frac{\partial^2 \phi_j}{\partial t^2} & = & E I_j \frac{\partial^2 \phi_j}{\partial x^2} + \kappa_j A g \left(\frac{\partial w_j}{\partial x} - \phi_j \right), \qquad j = 2,3 \end{array} \right.$$

where κ_j is a shape coefficient and I_j is the inertial moment of the cross-section with respect to axis q_3 for $j = 2$ and q_2 for $j = 3$. The displacements and the

[4]Messing up with the minus-signs will not only destroy your Green–Lagrange identity, but may result in unexpected catastrophic consequences, see [11].

orientations of the endpoints are given in global coordinates by

$$Q\begin{bmatrix} w_1(x)\\ w_2(x)\\ w_3(x)\end{bmatrix} \quad \text{and} \quad Q\begin{bmatrix} \phi_1(x)\\ -\phi_3(x)\\ \phi_2(x)\end{bmatrix}, \qquad x=0,l. \tag{15}$$

The force and moment inflicted to an endpoint of a beam are

$$\pm Q\begin{bmatrix} AE\frac{\partial w_1}{\partial x}\\ \kappa_2 Ag\left(\frac{\partial w_2}{\partial x}-\phi_2\right)\\ \kappa_3 Ag\left(\frac{\partial w_3}{\partial x}-\phi_3\right)\end{bmatrix} \quad \text{and} \quad \pm Q\begin{bmatrix} gJ_{tor}\frac{\partial \phi_1}{\partial x}\\ EI_3\frac{\partial \phi_3}{\partial x}\\ -EI_2\frac{\partial \phi_2}{\partial x}\end{bmatrix}, \qquad x=0,l$$

where the sign is $-$ if $x=0$ and $+$ when $x=l$. (Note that $\frac{\partial}{\partial x}$ is outer normal at $x=l$ and inner normal at $x=0$).

Now we write the beam equations in the form of a boundary control system. First the equations are transformed into a first order system by choosing the state vector as $z=[w_1,\phi_1,w_2,\phi_2,w_3,\phi_3,\dot w_1,\dot\phi_1,\dot w_2,\dot\phi_2,\dot w_3,\dot\phi_3]^T$. The solution space is defined as $\mathcal{Z}=\left(h^2[0,l]\times H^2[0,l]\right)^3\times\left(H^1[0,l]\right)^6$ and the state space as $\mathcal{X}=\left(h^1[0,l]\times H^1[0,l]\right)^3\times\left(L^2(0,l)\right)^6$. The inner product in $\mathcal{X}$ is defined by $\left\langle z_1,z_2\right\rangle_{\mathcal{X}} := \left\langle z_1,z_2\right\rangle_1+\left\langle z_1,z_2\right\rangle_2+\left\langle z_1,z_2\right\rangle_3$ where

$$\begin{aligned}\left\langle z_1,z_2\right\rangle_1 \;&:=\; \tfrac12 AE\int_0^l \frac{\partial w_{1,1}}{\partial x}\overline{\frac{\partial w_{2,1}}{\partial x}}\,dx+\tfrac12 gJ_{tor}\int_0^l \frac{\partial \phi_{1,1}}{\partial x}\overline{\frac{\partial \phi_{2,1}}{\partial x}}\,dx\\ &\quad+\tfrac12 A\rho\int_0^l \dot w_{1,1}\overline{\dot w_{2,1}}\,dx+\tfrac12 I_r\rho\int_0^l \dot\phi_{1,1}\overline{\dot\phi_{2,1}}\,dx\end{aligned}$$

and

$$\begin{aligned}\left\langle z_1,z_2\right\rangle_{2/3} \;&:=\; \tfrac12\kappa_{2/3}Ag\int_0^l\left(\frac{\partial w_{1,2/3}}{\partial x}-\phi_{1,2/3}\right)\overline{\left(\frac{\partial w_{2,2/3}}{\partial x}-\phi_{2,2/3}\right)}\,dx\\ &\quad+\tfrac12 EI_{2/3}\int_0^l\frac{\partial\phi_{1,2/3}}{\partial x}\overline{\frac{\partial\phi_{2,2/3}}{\partial x}}\,dx\\ &\quad+\tfrac12\rho A\int_0^l \dot w_{1,2/3}\overline{\dot w_{2,2/3}}\,dx+\tfrac12\rho I_{2/3}\int_0^l\dot\phi_{1,2/3}\overline{\dot\phi_{2,2/3}}\,dx.\end{aligned}$$

The norm induced by $\left\langle\cdot,\cdot\right\rangle_{\mathcal{X}}$ is the physical energy norm. In the space $\mathcal{Z}$ we use the norm

$$\|z\|^2_{\mathcal{Z}} := \sum_{j=1}^{3}\left(\left\|\frac{\partial z_{(2j-1)}}{\partial x}\right\|^2_{L^2(0,l)}+\left\|\frac{\partial^2 z_{(2j-1)}}{\partial x^2}\right\|^2_{L^2(0,l)}+\|z_{(2j)}\|^2_{H^2[0,l]}\right)+\sum_{j=7}^{12}\|z_j\|^2_{H^1[0,l]}.$$

Thus, the interior operator takes the form

$$L=\left[\begin{array}{c|c} 0 & I\\ \hline T & 0\end{array}\right]:\mathcal{Z}\to\mathcal{X}$$

where the block size is 6×6 and

$$T=\begin{bmatrix} \frac{E}{\rho}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{gJ_{tor}}{\rho I_r}\frac{\partial^2}{\partial x^2} & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{\kappa_2 g}{\rho}\frac{\partial^2}{\partial x^2} & -\frac{\kappa_2 g}{\rho}\frac{\partial}{\partial x} & 0 & 0\\ 0 & 0 & \frac{\kappa_2 Ag}{\rho I_2}\frac{\partial}{\partial x} & \left(\frac{E}{\rho}\frac{\partial^2}{\partial x^2}-\frac{\kappa_2 Ag}{\rho I_2}\right) & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{\kappa_3 g}{\rho}\frac{\partial^2}{\partial x^2} & -\frac{\kappa_3 g}{\rho}\frac{\partial}{\partial x}\\ 0 & 0 & 0 & 0 & \frac{\kappa_3 Ag}{\rho I_3}\frac{\partial}{\partial x} & \left(\frac{E}{\rho}\frac{\partial^2}{\partial x^2}-\frac{\kappa_3 Ag}{\rho I_3}\right)\end{bmatrix}.$$

When the beams are connected together into graph structures, it is natural to impose the following boundary conditions: *(i)* the connections are rigid in the sense that displacements and orientations of the connected beams, given by (15), coincide at the vertices, and *(ii)* the sums of all the forces and moments inflicted to each vertex vanish[5]. Thus, we define $G = \left[\begin{smallmatrix} G_1 \\ G_2 \end{smallmatrix}\right] : \mathcal{Z} \to \mathcal{U} = \mathbb{C}^{12}$ and $K = \left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right] : \mathcal{Z} \to \mathcal{Y} = \mathbb{C}^{12}$ where

$$G_1 z := \begin{bmatrix} Q[\dot{w}_1(0), \dot{w}_2(0), \dot{w}_2(0)]^T \\ Q[\dot{\phi}_1(0), -\dot{\phi}_3(0), \dot{\phi}_2(0)]^T \end{bmatrix}$$

and

$$K_1 z := -\begin{bmatrix} Q\left[AE\frac{\partial w_1}{\partial x}(0),\ \kappa_2 Ag\left(\frac{\partial w_2}{\partial x}(0) - \phi_2(0)\right),\ \kappa_3 Ag\left(\frac{\partial w_3}{\partial x}(0) - \phi_3(0)\right)\right]^T \\ Q\left[gJ_{tor}\frac{\partial \phi_1}{\partial x}(0),\ EI_3\frac{\partial \phi_3}{\partial x}(0),\ -EI_2\frac{\partial \phi_2}{\partial x}(0)\right]^T \end{bmatrix}.$$

The operators G_2 and K_2 are defined similarly, but at the other end we evaluate at $x = l$ and K_2 is without the minus-sign in front. The aforementioned boundary conditions take the form (8) and (9) when we connect many beams into one vertex.

The following theorem states that the BCS formulated above satisfies the assumptions in Lemmas 4.1, 4.2, and 4.3.

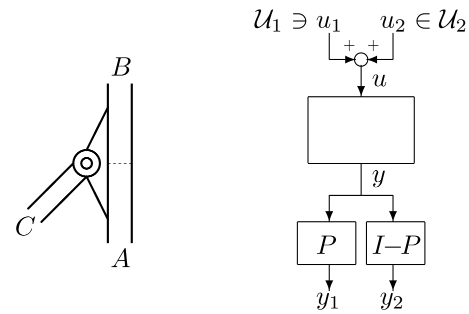
Theorem 5.3. *The colligation $\Xi = (G, L, K)$ on $\left(\mathbb{C}^{12}, \mathcal{X}, \mathbb{C}^{12}\right)$ defined above is an impedance conservative, internally well-posed, strong boundary node with the property that $\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right]$ is surjective.*

This together with Theorem 3.4 enables composing Timoshenko beams into lattice structures that define solvable and conservative dynamical equations.

Proof. It is easy to see that L (that consists on differential operators on Sobolev spaces) is closed on $\mathcal{X}$ with domain $\mathcal{Z}$, and G and K are continuous operators from $\mathcal{Z}$ to $\mathbb{C}^{12}$. Thus, Ξ is a strong colligation. Again, $\left[\begin{smallmatrix} G \\ K \end{smallmatrix}\right]$ is trivially surjective. Surjectivity of $(\pm 1 - L)|_{\mathcal{N}(G)}$ is shown similarly as in Theorem 5.1. Green–Lagrange identity (6) as an equality follows by partial integration, completing the proof by Proposition 2.4. $\square$

Remark 5.4. The aforementioned boundary conditions correspond to a rigid (welded) junction. Of course, there are other reasonable ways to connect beams, *e.g.*, by hinges as shown in Fig. 4a. Then, all degrees of freedom are not transmitted over the junction. Such connection can be composed using the steps presented above in the context of Theorem 3.4. More precisely, this is done by splitting the input and output signals to parts corresponding to those degrees of freedom that are transmitted and to those that are not at the vertex of interest: that is, we split an input signal u to parts $u_1 = Pu$ and $u_2 = (I - P)u$ and correspondingly $y_1 = Py$ and $y_2 = (I - P)y$ where P is an orthogonal projection.

[5]These are not the only possible reasonable boundary conditions; see Remarks 5.4 and 5.5.



(a) (b)

Figure 4: (a) A hinge junction; (b) Splitting of an input/output pair

This is illustrated in Fig. 4b. The input/output space is split accordingly to an orthogonal decomposition $\mathcal{U} = \mathcal{U}_1 \bigoplus \mathcal{U}_2$ where $\mathcal{U}_1 = P\mathcal{U}$ and $\mathcal{U}_2 = (I - P)\mathcal{U}$. These parts are then treated as independent inputs and outputs.

In the case of the hinge junction of Fig. 4a the details are as follows. Assume that the input operators are split into parts corresponding to displacement and orientation $G^{(j)} = \begin{bmatrix} G_d^{(j)} \\ G_o^{(j)} \end{bmatrix}$ and output operators into parts corresponding to force and moment $K^{(j)} = \begin{bmatrix} K_f^{(j)} \\ K_m^{(j)} \end{bmatrix}$ where $j = A, B, C$. For simplicity we only consider the parts of these operators corresponding to the ends that are to be connected by the hinge. Denote by h the unit vector parallel to the axle of the hinge. Now the connection is formed as follows by using the basic operations of Section 4:

- Connect systems $\Xi^{(A)}$ and $\Xi^{(B)}$ by step 1 (see Section 4.1) in order to produce a beam modelled by $\Xi^{(AB)}$ that has external variables for a connection in the middle.

- Split the orientation-parts of the inputs and the moment-parts of the outputs of systems $\Xi^{(AB)}$ and $\Xi^{(C)}$ by the projection $P = \begin{bmatrix} I & 0 \\ 0 & I - hh^T \end{bmatrix}$.

- Connect $\Xi^{(AB)}$ and $\Xi^{(C)}$ by step 1 so that $\mathrm{dom}(\Xi^{(ABC)}) = \left\{ \begin{bmatrix} z^{(AB)} \\ z^{(C)} \end{bmatrix} \in \begin{bmatrix} \mathrm{dom}(\Xi^{(AB)}) \\ \mathrm{dom}(\Xi^{(C)}) \end{bmatrix} \;\middle|\; PG^{(AB)}z^{(AB)} = PG^{(C)}z^{(C)} \right\}$.

Remark 5.5. There are situations where the junctions themselves may have a (dissipative) dynamics of their own. Consider, for example, a hinge with a spring or a damper where the dynamics is governed by finite-dimensional linear system. Such system can never be of boundary control form and therefore Theorem 3.4 is not directly applicable.

6 Remarks and conclusions

Remark 6.1. Recall that a transfer function $\mathcal{G}(\cdot)$ of any impedance passive system node (with $\mathcal{U} = \mathcal{Y} = \mathbb{C}$) is positive-real; *i.e.*, $\mathrm{Re}(\mathcal{G}(s)) \geq 0$ for all $s \in \mathbb{C}^+$. For boundary nodes this follows from the Green–Lagrange inequality (6), implying $\langle \mathcal{G}(s)u, u\rangle_{\mathcal{U}} \geq 2\mathrm{Re}(s)\|(s - A_{-1})^{-1}Bu\|^2$ for all $u \in \mathcal{U}$ and $s \in \rho(A)$where $\mathcal{G}(s) = K(s - A_{-1})^{-1}B$, A_{-1}, and B are as given in [19: Theorem 2.3]. Conversely, any positive-real analytic function is the transfer function of some impedance passive (even conservative) system node.

Theorem 3.4 can be seen as a state-space variant of the *Nyquist's stability criterion* for boundary control systems. For positive-real $\mathcal{G}(\cdot)$, the closed loop transfer function $\mathcal{G}_k(s) := \frac{\mathcal{G}(s)}{1+k\mathcal{G}(s)}$ is analytic in $\mathbb{C}^+$ for proportional negative feedback $k > 0$, and it is even positive-real by a simple calculation that leads to the proof of Nyquist's stability criterion. In particular, the feedback loop is admissible in the sense that $\mathcal{G}_k(\cdot)$ is the transfer function of some impedance passive system node. It takes, indeed, some energy that passive systems cannot afford to introduce closed loop objects that are not internally well-posed.

Remark 6.2. Assumption (10) is actually stronger than what was needed in Theorem 3.4. Indeed, it was only used in the last lines of the proof of Lemma 4.3. However, the minimal sufficient conditions are impossible to formulate in terms of the control/observation operators of the subsystems. Instead, we would have to consider the whole composed system. The requirement is that the control operator of the composed system has to remain surjective despite the couplings in the closed vertices.

Remark 6.3. All results in this paper require the colligations to be strong in the sense of Definition 2.1. As mentioned before, there are internally well-posed boundary nodes (in the sense of [20: Definition 2.2]) that are even impedance conservative and satisfy $\mathcal{U} = \mathcal{Y}$ but are not strong. One such example is given in terms of the boundary controlled wave equation on $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, see [20: Proposition 6.3]. However, the same PDE with the same boundary control can be written as a strong node at the cost of $\mathcal{U} \neq \mathcal{Y}$; these spaces are still a dual pair. Note that Theorem 3.4 can be applied also in this case even though the smoothness assumption on $\partial\Omega$ seriously restricts the possible couplings of this kind of systems.

Remark 6.4. Using the words input and output for Gz and Kz is somewhat misleading. In fact, since our coupling equations (8) and (9) include conditions for both Gz and Kz, we have to assume that also the flow-inverted system is solvable, that is, if G and K are interchanged. For many systems this is not a serious restriction and, in fact, the whole concept of *abstract boundary value spaces* (introduced in [8]) is based on the existence of such interchangeable pair of possible boundary values. See also [4] for a study of compositions of systems using such abstract boundary spaces and [13] for an introduction of *state/signal systems* that are based on equal treatment of inputs and outputs.

References

[1] Arendt, W., Batty, C., Hieber, M., and Neubrander, F. (**2001**). Vector-valued Laplace Transforms and Cauchy Problems, Birkhäuser Verlag.

[2] Cervera, J., van der Schaft, A. J., and Baños, A. (**2007**). "Interconnection of port-Hamiltonian systems and composition of Dirac structures," Automatica (J. of IFAC) **43**, 212–225.

[3] Curtain, R. F. and Zwart, H. (**1995**). An Introduction to Infinite-Dimensional Linear Systems Theory, Springer-Verlag, New York.

[4] Derkach, V., Hassi, S., Malamud, M., and de Snoo, H. (**2006**). "Boundary relations and their Weyl families," Transactions of the American Mathematical Society **358**, 5351–5400.

[5] Engel, K.-J., Kramar Fijavž, M., Nagel, R., and Sikolya, E. (**2008**). "Vertex control of flows in networks," Networks and Heterogeneous Media **3**, 709–722.

[6] Fattorini, H. (**1968**). "Boundary control systems," SIAM Journal of Control **6**, 349–385.

[7] Feng, K. and Shi, Z.-C. (**1996**). Mathematical Theory of Elastic Structures, Springer-Verlag.

[8] Gorbachuk, V. and Gorbachuk, M. (**1991**). Boundary Value Problems for Operator Differential Equations, Kluwer Academic Publishers.

[9] Greiner, G. (**1987**). "Perturbing the boundary conditions of a generator," Houston Journal of Mathematics **13**, 213–229.

[10] Hannukainen, A., Lukkari, T., Malinen, J., and Palo, P. (**2007**). "Vowel formants from the wave equation," Journal of Acoustical Society of America Express Letters **122**.

[11] Jutila, A. (**2010**). "Emeritusprofessori: Myllysilta oli tietoinen riski," http://yle.fi/.

[12] Kuchment, P. and Zeng, H. (**2001**). "Convergence of spectra of mesoscopic systems collapsing onto a graph," Journal of Mathematical Analysis and Applications **258**, 671–700.

[13] Kurula, M. (**2010**). Towards Input/Output-Free Modelling of Linear Infinite-Dimensional Systems in Continuous Time, Ph.D. thesis, Åbo Akademi.

[14] Kurula, M., Zwart, H., van der Schaft, A., and Behrndt, J. (**2010**). "Dirac structures and their composition on Hilbert spaces," Journal of Mathematical Analysis and Applications **372**, 402–422.

[15] Latushkin, Y. and Pivovarchik, V. (**2008**). "Scattering in a forked-shaped waveguide," Integral Equations and Operator Theory **61**, 365–399.

[16] Livšic, M. S. (**1973**). Operators, Oscillations, Waves (open systems), Translations of Mathematical Monographs, vol. 34, American Mathematical Society, Providence, Rhode Island.

[17] Lukkari, T. and Malinen, J. (**2010**). "Webster's model with curvature and dissipation," Manuscript, 21 pp .

[18] Malinen, J. (**2004**). "Conservativivity of time-flow invertible and boundary control systems," Tech. rep., Helsinki University of Technology, Institute of Mathematics, Research Reports, A479.

[19] Malinen, J. and Staffans, O. (**2006**). "Conservative boundary control systems," Journal of Differential Equations **231**, 290–312.

[20] Malinen, J. and Staffans, O. (**2007**). "Impedance passive and conservative boundary control systems," Complex Analysis and Operator Theory **1**, 279–300.

[21] Malinen, J., Staffans, O., and Weiss, G. (**2006**). "When is a linear system conservative," Quarterly of Applied Mathematics **65**, 61–91.

[22] Rubinstein, J. and Schatzman, M. (**2001**). "Variational problems on multiply connected thin strips I: Basic estimates and convergence of the Laplacian spectrum," Archive for Rational Mechanics and Analysis **160**, 271–308.

[23] Salamon, D. (**1987**). "Infinite dimensional linear systems with unbounded control and observation: A functional analytic approach," Transactions of the American Mathematical Society **300**, 383–431.

[24] Salamon, D. (**1989**). "Realization theory in Hilbert space," Mathematical Systems Theory **21**, 147–164.

[25] Staffans, O. (**2005**). Well-Posed Linear Systems, Cambridge University Press.

[26] Thomson, W. (**1993**). Theory of Vibration with Applications, 4th ed., Chapman & Hall.

[27] Villegas, J. (**2007**). A Port-Hamiltonian Approach to Distributed Parameter Systems, Ph.D. thesis, University of Twente.

[28] Weiss, G. (**1994**). "Regular linear systems with feedback," Mathematics of Control, Signals, and Systems **7**, 23–57.