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Abstract: *We give sufficient and necessary conditions for a boundary control system (in the sense of Salamon) to define a Livšic – Brodskiĭ operator node; that is, a linear (scattering) conservative system. This appears to be a special case of a more general result involving time-flow invertible linear systems. We show the following result of "entropy type": the dual system of a conservative boundary control system is always a boundary control system. Five PDE examples are given to illuminate how these operator theory techniques apply to (rather simple) concrete problems. In particular, a fairly complete example on the n-dimensional wave equation is given.*

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1 Introduction

In this paper, we give sufficient and necessary conditions in Theorems 7 and 8 for the conservativity of linear boundary control systems. Such systems are described by differential equations of form

$$\begin{cases} \dot{z}(t) = Lz(t), \\ Gz(t) = u(t), \\ y(t) = Kz(t) \quad \text{for all } t \geq 0. \end{cases} \tag{1.1}$$

All of the conditions in Theorems 7 and 8 are stated in terms of data given; namely the operators L, K, and G, together with the Hilbert spaces they are defined on. We shall give five PDE examples to indicate that these results are practically applicable in concrete problems. However, our abstract setting does not require any of the operators in (1.1) to be a partial differential operator.

What is a (scattering) conservative linear system? By general linear systems we mean *system/operator nodes*; see [13, 19] and the references therein, including the classical works [1, 6, 7, 8, 17, 18, 20]. We assume henceforth that the reader is familiar with such nodes; reading [13, Section 2] gives a sufficient background. A boundary control system of form (1.1) always defines an operator node, see Definition 1 and Section 2 for details.

Now, let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node. The (separable) Hilbert spaces U, Y, X, $X_1 := \mathcal{D}(A)$ and $X_{-1} := \mathcal{D}(A^*)^d$ are defined as usual for system nodes. By $A : X_1 \to X$, $B \in \mathcal{L}(U; X_{-1})$, and $C \in \mathcal{L}(X_1; Y)$ denote the main operator, input operator, and the output operator of S, respectively. Assume that the functions $u(\cdot) \in C^2(\mathbb{R}_+; U)$, $z(\cdot) \in C^1(\mathbb{R}_+; X)$, $y(\cdot) \in C^1(\mathbb{R}_+; Y)$ satisfy the differential equation associated to S:

$$\begin{cases} \dot{z}(t) = A_{-1}z(t) + Bu(t), \\ y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix} \quad \text{for all } t \geq 0; \end{cases} \tag{1.2}$$

see [13, Proposition 2.5] for details. We say that S is *energy preserving* if for any (sufficiently smooth) input $u(\cdot)$ and any (compatible) initial state $z(0) = z_0$, the unique solution of (1.2) satisfies the *energy balance equation* $\frac{d}{dt}\|x(t)\|_X^2 = \|u(t)\|_U^2 - \|y(t)\|_Y^2$, see [13, Definition 3.1]. That S is *conservative* means that both S and S^d are energy preserving. Here S^d denotes the *dual system node* of S as described in [13, Proposition 2.3].

This notion of conservativity is the "right one" in the sense that is a direct extension of the well-known finite dimensional case. Hence the definition of conservativity refers directly to S^d. Unfortunately, it is less than obvious to relate S^d to the operators appearing in (1.1) – the data of a typical boundary control problem. Solving these complications is the purpose of this paper.

Following [13], a conservative system node S is said to be *tory* (or a *Julia colligation*) if $\operatorname{Ker} B = \{0\}$ and $(\operatorname{Ran} C)^\perp = \{0\}$. A powerful characterisation of tory nodes is given in [13, Theorem 4.4]. The main results of this paper –

Theorems 3, 7 and 8 – are based on this theorem. These results are applied to a number of PDE examples. In particular, a fairly complete treatment (apart from the exponential stability) of the boundary controlled (scattering) conservative wave equation is given.

2 Background

We develop the required background results for boundary control nodes and show their equivalence to usual operator nodes (of boundary control type). We review the related Cauchy problem, too.

Definition 1. *Assume that U, X and Y are separable Hilbert spaces.*

(i) *Assume that Z is a Hilbert space, such that $Z \subset X$ with a bounded dense inclusion. Let $L \in \mathcal{L}(Z;X)$, $G \in \mathcal{L}(Z;X)$ and $K \in \mathcal{L}(Z;Y)$ be operators such that the following conditions hold for some $\alpha \in \overline{\mathbb{C}_+}$:*

   (a) $U = \mathrm{Ran}\ G$,

   (b) $\mathrm{Ker}\ G$ *is dense in* X,

   (c) $(\alpha - L)\mathrm{Ker}\ G = X$, *and*

   (d) $\mathrm{Ker}\ (\alpha - L) \cap \mathrm{Ker}\ G = \{0\}$.

   Then the triple $\Gamma = (L, G, K)$ is called a boundary control node. *The space Z is* the solution space *of* Γ.

(ii) *If both $\Gamma = (L, G, K)$ and $\Gamma^{\leftarrow} := (-L, K, G)$ are boundary control nodes, then Γ is called a* doubly *boundary control node.*

(iii) *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix} \right]$ be an operator node on spaces U, X and Y as in [13, Definition 2.2]. Then S is called an operator node of* boundary control type *(in the sense of Salamon), if $\rho(A) \cap \overline{\mathbb{C}_+} \neq \emptyset$,* $\mathrm{Ker}\ B = \{0\}$ *and* $BU \cap X = \{0\}$.

The spaces U, X, and Y are called input, state, *and* output *spaces of both Γ and S, respectively.*

Each boundary control node defines a Cauchy problem through equations (1.1). The assumptions on operators L, G, and K are such that this Cauchy problem is "correctly posed" in a sense related to "correct posedness" for operator nodes[1] and their Cauchy problems (1.2). However, the assumptions of Definition 1 alone do not imply the existence of a (weak, strong) solution $z(\cdot)$ of either (1.1) or (1.2) – something more involving the generation of a C_0-semigroup in X is required.

[1] In PDE applications, checking this "correct posedness " requires certain *a priori* estimates involving the partial differential operators and Sobolev spaces. The abstract functional analysis framework does not and cannot take part in this. See how the elliptic regularity theory is used in Subsection 7.4.

Suppose that $\Gamma = (L, G, K)$ and $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ satisfy the conditions of Definition 1. It is described in Subsections 2.1 and 2.2 that such Γ and S are in one-to-one correspondence. Moreover, the two Cauchy problems (1.1) and (1.2) will then have the same solutions. This translation is essentially the same as given in [15, 16] in a different notation (but, unfortunately, with a small mistake[2]). For an earlier and somewhat different approach dealing mainly with controllability questions, see [4].

2.1 Towards operator nodes

We shall now show that any boundary control node Γ defines an operator node S in the sense of [13, Definition 2.2]. Let us first make sense about the main operator A and its domain X_1.

Proposition 1. *Let $\Gamma = (L, G, K)$ be a boundary control node on Hilbert spaces U, Z, X and Y. Define the space X_1 and the linear mapping A by setting*

$$X_1 := \text{Ker } G \quad \text{ and } A := L|X_1 : X_1 \to X.$$

Let $\alpha \in \overline{\mathbb{C}_+}$ satisfy conditions (c) *and* (d) *of Definition 1. Then*

(i) *X_1 is a closed subspace of Z, and it inherits a Hilbert space norm $\|\cdot\|_{X_1'}$ from Z,*

(ii) *$A \in \mathcal{L}(X_1; X)$ when X_1 is given the norm $\|\cdot\|_{X_1'}$. Moreover, $A : \mathcal{D}(A) \subset X \to X$ is an unbounded, closed, densely defined linear operator with $\mathcal{D}(A) = X_1$ and $\alpha \in \rho(A)$, and*

(iii) *X_1 is a Hilbert space under the norm $\|x\|_{X_1} := \|(\alpha - A)x\|_X$, and this norm $\|\cdot\|_{X_1}$ is equivalent to norm $\|\cdot\|_{X_1'}$.*

Proof. Claim (i) follows because $G \in \mathcal{L}(Z; U)$, and we attack claim (ii). For clarity, let X_1' denote the Hilbert space Ker G equipped with $\|\cdot\|_{X_1'}$. Firstly, $\mathcal{D}(A)$ is dense in X by condition (b) of Definition 1. To show that $\alpha \in \rho(A)$, take any $x \in X_1'$. We have $A \in \mathcal{L}(X_1'; X)$ since

$$\|Ax\|_X = \|Lx\|_X \le \|L\|_{\mathcal{L}(Z;X)} \cdot \|x\|_Z = \|L\|_{\mathcal{L}(Z;X)} \cdot \|x\|_{X_1'}.$$

Because $Z \subset X$ with a bounded inclusion, we have $\|x\|_X \le C\|x\|_Z = C\|x\|_{X_1'}$ for any $x \in X_1'$. Hence, $X_1' \subset X$ with a bounded inclusion, and it follows that $\alpha - A \in \mathcal{L}(X_1'; X)$, too. By condition (c) of Definition 1, $\alpha - A : X_1' \to X$ is surjective. By condition (d), it is injective, too. Hence, there exists a bounded inverse $(\alpha - A)^{-1} : X \to X_1'$. Because $X_1' \subset X$ with a bounded inclusion, in fact $(\alpha - A)^{-1} \in \mathcal{L}(X)$ and $\alpha \notin \sigma(A)$. In particular, A is a densely defined operator on X, with domain $\mathcal{D}(A) = \text{Ran } (\alpha - A)^{-1} = X_1$.

[2] This mistake was independently discovered by G. Weiss and the author.

Now the last claim (iii). Because $(\alpha - A)^{-1} : X \to X_1'$ is a bounded bijection with a bounded inverse, it follows that

$$\frac{\|x\|_{X_1'}}{\|(\alpha - A)^{-1}\|_{\mathcal{L}(X;X_1')}} \leq \|(\alpha - A)x\|_X \leq \|\alpha - A\|_{\mathcal{L}(X_1';X)} \cdot \|x\|_{X_1'}.$$

Hence, the norm $\|\cdot\|_{X_1}$ is equivalent to the inherited norm $\|\cdot\|_{X_1'}$. $\square$

From now on, we always use the norm $\|x\|_{X_1} := \|(\alpha - A)x\|_X$ on X_1. By X_{-1} denote the completion of X in norm $\|x\|_{X_{-1}} := \|(\alpha - A)^{-1}x\|$. Regard X as a subspace of X_{-1} with the natural inclusion operator coming from the completion process. As is well known in the context of rigged Hilbert spaces, $A : X_1 \to X$ has a linear extension to an operator $A_{-1} : X \to X_{-1}$ satisfying $A_{-1} \in \mathcal{L}(X, X_{-1})$.

Next we extract the input operator $B \in \mathcal{L}(U; X_{-1})$ from Γ. We also show that the norm of Z is equivalent to another norm that can easily be expressed with the aid of A_{-1} and B.

Proposition 2. *Let $\Gamma = (L, G, K)$ be a boundary control node on Hilbert spaces U, Z, X and Y. Let X_1 and A be as in Proposition 1, and let $\alpha \in \rho(A)$ be arbitrary. Then*

(i) *there exists a unique operator $B \in \mathcal{L}(U; X_{-1})$ satisfying the equation*

$$Lz = (A_{-1}|Z)z + BGz \quad \text{for all } z \in Z, \tag{2.1}$$

(ii) *we have $(\alpha - A_{-1})^{-1}B \in \mathcal{L}(U; Z)$, $G(\alpha - A_{-1})^{-1}B = I$ and* Ker $B = \{0\}$,

(iii) *$X_1 \cap (\alpha - A_{-1})^{-1}BU = \{0\}$, $Z = X_1 \dot{+} (\alpha - A_{-1})^{-1}BU$, and the norm of Z is equivalent to the Hilbert space norm*

$$\|z\|^2_{X_1 + (\alpha - A_{-1})^{-1}BU} = \|x\|^2_{X_1} + \|u\|^2_U \quad \text{where } z = x + (\alpha - A_{-1})^{-1}Bu. \tag{2.2}$$

Proof. Because $G \in \mathcal{L}(Z; U)$ is surjective, there exists a right inverse $H \in \mathcal{L}(U; Z)$ such that $GH = I$ on all of Z. Define $B := (L - A_{-1}|Z)H$. Because $L \in \mathcal{L}(Z; X)$ and $X \subset X_{-1}$ with a bounded inclusion, it follows that $L \in \mathcal{L}(Z; X_{-1})$. Because $Z \subset X$ with a bounded inclusion, $A_{-1}|Z \in \mathcal{L}(Z; X_{-1})$ and hence $B \in \mathcal{L}(U; X_{-1})$. It is clear from construction that $BGz = (L - A_{-1}|Z)Gz$ for $z = Hu$, $u \in U$. Since $Z = \text{Ker } G \dot{+} \text{Ran } H$, equation (2.1) follows.

If there were two operators $B_1, B_2 \in \mathcal{L}(U; X_{-1})$ satisfying (2.1) with $B = B_1, B_2$, then their difference would satisfy $(B_1 - B_2)u = 0$ for all $u \in$ Ran $(G) = U$. Thus B is uniquely defined and does not depend on the particular choice of the right inverse H.

In order to prove (ii), let $\alpha \in \rho(A)$ and $u \in U$ be arbitrary. We start with the identity $(\alpha - A_{-1}|Z)\, Hu - (\alpha - L)\, Hu = Bu \in X_{-1}$. Now we have (at least formally)

$$
\begin{aligned}
&G\,(\alpha - A_{-1})^{-1}\,(\alpha - A_{-1}|Z)\, Hu - G\,(\alpha - A_{-1})^{-1}\,(\alpha - L)\, Hu \\
&= G\,(\alpha - A_{-1})^{-1}\, Bu.
\end{aligned}
$$

The first term in the left is well-defined for all $u \in U$. By cancelling the resolvents and recalling $GHu = u$, we get (still formally)

$$
u - G\,(\alpha - A_{-1})^{-1}\,(\alpha - L)\, Hu = G\,(\alpha - A_{-1})^{-1}\, Bu. \tag{2.3}
$$

This time $Hu \in Z$ and hence $(\alpha - A)^{-1}\,(\alpha - L)\, Hu \in X_1$, no matter what value $\alpha \in \rho(A)$ attains. So the second term on the left of (2.3) is well-defined, too, and this computation verifies that $G\,(\alpha - A_{-1})^{-1}\, Bu$ is a well-defined element of U.

But $X_1 = \operatorname{Ker} G$ by definition, and hence the identity $I = G\,(\alpha - A_{-1})^{-1}\, B$ follows from (2.3). The above computations show that $(\alpha - A_{-1})^{-1}\, BU \subset Z$. As $(\alpha - A_{-1})^{-1}\, B \in \mathcal{L}(U;X)$ and $Z \subset X$ with a dense inclusion, it follows (by the compatibility of Banach spaces Z and X) that $(\alpha - A_{-1})^{-1}\, B \in \mathcal{L}(U;Z)$.

It remains to establish claim (iii). Suppose $x \in X_1 \cap (\alpha - A_{-1})^{-1}BU$, $x \neq 0$. As $X_1 = \operatorname{Ker}\, G$, then $Gx = 0$. As $x = (\alpha - A_{-1})^{-1}Bu$ for some $u \neq 0$, we have by claim (ii) that $Gx = u \neq 0$, a contradiction.

For any $z \in Z$, define $u := Gz \in U$. Then $x_1 := z - (\alpha - A_{-1})^{-1}Bu$ satisfies $Gx_1 = Gz - G(\alpha - A_{-1})^{-1}Bu = Gz - u = 0$. Hence $x_1 \in \operatorname{Ker}\, G = X_1$, and trivially $z = x_1 + (\alpha - A_{-1})^{-1}Bu$. This proves that $Z \subset X_1 + (\alpha - A_{-1})^{-1}BU$. The converse inclusion follows as we have already proved $(\alpha - A_{-1})^{-1}BU \subset Z$ for claim (ii).

It is clear that (2.2) defines another Hilbert space norm for Z. By a short estimation, we learn that for all $z \in Z \subset X$

$$
\|z\|_X \leq \max\left(1, \|(\alpha - A_{-1})^{-1}B\|_{\mathcal{L}(U;X)}\right) \cdot \left(\|x_1\|_{X_1} + \|u\|_U\right) \tag{2.4}
$$

where $z = x_1 + (\alpha - A_{-1})^{-1}Bu$ is the unique decomposition of $z \in Z$ according to $Z = X_1 \dot{+} (\alpha - A_{-1})^{-1}BU$.

It follows from (2.4) that the inclusion $Z \subset X$ is bounded, when Z is given the norm in (2.2). It is an explicit assumption that the inclusion $Z \subset X$ in bounded, with the original norm of Z. Hence, these two differently normed versions of Z are *compatible* Banach spaces, and their norms are accordingly equivalent. □

Note that the spaces X_1 and $(\alpha - A_{-1})^{-1}BU$ are orthogonal in Z, when Z is given the norm (2.2).

Proposition 3. *Let $\Gamma = (L, G, K)$ be a boundary control node on Hilbert spaces U, Z, X and Y. Let the spaces X_1, X_{-1} and operators A, A_{-1}, B be as in Propositions 1 and 2. Define the vector space*

$$
V := \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] : A_{-1}x + Bu \in X \right\} \tag{2.5}
$$

and equip it with the Hilbert space norm

$$\left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|_V^2 := \|x\|_X^2 + \|u\|_U^2 + \|A_{-1}x + Bu\|_X^2. \tag{2.6}$$

Then

(i) $V \subset \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]$ *with a bounded inclusion,*

(ii) *the operator* $C\&D : V \to Y$ *defined by*

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} := Kx \tag{2.7}$$

satisfies $C\&D \in \mathcal{L}(V;Y)$*, and*

(iii) *the identity* $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ *holds.*

Proof. To prove (i), we show the following functional analytic fact: if H_1, H_2, and H_3 are Banach spaces, if $H_2 \subset H_3$ with a bounded inclusion, and if $T \in \mathcal{L}(H_1;H_3)$ with $\operatorname{Ran} T \subset H_2$, then $T \in \mathcal{L}(H_1;H_2)$. By the closed graph theorem, it is enough to show that T is closed as a mapping from H_1 to H_2. Suppose $g_j \to g$ in H_1 and $Tg_j \to h$ in H_2. Since $H_2 \subset H_3$ with a bounded inclusion, $Tg_j \to h$ in H_3, too. Because $T \in \mathcal{L}(H_1;H_3)$, it follows that $h = Tg$. Hence T is closed as required.

Recalling that $Z \subset X$ with a bounded inclusion, claim (i) follows by setting $H_1 = V$, $H_2 = \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]$, $H_3 = \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$, and letting T be the natural inclusion from V to $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. To prove (ii), estimate for $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$

$$\begin{aligned}
&\left\|C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|_Y = \left\| \begin{bmatrix} K & 0 \end{bmatrix} \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|_Y \le \left\| \begin{bmatrix} K & 0 \end{bmatrix} \right\|_{\mathcal{L}(\left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right];Y)} \cdot \left\| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|_{\left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]} \\
&\le \|K\|_{\mathcal{L}(Z;Y)} \cdot C \left\| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|_V,
\end{aligned}$$

where C is the norm of the inclusion $V \subset \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]$.

To prove claim (iii), note that $V := \left\{ \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right] \;\middle|\; A_{-1}x + Bu \in X \right\}$ by claim (i) and let $\alpha \in \rho(A)$. Now, as $(\alpha - A)^{-1} : X_1 \to X$ is a bounded bijection, we have for any $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]$

$$\begin{aligned}
& A_{-1}x + Bu \in X \\
\Leftrightarrow \quad & (\alpha - A)^{-1} A_{-1}x + (\alpha - A_{-1})^{-1} Bu \in X_1 \\
\Leftrightarrow \quad & -x + \alpha(\alpha - A)^{-1}x + (\alpha - A_{-1})^{-1} Bu \in X_1 \\
\Leftrightarrow \quad & -x + (\alpha - A_{-1})^{-1} Bu \in X_1,
\end{aligned}$$

where the last equivalence holds since $\alpha(\alpha - A_{-1})^{-1}x \in X_1$ as $x \in Z \subset X$. Using $X_1 := \operatorname{Ker} G$ and $G(\alpha - A_{-1})^{-1}Bu = u$ completes the proof. $\square$

Now we have all the ingredients to put up an operator node of boundary control type in the sense of Definition 1:

Theorem 1. *Let $\Gamma = (L, G, K)$ be a boundary control node on Hilbert spaces U, Z, X and Y. Let the spaces X_1, X_{-1}, V and operators A, A_{-1}, B, $C\&D$ be as in Propositions 1, 2 and 3. Define $A\&B := [A_{-1} \quad B]\,|V$. Then $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator node with $\mathcal{D}(S) = V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$. Moreover, S is of the boundary control type in the sense that* Ker $B = \{0\}$ *and* Ran $B \cap X = \{0\}$.

Proof. All this follows from the properties of an operator node in [13, Section 2], and Propositions 1, 2 and 3. □

2.2 Towards boundary control nodes

Now we shall go to the converse direction: we show that any operator node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ of *boundary control type* defines an unique boundary control node $\Gamma = (L, G, K)$, see Definition 1. In this subsection, the spaces $X_1 = \mathcal{D}(A)$, X_{-1} and the operators $A \in \mathcal{L}(X_1; X)$, $A_{-1} \in \mathcal{L}(X; X_{-1})$, $B \in \mathcal{L}(U; X_{-1})$ are defined as usual for the operator node S. Moreover, we define

$$Z := X_1 + (\alpha - A_{-1})^{-1} BU \tag{2.8}$$

for some $\alpha \in \rho(A)$. The Hilbert space $V = \mathcal{D}(S)$ is given by (2.5) and (2.6); see [13, Section 2] for details.

Proposition 4. *Assume $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator node on Hilbert spaces U, X and Y, such that $BU \cap X = \{0\}$ and* Ker $B = \{0\}$. *Then the following holds:*

(i) *There exists a unique linear map $G \in \mathcal{L}(Z; U)$ such that $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$. Moreover, $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] \in \mathcal{L}(Z; V)$,* Ran $G = U$, *and* Ker $G = X_1$.

(ii) *There exists a unique linear mapping $K \in \mathcal{L}(Z; Y)$ satisfying $K = C\&D \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right]$, $Kx = C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$, and $C\&D = \left[K \quad 0\right] |V$. We have* Ran $K = Y$ *if and only if* Ran $C\&D = Y$.

(iii) *The operator $L : Z \to X$, defined by*

$$Lz := (A_{-1}|Z)z + BGz \quad \text{for all } z \in Z,$$

*satisfies $L \in \mathcal{L}(Z; X)$, $(\alpha - L)$*Ker $G = X$, *and* Ker $(\alpha - L) \cap$ Ker $G = \{0\}$ *for all $\alpha \in \rho(A)$.*

Proof. Let us start with claim (i). Because $BU \cap X = \{0\}$ and Ker $B = \{0\}$, for each $z \in Z \subset X$ there exists this time a *unique* $u \in U$ such that $A_{-1}z + Bu \in X$. Namely, if there were two, say $u_1 \neq u_2$, then $B(u_1 - u_2) \in BU \cap X$. Hence $B(u_1 - u_2) = 0$ and thus $u_1 = u_2$. Let us call this (well-defined) mapping $Z \ni z \mapsto u \in U$ by G.

Such G is clearly linear, and so is $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] : Z \to \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$. It follows from the definition of G that $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z \subset V$. Conversely, if $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$, then $A_{-1}x + Bu \in X$ and hence $x \in Z$. Then $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] x \in V$ and $u = Gx$ by a similar uniqueness argument as given above. It now follows that $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z = V$ and that $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] : Z \to V$ is a bijection, since the operator is trivially injective.

We proceed to show that $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] \in \mathcal{L}(Z;V)$ and $G \in \mathcal{L}(Z;U)$. As Z is complete, it is enough to show that $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right]$ is closable. Let $z_j \to 0$ in the norm of Z and $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] z_j \to \left[\begin{smallmatrix} z_0 \\ u_0 \end{smallmatrix}\right] \in V$ in the norm of V. As $Z \subset X$ with bounded inclusion, it follows that $z_j \to 0$ in the norm of X. As $V \subset \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ with a bounded inclusion (see (2.5) and (2.6)), it follows that $z_j \to z_0$ in the norm of X. Hence $z_0 = 0$. Recalling that $\left[\begin{smallmatrix} 0 \\ u_0 \end{smallmatrix}\right] \in V$, we must have $Bu_0 \in BU \cap X = \{0\}$. Hence, $\left[\begin{smallmatrix} z_0 \\ u_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ and the closability of $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right]$ follows. Note that $G = \begin{bmatrix} 0 & I \end{bmatrix} |V \cdot \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right]$. Because $\begin{bmatrix} 0 & I \end{bmatrix} |V \in \mathcal{L}(V;U)$ by a simple estimate, it follows $G \in \mathcal{L}(Z;U)$.

We show next that Ran $G = U$. Because S is an operator node, there exists a $x_u \in Z$ such that $\left[\begin{smallmatrix} x_u \\ u \end{smallmatrix}\right] \in V$ for every $u \in U$. Indeed, take $x_u = (\alpha - A_{-1})^{-1}Bu \in Z$ and note that $A_{-1}x_u + Bu = \alpha(\alpha - A_{-1})^{-1}Bu \in Z \subset X$. By what we have already proved above, such x_u is unique and it satisfies $u = Gx_u$.

If $x \in \text{Ker } G$, then $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V$, $A_{-1}x \in X$, and thus $x \in \mathcal{D}(A)$. Conversely, let $x \in \mathcal{D}(A)$. Then $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V$, and it follows that $Gx = 0$ because $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$. Now claim (i) is proved.

That $K \in \mathcal{L}(Z;Y)$ follows from $C\&D \in \mathcal{L}(V;Y)$ and claim (i). Moreover, Ran K = Ran $C\&D$ and $Kx = C\&D \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$ follow as $\left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] : Z \to V$ is a bijection.

It remains to prove (iii). Let $z \in Z$ be arbitrary. Then $\left[\begin{smallmatrix} z \\ Gz \end{smallmatrix}\right] \in V$, $Lz = A_{-1}z + BGz \in X$ and thus $L : Z \to X$. Since $Z \subset X$ with a bounded inclusion, we conclude that $A_{-1}|Z \in \mathcal{L}(Z;X_{-1})$ and hence $L \in \mathcal{L}(Z;X_{-1})$, too. Because Ran $L \subset X$ and $X \subset X_{-1}$ with a bounded inclusion, we conclude that $L \in \mathcal{L}(Z;X)$ using the technique presented in the beginning of the proof of Proposition 3.

Let $\alpha \in \rho(A)$ and $x \in \text{Ker } (\alpha - L) \cap \text{Ker } G$. Clearly $(\alpha - L)\text{Ker } G = X$ is equivalent to the fact that $(\alpha - A)X_1 = X$. Furthermore, $x \in X_1$ and $0 = (\alpha - L)x = (\alpha - A)x = 0$, which implies $x = 0$. The proof is complete. $\square$

We have now proved the following theorem:

Theorem 2. *Assume $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator node of boundary control type on Hilbert spaces U, X and Y with $\mathcal{D}(S) = V$. Define the space Z by* (2.8)*, and let the operators $L \in \mathcal{L}(Z;X)$, $G \in \mathcal{L}(Z;U)$ and $K \in \mathcal{L}(Z;Y)$ be as in Proposition 4.*

Then $\Gamma = (L, G, K)$ is a boundary control node (in the sense of Definition 1) on Hilbert spaces U, X and Y, with the solution space Z.

By inspecting the translation procedures of Subsections 2.1 and 2.2, we see that the boundary control nodes $\Gamma = (L, G, K)$ and the operator nodes $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ of boundary control type are in one-to-one correspondence. In particular, the solution space Z for Γ is same as the space given by (2.8) for for the operator node S corresponding to Γ. For general operator nodes, we have yet another characterisation for the same space

$$\begin{aligned} Z &= X_1 + (\alpha - A_{-1})^{-1}BU \\ &= \{z \in X : \exists u \in U \text{ such that } \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)\}. \end{aligned} \tag{2.9}$$

It is a characterising property for operator nodes of *boundary control type* that we can write the direct sum decomposition $Z = X_1 \dot{+} (\alpha - A_{-1})^{-1}$ instead of (2.8). Analogously, it is true only for S of *boundary control type* that the space $V = \mathcal{D}(S)$ (given by (2.5)) can be written as in form $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ for some operator $G \in \mathcal{L}(Z; U)$.

2.3 The Cauchy problem

We now solve the Cauchy problem for the formal system (1.1). This is reduced to the corresponding Cauchy problem (1.2) *system* nodes, as presented in [13, Proposition 2.5].

Lemma 1. *Assume that $\Gamma = (L, G, K)$ is a boundary control node, such that $A = L|\mathrm{Ker}\ G : \mathcal{D}(A) \subset X \to X$ is a generator of a C_0-semigroup. Let $u \in C^2([0,\infty); U)$ and $z_0 \in Z$ be such that the compatibility condition $Gz_0 = u(0)$ is satisfied.*

Then the Cauchy problem (1.1) *has a unique classical solution $z(\cdot) \in C([0,\infty); Z) \cap C^1([0,\infty); X)$, such that $z(0) = z_0$ and $y(\cdot) \in C([0,\infty); Y)$.*

Proof. By $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ denote the operator node that is related to Γ as in Theorems 1 and 2. By V and Z denote the two common Hilbert spaces for Γ and S that have been described in Subsections 2.1 and 2.2. Since A is the generator of a C_0-semigroup it follows from [13, Proposition 2.5] that there exists a unique $z(\cdot) \in C^1([0,\infty); X) \cap C^2([0,\infty); X_{-1})$ such that (1.2) holds and $\left[\begin{smallmatrix} z(\cdot) \\ u(\cdot) \end{smallmatrix}\right] \in C([0,\infty); V)$. Since $V \subset \left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right]$ with a bounded inclusion by Proposition 3, it follows that $z(\cdot) \in C([0,\infty); Z)$ and $u(t) = Gz(t)$ for all $t \geq 0$. Since $L = A_{-1}|Z + BG$, (1.2) implies that for all $t \geq 0$

$$\dot{z}(t) = A_{-1}z(t) + Bu(t) = (A_{-1}|Z + BG)\, z(t) = Lz(t).$$

Since $C\&D$ and K are connected by (2.7), we conclude that $z(\cdot)$ solves (1.1). The uniqueness is checked by going a similar reasoning in reverse order. □

Theorem 1 gives a working interpretation to differential equation (1.1). Note that the trajectory $z(\cdot)$ is continuous in $Z \subset X$, but $\dot{z}(\cdot)$ is computed (as a limit of a differential quotient) in the norm of X.

3 Conservativity and time-flow inverses

For some system nodes $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, equations (1.2) can be solved backwards in time for smooth signals, if the input and output are interchanged by each other, too. For bounded B, C, D, and D^{-1}, the inverse dynamics can be obtained easily:

$$\begin{cases} \dot{z}(t) = \left(-A + BD^{-1}C\right)_{-1} z(t) - BD^{-1}y(t), \\ u(t) = -D^{-1}Cz(t) + D^{-1}y(t). \end{cases}$$

The general case is covered by a formal definition which unfortunately does not give much help for the verification time-flow invertibility:

Definition 2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an operator node with $V = \mathcal{D}(S)$. We say that S is* time-flow invertible, *if there exists an operator node $S^{\leftarrow} = \left[\begin{smallmatrix} [A\&B]^{\leftarrow} \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$ with domain $\mathcal{D}(S^{\leftarrow}) = V^{\leftarrow} \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ and the main operator $A^{\leftarrow}$, such that*

(i) *both $\rho(A) \cap \overline{C_+} \neq \emptyset$ and $\rho(A^{\leftarrow}) \cap \overline{C_+} \neq \emptyset$,*

(ii) $\begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix} : V \to V^{\leftarrow}$ *is a bounded bijection, and*

(iii) *we have on all of $V^{\leftarrow}$*

$$S^{\leftarrow} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix}^{-1}. \tag{3.1}$$

When these conditions hold for S and $S^{\leftarrow}$, we say that $S^{\leftarrow}$ is the time-flow inverse *of S.*

A boundary control node $\Gamma = (L, G, K)$ is time-flow invertible, if the operator node S obtained in Theorem 1 is time-flow invertible.

For a deeper treatment of time-flow invertibility, see [19, 21]. Whenever $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ has a time-flow inverse, we have $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$. It follows from this that $(S^{\leftarrow})^{\leftarrow} = S$. To understand the underlying symmetry in things, consider the following two propositions. From now on, $S^d = \left[\begin{smallmatrix} [A\&B]^d \\ [C\&D]^d \end{smallmatrix}\right]$ denotes the dual node of S, see [13, Proposition 2.3] for details.

Proposition 5. *Let S be a system node. Then S is conservative if and only if it is time-flow invertible and $S^d = S^{\leftarrow}$.*

Proof. Assume $S^d = S^{\leftarrow}$. Then by Definition 2 we have $V^d = V^{\leftarrow} = \left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right] V$ and (3.1) implies

$$\begin{bmatrix} [A\&B]^d \\ [C\&D]^d \end{bmatrix} \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \text{ on } V. \tag{3.2}$$

Now [13, Lemma 3.2] implies that S is energy preserving. Furthermore, $S^d = S^{\leftarrow} \Leftrightarrow (S^d)^d = (S^{\leftarrow})^d$ and with some good faith[3] $(S^{\leftarrow})^d = (S^d)^{\leftarrow}$, too. Thus $(S^d)^d = (S^d)^{\leftarrow}$ and by dualizing the above argument, also S^d is energy preserving. The conservativity of S follows.

Conversely, let S be conservative. Then (3.2) and its dual version give

$$\begin{aligned} \begin{bmatrix} I & 0 \\ [C\&D]^d \end{bmatrix} \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{on} \quad V \quad \text{and} \\ \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} I & 0 \\ [C\&D]^d \end{bmatrix} &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad \text{on} \quad V^d. \end{aligned}$$

Hence $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right] : V \to V^d$ is a bounded bijection from V onto V^d. Identity (3.2) implies directly $S^d = S^{\leftarrow}$, where the operator $S^{\leftarrow}$ is defined by (3.1)

[3]...or by a rather long computation...

on V^d. But now $S^{\leftarrow}$ is a system node with $\mathcal{D}(S^{\leftarrow}) = V^d$, since S^d is a system node. In particular, the main operator of $S^{\leftarrow}$ satisfies $A^{\leftarrow} = A^*$, and certainly $\rho(A^*) \cap \overline{\mathbb{C}_+} = \overline{\rho(A)} \cap \overline{\mathbb{C}_+} \neq \emptyset$, where $\overline{\rho(A)}$ denotes the conjugate set of $\rho(A)$. We conclude that S is time-flow invertible by Definition 2, and its time flow inverse satisfies $S^{\leftarrow} = S^d$. $\square$

Let us give another easy piece:

Proposition 6. *An energy preserving system node S is conservative if and only if it is time-flow invertible.*

Proof. Conservativity implies time-flow invertibility, by Proposition 5. For the converse direction, assume that S is both energy preserving and time-flow invertible with the operator node $S^{\leftarrow}$ given by (3.1). The time-flow invertibility implies that $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] V = V^{\leftarrow}$. Now identity (3.1) gives

$$S^{\leftarrow} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \text{ on all of } V.$$

By [13, Lemma 3.2], the energy-preserving property implies $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] V \subset V^d$ and

$$S^d \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \text{ on all of } V.$$

We conclude that $V^{\leftarrow} \subset V^d$, $S^{\leftarrow} = S^d | V^{\leftarrow}$ and $A^{\leftarrow} = A^d | X_1^{\leftarrow}$. It remains to show that $V^d = V^{\leftarrow}$.

Since both A^d and $A^{\leftarrow}$ generate a C_0-semigroup on X, it follows that $X_1^d = X_1^{\leftarrow}$, $A^{\leftarrow} = A^d$ and $X_{-1}^d = X_{-1}^{\leftarrow}$. Moreover, $S^{\leftarrow} = S^d | V^{\leftarrow}$ implies $[A\&B]^{\leftarrow} = [A_{-1}^* \quad C^*] | V^{\leftarrow}$, and hence $[A_{-1}^* \quad C^*] : \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to X_{-1}^d$ is a bounded extension of $[A\&B]^{\leftarrow}$. Since $V^{\leftarrow}$ is dense in $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ (see [13, equation (2.2)]), this is the only possible bounded extension on these spaces. We conclude that $[A_{-1}^{\leftarrow} \quad C^{\leftarrow}] = [A_{-1}^* \quad C^*]$ on all of $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, and $V = V^{\leftarrow}$ follows. $\square$

We give in Theorem 3 yet another characterisation for tory systems. The motivation for this result is the following: for boundary control nodes $\Gamma = (L, G, K)$ associated to dynamics (1.1), the time-flow inverse is very easy to guess. Indeed, as will be seen in Theorem 6, it is $\Gamma^{\leftarrow} = (-L, K, G)$ whenever such $\Gamma^{\leftarrow}$ satisfies the axioms (a) – (d) of Definition 1. On the other hand, computing the dual system Γ^d is quite difficult [4]. The following proposition contains the trick involved.

Proposition 7. *Assume $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a time-flow invertible system node. Let $A^{\leftarrow} : X_1^{\leftarrow} \to X$ be the main operator and $C^{\leftarrow}$ the output operator of the time-flow inverse $S^{\leftarrow}$. Assume that the dual cross-term equation holds*

$$C\&D \begin{bmatrix} I \\ B^* \end{bmatrix} = 0 \quad \textit{on} \quad X_1^d,$$

[4] Conversely, the adjoint node S^d is almost trivial to obtain for an operator node S, but the time-flow inverse $S^{\leftarrow}$ is given by the rather difficult formula (3.1) in Definition 2.

and $A^* = A^{\leftarrow}$ (with equal domains). Then

$$A_{-1} + A^* + BB^* = 0 \text{ on } X_1^d \quad \text{and} \quad C^{\leftarrow} = B^* \text{ on } X_1^d.$$

Proof. Because $A^* = A^{\leftarrow}$, we have $X_1^{\leftarrow} = X_1^d$. Hence $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^{\leftarrow}$ for all $x \in X_1^d$. Because $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] : V \to V^{\leftarrow}$ is a bounded bijection (by the existence of the time-flow inverse), there exists for any $x \in X_1^d$ a *unique* vector $\left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right] \in V$ such that

$$\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \multicolumn{2}{c}{C\&D} \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}.$$

By using the assumed dual cross-term equation, we see that in fact $x_1 = x$ and $u_1 = B^*x$. Hence, for any $x \in X_1^d$

$$\begin{aligned} \begin{bmatrix} A^{\leftarrow}x \\ C^{\leftarrow}x \end{bmatrix} &= S^{\leftarrow} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \multicolumn{2}{c}{C\&D} \end{bmatrix}^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ B^*x \end{bmatrix} = \begin{bmatrix} -A_{-1}x - BB^*x \\ B^*x \end{bmatrix}. \end{aligned}$$

But $A^{\leftarrow}x = A^*x$ by assumption, and the claim follows. □

We can now characterise tory systems without referring to the dual system at all:

Theorem 3. *Assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a time-flow invertible operator node. By $A^{\leftarrow} : X_1^{\leftarrow} \to X$ denote the main operator of the time-flow inverse $S^{\leftarrow}$. Then S is tory*[5] *if and only if*

(i) $\operatorname{Ker} B = \{0\}$,

(ii) $A + A^*_{-1} = -C^*C$ *on* X_1,

(iii) $C\&D \begin{bmatrix} I \\ B^* \end{bmatrix} = 0$ *on* X_1^d, *and*

(iv) *We have $A^{\leftarrow} = A^*$ with equal domains, i.e. $X_1^{\leftarrow} = X_1^d$.*

Proof. Conditions (i) – (iii) are necessary for toryness, again by [13, Theorem 4.4]. By Proposition 5, tory systems satisfy $S^d = S^{\leftarrow}$, and (iv) follows, too.

Assume that conditions (i) – (iv) hold. Then the dual Liapunov equation is given by Proposition 7, and S is tory by [13, Theorem 4.4] provided we can show that $\operatorname{Ker} C^* = \{0\}$. Following [13, Proposition 2.4], decompose the space Y orthogonally $Y = \left[\begin{smallmatrix} Y_1 \\ Y_0 \end{smallmatrix}\right]$ where $Y_1 = \overline{\operatorname{Ran} C}$ and $Y_0 = Y_1^{\perp}$. The induced decomposition of S is then given by

$$S = \begin{bmatrix} [A\&B]_r \\ [C\&D]_r \\ 0 \quad D_{01} \end{bmatrix} : V \to \begin{bmatrix} X \\ Y_1 \\ Y_0 \end{bmatrix} \quad \text{with} \quad S_r := \begin{bmatrix} [A\&B]_r \\ [C\&D]_r \end{bmatrix};$$

here S_r is the reduced operator node with output space Y_1, the domains satisfy $V = \mathcal{D}(S) = \mathcal{D}(S_r)$, and $D_{01} \in \mathcal{L}(U; Y_0)$ is nonzero if and only if Y_0 is

[5] I.e. a conservative system node with $\operatorname{Ker} B = \{0\}$ and $\operatorname{Ker} C^* = \{0\}$.

nontrivial. Since $B = B_r$, $C = \left[\begin{smallmatrix} C_r \\ 0 \end{smallmatrix}\right]$, and $C^* = \begin{bmatrix} C_r^* & 0 \end{bmatrix}$, we conclude (using Proposition 7) that $A + A^*_{-1} = -C_r^* C_r$ on X_1, together with $A_{-1} + A^* = -B_r B_r^*$ and $[C\&D]_r \left[\begin{smallmatrix} I \\ B_r^* \end{smallmatrix}\right] = 0$ on X_1^d.

It follows from [13, Theorem 4.4] that S_r is a tory node, and it is thus time-flow invertible with $S_r^{\leftarrow} = S_r^d = \left[\begin{smallmatrix} [A\&B]_r^d \\ [C\&D]_r^d \end{smallmatrix}\right]$; see Proposition 5. In particular, $\left[\begin{smallmatrix} I & 0 \\ \;[C\&D]_r & \end{smallmatrix}\right] : V \to V_r^d = \mathcal{D}(S_r^d)$ is a bijection with the inverse $\left[\begin{smallmatrix} I & 0 \\ \;[C\&D]_r^d & \end{smallmatrix}\right]$, and $\left[\begin{smallmatrix} I & 0 \\ C\&D & \end{smallmatrix}\right] = \left[\begin{smallmatrix} I \;\; 0 \\ [C\&D]_r \\ 0 \;\; D_{01} \end{smallmatrix}\right]$. Because also S is time-flow invertible, we get

$$V^{\leftarrow} = \begin{bmatrix} I \;\; 0 \\ [C\&D]_r \\ 0 \;\; D_{01} \end{bmatrix} V = \begin{bmatrix} \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \\ \begin{bmatrix} 0 & D_{01} \end{bmatrix} \begin{bmatrix} I \;\; 0 \\ [C\&D]_r \end{bmatrix}^{-1} \end{bmatrix} V_r^d = \begin{bmatrix} \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right] \\ D_{01}\,[C\&D]_r^d \end{bmatrix} V_r^d \tag{3.3}$$

and

$$\begin{bmatrix} I & 0 \\ C\&D & \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 & 0 \\ [C\&D]_r^d & & 0 \end{bmatrix} \Big| V^{\leftarrow}.$$

But now we obtain $S^{\leftarrow} = \left[\begin{smallmatrix} [A\&B]_r^d & 0 \\ [C\&D]_r^d & 0 \end{smallmatrix}\right]$ on all of $V^{\leftarrow}$ by (3.1). Because both S^d and $S^{\leftarrow}$ are operator nodes, it follows that $V^{\leftarrow} = \left[\begin{smallmatrix} V_r^d \\ Y_0 \end{smallmatrix}\right]$ which contradicts (3.3) unless $D_{01} = 0$. This completes the proof. $\square$

4 Construction of the time-flow inverse

In this section, we show that the time-flow invertibility of an operator node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ (in the sense of Definition 2) *almost* follows if it is known that S is of boundary control type (see Definition 1). Indeed, only one extra assumption is needed on the "time-flow-inverted" main operator $A^{\leftarrow}$.

In this section, we make it a standing assumption that S is an operator node of boundary control type. We further assume that S and $\Gamma = (L, G, K)$ are related to each other as in Theorems 1 and 2. In particular, the operators L, G and K are given by Proposition 4. The spaces Z and $V = \mathcal{D}(S)$ are described (unambiguously) by (2.5), (2.9), and claim (iii) of Proposition 3. Our approach leads – step by step – to the construction of the time-flow inverse $S^{\leftarrow}$ in Theorem 4.

Let us first define a Banach space $V^{\leftarrow}$ which is finally going to be the domain of $S^{\leftarrow}$ *in spe*. Motivated by Definition 2, we set plainly

$$V^{\leftarrow} := \begin{bmatrix} I & 0 \\ C\&D & \end{bmatrix} V \subset \begin{bmatrix} X \\ Y \end{bmatrix}. \tag{4.1}$$

A Banach space norm for $V^{\leftarrow}$ is defined by

$$\| \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \|_{V^{\leftarrow}} := \| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \|_V \quad \text{where} \quad \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ C\&D & \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]. \tag{4.2}$$

With this choice of norm, the operator $\left[\begin{smallmatrix} I & 0 \\ C\&D & \end{smallmatrix}\right] : V \to V^{\leftarrow}$ becomes an isometry with an isometric inverse.

Not surprisingly, in the boundary control context we have the relations

$$V = \begin{bmatrix} I \\ G \end{bmatrix} Z \quad \text{and} \quad V^{\leftarrow} = \begin{bmatrix} I \\ K \end{bmatrix} Z \tag{4.3}$$

Hence G and K are expected to play dual roles with respect to the time-flow inversion, and (4.1) can be replaced by $V^{\leftarrow} := \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right] Z$. Indeed, the latter equality in (4.3) follows from the former by using the identity

$$\begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} I \\ G \end{bmatrix} = \begin{bmatrix} I \\ K \end{bmatrix} \quad \text{on all of} \quad Z. \tag{4.4}$$

It is also instructive to note that (under the assumptions of Proposition 4)

$$\begin{bmatrix} I \\ G \end{bmatrix} \operatorname{Ker} K = \operatorname{Ker} C\&D \quad \text{and} \quad \begin{bmatrix} I \\ G \end{bmatrix} X_1 = \begin{bmatrix} X_1 \\ \{0\} \end{bmatrix}.$$

Note that in the boundary control case $BU \cap X = \{0\}$, the upper component of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ determines the lower. Conversely, the lower component determines the upper only modulo the space $X_1 = \operatorname{Ker} G$.

The symmetry in equalities (4.3) becomes even more pronounced once we discover that the solution space Z remains unchanged under the time-flow inversion; see (2.9) for the motivation of (4.5):

Proposition 8. *Make the same assumptions and use the same notations as in Proposition 4. Define $V^{\leftarrow}$ by (4.1). Then $Z^{\leftarrow} = Z$ where*

$$Z^{\leftarrow} := \{x \in X : \exists y \in Y \text{ such that } \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in V^{\leftarrow}\}. \tag{4.5}$$

Proof. By the definition of $V^{\leftarrow}$, we have $Z^{\leftarrow} \subset Z$. Conversely, if $x \in Z$, then $\left[\begin{smallmatrix} x \\ Gx \end{smallmatrix}\right] \in V$ and $\left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ Gx \end{smallmatrix}\right] \in V^{\leftarrow}$ by (4.4). Hence $x \in Z^{\leftarrow}$. □

Also the space $V^{\leftarrow}$ is seen to have some of its expected properties:

Proposition 9. *Assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is an operator node of boundary control type with* Ran $C\&D = Y$. *Define Z by (2.9) and $V^{\leftarrow}$ by (4.1). Then the following holds:*

(i) *For all $y \in Y$ there exists $x \in Z (= Z^{\leftarrow})$ such that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in V^{\leftarrow}$.*

(ii) *The inclusion $V^{\leftarrow} \subset \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right]$ is bounded.*

(iii) Ker $K = \{x \in X : \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^{\leftarrow}\}$.

Proof. Denote $V = \mathcal{D}(S)$ and fix $y \in Y$. Then for any $x \in Z$ the equality

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} x_1 \\ u \end{bmatrix} \quad \text{for} \quad \begin{bmatrix} x_1 \\ u \end{bmatrix} \in V$$

is equivalent to

$$x = x_1 \quad \text{and} \quad y = C\&D \begin{bmatrix} x \\ u \end{bmatrix} \text{ for } \begin{bmatrix} x \\ u \end{bmatrix} \in V.$$

Because Ran $C\&D = Y$, then there exists such a $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in V$ with $x \in Z$. To prove claim (ii), we first estimate the norm of $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$:

$$
\begin{aligned}
\| \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \|_{\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]} &\leq (\|x\|_X + \|y\|_Y) \leq \left(\|x\|_X + \|C\&D\|_{\mathcal{L}(V;Y)} \| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \|_V\right) \\
&\leq \max\left(1, \|C\&D\|_{\mathcal{L}(V;Y)}\right) \cdot (\|x\|_X + \|x\|_X + \|u\|_U + \|A_{-1}x + Bu\|_X) \\
&\leq 2\max\left(1, \|C\&D\|_{\mathcal{L}(V;Y)}\right) \cdot (\|x\|_X + \|u\|_U + \|A_{-1}x + Bu\|_X) \\
&\leq 6\max\left(1, \|C\&D\|_{\mathcal{L}(V;Y)}\right) \cdot \| \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \|_V.
\end{aligned}
$$

Now, the boundedness of the inclusion $V^{\leftarrow} \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ follows from (4.2). By definition, we have $V^{\leftarrow} \subset \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right]$. As in the beginning of the proof of Proposition 3, we see that also the inclusion $V^{\leftarrow} \subset \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right]$ is bounded and (ii) follows.

To verify the last claim (iii), recall that $V^{\leftarrow} = \left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] V$ and $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$. We have $x \in X$ with $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^{\leftarrow}$ if and only if $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right] \left[\begin{smallmatrix} z \\ Gz \end{smallmatrix}\right]$ for some $z \in Z$ if and only if for some $z \in Z$

$$
x = z \quad \text{and} \quad 0 = C\&D \begin{bmatrix} I \\ G \end{bmatrix} z = Kz
$$

if and only if $x \in \operatorname{Ker} K$. $\square$

To get ahead, we must assume that Ker K is dense[6] in X.

Proposition 10. *Assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is an operator node of boundary control type with* Ran $C\&D = Y$. *Define* $V^{\leftarrow}$ *by* (4.1), *and assume that* Ker K *is dense in* X. *Then* $V^{\leftarrow}$ *is dense in* $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$.

Proof. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ be arbitrary. As Ran $C\&D = Y$, there exists $\left[\begin{smallmatrix} z \\ v \end{smallmatrix}\right] \in V = \mathcal{D}(S)$ such that $y = C\&D \left[\begin{smallmatrix} z \\ v \end{smallmatrix}\right]$. Because Ker K is dense in X, there is a sequence $\{x_j\}_{j \geq 0} \subset \operatorname{Ker} K$ such that $x_j \to x - z \in X$ in the norm of X. Now,

$$
C\&D \begin{bmatrix} z + x_j \\ v + Gx_j \end{bmatrix} = C\&D \begin{bmatrix} z \\ v \end{bmatrix} + C\&D \begin{bmatrix} I \\ G \end{bmatrix} x_j = y + Kx_j = y.
$$

Using this gives

$$
V^{\leftarrow} \ni \begin{bmatrix} I & 0 \\ C\&D \end{bmatrix} \begin{bmatrix} z + x_j \\ v + Gx_j \end{bmatrix} = \begin{bmatrix} z + x_j \\ y \end{bmatrix} \to \begin{bmatrix} x \\ y \end{bmatrix}
$$

in the norm of $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ since $z + x_j \to x$ in the norm of X. $\square$

Under the assumptions of Proposition 10, the linear mapping

$$
S^{\leftarrow} := \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix}^{-1} : V^{\leftarrow} \subset \begin{bmatrix} X \\ Y \end{bmatrix} \to \begin{bmatrix} X \\ U \end{bmatrix} \tag{4.6}
$$

is densely defined. We next establish that $S^{\leftarrow}$ is an operator node, so as to verify that S is time-flow invertible in the sense of Definition 2. For this purpose, we need to define some new objects:

[6] If S was already known to be time-flow invertible, this would be a necessary condition for $S^{\leftarrow}$ to be of boundary control type; see (4.3) together with Proposition 8. So, we do not regret making this assumption at all.

Definition 3. *Assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is an operator node of boundary control type with* Ran $C\&D = Y$*, and let the boundary control node* $\Gamma = (L, G, K)$ *be given by Theorem 2. Assume that* Ker K *is dense in* X*.*

(i) *The mapping* $A^{\leftarrow} : \mathrm{Ker}\ K \to X$ *is defined by* $A^{\leftarrow} := -L|\mathrm{Ker}\ K$*.*

(ii) *The mapping* $C^{\leftarrow} : \mathrm{Ker}\ K \to Y$ *is defined by* $C^{\leftarrow} := G|\mathrm{Ker}\ K$*.*

Definition 4. *Make the same assumptions and use the same notations as in Definition 3. Assume, in addition, that* $\rho(A^{\leftarrow}) \cap \overline{\mathbb{C}_+} \neq \emptyset$ *with* $\mathcal{D}(A^{\leftarrow}) = \mathrm{Ker}\ K$*.*

(i) *Denote by* $X_{-1}^{\leftarrow}$ *the completion*[7] *of* X *in norm* $\|x\|_{X_{-1}^{d}} := \|(\alpha - A^{\leftarrow})^{-1} x\|$ *for* $\alpha \in \rho(A^{\leftarrow}) \cap \overline{\mathbb{C}_+}$*.*

(ii) *Define* $B^{\leftarrow} : Y \to X_{-1}^{\leftarrow}$ *by setting for all* $x \in Z$

$$B^{\leftarrow} K x := -Lx - A_{-1}^{\leftarrow} x, \tag{4.7}$$

where $A_{-1}^{\leftarrow} \in \mathcal{L}(X; X_{-1}^{\leftarrow})$ *is the Yosida extension of* $A^{\leftarrow}$*.*

The linear mapping $B^{\leftarrow}$ in part (ii) of Definition 4 is well-defined. Note that $Lx \in X \subset X_{-1}^{\leftarrow}$ in (4.7) because $L \in \mathcal{L}(Z; X)$ by Proposition 4. Hence the right hand side of (4.7) defines a unique element of $X_{-1}^{\leftarrow}$. The $B^{\leftarrow}$ mapping is also uniquely defined: if $y = Kx_1 = Kx_2$, then $x_1 - x_2 \in \mathrm{Ker}\ K$; but both $-L$ and $A_{-1}^{\leftarrow}$ are extensions of $A^{\leftarrow}$ defined on Ker K. The operator $B^{\leftarrow}$ is defined on all of Y, since Ran K = Ran $C\&D = Y$ by Proposition 4.

Proposition 11. *Assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is an operator node of boundary control type with* Ran $C\&D = Y$*. Define* $V^{\leftarrow}$ *by* (4.1) *and the operators* $A^{\leftarrow}$*,* $B^{\leftarrow}$ *by Definition 3. Assume that* Ker K *is dense in* X *and* $\rho(A^{\leftarrow}) \cap \overline{\mathbb{C}_+} \neq \emptyset$*. Then the following holds:*

(i) $B^{\leftarrow} \in \mathcal{L}(Y; X_{-1}^{\leftarrow})$ *and* Ker $B^{\leftarrow} = \{0\}$*.*

(ii) *The space* $V^{\leftarrow}$ *satisfies*

$$V^{\leftarrow} = \left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] : A_{-1}^{\leftarrow} x + B^{\leftarrow} y \in X \right\} \tag{4.8}$$

and the norm (4.2) *for* $V^{\leftarrow}$ *is equivalent to*

$$\| \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \|_{V^{\leftarrow}}^2 := \|x\|_X^2 + \|y\|_Y^2 + \|A_{-1}^{\leftarrow} x + B^{\leftarrow} y\|_X^2. \tag{4.9}$$

(iii) *The operator* $[A\&B]^{\leftarrow} := \begin{bmatrix} A_{-1}^{\leftarrow} & B^{\leftarrow} \end{bmatrix} |V^{\leftarrow}$ *is closed from* $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ *to* X*, with domain* $\mathcal{D}([A\&B]^{\leftarrow}) = V^{\leftarrow}$*.*

[7] See the discussion following Proposition 1.

Proof. We show that $B^{\leftarrow} y_j \to 0$ in $X_{-1}^{\leftarrow}$ for all sequences $y_j \to 0$ in Y. As $K \in \mathcal{L}(Z;Y)$ and $\text{Ran } K = Y$ by claim (ii) Proposition 4, there exists a sequence $\{x_j\}_{j\geq 0} \subset Z \ominus \text{Ker } K$ (orthogonality taken in the sense of (2.2)) and $y_j = Kx_j$. Because $K|(Z \ominus \text{Ker } K)$ has a bounded inverse $Y \to Z \ominus \text{Ker } K$, it follows that $x_j \to 0$ in Z and in the weaker norm of X, too. As $A_{-1}^{\leftarrow} \in \mathcal{L}(X;X_{-1}^{\leftarrow})$, it follows that $A_{-1}^{\leftarrow} x_j \to 0$ in $X_{-1}^{\leftarrow}$. As $L \in \mathcal{L}(Z;X)$, it follows that $Lx_j \to 0$ in X and hence in $X_{-1}^{\leftarrow}$, too. By equation (4.7), $B^{\leftarrow} K x_j = B^{\leftarrow} y_j \to 0$ in $X_{-1}^{\leftarrow}$.

We prove next that $\text{Ker } B^{\leftarrow} = \{0\}$. Assume that $B^{\leftarrow} y = 0$ for some $y = Kx$, $x \in Z$. Then $A_{-1}^{\leftarrow} x = -Lx \in X$ by (4.7). It follows that $x \in \mathcal{D}(A^{\leftarrow}) = \text{Ker } K$ and $y = Kx = 0$. Thus claim (i) holds.

Claim (ii) is treated next. Let $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in V^{\leftarrow}$ be arbitrary, and note that $x \in Z$ and $y = Kx$ by (4.3). Rewriting (4.7) we get $A_{-1}^{\leftarrow} x + B^{\leftarrow} y = -Lx \in X$, since $L \in \mathcal{L}(Z;X)$. To prove the converse inclusion in (4.8), assume that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ satisfies $A_{-1}^{\leftarrow} x + B^{\leftarrow} y \in X$. As $\text{Ran } K = Y$ by Proposition 4, we have $y = Kz$ for some $z \in Z$. Now

$$
\begin{aligned}
&X \ni A_{-1}^{\leftarrow} x + B^{\leftarrow} y = \left(A_{-1}^{\leftarrow} + B^{\leftarrow} K\right) z + A_{-1}^{\leftarrow}(x - z) \\
&= -Lz + A_{-1}^{\leftarrow}(x - z),
\end{aligned}
\tag{4.10}
$$

where we have used (4.7) again. Because $-Lz \in X$, equation (4.10) implies $A_{-1}^{\leftarrow}(x - z) \in X$, and thus $x - z \in \mathcal{D}(A^{\leftarrow}) = \text{Ker } K$. We conclude that $y = Kz = Kx$ and so $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right] \in \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right] Z = V^{\leftarrow}$ follows.

It is clear that $V^{\leftarrow}$ with norm (4.9) is a Banach space, and $V^{\leftarrow} \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ with a bounded (even dense) inclusion. Recall that $V^{\leftarrow}$ with norm (4.2) is a Banach space, and also then the inclusion $V^{\leftarrow} \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ is bounded, by claim (ii) of Proposition 9. Hence, these two differently normed versions of $V^{\leftarrow}$ are *compatible* Banach spaces (in the sense of interpolation theory) and their norms are accordingly equivalent.

To prove claim (iii), note that $\left[A_{-1}^{\leftarrow} \quad B^{\leftarrow}\right] \in \mathcal{L}(\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]; X_{-1}^{\leftarrow})$ by claim (i) and the fact that $A_{-1}^{\leftarrow} \in \mathcal{L}(X;X_{-1}^{\leftarrow})$. Now, $[A\&B]^{\leftarrow}$ is closed, as it is the restriction of bounded $\left[A_{-1}^{\leftarrow} \quad B^{\leftarrow}\right]$ to its natural domain $V^{\leftarrow}$, when the range is restricted to a subset of X. □

Now comes the main result of this section;

Theorem 4. *Assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is an operator node of boundary control type with* $\text{Ran } C\&D = Y$, *and let the boundary control node* $\Gamma = (L, G, K)$ *be given by Theorem 2. Define* $V^{\leftarrow} := \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right] Z$ *and* $S^{\leftarrow}$ *by* (4.6). *Assume that* $-L|\text{Ker } K$ *is a densely defined operator on* X, *with* $\rho(-L|\text{Ker } K) \cap \overline{\mathbb{C}_+} \neq \emptyset$.

Define the operators $A^{\leftarrow}$, $B^{\leftarrow}$, *and* $C^{\leftarrow}$ *by Definition 3. Then the following holds:*

(i) $S^{\leftarrow} : \mathcal{D}(S^{\leftarrow}) \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ *is an operator node with* $\mathcal{D}(S^{\leftarrow}) = V^{\leftarrow}$.

The main operator of $S^{\leftarrow}$ *is* $A^{\leftarrow}$ *with domain* $\mathcal{D}(A^{\leftarrow}) = \text{Ker } K$. *The operator* $B^{\leftarrow}$ *is the input operator of* $S^{\leftarrow}$, *and the combined feedthrough/output* *operator* $[C\&D]^{\leftarrow}$ *of* $S^{\leftarrow}$ *satisfies*

$$[C\&D]^{\leftarrow}\begin{bmatrix}x\\ y\end{bmatrix}=Gx \quad \textit{for all} \quad \begin{bmatrix}x\\ y\end{bmatrix}\in V^{\leftarrow}. \tag{4.11}$$

(ii) *The operator node* S *is time-flow invertible, and its time-flow inverse equals* $S^{\leftarrow}$.

Proof. The operator $B^{\leftarrow}$ lies in $\mathcal{L}(Y;X^{\leftarrow}_{-1})$ by claim (i) of Proposition 11. The operator $[A\&B]^{\leftarrow}:=\begin{bmatrix}A^{\leftarrow}_{-1} & B^{\leftarrow}\end{bmatrix}|V^{\leftarrow}$ is closed and densely defined with $\mathcal{D}([A\&B]^{\leftarrow})=V^{\leftarrow}$ by Propositions 10 and 11. Define $[C\&D]^{\leftarrow}:V^{\leftarrow}\to U$ by (4.11), and note that it is well defined by Proposition 9. Let us now estimate

$$\|[C\&D]^{\leftarrow}\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\|_U\le\|G\|_{\mathcal{L}(Z;U)}\|\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\|_{\left[\begin{smallmatrix}Z\\ Y\end{smallmatrix}\right]}\le\|G\|_{\mathcal{L}(Z;U)}\cdot C\|\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\|_V,$$

since the inclusion $V^{\leftarrow}\subset\left[\begin{smallmatrix}Z\\ Y\end{smallmatrix}\right]$ is bounded by constant C, see Proposition 9. We conclude that $S':=\left[\begin{smallmatrix}[A\&B]^{\leftarrow}\\ [C\&D]^{\leftarrow}\end{smallmatrix}\right]$ is an operator node with $\mathcal{D}(S')=V^{\leftarrow}$.

We proceed to show that $S^{\leftarrow}=S'$. For all $\left[\begin{smallmatrix}x\\ Kx\end{smallmatrix}\right]\in\left[\begin{smallmatrix}I\\ K\end{smallmatrix}\right]Z=V^{\leftarrow}$ (in other words, for all $x\in Z$) we have

$$\begin{aligned}\begin{bmatrix}[A\&B]^{\leftarrow}\\ [C\&D]^{\leftarrow}\end{bmatrix}\begin{bmatrix}x\\ Kx\end{bmatrix}&=\begin{bmatrix}\left(A^{\leftarrow}_{-1}+B^{\leftarrow}K\right)x\\ Gx\end{bmatrix}=\begin{bmatrix}-\left(A_{-1}+BG\right)x\\ Gx\end{bmatrix}\\ &=\begin{bmatrix}-A_{-1} & -B\\ 0 & I\end{bmatrix}\begin{bmatrix}x\\ Gx\end{bmatrix}=\begin{bmatrix}-A_{-1} & -B\\ 0 & I\end{bmatrix}\begin{bmatrix}1 & 0\\ C\&D\end{bmatrix}^{-1}\begin{bmatrix}x\\ Kx\end{bmatrix},\end{aligned}$$

where the second equality follows from (4.7). By Definition 2, $S^{\leftarrow}=\left[\begin{smallmatrix}[A\&B]^{\leftarrow}\\ [C\&D]^{\leftarrow}\end{smallmatrix}\right]$, and the proof is complete. □

Corollary 1. *Make the same assumptions as in Theorem 4. Then the time-flow inverse* $S^{\leftarrow}$ *is an operator node of boundary control type satisfying* Ran $[C\&D]^{\leftarrow}=U$.

Proof. Recall that Ker $B^{\leftarrow}=\{0\}$ by claim (i) of Proposition 11. It follows directly from (4.11) that Ran $[C\&D]^{\leftarrow}=U$, as Ran $G=U$ by claim (i) of Proposition 4.

If $\tilde z\in B^{\leftarrow}Y\cap X$, $\tilde z\neq 0$, then $\tilde z=B^{\leftarrow}\tilde y$ for some $\tilde y\neq 0$. Suppose $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\in V^{\leftarrow}$. Then we have both $A^{\leftarrow}_{-1}x+B^{\leftarrow}y\in X$ and $A^{\leftarrow}_{-1}x+B^{\leftarrow}(y+\tilde y)\in X$, implying that both $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\in V^{\leftarrow}$ and $\left[\begin{smallmatrix}x\\ y+\tilde y\end{smallmatrix}\right]\in V^{\leftarrow}$. It now follows that the space $V^{\leftarrow}$ cannot be of graph form $\left[\begin{smallmatrix}I\\ K\end{smallmatrix}\right]Z$ for any linear mapping $K:Z\to U$. This contradiction proves that $B^{\leftarrow}Y\cap X=\{0\}$. □

5 Duals of conservative boundary control systems

In contrast to the previous section, the operators $A^{\leftarrow}$, ă$B^{\leftarrow}$, and $C^{\leftarrow}$ are no longer defined *a priori* by Definition 3. Instead, now they denote the main, input, and output operators of the time-flow inverse $S^{\leftarrow} = \left[\begin{smallmatrix} [A\&B]^{\leftarrow} \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$ of S; existence of $S^{\leftarrow}$ is assumed *a* priori. The next proposition is a partial converse result to Theorem 4, and it will be needed in the proof of Theorem 5 and Lemma 2.

Proposition 12. *Assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is a time-flow invertible operator node of boundary control type, with* Ran $C\&D = Y$. *Let the associated boundary control node* $\Gamma = (L, G, K)$ *be given by Theorem 2. Denote by* $A^{\leftarrow}$ *the main operator and by* $C^{\leftarrow}$ *the output operator of time-flow inverse* $S^{\leftarrow}$.

Then $\mathcal{D}(A^{\leftarrow}) = \mathrm{Ker}\ K$*,* $A^{\leftarrow} = -L|\mathrm{Ker}\ K$*, and* $C^{\leftarrow} = G|\mathrm{Ker}\ K$.

Proof. By the standard theory of operator nodes, we have $\mathcal{D}(S^{\leftarrow}) = V^{\leftarrow}$ and $\mathcal{D}(A^{\leftarrow}) = \{x \in X : \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^{\leftarrow}\}$ where $V^{\leftarrow}$ is defined by (4.8). By the time-flow invertibility of S, we have $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right] V = V^{\leftarrow}$, and the operator $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]$ is a bounded bijection from $V := \mathcal{D}(S)$ onto $V^{\leftarrow}$. Recalling the reasoning leading to (4.3), we have $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ and $V^{\leftarrow} = \left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right] Z$. Now $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in V^{\leftarrow}$ if and only if $x \in Z$ and $Kx = 0$ if and only if $x \in \mathrm{Ker}\ K$. Hence $\mathcal{D}(A^{\leftarrow}) = \mathrm{Ker}\ K$. To complete the proof, we compute by using (3.1)

$$\begin{aligned} \begin{bmatrix} A^{\leftarrow}x \\ C^{\leftarrow}x \end{bmatrix} &= S^{\leftarrow} \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ C\&D \end{bmatrix}^{-1} \begin{bmatrix} x \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -A_{-1} & -B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ Gx \end{bmatrix} = \begin{bmatrix} -Lx \\ Gx \end{bmatrix} \end{aligned} \tag{5.1}$$

for any $x \in \mathcal{D}(A^{\leftarrow})$, where we have once again used the fact that $\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right] \in V^{\leftarrow}$ implying $\left[\begin{smallmatrix} I & 0 \\ C\&D \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} x \\ Kx \end{smallmatrix}\right] = \left[\begin{smallmatrix} x \\ Gx \end{smallmatrix}\right] \in V$. □

Dual systems of tory boundary control systems are boundary control systems themselves:

Theorem 5. *Assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is a tory operator node of boundary control type, with* Ran $C\&D = Y$. *Let the associated boundary control node* $\Gamma = (L, G, K)$ *be given by Theorem 2. By* S^d *denote the dual node of* S*, with main operator* $A^* \in \mathcal{L}(X_1^d; X)$*. Then the dual system* S^d *is of boundary control type, and its solution space satisfies* $Z^d = Z$.

Proof. By Proposition 5, S is time-flow invertible, $S^{\leftarrow} = S^d$, $A^{\leftarrow} = A^*$, and $\mathcal{D}(A^{\leftarrow}) = X_1^d$; here A^* is a generator of a C_0-semigroup of contractions on X. By Proposition 12, we have $\mathcal{D}(A^{\leftarrow}) = \mathrm{Ker}\ K$ and $-L|\mathrm{Ker}\ K = A^{\leftarrow}$. Because now $-L|\mathrm{Ker}\ K = A^*$, we conclude that $\mathrm{Ker}\ K = X_1^d$ is dense in X and $\rho(-L|\mathrm{Ker}\ K) \cap \overline{\mathbb{C}_+} \neq \emptyset$. Now the claim follows from Corollary 1 because all of the assumptions of Theorem 4 are satisfied. □

6 Time-flow invertibility and conservativity of boundary control nodes

We are now ready to apply all the previous results to conservative boundary control systems. First comes an adaptation of Theorem 3 to the boundary control context.

Lemma 2. *Assume* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is an operator node of boundary control type with* $\mathrm{Ran}\ C\&D = Y$*, and let the associated boundary control node* $\Gamma = (L, G, K)$ *be given by Theorem 2. Then* S *is tory if and only if*

(i) *the primal Liapunov equation* $A + A_{-1}^* = -C^*C$ *holds on* X_1*,*

(ii) *we have* $Gx = B^*x$ *for all* $x \in X_1^d := \mathcal{D}(A^*)$*, and*

(iii) *the identity* $-L|\mathrm{Ker}\ K = A^*$ *holds (with equal domains).*

Proof. We start from the more interesting "sufficiency" part. It is clear that condition (i) of Theorem 3 always holds for boundary control systems. Conditions (ii) and (iv) of Theorem 3 are same as condition (i) and (iii) of this lemma. By condition (iii), we have $X_1^d = \mathrm{Ker}\ K \subset Z$. By condition (ii) we have $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] x = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] x \subset \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z = V$ for all $x \in X_1^d$, and hence $C\&D \left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] x \in Y$ is well defined; see claim (iii) of Proposition 3. Now, by the definition of operator K (see claim (ii) of Proposition 4), we obtain $C\&D \left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] x = Kx = 0$ for all $x \in X_1^d$. This is condition (iii) of Theorem 3, namely the dual cross-term equation. Time-flow invertibility of S follows from condition (iii) and Theorem 4 since $-L|\mathrm{Ker}\ K = A^*$ and $\rho(A) \cap \overline{\mathbb{C}_+} \neq \emptyset$.

To prove the "necessity" part, assume that S is tory. Such S is time-flow invertible by Proposition 5, $S^{\leftarrow} = S^d$, and all the conditions of Theorem 3 hold; including conditions (i) and (iii) of this lemma hold, too.

By [13, Theorem 4.4], the dual Liapunov equation holds in the form

$$\begin{bmatrix} A_{-1} & B \end{bmatrix} \begin{bmatrix} I \\ B^* \end{bmatrix} x = -A^*x \in X \quad \text{for all} \quad x \in X_1^d = \mathrm{Ker}\ K,$$

and hence $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] \mathrm{Ker}\ K \subset V = \mathcal{D}(S)$. But because S satisfies the conditions Proposition 4, we have $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$. Now the inclusion $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] \mathrm{Ker}\ K \subset \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ implies condition (ii) of this lemma. □

We have actually proved above that condition (ii) of Lemma 2 can be replaced by the inclusion $\left[\begin{smallmatrix} I \\ B^* \end{smallmatrix}\right] \mathrm{Ker}\ K \subset V$.

It is now time to turn attention to boundary control nodes $\Gamma = (L, G, K)$. We show first that *doubly* boundary control nodes can, indeed, be time-flow inverted as expectedly.

Theorem 6. *Let* $\Gamma = (L, G, K)$ *be a doubly boundary control node, and assume that* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *is the associated operator node given by Theorem 1. Then* S *is time-flow invertible,* $\mathrm{Ran}\ C\&D = Y$*, and the time-flow inverse* $S^{\leftarrow}$ *is of boundary control type. Moreover,* $S^{\leftarrow}$ *is the operator node associated to* $\Gamma^{\leftarrow} := (-L, K, G)$ *in the sense of Theorem 2*

In other words, it is right to call $\Gamma^{\leftarrow}$ the time-flow inverse of Γ.

Proof. Because $\Gamma^{\leftarrow} = (-L, K, G)$ is a boundary control node, Ker K is dense in X and $Y =$ Ran K. Since Γ is a boundary control node, it follows now that Ran $C\&D = Y$, see Proposition 4. Applying Proposition 1 to $\Gamma^{\leftarrow}$ shows that $\rho(-L|\text{Ker } K) \cap \overline{\mathbb{C}_+} \neq \emptyset$. Thus S is time-flow invertible by Theorem 4.

By Corollary 1, $S^{\leftarrow}$ is of boundary control type, and so it corresponds to some boundary control node $\Gamma' := (L', G', K')$. Clearly Γ' has a common solution space Z with S and $S^{\leftarrow}$, see Proposition 8. Moreover, $V^{\leftarrow} = \mathcal{D}(S^{\leftarrow})$ satisfies (4.3), and hence $G' = K$. By using the symmetry $(S^{\leftarrow})^{\leftarrow} = S$, also $G = K'$ follows.

Denoting by $A^{\leftarrow}$, $A^{\leftarrow}_{-1}$ the main operator of $S^{\leftarrow}$ and its Yosida extension, we have $A^{\leftarrow}_{-1}|Z + B^{\leftarrow}K = -L$ on all of Z; see Theorem 4 and equation (4.7). Applying claim (iii) of Proposition 4 to $S^{\leftarrow}$, we conclude that $L' = A^{\leftarrow}_{-1}|Z + B^{\leftarrow}K$. Hence $L' = -L$, and the proof is complete. □

Now come the main results of this paper.

Theorem 7. *Let $\Gamma = (L, G, K)$ be a doubly boundary control node, and assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the associated operator node given in Theorem 1. Then S is conservative (hence, tory) if and only if*

(i) $2\text{Re}\,\langle x, Lx\rangle_X = -\|Kx\|_Y^2$ *for all* $x \in \text{Ker } G$,

(ii) $\langle z, Lx\rangle_X + \langle Lz, x\rangle_X = \langle Gz, Gx\rangle_U$ *for all* $z \in Z$ *and* $x \in \text{Ker } K$.

Proof. Since Γ is is a doubly boundary control node, the time-flow inverse $S^{\leftarrow}$ exists by Theorem 6, and it is of boundary control type. For the usual spaces and operators involving S and $S^{\leftarrow}$, we have the identities $X_1 = \text{Ker } G$, $A = L|\text{Ker } G$, $C = K|\text{Ker } G$, $X_1^{\leftarrow} = \text{Ker } K$, $A^{\leftarrow} = -L|\text{Ker } K$, and $C^{\leftarrow} = G|\text{Ker } K$. Then (i) is same as $2\text{Re}\,\langle x, Ax\rangle_X = -\|Cx\|_Y^2$ for all $x \in X_1$, which is (by polarisation) equivalent to condition (i) of Lemma 2. Condition (ii) of Lemma 2 holds if and only if

$$-\langle z, A^*x\rangle_X + \langle Lz, x\rangle_X = \langle Gz, Gx\rangle_U \text{ for all } z \in Z \text{ and } x \in \mathcal{D}(A^*), \quad (6.1)$$

since Ran $G = U$ and $BGz = -A_{-1}z + Lz$. This together with condition (iii) of Lemma 2 imply condition (ii).

Because X_1 is dense in X, condition (iii) of Lemma 2 holds if and only if $X_1^{\leftarrow} = \mathcal{D}(A^*)$ and $\langle z, A^{\leftarrow}x\rangle_X = \langle z, A^*x\rangle_X$ for all $z \in X_1, x \in \mathcal{D}(A^*)$ if and only if

$$\langle z, Lx\rangle_X + \langle Lz, x\rangle_X = 0 \text{ for all } z \in \text{Ker } G \text{ and } x \in \text{Ker } K. \quad (6.2)$$

Clearly (ii) implies (6.2), and hence it implies condition (iii) of Lemma 2, too. Finally note that (ii) together with condition (iii) of Lemma 2 imply (6.1) and thus condition (ii) of Lemma 2. □

Note that condition (ii) of Theorem 7 implies $2\mathrm{Re}\,\langle x, -Lx\rangle_X = -\|Gx\|_U^2$ for all $x \in \mathrm{Ker}\ G$, which is equivalent to the (primal) Liapunov equation of the time-flow inverse $S^{\leftarrow}$.

There is another variant of Theorem 7 whose formulation is more symmetric but slightly weaker.

Theorem 8. *Let $\Gamma = (L, G, K)$ be a doubly boundary control node, and assume that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is the associated operator node given in Theorem 1. Then S is conservative (hence, tory) if and only if the* Green–Lagrange identity

$$2\mathrm{Re}\,\langle z_0, Lz_0\rangle_X = \|Gz_0\|_U^2 - \|Kz_0\|_Y^2 \tag{6.3}$$

holds for all $z_0 \in Z$.

Proof. By polarisation identity, (6.3) implies for all $z_1, z_2 \in Z$ the identity $\langle z_1, Lz_2\rangle_X + \langle Lz_1, z_2\rangle_X = \langle Gz_1, Gz_2\rangle_U - \langle Kz_1, Kz_2\rangle_U$. It is trivial that both the conditions (i) and (ii) of Theorem 7 follow from this.

Conversely, assume that S is conservative. Let $z_0 \in Z$ be arbitrary and $u \in C^2([0,\infty); U)$ such that $Gz_0 = u(0)$. By Lemma 1, there exists a solution $z(\cdot) \in C([0,\infty); Z) \cap C^1([0,\infty); X)$ of (1.1) that satisfies $z(0) = z_0$ and $\frac{d}{dt}\|z(t)\|_X^2 = \|u(t)\|_U^2 - \|y(t)\|_Y^2$. Differentiating and using (1.1) gives

$$\langle z(t), Lz(t)\rangle_X + \langle Lz(t), z(t)\rangle_X = \langle Gz(t), Gz(t)\rangle_U - \langle Kz(t), Kz(t)\rangle_Y$$

for all $t > 0$. Since all the operators L, G and K are bounded from space Z and $z(\cdot) \in C([0,\infty); Z)$, we may take the limit as $t \to 0+$. Now (6.3) follows because $z_0 \in Z$ was arbitrary. $\square$

7 Five examples

We review the five easiest, well-known PDE examples of conservative boundary control systems, and check how our techniques work for them.

7.1 Delay line

We consider the delay line system S on state space $X = L^2(0,1)$. The Lax-Phillips group of the system is the unitary right (forward) shift on $L^2(\mathbb{R})$, and hence S is a conservative system with a nilpotent semigroup. The system S is given in PDE form as follows:

$$\begin{cases} z_t(t,\xi) = -z_\xi(t,\xi) \text{ for all } t \geq 0 \text{ and } \xi \in (0,1), \\ z(t,0) = u(t) \text{ and } z(t,1) = y(t) \text{ for all } t \geq 0, \\ u(0,\xi) = u_0(\xi) \text{ for all } \xi \in (0,1); \end{cases}$$

The system theory of such equations has been treated e.g. in [2], [11] in a more general setting. The input (output) end of the delay line is at $\xi = 0$ ($\xi = 1$, respectively). Hence $L = -\frac{d}{d\xi}$, $Gz = z(0)$ and $Kz = z(1)$, and the solution space is $Z = H^1(0,1)$. It is easy to check that $\Gamma := (L, G, K)$ is a doubly boundary control node.

Let us check that Γ satisfies the conditions of Theorem 7. Verifying (i) amounts to computing the integral

$$2\mathrm{Re} \int_0^1 \overline{x(\xi)}\,(-x'(\xi))\; d\xi = - \int_0^1 \frac{d}{d\xi} |x(\xi)|^2 \, d\xi = -|x(1)|^2$$

since $x(0) = 0$ in Ker G. To prove (ii), integrate partially

$$\int_0^1 \overline{z(\xi)}\,(-x'(\xi))\; d\xi + \int_0^1 \overline{z'(\xi)}\,(-x(\xi))\; d\xi$$
$$= -\overline{z(1)}x(1) + \overline{z(0)}x(0) = \overline{z(0)}x(0)$$

since now $x(1) = 0$ in Ker K.

7.2 Vibrating string

Consider the system S described by the wave equation on interval $[0, 1]$ with endpoint control and observation:

$$\begin{cases} z_{tt}(t,\xi) = z_{\xi\xi}(t,\xi) \quad \text{for } \xi \in (0,1) \text{ and } t \geq 0, \\ -z_t(t,1) - z_\xi(t,1) = \sqrt{2}\, u(t) \quad \text{for } t \geq 0, \\ \sqrt{2}\, y(t) = -z_t(t,1) + z_\xi(t,1) \quad \text{for } t \geq 0, \\ z(t,0) = 0 \quad \text{for } t \geq 0, \text{ and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in (0,1). \end{cases} \tag{7.1}$$

Equations (7.1) can be cast into form of (1.1) by using the rule

$$z_{tt} = z_{\xi\xi} \quad \hat{=} \quad \frac{d}{dt} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\frac{d^2}{d\xi^2} & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}.$$

Henceforth let $L := \left[\begin{smallmatrix} 0 & -1 \\ -\frac{d^2}{d\xi^2} & 0 \end{smallmatrix}\right] : Z \to X$, together with

$$Z := \left(H^1_{\{0\}}(0,1) \cap H^2(0,1)\right) \times H^1_{\{0\}}(0,1), \quad X := H^1_{\{0\}}(0,1) \times L^2(0,1)$$
$$\text{where} \quad H^1_{\{0\}}(0,1) := \left\{ z \in H^1_{\{0\}}(0,1) : z(0) = 0 \right\}.$$

It follows directly that $Z = \{z \in X : Lz \in X\}$ and $X = LZ$. The Hilbert spaces X and Z are equipped with their direct sum inner products for now but another norm for X will be given in Proposition 7.2. Then $Z \subset X$ with a bounded inclusion and $L \in \mathcal{L}(Z; X)$.

The (restriction of the distribution) derivative of $z \in H^1(0,1)$ is denoted by $z' \in L^2(0,1)$[8]. The operators $G : Z \to \mathbb{C}$ and $K : Z \to \mathbb{C}$ are defined by

$$G \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}} \left(w_0(1) - z_0'(1)\right) \text{ and } K \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}} \left(w_0(1) + z_0'(1)\right).$$

[8]But the time derivative is always denoted by subindex t.

Clearly Ker $G = \{\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in Z : w_0(1) = z_0'(1)\}$. Since point evaluations (also on the boundary point 1) are continuous in $H^1(0,1)$, it follows that $G, K \in \mathcal{L}(Z;\mathbb{C})$. By approximating the components of $z_0 \in X$ by C^2-functions, it follows that Ker G is dense in X.

It is easy to see that Ker $L = \left\{\left[\begin{smallmatrix} c\phi \\ 0 \end{smallmatrix}\right] : c \in \mathbb{C}\right\}$ where $\phi(\xi) = \xi$ for $\xi \in (0,1)$. We show next that the conditions (c) and (d) of Definition 1 hold for $\alpha = 0$. Trivially Ker $L \cap$ Ker $G = \{0\}$. Also LKer $G = X$, as for any $x \in X$ there exists $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in Z$ so that $L\left[\begin{smallmatrix} z_0 + c\phi \\ w_0 \end{smallmatrix}\right] = x$ for all $c \in \mathbb{C}$. Choosing $c = w_0(1) - z_0'(1)$, we see that $\left[\begin{smallmatrix} z_0 + c\phi \\ w_0 \end{smallmatrix}\right] \in$ Ker G. We have now:

Proposition 13. *Let the operators L, G, K and spaces Z, X be defined as earlier in this subsection.*

(i) *The triple $\Gamma = (L, G, K)$ is a boundary control node in the sense of Definition 1. The domain space $V = \left[\begin{smallmatrix} I \\ G \end{smallmatrix}\right] Z$ for the associated operator node is given by*

$$V = \left\{ \begin{bmatrix} z_0 \\ w_0 \\ \frac{1}{\sqrt{2}}\left(w_0(1) - z_0'(1)\right) \end{bmatrix} : z_0 \in H^1_{\{0\}}(0,1) \cap H^2(0,1) \text{ and } w_0 \in H^1_{\{0\}}(0,1) \right\}.$$

(ii) *For any $u \in C^2([0,\infty))$ and $\begin{bmatrix} z_0 & w_0 & u(0) \end{bmatrix}^T \in V$, there exists a unique classical solution*

$$z(\cdot) \in C([0,\infty); H^2(0,1)) \cap C^1([0,\infty); H^1_{\{0\}}(0,1)) \cap C^2([0,\infty); L^2(0,\infty))$$

of (7.1) *satisfying the initial conditions $z(0) = z_0$ and $z_t(0) = w_0$.*

The requirement $\begin{bmatrix} z_0 & w_0 & u(0) \end{bmatrix}^T \in V$ is known as a *compatibility condition* in PDE literature.

Proof. Only (ii) has not been proved yet. If we show that $L|$Ker G is a dissipative operator (which will be omitted now, as it follows from Proposition 13 anyway), then there exists a unique solution

$$\begin{bmatrix} z(\cdot) \\ -z_t(\cdot) \end{bmatrix} \in C([0,\infty); Z) \cap C^1([0,\infty); X)$$

for (1.1) by Lemma 1. Then $z(\cdot)$ solves (7.1) (in the sense of distributions), and it has the other required properties, too. $\square$

Let us treat the energy balance questions next. From equations (7.1) we see that $z_t(t,1) = \frac{1}{\sqrt{2}}\left(u(t) + y(t)\right)$ and $z_\xi(t,1) = \frac{1}{\sqrt{2}}\left(u(t) - y(t)\right)$. By partial integration, we get (at least formally) for solutions of (7.1)

$$\frac{d}{dt}\int_0^1 |z_\xi(t,\xi)|^2 \, d\xi = 2\mathrm{Re}\, \overline{z_t(t,1)} z_\xi(t,1) - \frac{d}{dt}\int_0^1 |z_t(t,\xi)|^2 \, d\xi.$$

Thus $\frac{d}{dt}E(z;t) = |u(t)|^2 - |y(t)|^2$ where the energy functional is defined by

$$E(z;t) := \int_0^1 \left(|z_\xi(t,\xi)|^2 + |z_t(t,\xi)|^2\right) \, d\xi = \|z'(t)\|^2_{L^2(0,1)} + \|z_t(t)\|^2_{L^2(0,1)}.$$

This energy functional is associated to a norm on the state space X, which makes S an energy preserving system:

Proposition 14. *The expression*

$$\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_X := \|z_0'\|^2_{L^2(0,1)} + \|w_0\|^2_{L^2(0,1)} \tag{7.2}$$

defines a Hilbert space norm for X such that $E(z;t) = \| \left[\begin{smallmatrix} z(t) \\ z_t(t) \end{smallmatrix}\right] \|^2_X$ for all all solutions $z(\cdot)$ of (7.1) satisfying the conditions of Proposition 13.

Proof. Equation (7.2) defines clearly a norm on X, and we have

$$\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_X < \|z_0\|^2_{L^2(0,1)} + \| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_X = \| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_{H^1_{\{0\}}(0,1)\times L^2(0,1)}.$$

The elementary form of the Poincaré inequality $\|z_0\|_{L^2(0,1)} \leq \|z_0'\|_{L^2(0,1)}$ is easy to show for $z_0 \in H^1_{\{0\}}(0,1)$, and it implies the converse inequality $\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_{H^1_{\{0\}}(0,1)\times L^2(0,1)} \leq 2\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_X$. The rest is clear from Proposition 13. □

Proposition 15. *Let the operators L, G, K and spaces Z, X be defined as earlier in this subsection. Use the energy norm (7.2) for X. Then $\Gamma = (L, G, K)$ describes a conservative system, associated to wave equation (7.1).*

Proof. It is a matter of changing a few signs in the earlier computations of this subsection to verify that $\Gamma^{\leftarrow} = (-L, K, G)$ is a boundary control node. For an arbitrary $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{Ker} G$, integrate partially to obtain

$$\begin{aligned}
&- 2\mathrm{Re}\, \langle \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], L \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \rangle_X = 2\mathrm{Re}\, \langle \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_0 \\ z_0'' \end{smallmatrix}\right] \rangle_X \\
&= \langle z_0'', w_0 \rangle_{L^2(0,1)} + \langle z_0', w_0' \rangle_{L^2(0,1)} + \langle w_0, z_0'' \rangle_{L^2(0,1)} + \langle w_0', z_0' \rangle_{L^2(0,1)} \\
&= 2\mathrm{Re}\, \left(\overline{z_0'(1)} w_0(1) - \overline{z_0'(0)} w_0(0) \right) = 2|w_0(1)|^2 = |K \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]|^2,
\end{aligned}$$

where the second to last equality follows from $w_0(0) = 0$ (since $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in Z$), and $w_0(1) = z_0'(1)$ (since $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{Ker} G$). Hence condition (i) of Theorem 7 follows.

To establish condition (ii), let $\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] \in \operatorname{Ker} K$. Then $w_1(0) = w_2(0) = 0$, $z_2'(1) = -w_2(1)$, and $G \left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] = \sqrt{2}\, w_2(1)$. By partial integration and using the boundary conditions, we get

$$\begin{aligned}
&\langle \left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right], L \left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] \rangle_X + \langle L \left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] \rangle_X \\
&= - \langle z_1', w_2' \rangle_{L^2(0,1)} - \langle z_1'', w_2 \rangle_{L^2(0,1)} - \langle w_1', z_2' \rangle_{L^2(0,1)} - \langle w_1'', z_2 \rangle_{L^2(0,1)} \\
&= \overline{z_1'(1)} w_2(1) + \overline{w_1(1)} z_2'(1) = \left(\overline{z_1'(1)} - \overline{w_1(1)} \right) w_2(1) = \overline{G \left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right]} G \left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right].
\end{aligned}$$

This completes the proof. □

7.3 Telegraph equation

A slight generalisation of the vibrating string is given by the telegraph equation for parameter $k \in \mathbb{R}$:

$$\begin{cases} z_{tt}(t,\xi) = k^2 z(t,\xi) - z_{\xi\xi}(t,\xi) \quad \text{for } \xi \in (0,1) \text{ and } t \geq 0, \\ -z_t(t,1) - z_\xi(t,1) = \sqrt{2}\, u(t) \quad \text{for } t \geq 0, \\ \sqrt{2}\, y(t) = -z_t(t,1) + z_\xi(t,1) \quad \text{for } t \geq 0, \\ z(t,0) = 0 \quad \text{for } t \geq 0, \text{ and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in (0,1). \end{cases} \tag{7.3}$$

The analysis of this example is analogous to that in Subsection 7.2, and only some differences are indicated. The operator L is this time given by $L := \left[\begin{smallmatrix} 0 & -1 \\ k^2 - \frac{d^2}{d\xi^2} & 0 \end{smallmatrix}\right]$. The spaces Z and X, together with the operators G and K are same as for the vibrating string. With these definitions, the triple $\Gamma = (L, G, K)$ appears to be a doubly boundary control node. If the energy norm is defined by

$$\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|_X^2 := \|z_0'\|_{L^2(0,1)}^2 + k^2 \|z_0\|_{L^2(0,1)}^2 + \|w_0\|_{L^2(0,1)}^2,$$

node Γ is seen to describe a conservative system, by almost same computations as in the proof of Proposition 15.

7.4 Reflecting mirror

This example is very much like the vibrating string, and for that reason we discuss in detail only the new aspects that emerge. We shall review the more complicated structure of Sobolev spaces and the elliptic regularity theory. A more general version has been treated in terms of "thin air" systems in [24, Section 7]; a construction that bears some resemblance to feedback techniques appearing in [23]. Our approach resembles the techniques of [9].

Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is an open bounded set with C^2-boundary $\partial\Omega$. We assume that $\partial\Omega$ is the union of two sets Γ_0 and Γ_1 with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ [9]. System S is described by the exterior problem

$$\begin{cases} z_{tt}(t,\xi) = \Delta z(t,\xi) \quad \text{for } \xi \in \Omega \text{ and } t \geq 0, \\ -z_t(t,\xi) - \frac{\partial z}{\partial \nu}(t,\xi) = \sqrt{2}\, u(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geq 0, \\ \sqrt{2}\, y(t,\xi) = -z_t(t,\xi) + \frac{\partial z}{\partial \nu}(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geq 0, \\ z(t,\xi) = 0 \quad \text{for } \xi \in \Gamma_0 \text{ and } t \geq 0, \text{ and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in \Omega. \end{cases} \tag{7.4}$$

We obtain equations of form (1.1) by using the rule

$$z_{tt} = \Delta z \quad \hat{=} \quad \frac{d}{dt} \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -\Delta & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix}.$$

[9] The sets Γ_1 and Γ_0 are allowed to have zero distance in [24]. This is possible because stronger background results from [14] are used there.

In analogy with the vibrating string, let $L := \left[\begin{smallmatrix} 0 & -1 \\ -\Delta & 0 \end{smallmatrix}\right] : Z \to X$ with

$$
\begin{aligned}
&Z := Z_0 \times H^1_{\Gamma_0}(\Omega) \text{ and } X := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega) \\
&\text{where} \quad Z_0 := \left\{ z \in H^1_{\Gamma_0}(\Omega) \cap H^{3/2}(\Omega) : \Delta z \in L^2(\Omega) \right\}.
\end{aligned}
$$

The norm for Z_0 is given by

$$\|z_0\|^2_{Z_0} := \|z_0\|^2_{H^1(\Omega)} + \|z_0\|^2_{H^{3/2}(\Omega)} + \|\Delta z_0\|^2_{L^2(\Omega)}.$$

For space X, we use the energy norm

$$\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \|^2_X := \||\nabla z_0|\|^2_{L^2(\Omega)} + \|w_0\|^2_{L^2(\Omega)}. \tag{7.5}$$

As is well known, it follows from Poincaré inequality $\|z_0\|_{L^2(\Omega)} \leq K\||\nabla z_0|\|_{L^2(\Omega)}$ for $z_0 \in H^1_{\Gamma_0}(\Omega)$ that this norm is equivalent to the direct sum norm of X, see e.g. [9, p. 168]. Thus $Z \subset X$ with a bounded inclusion and $L \in \mathcal{L}(Z; X)$.

Let us review the properties of Sobolev spaces and the trace mappings. The spaces $H^s(\Omega) := W^2_2(\Omega)$ for $s = 1, 3/2, 2$, and the boundary spaces $H^{1/2}(\partial\Omega)$, $H^{1/2}(\Gamma_0)$, and $H^{1/2}(\Gamma_1)$ are defined as usual, see [5, Definition 1.3.2.1]. Note that (by extending functions by zero on the other component) $L^2(\partial\Omega) = L^2(\Gamma_0) \oplus L^2(\Gamma_1)$. Because $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, we have (by locality) $H^{1/2}(\partial\Omega) = H^{1/2}(\Gamma_0) \oplus H^{1/2}(\Gamma_1)$, too. By [5, Theorem 1.5.1.3] the (Dirichlet) trace operator γ maps

$$H^1(\Omega) \ni g \overset{\gamma}{\mapsto} g|\partial\Omega \in H^{1/2}(\partial\Omega) \subset L^2(\partial\Omega),$$

and thus $\gamma \in \mathcal{L}(H^1(\Omega); L^2(\partial\Omega))$. Now, let π be the orthogonal projection of $L^2(\partial\Omega)$ onto $L^2(\Gamma_1)$; the latter regarded as a subspace of $L^2(\partial\Omega)$ in a natural way. With a slight misuse of notation, we write $\pi g = g|\Gamma_1$ and $(I - \pi)g = g|\Gamma_0$. Since now $(I - \pi)\gamma \in \mathcal{L}(H^1(\Omega); L^2(\partial\Omega))$, the space

$$H^1_{\Gamma_0}(\Omega) := \mathrm{Ker}\ (I - \pi)\gamma = \left\{ g \in H^1(\Omega) : g|\Gamma_0 = 0 \right\}$$

is a closed subspace of $H^1(\Omega)$. So $\gamma_0 := \pi\gamma|H^1_{\Gamma_0}(\Omega) \in \mathcal{L}(H^1_{\Gamma_0}(\Omega); L^2(\Gamma_1))$ and we abbreviate $\gamma_0 g = g|\Gamma_1$.

In the same manner, Z_0 is a closed subspace of $H^{3/2}(\Omega)$ since $Z_0 \subset H^1(\Omega) \subset H^{3/2}(\Omega)$ with continuous inclusions. By [5, Theorem 1.5.1.2], the (Neumann) trace operator $\gamma\frac{\partial}{\partial\nu} \in \mathcal{L}(H^{3/2}(\Omega); L^2(\partial\Omega))$ for Ω has a C^2-boundary. Now $\gamma_1 := \pi\gamma\frac{\partial}{\partial\nu}|Z_0 \in \mathcal{L}(Z_0; L^2(\Gamma_1))$, and we write $\gamma_1 g = \frac{\partial g}{\partial\nu}|\Gamma_1$.

Defining $U = Y := L^2(\Gamma_1)$, we get $G \in \mathcal{L}(Z; U)$ and $K \in \mathcal{L}(Z; Y)$ where

$$G\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\left(-\frac{\partial z_0}{\partial\nu}|\Gamma_1 + w_0|\Gamma_1\right) \text{ and } K\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\left(\frac{\partial z_0}{\partial\nu}|\Gamma_1 + w_0|\Gamma_1\right).$$

We shall require some facts from the elliptic regularity theory. Following [23, p. 444], we denote the Neumann mapping $\tilde{N}$ by

$$h = \tilde{N}g \Leftrightarrow \begin{cases} \Delta h = 0 \text{ in } \Omega, \\ h|\Gamma_0 = 0 \text{ in } \Gamma_0, \\ \frac{\partial h}{\partial\nu}|\Gamma_1 = g \text{ in } \Gamma_1, \end{cases} \tag{7.6}$$

where $h \in H^1_{\Gamma_0}(\Omega)$ is the unique variational solution. By the elliptic regularity theory, $\tilde{N} \in \mathcal{L}(L^2(\Gamma_1); H^{3/2}(\Omega)) \cap \mathcal{L}(H^{1/2}(\Gamma_1); H^2(\Omega))$. Moreover, if $z_0 \in H^1_{\Gamma_0}(\Omega)$ is the unique variational solution of

$$\Delta h = f \in L^2(\Omega), \quad h|\Gamma_0 = 0, \quad \frac{\partial z_0}{\partial \nu}|\Gamma_1 = 0,$$

then $h \in H^2(\Omega)$, see [9, Section 4]. Hence, the unique variational solution of

$$\Delta h = f \in L^2(\Omega), \quad h|\Gamma_0 = 0, \quad \frac{\partial h}{\partial \nu}|\Gamma_1 = g$$

belongs to $H^{3/2}(\Omega)$ $(H^2(\Omega))$ if $g \in L^2(\Gamma_1)$ $(H^{1/2}(\Gamma_1)$, respectively).

It is worth mentioning that the space Z_0 is given in another equivalent form [24, Section 7]:

Proposition 16. *Under the standing assumptions on Γ_1 and Γ_2, the space Z_0 satisfies*

$$Z_0 = \{z_0 \in H^1_{\Gamma_0}(\Omega) : \Delta z_0 \in L^2(\Omega) \textit{ and } \frac{\partial z_0}{\partial \nu}|\Gamma_1 \in L^2(\Gamma_1)\}.$$

Proof. If $z_0 \in H^{3/2}(\Omega)$, then $\frac{\partial z_0}{\partial \nu}|\Gamma_1 \in L^2(\Gamma_1)$ by [5, Theorem 1.5.1.2]. Conversely, if $z_0 \in H^1(\Omega)$ is the variational solution of

$$\Delta z_0 = f \in L^2(\Omega), \quad z_0|\Gamma_0 = 0, \quad \frac{\partial z_0}{\partial \nu}|\Gamma_1 = g \in L^2(\Gamma_1),$$

then $z_0 \in H^{3/2}(\Omega)$ by what has been said above about elliptic regularity. □

There is another consequence of elliptic regularity that depends on the assumption that $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$:

Proposition 17. *Under the standing assumptions on Γ_1 and Γ_2, we have*

$$\operatorname{Ker} G = \left\{ \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix} \right] \in \left(H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) : \frac{\partial z_0}{\partial \nu}|\Gamma_1 = w_0|\Gamma_1 \right\}.$$

Proof. If $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix} \right] \in \operatorname{Ker} G$, then $w_0 \in H^1(\Omega)$ and hence $w_0|\Gamma_1 \in H^{1/2}(\Gamma_1)$. But then z_0 is the variational solution of

$$\Delta z_0 = f \in L^2(\Omega), \quad z_0|\Gamma_0 = 0, \quad \frac{\partial z_0}{\partial \nu}|\Gamma_1 = w_0|\Gamma_1 \in H^{1/2}(\Gamma_1),$$

and so $z_0 \in H^2(\Omega)$ by elliptic regularity. □

Note that $Z_0 \subset H^2(\Omega)$ never holds because this would contradict the fact that $\gamma_1 Z_0 = L^2(\Gamma_1)$, as given in the proof of the following:

Proposition 18. *Let the operators L, G, K and spaces Z, X be defined as above. Then $\Gamma = (L, G, K)$ is a doubly boundary control node.*

Proof. Since $\tilde{N} \in \mathcal{L}(L^2(\Gamma_1); H^{3/2}(\Omega))$, we have $\tilde{N}L^2(\Gamma_1) \subset Z_0$. Furthermore, for any $g \in L^2(\Gamma_1)$ we have $\gamma_1 \tilde{N} g = g$. Thus $\gamma_1 Z_0 = L^2(\Gamma_1)$, and condition (a) of Definition 1 is satisfied. It is not difficult to see, using Proposition 17, that Ker G is dense in $X = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$: let $\epsilon > 0$, $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in X$ and choose $\left[\begin{smallmatrix} \tilde{z} \\ \tilde{w} \end{smallmatrix}\right] \in \left(H^1_{\Gamma_0}(\Omega) \cap C^\infty(\overline{\Omega})\right) \times H^1_{\Gamma_0}(\Omega)$ with $\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \tilde{z} \\ \tilde{w} \end{smallmatrix}\right] \|_X < \epsilon$. It is possible to construct $\hat{w} \in H^1_{\Gamma_0}(\Omega)$ satisfying $\|\hat{w}\|_{L^2(\Omega)} < \epsilon$ and $\hat{w}|\Gamma_1 = \tilde{w}|\Gamma_1 - \frac{\partial \tilde{z}}{\partial \nu}|\Gamma_1$; indeed, such $\hat{w}$ could be made to vanish in almost all of Ω except for points very close to Γ_1 by using a suitable smooth "mollifier". Now $\left[\begin{smallmatrix} \tilde{z}_0 \\ \tilde{w}_0 \end{smallmatrix}\right] := \left[\begin{smallmatrix} \tilde{z} \\ \tilde{w} \end{smallmatrix}\right] - \left[\begin{smallmatrix} 0 \\ \hat{w} \end{smallmatrix}\right] \in$ Ker G and $\| \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \tilde{z}_0 \\ \tilde{w}_0 \end{smallmatrix}\right] \|_X < 2\epsilon$.

Now, let $\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right] \in X$ be arbitrary. By Proposition 17, $\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right] = L \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -w_0 \\ -\Delta z_0 \end{smallmatrix}\right]$ for $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in$ Ker G if and only if $w_0 = -z_1$ and the variational solution $z_0 \in H^1_{\Gamma_0}(\Omega)$ of the problem

$$\Delta z_0 = -w_1, \quad z_0|\Gamma_0 = 0, \quad \frac{\partial z_0}{\partial \nu}|\Gamma_1 = -z_1|\Gamma_1,$$

satisfies $z_0 \in H^2(\Omega)$. Since $w_1 \in L^2(\Omega)$ and $z_1|\Gamma_1 \in H^{1/2}(\Gamma_1)$, this follows from the same elliptic regularity result as Proposition 17.

Finally, $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in$ Ker $L \cap$ Ker G if and only if $w_0 = 0$ together with $z_0 \in H^2(\Omega)$, $\Delta z_0 = 0$, $z_0|\Gamma_0 = 0$ and $\frac{\partial z_0}{\partial \nu}|\Gamma_1 = w_0|\Gamma_1 = 0$ if and only if $w_0 = 0$ and $z_0 = \tilde{N}0 = 0$ in (7.6). Conditions of Definition 1 are satisfied with $\alpha = 0$, and thus $\Gamma = (L, G, K)$ is a boundary control node. That also $\Gamma^{\leftarrow} = (-L, K, G)$ is such a node, is proved by a similar argument. $\square$

Lemma 3. *Let the operators L, G, K and spaces Z, X be defined as earlier in this subsection. Use the energy norm (7.5) for X.*

(i) *The boundary control node $\Gamma = (L, G, K)$ associated to wave equation (7.4) describes a (tory) conservative system $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ through Theorem 1.*

(ii) *The transfer function $\mathbf{G}(\cdot)$ of S is inner from both sides and analytic in an open set containing $\overline{\mathbb{C}_+}$. The semigroups of S and the dual system S^d are strongly stable in the reducing subspace $X_{cnu} := \left(\text{Ker } (\mathcal{C}^d)^* \cap \text{Ker } \mathcal{C}\right)^\perp$, where $\mathcal{C}$ ($\mathcal{C}^d$) denotes the observability map of S (S^d, respectively).*

(iii) *Assume, in addition, that Ω is connected. Then S is exactly controllable and observable in infinite time, and the semigroups of S and S^d are strongly (asymptotically) stable.*

Proof. For an arbitrary $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in$ Ker G, the Green's formula [5, Lemma 1.5.3.8] implies

$$
\begin{aligned}
&-2\mathrm{Re}\,\langle\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], L\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]\rangle_X = 2\mathrm{Re}\,\langle\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} w_0 \\ \Delta z_0 \end{smallmatrix}\right]\rangle_X \\
&= 2\mathrm{Re}\left(\langle\Delta\overline{z_0}, w_0\rangle_{L^2(\Omega)} + \int_\Omega \nabla\overline{z_0}\cdot\nabla w_0\, d\Omega\right) \\
&= 2\mathrm{Re}\left(\int_{\Gamma_0\cup\Gamma_1} \frac{\partial\overline{z_0}}{\partial\nu} w_0\, d\omega\right) = 2\|w_0|\Gamma_1\|^2_{L^2(\Gamma_1)}
\end{aligned}
$$

because $\frac{\partial z_0}{\partial\nu}|\Gamma_1 = w_0|\Gamma_1$. Clearly $K\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] = \sqrt{2}w_0|\Gamma_1$ for all $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathrm{Ker}\ G$, and condition (i) of Theorem 7 holds. Similarly,

$$
\langle\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], L\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]\rangle_X + \langle L\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]\rangle_X = -\int_{\Gamma_1} \frac{\partial\overline{z_0}}{\partial\nu} y_0\, d\omega - \int_{\Gamma_1} \overline{w_0}\frac{\partial x_0}{\partial\nu}\, d\omega \tag{7.7}
$$

for any $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in \mathrm{Ker}\ K$. On the other hand,

$$
\begin{aligned}
&\langle G\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], G\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]\rangle_{L^2(\Gamma_1)} \\
&= -\frac{1}{\sqrt{2}}\left\langle \frac{\partial z_0}{\partial\nu}|\Gamma_1, G\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]\right\rangle_{L^2(\Gamma_1)} + \frac{1}{\sqrt{2}}\langle w_0|\Gamma_1, G\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right]\rangle_{L^2(\Gamma_1)}.
\end{aligned} \tag{7.8}
$$

Since $G\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] = \sqrt{2}y_0|\Gamma_1 = -\sqrt{2}\frac{\partial x_0}{\partial\nu}|\Gamma_1$ for any $\left[\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right] \in \mathrm{Ker}\ K$, condition (ii) of Theorem 7 follows from (7.7) and (7.8).

Let us prove claim (ii) by using the theory of conservative systems and the classical Sz.-Nagy – Foiaş model for contractions. By $X_u \subset X$ denote the largest reducing subspace of the semigroup $S(t)$ of S (generated by $A = L|\mathrm{Ker}\ G$), such that $S(t)|X_u$ is a unitary group. By a continuous time analogue of [12, Proposition A.2], we have $X_u^\perp = \left(\mathrm{Ker}\ (\mathcal{C}^d)^* \cap \mathrm{Ker}\ \mathcal{C}\right)^\perp = X_{cnu}$. By reducing the unobservable and uncontrollable subspace X_u away from the state space X of S, we obtain another *simple* conservative system S' whose transfer transfer function is same $\mathbf{G}(\cdot)$ as that of S. The c.n.u. semigroup of S' is $S(t)|X_{cnu}$ with generator $A_{cnu} = L|\left(\mathrm{Ker}\ G \cap X_{cnu}\right)$.

Because the inclusion Ker $G \subset X$ is compact, the resolvent of the generator A is compact with $\sigma(A) = \sigma_p(A)$. Because the same holds for A_{cnu}, the intersection $\sigma(A_{cnu}) \cap i\mathbb{R}$ can have only $\pm i\infty$ as limit points. It follows then that $\mathbf{G}(\alpha)^*\mathbf{G}(\alpha) = I$ for almost all $\alpha \in i\mathbb{R}$ by [13, Lemma 3.2(v)] or [21, Corollary 7.3]. Since all this holds also for the dual system $S^d = S^{\leftarrow}$ by symmetry, we conclude that the H^∞-function $\mathbf{G}(\cdot)$ is inner from both sides.

Since S' is a tory system, the Sz.-Nagy – Foiaş characteristic function of A_{cnu} satisfies $\theta(\cdot) = V_1\mathbf{G}(\cdot)V_2$ where V_1 and V_2 identify unitarily the input and output spaces U and Y with the defect spaces of A_{cnu}. Then $\theta(\cdot)$ is inner from both sides, and the Sz.-Nagy – Foiaş operator model [22, formula (a) on p. 279] for $S(t)|X_{cnu}$ reduces to the more simple Hankel range form [22, formula (a') on p. 279]. From this it follows easily that $S(t)|X_{cnu}$ is strongly stable[10] on X_{cnu}, $\sigma(A_{cnu}) \cap i\mathbb{R} = \emptyset$ by the compact resolvent, and thus $\mathbf{G}(\cdot)$ is analytic outside $\sigma(A_{cnu}) \subset \mathbb{C}_-$.

It remains to prove claim (iii). Suppose we had shown that $\dim X_u = 0$. Then the semigroups of S and S^d are strongly stable, and that S itself is a simple conservative system. As $\mathbf{G}(\cdot)$ is inner from both sides, its Hankel operator has closed range, and the canonical (simple conservative) Hankel range realization of $\mathbf{G}(\cdot)$ is exactly controllable in infinite time. The same holds for S by the well-known state space isomorphism theorem for simple conservative systems, see e.g. [19, Chapter 11]. By considering the dual system S^d, the exact observability of S in infinite time follows.

We proceed to show that $\sigma_p(A) \cap i\mathbb{R} = \emptyset$ which clearly implies $\dim X_u = 0$. We already know that $0 \notin \sigma(A)$ from the proof of Proposition 18. If $(ir - L)\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] = 0$ for $r \in \mathbb{R} \setminus \{0\}$, then $w_0 = -irz_0 \in H^2(\Omega)$, $(r^2 + \Delta)z_0 = 0$, $z_0|\Gamma_0 = 0$, $\frac{\partial z_0}{\partial \nu}|\Gamma_1 = -irz_0|\Gamma_1$. But then Green's formula implies

$$
\begin{aligned}
-r^2\|z_0\|_{L^2(\Omega)} &= \langle \Delta z_0, z_0 \rangle_{L^2(\Omega)} = -\||\nabla z_0|\|_{L^2(\Omega)} + \int\limits_{\Gamma_0 \cup \Gamma_1} \frac{\partial \overline{z_0}}{\partial \nu} z_0 \, d\omega \\
&= -\||\nabla z_0|\|^2_{L^2(\Omega)} + ir\|z_0\|^2_{L^2(\Gamma_1)}.
\end{aligned}
$$

We conclude that z_0 solves the Helmholtz equation

$$
(r^2 + \Delta)z_0 = 0 \text{ on } \Omega, \quad z_0|\partial\Omega = 0, \quad \frac{\partial z_0}{\partial \nu}|\Gamma_1 = 0; \tag{7.9}
$$

compare this with [23, proof of Lemma 2.1(iii)]. Conversely, any solution $z_0 \in H^2(\Omega)$ of (7.9) satisfies $(ir - L)\left[\begin{smallmatrix} z_0 \\ -irz_0 \end{smallmatrix}\right] = 0$. Note that any solution of $\Delta z_0 = -r^2 z_0$, $z_0|\partial\Omega = 0$ in $H^1(\Omega)$ satisfies $z_0 \in \cap_{s>0} H^s(\Omega) \subset C_0^\infty(\Omega)$ as can be seen by using the elliptic regularity result iteratively, see e.g. [10].

To complete the proof, we shall show that (7.9) implies[11] $z_0 = 0$. Extend the set Ω to a larger open set Ω' by "glueing" an additional set Ω'' (with a nonempty interior) to the Γ_1-part of $\partial\Omega$. This extension can be carried out so that Ω' is connected, it has a C^2-boundary, $\partial\Omega' = \Gamma_0 \cap \Gamma_1'$, $\overline{\Gamma_0} \cap \overline{\Gamma_1'} = \emptyset$, and $\Gamma_1' \subset \Gamma_1 \cup \partial\Omega''$. Suppose that $z_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfies (7.9), and define the extended functions

$$
u(\xi) := \begin{cases} z_0(\xi) & \text{for } \xi \in \Omega, \\ 0 & \text{for } \xi \in \Omega' \setminus \Omega, \end{cases} \qquad u_j(\xi) := \begin{cases} \frac{\partial z_0}{\partial \xi_j}(\xi) & \text{for } \xi \in \Omega, \\ 0 & \text{for } \xi \in \Omega' \setminus \Omega, \end{cases}
$$

$$
\text{and} \quad g(\xi) := \begin{cases} \Delta z_0(\xi) & \text{for } \xi \in \Omega, \\ 0 & \text{for } \xi \in \Omega' \setminus \Omega \end{cases}
$$

[10] By the Sz.-Nagy – Foiaş operator model [22, formula (a) on p. 279], $S(t)|X_{cnu}$ is seen to be weakly stable. This together with the compact resolvent property implies the strong stability; the argument appearing in [23].

[11] Note that this implication does not hold, if Ω has a component Ω_0 such that $\partial\Omega_0 \cap \Gamma_1 = \emptyset$. Indeed, the spectrum of the "Dirichlet Laplacian" on a bounded connected set Ω_0 is always nonempty, see e.g. [3].

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. Then for any test function $\phi \in \mathcal{D}(\Omega')$ we have

$$\int\limits_{\Omega'} \phi u_j \, d\Omega = \int\limits_{\Omega} \phi \frac{\partial z_0}{\partial \xi_j} \, d\Omega = - \int\limits_{\Omega} \frac{\partial \phi}{\partial \xi_j} z_0 \, d\Omega = - \int\limits_{\Omega'} \frac{\partial \phi}{\partial \xi_j} u \, d\Omega$$

where the middle equality holds by [5, Theorem 1.5.3.1] because $z_0|\partial\Omega = 0$. It follows that each partial (distributional) derivative of u satisfies $\frac{\partial u}{\partial \xi_j} = u_j$. Since $u, u_j \in L^2(\Omega')$, we conclude that $u \in H^1(\Omega')$. Because $\Gamma_1' \subset \Gamma_1 \cup \partial\Omega''$, we get $u|\partial\Omega' = 0$ and $\frac{\partial u}{\partial \nu}|\Gamma_1' = 0$, too.

Since $z_0 \in H^2(\Omega)$, we have $g \in L^2(\Omega)$. Again, for any $\phi \in \mathcal{D}(\Omega')$ we get

$$\begin{aligned} \int\limits_{\Omega'} \phi g \, d\Omega &= \int\limits_{\Omega} \phi \Delta z_0 \, d\Omega = \int\limits_{\Gamma_0 \cup \Gamma_1} \phi \frac{\partial z_0}{\partial \nu} \, d\omega - \int\limits_{\Omega} \nabla \phi \cdot \nabla z_0 \, d\Omega \\ &= - \int\limits_{\partial\Omega} \frac{\partial \phi}{\partial \nu} z_0 \, d\omega + \int\limits_{\Omega} \Delta \phi \cdot z_0 \, d\Omega = \int\limits_{\Omega'} \Delta \phi \cdot u \, d\Omega \end{aligned}$$

where both boundary terms vanish since $\phi|\Gamma_0 = 0$, $\frac{\partial z_0}{\partial \nu}|\Gamma_1 = 0$, and $z_0|\partial\Omega = 0$. We conclude that $\Delta u = g \in L^2(\Omega')$ in the sense of distributions.

Since $u \in H^1(\Omega')$ and $\Delta u \in L^2(\Omega')$, the (generalised) Green's formula [5, Theorem 1.5.3.11] can be used as follows: for any $\phi \in \mathcal{D}(\Omega')$

$$\begin{aligned} \int\limits_{\Omega'} \phi \Delta u \, d\Omega &= \int\limits_{\Gamma_0 \cup \Gamma_1'} \phi \frac{\partial u}{\partial \nu} \, d\omega - \int\limits_{\Omega'} \nabla \phi \cdot \nabla u \, d\Omega = - \int\limits_{\Omega} \nabla \phi \cdot \nabla z_0 \, d\Omega \\ &= - \int\limits_{\Gamma_0 \cup \Gamma_1} \phi \frac{\partial z_0}{\partial \nu} \, d\omega + \int\limits_{\Omega} \phi \Delta z_0 \, d\Omega = -r^2 \int\limits_{\Omega} \phi z_0 \, d\Omega = -r^2 \int\limits_{\Omega'} \phi u \, d\Omega. \end{aligned}$$

Indeed, the second equality follows from the facts that $\phi|\Gamma_0 = 0$, $\frac{\partial u}{\partial \nu}|\Gamma_1' = 0$, and that $\nabla u(\xi) = 0$ vanishes in the interior of $\Omega' \setminus \Omega$; the second to the last equality holds since $\phi|\Gamma_0 = 0$ and z_0 solves (7.9). We have now proved that $u \in H_0^1(\Omega')$ is a (distributional) solution for the extended domain Helmholtz problem

$$(r^2 + \Delta) u = 0, \quad u|\partial\Omega' = 0, \quad \frac{\partial u}{\partial \nu}|\Gamma_1' = 0.$$

As noted earlier after (7.9), it follows that $u \in C_0^\infty(\Omega')$. By using e.g. [3, Theorem 3.5] locally, we see that u is real analytic in Ω'. By construction, u vanishes in the nonempty interior of the set $\Omega'' \subset \Omega'$. Since Ω' is connected, u vanishes in all of Ω'. Hence (7.9) has only the trivial solution in $H_0^1(\Omega)$ for all $r \in \mathbb{R}$, and the proof is complete. □

The exponential stability of the system S in Lemma 3 has been proved in [9, 23] under an additional geometric condition on Ω.

7.5 Kirchhoff beam

We next consider the system S associated to the Kirchhoff beam on interval $[0,1]$. The beam is clamped at the end $\xi = 0$, and we apply endpoint control and observation at the other end $\xi = 1$. The system is described by the following PDE:

$$\begin{cases} z_{tt}(t,\xi) = -z_{\xi\xi\xi\xi}(t,\xi) \quad \text{for } \xi \in (0,1) \text{ and } t \geq 0, \\ \begin{bmatrix} z_{\xi t}(t,1)+z_{\xi\xi}(t,1) \\ z_t(t,1)-z_{\xi\xi\xi}(t,1) \end{bmatrix} = \sqrt{2}\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \quad \text{for } t \geq 0, \\ \sqrt{2}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} z_{\xi t}(t,1)-z_{\xi\xi}(t,1) \\ z_t(t,1)+z_{\xi\xi\xi}(t,1) \end{bmatrix} \quad \text{for } t \geq 0, \\ z(t,0) = z_\xi(t,0) = 0 \quad \text{for } t \geq 0, \quad \text{and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in (0,1). \end{cases} \tag{7.10}$$

Again, we obtain equations of form (1.1) by using the rule

$$z_{tt} = -z_{\xi\xi\xi\xi} \quad \hat{=} \quad \frac{d}{dt}\begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{d^4}{d\xi^4} & 0 \end{bmatrix}\begin{bmatrix} z \\ w \end{bmatrix}.$$

Consequently, we define $L := \begin{bmatrix} 0 & 1 \\ -\frac{d^4}{d\xi^4} & 0 \end{bmatrix} : Z \to X$ together with

$$Z := \left(H^2_{\{0\}}(0,1) \cap H^4(0,1)\right) \times H^2_{\{0\}}(0,1) \text{ and } X := H^2_{\{0\}}(0,1) \times L^2(0,1),$$

where $H^2_{\{0\}}(0,1) := \left\{z \in H^1_{\{0\}}(0,1) \cap H^2(0,1) : z'(0) = 0\right\}$. The input and output operators are clearly given by

$$G\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\begin{bmatrix} w_0'(1)+z_0''(1) \\ w_0(1)-z_0'''(1) \end{bmatrix} \text{ and } K\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] := \frac{1}{\sqrt{2}}\begin{bmatrix} w_0'(1)-z_0''(1) \\ w_0(1)+z_0'''(1) \end{bmatrix}.$$

We leave it for an interested reader to carry out the similar arguments as in Subsection 7.2 for the wave equation, to verify that $\Gamma := (L, G, K)$, indeed, is a doubly boundary control node. For space X, we shall from now use the following norm

$$\|\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]\|_X^2 := \|z_0''\|^2_{L^2(0,1)} + \|w_0\|^2_{L^2(0,1)}. \tag{7.11}$$

Analogously to Proposition 14, this norm is equivalent to the natural cartesian product norm of X.

Proposition 19. *Let the operators L, G, K and spaces Z, X be defined as as earlier in this subsection. Use the Hilbert space norm* (7.11) *for X. Then $\Gamma = (L, G, K)$ is a conservative system, associated to the beam equation* (7.10).

Proof. As we said, showing that Γ is a doubly boundary control node will be left as an exercise to an interesting reader. Let $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{Ker} G$; i.e. $w_0(0) = w_0'(0) = 0$, $w_0'(1) = -z_0''(1)$ and $w_0(1) = z_0'''(1)$. Then

$$\begin{aligned} &-2\mathrm{Re}\,\langle\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], L\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]\rangle_X = 2\mathrm{Re}\left\langle\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -w_0 \\ z_0'''' \end{smallmatrix}\right]\right\rangle_X \\ &= \langle z_0'''', w_0\rangle_{L^2(0,1)} - \langle z_0'', w_0''\rangle_{L^2(0,1)} + \langle w_0, z_0''''\rangle_{L^2(0,1)} - \langle w_0'', z_0''\rangle_{L^2(0,1)} \\ &= 2\mathrm{Re}\left(\overline{z_0'''(1)}w_0(1) - \overline{z_0''(1)}w_0'(1)\right) = 2\left(|w_0(1)|^2 + |z_0''(1)|^2\right) \\ &= |K\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right]|^2, \end{aligned}$$

where the last equality follows since $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \operatorname{Ker} G$. So condition (i) of Theorem 7 is satisfied.

Now, let $\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right] \in Z$ and $\left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] \in \operatorname{Ker} K$. Then $z_1(0) = z_1'(0) = w_1(0) = w_1'(0) = 0$, $z_2(0) = z_2'(0) = w_2(0) = w_2'(0) = 0$, $w_2'(1) = z_2''(1)$, and $w_2(1) = -z_2'''(1)$. Using these gives by partial integration

$$\begin{aligned}
&\langle \left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right], L\left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right]\rangle_X + \langle L\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right]\rangle_X \\
&= -\langle z_1'''', w_2\rangle_{L^2(0,1)} + \langle z_1'', w_2''\rangle_{L^2(0,1)} - \langle w_1, z_2''''\rangle_{L^2(0,1)} + \langle w_1'', z_2''\rangle_{L^2(0,1)} \\
&= -\overline{z_1'''(1)}w_2(1) + \overline{z_1''(1)}w_2'(1) - \overline{w_1(1)}z_2'''(1) + \overline{w_1'(1)}z_2''(1) \\
&= \left(\overline{w_1'(1)} + \overline{z_1''(1)}\right) w_2'(1) + \left(\overline{w_1(1)} - \overline{z_1'''(1)}\right) w_2(1) \\
&= \langle G\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right], G\left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right]\rangle_{\mathbb{C}^2},
\end{aligned}$$

since $G\left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] = \sqrt{2}\begin{bmatrix} w_2'(1) \\ w_2(1) \end{bmatrix}$ for $\left[\begin{smallmatrix} z_2 \\ w_2 \end{smallmatrix}\right] \in \operatorname{Ker} K$. Hence condition (ii) of Theorem 7 follows, and the proof is complete. $\square$

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