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A Lower Bound for the Differences of Powers of Linear Operators

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Abstract Let T be a bounded linear operator in a Banach space, with $\sigma(T) = \{1\}$. In 1983, Esterle–Berkani' s conjecture was proposed for the decay of differences $(I - T)\,T^n$ as follows: Either

$$\liminf_{n\to\infty} (n+1)\|\,(I-T)\,T^n\| \geq 1/e$$

or $T = I$. We prove this claim and discuss some of its consequences.

Keywords Esterle–Berkani's conjecture, Quasi-nilpotent linear operator, Differences of powers, Decay
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1 Introduction

Let $T \in \mathscr{L}(X)$, a bounded linear operator in a (complex) Banach space X. The following result by Esterle holds, see [1, Corollary 9.5]:

Proposition 1 *Let* $T \in \mathscr{L}(X)$ *satisfy* $\sigma(T) = \{1\}$. *If* $T \neq I$ *then* $\liminf_{n\to\infty}(n+1)\|(I-T)\,T^n\| \geq \frac{1}{96}$.

Berkani improved the lower bound to $1/12$, and he conjectured that the best lower bound is $1/e$, see [2]. That $1/e$ has a special role in related estimates can also be seen in the following remark by Nevanlinna, see [3, Theorem 4.5.1]:

Proposition 2 *Assume that there exists* $\{\lambda_j\} \subset \sigma(T)$ *such that* $|\lambda_j| < 1$ *and* $|\lambda_j| \to 1$ *as* $j \to \infty$. *Then* $\limsup_{n\to\infty} (n+1)\|(I-T)T^n\| \geq \frac{1}{e}$.

The constant $1/e$ also appears in the well-known "continuous time" case [4, Theorem 10.3.6].

In this paper, we show that Berkani's and Esterle's conjecture is right in the sense that Proposition 1 holds with $1/96$ replaced by $1/e$. We use a related but more careful analysis that has already been used in [1], involving the univalent functions $g_n(z) = z(1-z)^n$. Also we give another variant of Proposition 2 without restrictions on $\sigma(T)$.

All of these results were first presented in [5] (Yuan, 2002) with somewhat longer proofs. That $1/e$ in Proposition 1 is a valid lower bound, is also proved in [6] (Kalton, Montgomery-Smith, Oleszkiewicz, and Tomilov, 2002) by quite different means. Both of the existing approaches can be generalized to a larger class of results, but these respective classes are different (and we shall not discuss these generalizations here). An example is given in [6], indicating that the constant $1/e$ is the best possible. The construction is a modification of an example given in [7] (Lyubich, 2001).

2 Estimating $\liminf_{n\to\infty} (n+1)\|(I-T)T^n\|$

Denote $\mathbb{D}(R) := \{z \in \mathbb{C} : |z| < R\}$, and let $g : \mathbb{D}(R) \to \mathbb{C}$ be an analytic function satisfying $g(0) = 0$ and $g'(0) \neq 0$. Then there exists a maximal radius R_u, $0 < R_u \leq R$, such that g is a *univalent* (i.e. an injective analytic) function on the disk $\mathbb{D}(R_u)$. It is then easy to see that

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the image of $g(\mathbb{D}(R_u))$ contains an open disc, centered at origin. Let $0 < c < \infty$ be the largest radius such that $\mathbb{D}(c) \subset g(\mathbb{D}(R_u))$. Then there exists an analytic function $f : \mathbb{D}(c) \to \mathbb{D}(R_u)$ such that

$$(g \circ f)(z) := g(f(z)) = z \quad \text{for all} \quad z \in \mathbb{D}(c). \tag{1}$$

We denote the spectral radius of $L \in \mathscr{L}(X)$ by $\rho(L)$. If $\rho(L) = 0$, then L is called *quasi-nilpotent*. With these notations, we can prove the following proposition:

Proposition 3 *Let $g : \mathbb{D}(R) \to \mathbb{C}$ be an analytic function such that $g(0) = 0$ and $g'(0) \neq 0$. Let the constants c and R_u be as above. Then, for all $0 < \eta < 1$,*

$$\inf\{\|g(L)\| : L \in \mathscr{L}(X), \rho(L) = 0, \|L\| \geq R_u\eta(1-\eta)^{-1}\} \geq \eta c.$$

Proof The proof is carried out by showing that the set

$$\{L \in \mathscr{L}(X) : \rho(L) = 0, \|g(L)\| < \eta c, \|L\| \geq R_u\eta(1-\eta)^{-1}\}$$

is empty for all $0 < \eta < 1$. This is achieved by using the Cauchy estimates for the function f defined in (1). Denote the power series representations by $f(z) = \sum_{j\geq 1} a_j z^j$ and $g(z) = \sum_{j\geq 1} b_j z^j$. Clearly $f : \mathbb{D}(c) \to \mathbb{D}(R_u)$ means that $\sup_{|z|<c} |f(z)| \leq R_u$, and then the Cauchy estimates give $|a_j|r^j \leq R_u$ for each $r < c$ and $j \geq 1$. Letting $r \to c-$, we get that $|a_j|c^j \leq R_u$ for all $j \geq 1$.

Let $L \in \mathscr{L}(X)$ be an arbitrary quasi-nilpotent operator. Then $g(L)$ is quasi-nilpotent by the spectral mapping theorem, as $g(0) = 0$. Similarly $Y := f(g(L))$ is also quasi-nilpotent. Now let $0 < \eta < 1$, and assume that $\|g(L)\| < \eta c$. It now follows from the above Cauchy estimates that

$$\|Y\| \leq \sum_{j\geq 1} |a_j| \cdot \|g(L)\|^j < \sum_{j\geq 1} |a_j|c^j \cdot \eta^j \leq R_u\eta(1-\eta)^{-1};$$

hence $\|Y\| < R_u\eta(1-\eta)^{-1}$.

We proceed to show that $Y = L$. Since Y is quasi-nilpotent, $g(Y)$ is well-defined. By the associativity $g(Y) = g[f(g(L))] = g(f[g(L)]) = (g \circ f)(g(L)) = g(L)$ because $(g \circ f)(z) = z$ for any $z \in \mathbb{D}(c)$. As $g(0) = 0$, it follows that $\sigma(g(L)) = \{0\} \subset \mathbb{D}(c)$. Using the power series of g, we get

$$\begin{aligned} 0 = g(Y) - g(L) &= \sum_{j\geq 1} b_j Y^j - \sum_{j\geq 1} b_j L^j \\ &= (Y - L)\Big(b_1 I + \sum_{j\geq 2} b_j \left[Y^{j-1} + Y^{j-2}L + \cdots + L^{j-1}\right]\Big) \\ &= (Y - L)(b_1 I + U), \end{aligned} \tag{2}$$

where $b_1 = g'(0) \neq 0$ and $U := \sum_{j\geq 2} b_j \left[Y^{j-1} + Y^{j-2}L + \cdots + L^{j-1}\right]$.

We know that $Y = f(g(L))$ is quasi-nilpotent, and it is actually a function of L. We now consider function h defined in $\mathbb{D}(R_u)$ as follows:

$$h(z) := \sum_{j\geq 2} b_j \left[f(g(z))^{j-1} + f(g(z))^{j-2}z + \cdots + z^{j-1}\right].$$

Then $h(z)$ is analytic in $\mathbb{D}(R_u)$ and $h(0) = 0$. So $h(L)$ is well defined and $U = h(L)$. Since both L and Y are quasi-nilpotent, we see that U is quasi-nilpotent. Therefore $b_1 I + U$ is boundedly invertible. This together with (2) implies that $Y = L$. Hence, for any $0 < \eta < 1$ and any quasi-nilpotent $L \in \mathscr{L}(X)$, $\|g(L)\| < \eta c \Rightarrow \|L\| = \|Y\| < R_u\eta(1-\eta)^{-1}$. This proves the claim.

A somewhat analogous result to the previous proposition is [6, Theorem 4.5]. We proceed to study the functions

$$g_n(z) := (1 - z)^n z \quad \text{for } n \geq 1, \tag{3}$$

that also made their appearance in Esterle's original argument. We shall make use of the constants $R_u^{(n)}$ and $c^{(n)}$ defined as follows:

1) $R_u^{(n)} > 0$ is the largest radius of an open disc $\mathbb{D}(R_u^{(n)})$ such that $g_n(z)$ is univalent in $\mathbb{D}(R_u^{(n)})$;

2) $c^{(n)} > 0$ is the largest radius of an open disc $\mathbb{D}(c^{(n)})$ such that $\mathbb{D}(c^{(n)}) \subset g_n(\mathbb{D}(0, R_u^{(n)}))$.

Because $g_n'(z) = (1-z)^{n-1}(1-(n+1)z)$ and hence $g_n'(1/(n+1)) = 0$, it follows, by the elementary function theory, that $R_u^{(n)} \le 1/(n+1)$. The next proposition shows that equality holds here.

Proposition 4 *The functions* $g_n(z) = (1-z)^n z$ *are univalent in the disc* $\mathbb{D}(1/(n+1))$ *for all* $n \ge 1$.

Proof Let $z = re^{i\phi} \in \mathbb{C}$, where $0 \le r < 1/(n+1)$ and $\phi \in \mathbb{R}$. Now $g_n(z) = R(r,\phi)\ e^{i\Phi(r,\phi)}$, where $r_\phi = \sqrt{1 - 2r\cos(\phi) + r^2}$, $\Phi(r,\phi) = \phi - n\arcsin(r\sin(\phi)/r_\phi)$ and $R(r,\phi) = r \cdot r_\phi^n$; note that $\arcsin : [-1,1] \to [-\pi/2, \pi/2]$ is the inverse function of $\sin : [-\pi/2, \pi/2] \to [-1,1]$. Mapping $\phi \mapsto \Phi(r,\phi)$ is injective on $\mathbb{R}$, because, by writing $t = \cos(\phi)$,

$$\begin{aligned}\frac{\partial \Phi(r,\phi)}{\partial \phi} &= \left(1 - (n+2)rt + (n+1)r^2\right)\left(1 - 2rt + r^2\right)^{-1} \\ &\ge \left(1 - (n+2)r + (n+1)r^2\right)\left(1 - 2rt + r^2\right)^{-1} \\ &= (1-r)\left(1-(n+1)r\right)\left(1 - 2rt + r^2\right)^{-1} > 0,\end{aligned}$$

where the last estimate follows as $r < 1/(n+1)$. Notice, furthermore, that $\Phi(r, 2\pi k) = 2\pi k$ for every $k \in \mathbb{Z}$. Moreover, if ϕ is fixed then $\frac{\partial R(r,\phi)}{\partial r} = \frac{\partial \Phi(r,\phi)}{\partial \phi}\left(1 - 2rt + r^2\right)^{n/2} > 0$. Hence $r \mapsto R(r,\phi)$ is injective on $[0, 1/(n+1))$, and the claim follows.

In other words, we have now proved that $R_u^{(n)} = 1/(n+1)$ for all $n \ge 1$. The other sequence of constants can be determined easily.

Proposition 5 *The constants* $c^{(n)}$ (*as introduced earlier*) *satisfy* $c^{(n)} = \frac{1}{n+1}(1 - \frac{1}{n+1})^n$, *for all* $n \ge 1$.

Proof Clearly, for any fixed n, $c^{(n)} = \inf_{z \in \partial\mathbb{D}(R_u^{(n)})} |g_n(z)|$. Since $|(1-z)^n z| \ge (1 - R_u^{(n)})^n R_u^{(n)}$, for all z satisfying $|z| = R_u^{(n)}$, we get $c^{(n)} \ge \frac{1}{n+1}(1 - \frac{1}{n+1})^n$ as $R_u^{(n)} = 1/(n+1)$ by Proposition 4. By choosing $z = R_u^{(n)}$, we see that even the equality holds.

Now we are prepared to prove our main result. The required improvement of Proposition 1 follows by taking $L = I - T$ in the following theorem:

Theorem 1 *Let* $L \in \mathscr{L}(X)$, $L \neq 0$, *be quasi-nilpotent. Then* $\liminf_{n\to\infty}(n+1)\|(I-L)^n L\| \ge \frac{1}{e}$.

Proof Define the functions g_n and the constants $R_u^{(n)}$, $c^{(n)}$ as earlier. Let $0 < \eta < 1$ be arbitrary. Since by Proposition 4, $R_u^{(n)}\eta(1-\eta)^{-1} = \frac{1}{n+1} \cdot \eta(1-\eta)^{-1} \to 0$ as $n \to \infty$, there exists $N(\eta) < \infty$, such that, for all $n \ge N(\eta)$, we have $\|L\| \ge R_u^{(n)}\eta(1-\eta)^{-1}$. By Proposition 3 (with $g = g_n$) and Proposition 5, we have, for all $n \ge N(\eta)$,

$$\|(I-L)^n L\| \ge \eta c^{(n)} = \eta\,\frac{1}{n+1}\left(1 - \frac{1}{n+1}\right)^{n+1}.$$

Since $\lim_{n\to\infty}(1 - \frac{1}{n+1})^{n+1} = 1/e$, we get, by letting $n \to \infty$, $\liminf_{n\to\infty}(n+1)\|(I-L)^n L\| \ge \eta/e$. Because $0 < \eta < 1$ is arbitrary, the claim follows by letting $\eta \to 1$.

3 Estimating $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\|$

Theorem 2 *For any* $T \in \mathscr{L}(X)$, *either*

(i) $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| \ge 1/e$; *or,*

(ii) $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| = 0$ *holds.*

Proof If $\limsup_{n\to\infty}(n+1)\|(I-T)T^n\| = \infty$ or $T = I$, then the claim holds. It remains to consider the case when $\sup_{n\ge 0}(n+1)\|(I-T)T^n\| < \infty$ and $T \neq I$. By [3, Theorem 4.2.2],

$\sigma(T) \subset \mathbb{D}(1) \cup \{1\}$.

If $1 \notin \sigma(T)$, then $\|T^n\| \le M r^n$ for some $0 \le r < 1$, and (ii) follows. If 1 is an accumulation point of $\sigma(T)$, then (i) holds, by Proposition 2. If 1 is an isolated point, then either $\sigma(T) = \{1\}$ or there is a positive distance between 1 and $\sigma(T) \setminus \{1\}$. If $\sigma(T) = \{1\}$, then (i) holds by Theorem 1.

To complete the proof, we can assume $\operatorname{dist}(1, \sigma(T) \setminus \{1\}) > 0$. There exist closed, nonintersecting curves Γ_1 and Γ_2 with the following properties: Γ_1 lies strictly inside the open unit disc $\mathbb{D}(1)$ and it surrounds the set $\sigma(T) \setminus \{1\}$; Γ_2 surrounds point 1. Define the bounded spectral projections P_1 and P_2, together with the corresponding closed subspaces $P_1 := \frac{1}{2\pi i} \int_{\Gamma_1} (\lambda - T)^{-1} d\lambda$, $P_2 := \frac{1}{2\pi i} \int_{\Gamma_2} (\lambda - T)^{-1} d\lambda$, $X_1 := P_1 X$ and $X_2 := P_2 X$.

Both X_1 and X_2 are invariant for T, $X_1 \cap X_2 = \{0\}$ and $X = X_1 + X_2$. They inherit their norms from X, and X itself is isometrically isomorphic to the exterior direct sum $\begin{smallmatrix} X_1 \\ \times \\ X_2 \end{smallmatrix}$, equipped with the norm $\|[x_1 \quad x_2]^T\|_{X_1 \times X_2} := \|x_1 + x_2\|$ for all $x_1 \in X_1$, $x_2 \in X_2$. Define the bounded operators L and M by $L := T|X_1 \in \mathscr{L}(X_1)$ and $M := T|X_2 \in \mathscr{L}(X_2)$. Then T is isometrically equivalent to the block matrix $\left[\begin{smallmatrix} L & 0 \\ 0 & M \end{smallmatrix}\right] : \begin{smallmatrix} X_1 \\ \times \\ X_2 \end{smallmatrix} \to \begin{smallmatrix} X_1 \\ \times \\ X_2 \end{smallmatrix}$, and $(I - T)T^n$ is represented (apart from an isometric isomorphism) by $\left[\begin{smallmatrix} (I_{X_1} - L)L^n & 0 \\ 0 & (I_{X_2} - M)M^n \end{smallmatrix}\right]$. By the triangle inequality

$$\begin{aligned} \|(I - T)T^n\| &= \left\| \begin{bmatrix} (I_{X_1} - L)L^n & 0 \\ 0 & (I_{X_2} - M)M^n \end{bmatrix} \right\|_{\mathscr{L}(X_1 \times X_2)} \\ &\ge \left\| \begin{bmatrix} 0 & 0 \\ 0 & (I_{X_2} - M)M^n \end{bmatrix} \right\|_{\mathscr{L}(X_1 \times X_2)} - \left\| \begin{bmatrix} (I_{X_1} - L)L^n & 0 \\ 0 & 0 \end{bmatrix} \right\|_{\mathscr{L}(X_1 \times X_2)} \\ &= \|(I_{X_2} - M)M^n\|_{\mathscr{L}(X_2)} - \|(I_{X_1} - L)L^n\|_{\mathscr{L}(X_1)}. \end{aligned} \tag{4}$$

The spectra of L and M satisfy $\sigma(L) = \sigma(T) \setminus \{1\} \subset \mathbb{D}(1)$ and $\sigma(M) = \{1\}$. It follows again immediately that $\lim_{n\to\infty} (n+1)\|(I_{X_1} - L)L^n\|_{\mathscr{L}(X_1)} = 0$. By Theorem 1, $\limsup_{n\to\infty} (n+1)\|(I_{X_2} - M)M^n\|_{\mathscr{L}(X_2)} \ge 1/e$. Therefore (4) implies

$$\limsup_{n\to\infty} (n+1)\|T^n(T-1)\| \ge \limsup_{n\to\infty} (n+1)\|(I_{X_2} - M)M^n\|_{\mathscr{L}(X_2)} \ge 1/e,$$

and the proof is completed.

The lower bound $1/e$ in Theorem 2 can be reached, see [3, Example 4.5.2].

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