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Abstract: *This is the second part of a two-part work [26], [27], on the self-adjoint solutions P of the discrete time algebraic Riccati operator equation (DARE), associated to the discrete time linear system (DLS) $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$*

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\ \Lambda_P = D^*JD + B^*PB, \quad \Lambda_P K_P = -D^*JC - B^*PA, \end{cases}$$

where the indicator operator Λ_P is required to have a bounded inverse. We work under the standing hypothesis that the transfer function $\mathcal{D}_\phi(z) := D + zC(I - zA)^{-1}B$ belongs to $H^\infty(\mathbf{D}; \mathcal{L}(U, Y))$, in which case we call the Riccati equation H^∞DARE. The cost operator J is nonnegative, or at least the Popov operator $\mathcal{D}^*J\mathcal{D}$ satisfies $\mathcal{D}^*J\mathcal{D} \geq \epsilon\mathcal{I} > 0$. We occasionally require the input operator B to be a compact Hilbert–Schmidt operator, and the DLS ϕ be approximately controllable.
The algebraic structure of the DARE is studied with the aid of inner DLS ϕ^P and spectral DLS ϕ_P, associated to each solution P of the DARE. Also two chains of DAREs are defined, associated to DLSs ϕ_P and ϕ^P. The nonnegative solutions of the DARE are studied by the Liapunov methods. The I/O-map $\mathcal{D}_\phi$ is factorized into two stable factors, corresponding to ϕ_P and ϕ^P, where $P \in ric_0(\phi, J)$ is any nonnegative, regular H^∞ solution of DARE. A converse result is given, too.
An order-preserving correspondence between the set $ric_0(\phi, J)$, the partial inner factors of the I/O-map $\mathcal{D}_\phi$, and the shift invariant subspaces is established. The solution set $ric_0(\phi, J)$ is characterized order-theoretically in the full solution set of the DARE. Finally, we consider the regular H^∞ solutions of the two DAREs, associated to inner and spectral DLS ϕ_P and ϕ^P.

AMS subject classifications: 30D55, 47A68, 47N70, 93B28, 93C55.

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8 Introduction

This is the second part of a two-part study on the input-output stable (I/O stable) discrete time linear system (DLS) $\phi := \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and the associated algebraic Riccati equation (DARE)

$$
\left\{
\begin{array}{l}
A^*PA - P + C^*JC = K_P^*\Lambda_P K_P, \\
\Lambda_P = D^*JD + B^*PB, \\
\Lambda_P K_P = -D^*JC - B^*PA,
\end{array}
\right.
\tag{52}
$$

denoted, together with its solution set, by $Ric(\phi, J)$. We assume that the reader has access to and familiarity with the first part [26] of this work. However, we briefly remind the most important assumptions, notions and notations. The input, state and output spaces of the DLS ϕ are separable Hilbert spaces, and they are possibly (but not necessarily) infinite dimensional. The self-adjoint cost operator J is assumed to be nonnegative throughout most of this second part — this is in contrast to [26] where many results are valid also for an indefinite cost operator J. If the DLS ϕ is output stable and I/O stable, then the associated DARE (52) is called an H^∞DARE. It appears in [26] that certain solutions of an H^∞DARE are more interesting than others; these are the H^∞ solutions $P \in ric(\phi, J) \subset Ric(\phi, J)$ and the regular H^∞ solutions $P \in ric_0(\phi, J) \subset ric(\phi, J)$, see [26, Definitions 20 and 21]). In the first part [26], the regular H^∞ solutions $P \in ric_0(\phi, J)$ are associated to the stable spectral factorizations of the Popov operator $\mathcal{D}^*J\mathcal{D}$, where $\mathcal{D}$ denotes the I/O-map of ϕ. The main theme of this latter part is to connect $ric_0(\phi, J)$ to the factorizations of the I/O-map $\mathcal{D}$ into causal, shift-invariant and I/O stable factors. This work, together with [26], constitutes a theory of the regular H^∞ solutions of a H^∞DARE and simultaneously, an inner-outer type state space factorization theory for operator-valued bounded analytic functions.

Why is the algebraic Riccati equation interesting in the first place? What makes the special algebraic Riccati equation, namely the H^∞DARE of type (52), interesting? A traditional system theoretic application of the algebraic Riccati equation, associated to unstable systems, is to find a (nonnegative) solution, such that the associated (semigroup of the) closed loop system is (at least partially) (exponentially) stabilized; see e.g. [2], [4], and [55], to mention a few possible references. The algebraic Riccati equation appears (in an adjoint form) in the theory of the Kalman filter for the stochastic state estimation. For further information about this, see [1, Chapter 10] which is a nice overview of the various types and applications of the (matrix) Riccati equations, both in continuous and discrete time. Furthermore, the algebraic Riccati equation has an important application in the canonical and spectral factorization of rational matrix-valued functions by the state space methods, see [15, Chapter 19]. The state space factorization methods can be extended to the co-analytic–analytic type factorizations for classes of nonrational unstable operator-valued functions, see [7], [10] and the references therein.

Our view into the Riccati equation $Ric(\phi, J)$ in (52) is of this latter kind. Because of our standing I/O stability assumption of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the connections to the operator-valued function theory become very important. We remark that the theory of H^∞DAREs, as developed here, is richer but less general than that of DAREs without such stability assumptions. In the light of the present work, the feedback stabilization of (the semigroup or the I/O-map of) an unstable DLS is seen as a separate problem, to be discussed elsewhere. We regard our DLS ϕ as something "already output and I/O-stabilized" by some means — not necessarily by the state feedback law, induced by some (nonnegative, stabilizing, maximal nonnegative) solution of the DARE. In the applications, there exists genuinely I/O stable discrete time processes that need not be stabilized; consider, for example, a discrete time Lax-Phillips scattering where the scattering process is usually described by (a DLS that has) an inner H^∞ transfer function. Our aim is to develop a sufficiently general algebraic Riccati equation theory that is able to deal with these situations.

8.1 Outline of the paper

We start by giving a short outline of the results presented here. To each solution $P \in Ric(\phi, J)$, two families of algebraic Riccati equations are introduced in Section 9. These are associated to the spectral DLS ϕ_P and the inner DLS ϕ^P, centered at the solution $P \in Ric(\phi, J)$. For the definition of ϕ_P and ϕ^P, see [26, Definition 19]. The spectral DARE $Ric(\phi_P, \Lambda_P)$ is the DARE associated to the ordered pair (ϕ_P, Λ_P), where the cost operator $\Lambda_P := D^*JD + B^*PB$ is the indicator of the solution P. Analogously, the inner $Ric(\phi^P, J)$ is associated to the ordered pair (ϕ^P, J). The solution sets of spectral and inner DAREs have natural relations to the solution set $P \in Ric(\phi, J)$ of the original DARE, see Lemmas 64 and 65. The transitions from the original DLS ϕ to the inner DLS ϕ^P and the spectral DLS ϕ_P are basic operations that we use in Section 13 to obtain order-theoretic descriptions of the solution (sub)set $ric_0(\phi, J) \subset Ric(\phi, J)$. The results of Section 9 are proved by algebraic manipulations, and do not require DARE (52) to be a H^∞DARE.

We remark that if the spectral DLS ϕ_P, (the inner DLS ϕ^P) is I/O stable and output stable, then the DARE $Ric(\phi_P, J)$, $(Ric(\phi^P, J))$ is a H^∞DARE, and it is associated to the minimax problem of DLS ϕ_P with the cost operator Λ_P, (DLS ϕ^P with the cost operator J, respectively). The conditions for this to happen appear to be quite central in our study. Recall that for $P \in Ric(\phi, J)$, ϕ_P is I/O stable and output stable if and only if P is a H^∞ solution, by Definition 20. For this reason it is important that, under technical assumptions, all "reasonable" solutions $P \in Ric(\phi, J)$ are shown to be (even regular) H^∞ solutions, see [26, Corollary 47 and Equation 35]. We conclude that the question whether the spectral DARE $Ric(\phi_P, \Lambda_P)$ is an H^∞DARE has already been settled in [26]. It requires further study to give analogous conditions for the inner DARE $Ric(\phi^P, J)$ to be a H^∞DARE.

This study is carried out in the present paper. When this is done, we have shown that the general class of H^∞DAREs is closed under the transitions to spectral and inner DAREs.

A fair amount of stability theory for DLSs is needed for the further results. This is provided by the scratch of an infinite-dimensional Liapunov equation theory that we develop in Section 10. An essential part of the Liapunov theory is based on monotonicity techniques, requiring the nonnegativity of the cost operator J, or some closely related assumption. By Corollary 75, we conclude that ϕ^P is output stable if $P \in Ric(\Phi, J)$ is nonnegative and the cost operator $J > 0$ has a bounded inverse, under quite general assumptions. It requires more work (and stronger assumptions) to make the inner DLS ϕ^P I/O stable and $Ric(\phi^P, J)$ an H^∞DARE.

The first main results of this paper are given in Section 11. We conclude that each *nonnegative* regular H^∞ solution $P \in ric_0(\phi, J)$ gives a factorization of the I/O-map

$$J^{\frac{1}{2}}\mathcal{D}_\phi = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P}. \tag{53}$$

The causal, shift-invariant factor $J^{\frac{1}{2}}\mathcal{D}_{\phi^P} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is densely defined, not necessarily I/O stable, but always strongly H^2 stable. This means that the I/O-map $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ has a bounded impulse response, and the mapping $J^{\frac{1}{2}}\mathcal{D}_{\phi^P} : \ell^1(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is bounded. If the input operator B of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is a compact Hilbert–Schmidt operator, then this factorization becomes a partial inner-outer factorization where all factors are I/O stable, see Lemma 79 and Theorem 81. In particular, the (properly normalized) inner DARE $Ric(J^{\frac{1}{2}}\phi^P, I)$ (which is equivalent to the inner DARE $Ric(\phi^P, J)$) becomes now a H^∞DARE, provided $P \in ric_0(\phi, J)$.

A generalized H^2 factorization is considered in Lemma 82. Furthermore, finite increasing chains of solution in $ric_0(\phi, J)$ give factorizations of the I/O-map of Blaschke–Potapov product type, as stated in Theorem 83. However, neither the zeroes nor the singular inner factor of the transfer function $\mathcal{D}_\phi(z)$ (whatever these would mean in our generality) play any explicit role in this construction.

In Section 12, we consider converse results to those given in the previous Section 11. In Lemma 89 we show that for $P \in ric_0(\phi, J)$, the I/O stability of $J^{\frac{1}{2}}\phi^P$ implies that $P \geq 0$. Here, an approximate controllability assumption $\overline{\text{range}(\mathcal{B}_\phi)} = H$ is made. Theorem 90 is a combination of results given in Sections 11 and 12. It states, under restrictive technical assumptions, that among the state feedbacks associated to solutions $P \in ric_0(\phi, J)$, it is exactly the nonnegative solutions which output stabilize and I/O-stabilize the (normalized closed loop) inner DLS $J^{\frac{1}{2}}\phi^P$. In other words, among the H^∞ solutions of the DARE $ric(\phi, J)$, it is exactly the nonnegative $P \in ric_0(\phi, J)$ which give the factorization (53) of the I/O-map $\mathcal{D}_\phi$ so that all the factors are I/O stable.

In Section 13, we study the partial ordering of the elements of $ric_0(\phi, J)$, as self-adjoint operators. The maximal nonnegative solution in the set $ric_0(\phi, J)$

is considered in Corollary 94, and seen to be the unique regular critical solution $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}}$, if the approximate controllability $\overline{\text{range}(\mathcal{B}_\phi)} = H$ is assumed. An order-preserving correspondence between the set $ric_0(\phi, J)$ and a set of certain closed shift-invariant subspaces of $\ell^2(\mathbf{Z}_+; U)$ is given in Theorem 95, in the spirit of the classical Beurling–Lax–Halmos Theorem. An order-theoretic characterization of the nonnegative elements of $ric_0(\phi, J)$ is given in Theorem 96.

In Section 14 we consider the conditions when the spectral DARE $Ric(\phi_P, \Lambda_P)$ and the inner DARE $Ric(\phi^P, J)$ are H^∞DAREs. The reason why this in interesting is discussed in Subsection 8.2.3 of this Introduction. Also the regular H^∞ solutions and the regular critical solutions of both the spectral and inner DAREs are described. Our technical assumptions include approximate controllability $\overline{\text{range}(\mathcal{B}_\phi)} = H$ and the Hilbert–Schmidt compactness of the input operator B of the DLS ϕ. The case of the spectral DARE is dealt in Lemma 97 and Corollary 98. As a byproduct, we see that the set $ric_0(\phi, J)$ is an order-convex subset of $Ric(\phi, J)$ in the following sense: if $P_1, P_2 \in ric_0(\phi, J)$ with $P_2 \leq P_1$, then all $P \in Ric(\phi, J)$ such that $P_2 \leq P \leq P_1$ satisfy $P \in ric_0(\phi, J)$. In Lemma 100 it is shown that the inner DARE $Ric(\phi^P, J)$ is an H^∞DARE if $P \in ric_0(\phi, J)$ is nonnegative and the cost operator $J > 0$ has a bounded inverse — in this case the same P is also the regular critical solution of DARE $ric(\phi^P, J)$. The full description of the regular H^∞ solutions $ric_0(\phi^P, J)$ of the inner DARE is given in Lemma 101.

In the final section, it is shown that the structure of the H^∞DARE $ric(\phi, J)$ and its inner DARE $ric(\phi^{P_0^{\text{crit}}}, J)$ is similar, where $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J \mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ is the regular critical solution. This means that the outer factor of the I/O-map $\mathcal{D}_\phi$ is nonessential, from the H^∞DARE point of view. The treatment is similar to that given in Lemmas 100 and 101 for general nonnegative $P \in ric_0(\phi, J)$ but now the cost operator $J \geq 0$ is not required to be boundedly invertible. This result has an application in [25, Section 7].

8.2 Connections to existing DARE theories

We proceed to discuss the similarities and differences of the present work to previous works by other authors.

8.2.1 Different DAREs appearing in literature

It is quite necessary to comment why we use the more general DARE (2) instead of the conventional LQDARE

$$\text{(54)} \qquad \begin{cases} A^*PA - P + C^*JC = A^*PB \cdot \Lambda_P^{-1} \cdot B^*PA \\ \Lambda_P = D^*JD + B^*PB, \end{cases}$$

that appears in Least Quadratic type of problems, and is traditionally discussed (together with its continuous time analogue) in the literature.

As the reader can see, the difference between DAREs (52) and (54) is the absense of a cross term of form D^*JC in (54). It is well known that by the preliminary static state feedback

$$u_j = -(D^*JD)^{-1}D^*JCx_j, \tag{55}$$

(if it makes sense) equation (52) can always be cast in the form of (54) without changing the structure of the full solution set, see [15, Proposition 12.1.1]. We remark that the feedback in (55) can be "formally" associated to an artificial zero solution of DARE (52), and this feedback can be given a optimization theoretic interpretation: it minimizes the cost of the first step. However, the cost for the future steps (in the closed loop) can be very expensive for some initial states $x_0 \in H$. The range of the observability map of the closed loop system is orthogonal to the feed-through operator.

In particular, if the feed-through operator D of the original DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ has a bounded inverse, then 0 solves $Ric(\Phi, J)$, and the (well defined) inner DARE $Ric(\phi^0, J)$ is of form (54). In fact, now the closed loop I/O-map $\mathcal{D}_{\phi^0}$ is a static constant operator D, and the inner DARE $Ric(\phi^0, J)$ "lives" in the undetectable subspace, equalling all of the state space H. Because DAREs $Ric(\phi^0, J)$ and $Ric(\phi, J)$ have the same solution sets, this can be used to check that the Riccati equation theory presented here is in harmony with the (usually finite dimensional) LQDARE and LQ-CARE theories presented in the literature.

Now, if the modified LQDARE $Ric(\phi^0, J)$ describes completely the solution set $Ric(\phi, J)$, why do not we always normalize the cross term to zero by the preliminary feedback (55)? We first remark that as a H^∞DARE, $ric(\phi^0, J)$ is trivial because it has no nontrivial nonnegative H^∞ solutions, by Lemma 101. This is, of course, to be expected, because a nontrivial H^∞ solution would have to factorize the static I/O-map D, see Lemma 79. We further remark that the modified LQDARE $Ric(\phi^0, J)$ is no longer directly connected to a factorization of an I/O-map — this is somewhat unfortunate if our interest in DARE comes from such factorizations. If the semigroup generator A of the original DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ is e.g. strongly stable, the same is not true for the semigroup $A_0 = A - B(D^*JD)^{-1}D^*JC$ of ϕ^0, unless $\mathcal{D}_\phi$ is outer. Then the DLS ϕ^0 would have "undetectable unstable modes" which could be inconvenient.

Because these comments alone do not seem to be a sufficient motivation not to use the preliminary feedback (55), we try to discuss this question from several other directions, too.

8.2.2 Application oriented reasons

Riccati equations are associated to cost minimization problems, and even to minimax problems and game theoretic problems, if the cost operator J is allowed to be indefinite. The information structure of such a problem is reflected by the form of the associated DARE. The information structure of DARE (52) is more general that that of (54), and the theory can be directly applied to several minimax game problems with different information structures, without making the preliminary feedback (55) which changes (one might even say: confuses) the information structure. Clearly, we get he information structure of LQDARE (54) in the special case when a direct cost is applied on the input of the system.

In particular, if we want to factorize a transfer function $\mathcal{D}_{\phi'}(z)$ such that $D := \mathcal{D}_{\phi'}(0)$ has dense range in the output space Y, then the cross term vanishes if and only if $\mathcal{D}_{\phi'}(z) = D$ identically. We remark that the transfer functions of the spectral DLSs have always identity operator as their feedthrough part, and thus the theory of LQDARE is not directly applicable, except in a trivial case. The more general matrix DARE (52) is considered in [15, Chapter 12 and 13]. Furthermore, in the continuous time works [36], [37], [38], [39], [40], [41], [42], [43], [44], [45] (O. J. Staffans) and [29] (K. Mikkola), the presented CAREs for the regular well-posed system always have nontrivial cross terms. We conclude that if we want to make a discrete time Riccati equation theory that can be easily compared to the above mentioned works, we must retain the cross terms.

8.2.3 Internal self-similarity of the DARE theory

In claim (iv) of Lemma 79 we introduce the factorization of the I/O-map as a composition of two I/O stable I/O-maps

$$J^{\frac{1}{2}}\mathcal{D}_{\phi} = J^{\frac{1}{2}}\mathcal{D}_{\phi^{\tilde{P}}} \cdot \mathcal{D}_{\phi_{\tilde{P}}},$$

for any $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$. The left (I, Λ_P)-inner factor $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is related to the inner DLS ϕ^P, and this inner factor can be further factorized by nonnegative solutions of the inner H^∞DARE $P \in ric_0(\phi^{\tilde{P}}, J)$, at least if J is boundedly invertible. We remark that even if the whole solution set satisfies $Ric(\phi^{\tilde{P}}, J) = Ric(\phi, J)$, the set of regular H^∞ solutions $ric_0(\phi^{\tilde{P}}, J)$ is smaller than the original $ric_0(\phi, J)$ by Lemma 101. This is roughly related to the fact that the transfer function $J^{\frac{1}{2}}\mathcal{D}_{\phi^{\tilde{P}}}$ has less "zeroes" than $J^{\frac{1}{2}}\mathcal{D}_{\phi}(z)$ because some of them belong to the factor $\mathcal{D}_{\phi_{\tilde{P}}}$.

A similar consideration can be given for the right factor $\mathcal{D}_{\phi_{\tilde{P}}}$, which is a spectral factor of the Popov operator $\mathcal{D}_{\phi}^* J \mathcal{D}_{\phi}$: nonnegative solutions of the spectral DARE $P \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ factorize $\mathcal{D}_{\phi_{\tilde{P}}}$ into stable factors. We remark that the "cardinality" of nonnegative solutions in $ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is diminished from that of the original $ric_0(\phi, J)$ because a "shift" by $\tilde{P} \geq 0$ appears, as described in Lemma 97. We further remark that each inner and spectral DARE $ric_0(\phi^{\tilde{P}}, J)$, $ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is associated to a cost minimization problem in a natural way. This gives a system theoretic interpretation to each of the various DAREs.

We conclude that our DARE theory and factorization theory are fully recursive in the sense explained above. It is clear that the multiplicative factorization in any associative algebra (or factorial monoid) is recursive in the following sense: One would like to go on factoring the previous factors, until an irreducible element has been reached. Because the Riccati equation is related to such multiplicative factorization, we feel that the Riccati equation theory should be presented in a way that does not hide the recursive nature of things. For this to be possible, we need to have a class of DAREs that is large enough to be closed under passage to inner and spectral DAREs at solutions of interest. In fact, many of our proofs rely on a recursive application of the same DARE theory to inner or spectral DLSs and DAREs. It is very exceptional that an inner or spectral DARE has a vanishing cross term, and the cross term free class of equations (54) is not large enough. Introducing the preliminary feedback would destroy this overall image, and confuse the meaning of the various Riccati equations.

8.3 Parameterizations of nonnegative solutions

Assume that ϕ is I/O stable, output stable, and $J \geq 0$. Let us return to the preliminary feedback (55) for a moment, and assume that we have both the zero solution and the regular critical solution P_0^{crit}. Clearly, both are in the set $ric_0(\phi, J)$ of the regular H^∞ solutions. We compare now the H^∞DAREs $ric(\phi^0, J)$ and $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ whose full solution sets equal that of the original $Ric(\phi, J)$.

As already has been pointed out, the factorization of the I/O-map $\mathcal{D}_{\phi^0}$ as a product of nontrivial causal, shift-invariant and I/O stable operators is not a sensible task, because the I/O-map of the inner DLS $\phi^0 = \left(\begin{smallmatrix} A-BD^{-1}C & B \\ 0 & D \end{smallmatrix} \right)$ is a static constant D. It is in the nature of $Ric(\phi^0, J)$ that the DARE "operates" in the unobservable part of the state space, and there are not connections to the I/O-map. When the nonnegative solutions of such DARE are to be considered, we would have to consider the $A_0 := A - BD^{-1}C$-invariant, unobservable unstable subspaces of the state space, as has been done in the matrix DARE works [16] and [55]. When the state space is finite dimensional, such an approach is very succesful because the structure of generalized eigenspaces of the semigroup A_0 is available. For obvious reasons, no "fully general" infinite-dimensional Riccati equation theory can follow these lines, even though such an approach can be quite pleasing and even satisfactory from the applications point of view. For references, see the continuous time results [2] and [4], the latter of which contains a nice example of infinite-dimensional, exponentially stabilizable system, built around the heat equation. As already stated, it is quite instructive to compare our results to the existing matrix results with the aid of the preliminary state feedback (55).

The inner DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ is the other extreme when compared to $ric(\phi^0, J)$: the I/O-map of $\phi^{P_0^{\mathrm{crit}}}$ is the full $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of the original $\mathcal{D}_\phi$, and the equation has a nonvanishing cross term (apart from trivial cases). The following consideration could be carried out as well for the original DLS ϕ and its I/O-map $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$, but we consider the inner DLS $\phi^{P_0^{\mathrm{crit}}}$ and the factor I/O-map $\mathcal{N}$ instead.

The state space of DLS $\phi^{P_0^{\mathrm{crit}}}$ is, in a sense, "critically visible" to include all “zeroes” of $\mathcal{D}_\phi$ to $\mathcal{N} = \mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$, but not to generate any extra “zeroes” to $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$ that are not zeroes of the original I/O-map $\mathcal{D}_\phi$. This makes it possible to associate a Blaschke–Potapov type factorization of the I/O-map $\mathcal{N}$ to each nonnegative $P \in ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$. One immediately gets the idea that the nonnegative solutions of DARE could be parameterized up to their order structure, by using these factorizations and not having to assume excessively from the DLS in question. To some extent this vision is right but a disappointment appears, as will be discussed in the following.

Some of the factorizations of the I/O-map $\mathcal{N}$ are connected to a nonnegative $P \in ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$, see [26, claim (ii) of Theorem 50]. The problem here is that the factor in question must have a particular kind of realization, before it can be connected to some solution $P \in ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$ of the H^∞DARE. When this has been done, we necessarily have $P \geq 0$, by Theorem 95.

In other words, we have trouble in identifying which factors of $\mathcal{N}$ (if not all) are accounted by the solutions of the DARE in the first place. One approach to circumvent this is to show that certain “canonical” or “minimal” realization ϕ_C of the same I/O-map (characteristically constructed around a unilateral or bilateral shift operator) have a state space (and the DARE) “complicated enough” so that each factor of the I/O-map is associated to some solution of $Ric(\phi_C, J)$. Under very restrictive structural assumptions (such as the exact (infinite time) controllability), all such canonical or minimal realizations would have an isomorphic state spaces, and then the DAREs $Ric(\phi_C, J)$ and $Ric(\phi, J)$ would have the same structure. This would associate a solution of DARE $Ric(\phi, J)$ to each factor of $\mathcal{N}$, at the expence of additional restrictions on the data. We return to these considerations in our later works. We remark that it has been well known fact for quite a long time that general infinite dimensional state space systems do not have state space isomorphism, see [9, Chapter 3]. For positive (two-directional) results in this direction, see [10], and in particular the discrete time result [11, Theorem 4.1].

8.4 Notations

We use the following notations throughout the paper: $\mathbf{Z}$ is the set of integers. $\mathbf{Z}_+ := \{j \in \mathbf{Z} \quad | \quad j \geq 0\}$. $\mathbf{Z}_- := \{j \in \mathbf{Z} \quad | \quad j < 0\}$. $\mathbf{T}$ is the unit circle and $\mathbf{D}$ is the open unit disk of the complex plane $\mathbf{C}$. If H is a Hilbert space, then $\mathcal{L}(H)$ denotes the bounded and $\mathcal{LC}(H)$ the compact linear operators in H. Elements of a Hilbert space are denoted by upper case letters; for example $u \in U$. Sequences in Hilbert spaces are denoted by $\tilde{u} = \{u_i\}_{i \in I} \subset U$, where I is the index set. Usually $I = \mathbf{Z}$ or $I = \mathbf{Z}_+$. Given a Hilbert space Z, we define the sequence spaces

$$
\begin{aligned}
Seq(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \mid z_i \in Z \quad \text{and} \quad \exists I \in \mathbf{Z} \quad \forall i \leq I : z_i = 0\big\}, \\
Seq_+(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \mid z_i \in Z \quad \text{and} \qquad \forall i < 0 : z_i = 0\big\}, \\
Seq_-(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \in Seq(Z) \mid z_i \in Z \quad \text{and} \qquad \forall i \geq 0 : z_i = 0\big\}, \\
\ell^p(\mathbf{Z}; Z) &:= \Big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z \mid \sum_{i\in\mathbf{Z}} ||z_i||_Z^p < \infty\Big\} \quad \text{for} \quad 1 \leq p < \infty, \\
\ell^p(\mathbf{Z}_+; Z) &:= \Big\{\{z_i\}_{i\in\mathbf{Z}_+} \subset Z \mid \sum_{i\in\mathbf{Z}_+} ||z_i||_Z^p < \infty\Big\} \quad \text{for} \quad 1 \leq p < \infty, \\
\ell^\infty(\mathbf{Z}; Z) &:= \Big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z \mid \sup_{i\in\mathbf{Z}} ||z_i||_Z < \infty\Big\}.
\end{aligned}
$$

The following linear operators are defined for $\tilde{z} \in Seq(Z)$:

- the projections for $j, k \in \mathbf{Z} \cup \{\pm\infty\}$

$$
\begin{aligned}
&\pi_{[j,k]}\tilde{z} := \{w_j\}; \quad w_i = z_i \quad \text{for} \quad j \leq i \leq k, \quad w_i = 0 \quad \text{otherwise}, \\
&\pi_j := \pi_{[j,j]}, \quad \pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]}, \\
&\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,
\end{aligned}
$$

- the bilateral forward time shift τ and its inverse, the backward time shift τ^*

$$
\begin{aligned}
\tau\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j-1}, \\
\tau^*\tilde{u} &:= \{w_j\} \quad \text{where} \quad w_j = u_{j+1}.
\end{aligned}
$$

Other notations are introduced when they are needed. We also use some notations that have already been introduced in [26].

9 The algebraic properties of DARE

In this section, we write down a number of algebraic properties associated to iterated transitions to inner and spectral minimax nodes, DLSs and DLSs. The algebraic Riccati equation, together with the spectral DLS ϕ_P and the inner DLS ϕ^P, has already been introduced in [26, Section 3]. The spectral DLS ϕ_P has been extensively used in [26] because its I/O-map gives spectral factors for the Popov operator $\bar{\pi}_+\mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$. For the inner DLS ϕ^P we have not had much application until now. The results of this section are proved by purely algebraic manipulations, and do not require input, output or I/O stability of any of the DLSs considered. The definiteness of the cost operator J does not play any role, either. Later, in Sections 14 and 15, the analogous structure of the H^∞DARE is considered, for $J \geq 0$.

We associate two chains of DAREs to a given DARE $Ric(\phi, J)$. The elements of these chains are called the spectral and inner DAREs. Both the chains are indexed by the solutions $P \in Ric(\phi, J)$. These new DAREs make it easy to "move" in the solution set $Ric(\phi, J)$ of the original DARE, provided we can solve these Riccati equations. The presented structure (in some form) are well known to specialists in Riccati equations, but they are hard to locate in the literature. For us, the presented chains of DAREs are invaluable tools in sections 11 and 13.

Because DARE $Ric(\phi, J)$ does not solely depend on the DLS but also on the cost operator J, it is not sufficient to consider the DLS ϕ alone in this section. Instead, we have to consider the pairs (ϕ, J) that we call minimax nodes. Each minimax node defines a cost optimization problem, as defined in [19] for I/O stable DLSs. To this cost optimization problem, a Riccati equation is associated in a natural way. We first define two operations on the minimax nodes, and give their basic properties. The DARE in inntroduces in the familiar form in Definition 61.

Definition 57. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS with input space U, the state space H and output space Y. Let $J = J^* \in \mathcal{L}(Y)$ be a cost operator. Let $P = P^* \in \mathcal{L}(H)$ be arbitrary, such that the operator $\Lambda_P := D^*JD + B^*PB$ has a bounded inverse.*

(i) The ordered pair (ϕ, J) is called the minimax node, associated to the DLS ϕ and cost operator J.

(ii) The spectral minimax node of (ϕ, J) at P is defined by

$$(\phi, J)_P := \left(\begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}, \Lambda_P \right),$$

where $\Lambda_P := D^*JD + B^*PB$ and $\Lambda_P K_P := -D^*JC - B^*PA$. The operator Λ_P is called the indicator of P, and K_P is called the feedback operator of P.

(iii) The inner minimax node of (ϕ, J) at P is defined by

$$(\phi, J)^P := \left(\begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}, J \right),$$

where $A_P := A + BK_P$, $C_P = C + DK_P$, and K_P is as above. The operator A_P is called the (closed loop) semigroup generator of P, and C_P is called the (closed loop) output operator of P.

We call two DLSs equal, if their defining ordered operator quadruples (in difference equation form) are equal. Two minimax nodes are equal, if their DLSs are equal, and the cost operators are equal. In this case we write $(\phi_1, J_1) \equiv (\phi_2, J_2)$.

To each self-adjoint operator $P \in \mathcal{L}(H)$, two additional DLSs are associated:

Definition 58. *Let (ϕ, J), K_P, A_P and C_P be as in Definition 57. Let $P = P^* \in \mathcal{L}(H)$ be arbitrary, such that $D^*JD + B^*PB$ has a bounded inverse.*

(i) The DLS

$$\phi_P := \begin{pmatrix} A & B \\ -K_P & I \end{pmatrix}$$

is the spectral DLS, associated to the minimax node (ϕ, J), and centered at P.

(ii) The DLS

$$\phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix}$$

is called the inner DLS, associated to the minimax node (ϕ, J), and centered at P.

So, we can write (by definitions)

$$(\phi, J)_P = (\phi_P, \Lambda_P), \quad (\phi, J)^P = (\phi^P, J),$$

instead of formulae appearing in parts (ii) and (iii) of Definition 57. The iterated transitions to inner and spectral minimax nodes behave as follows.

Proposition 59. *Let (ϕ, J) be a minimax node. Then the following holds for $P_1 = P_1^* \in \mathcal{L}(H)$, $P_2 = P_2^* \in \mathcal{L}(H)$ and $\Delta P := P_2 - P_1$.*

$$\left((\phi, J)^{P_1}\right)_{P_2} \equiv \left(\phi^{P_1}, J\right)_{P_2} \equiv \left(\begin{pmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{pmatrix}, \Lambda_{P_2} \right), \tag{56}$$

$$\left((\phi, J)^{P_1}\right)^{P_2} \equiv \left(\phi^{P_1}, J\right)^{P_2} \equiv \left(\phi^{P_2}, J\right), \tag{57}$$

$$\left((\phi, J)_{P_1}\right)_{\Delta P} \equiv \left(\phi_{P_1}, \Lambda_{P_1}\right)_{\Delta P} \equiv \left(\phi_{P_2}, \Lambda_{P_2}\right), \tag{58}$$

$$\left((\phi, J)_{P_1}\right)^{\Delta P} \equiv \left(\phi_{P_1}, \Lambda_{P_1}\right)^{\Delta P} \equiv \left(\begin{pmatrix} A_{P_2} & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix}, \Lambda_{P_2} \right). \tag{59}$$

Proof. As before, denote by Λ_P, K_P the indicator and feedback operator, associated to the minimax node (ϕ, J) and $P \in \mathcal{L}(H)$. We start with proving equation (56). By $\tilde{\Lambda}_{P_2}$ and $\tilde{K}_{P_2}$ denote the indicator and feedback operator, associated to the minimax node (ϕ^{P_1}, J) and $P_2 \in \mathcal{L}(H)$. It is easy to see that $\tilde{\Lambda}_{P_2} = \Lambda_{P_2}$. The feedback operator of the inner DLS ϕ^{P_1} at P_2 satisfies $\tilde{K}_{P_2} = K_{P_2} - K_{P_1}$ because

$$\begin{aligned}(60)\qquad \tilde{K}_{P_2} &= \Lambda_{P_2}^{-1}\left(-D^*JC_{P_1} - B^*P_2A_{P_1}\right)\\ &= \Lambda_{P_2}^{-1}\left((-D^*JC - B^*P_2A) - (D^*JD + B^*P_2B)K_{P_1}\right)\\ &= \Lambda_{P_2}^{-1}\left(\Lambda_{P_2}K_{P_2} - \Lambda_{P_2}K_{P_1}\right) = K_{P_2} - K_{P_1},\end{aligned}$$

where $A_{P_1} = A + BK_{P_1}$ and $C_{P_1} = C + DK_{P_1}$, by part (ii) of Definition 57. Now (56) follows.

We proceed to prove equality (57). By part (iii) of Definition 57, we have

$$(\phi^{P_1}, J)^{P_2} \equiv \left(\begin{pmatrix} \tilde{A}_{P_2} & B \\ \tilde{C}_{P_2} & I \end{pmatrix}, J\right),$$

where the semigroup generator satisfies

$$\tilde{A}_{P_2} = A_{P_1} + B\tilde{K}_{P_2} = (A + BK_{P_1}) + B(K_{P_2} - K_{P_1}) = A + BK_{P_2} = A_{P_2},$$

and for the output operator we have

$$\tilde{C}_{P_2} = C_{P_1} + D\tilde{K}_{P_2} = (C + DK_{P_1}) + D(K_{P_2} - K_{P_1}) = C + DK_{P_2} = C_{P_2}$$

because $\tilde{K}_{P_2} = K_{P_2} - K_{P_1}$, as already shown in the proof of claim (56). This proves claim (57).

From now on, let $\tilde{\Lambda}_{\Delta P}$ and $\tilde{K}_{\Delta P}$ denote the indicator and feedback operator, associated to the spectral minimax node (ϕ_{P_1}, J). Denote also $\Delta P := P_2 - P_1$. Then

$$\begin{aligned}(61)\qquad \tilde{\Lambda}_{\Delta P} &= I^* \cdot \Lambda_{P_1} \cdot I + B^*\Delta P B\\ &= D^*JD + B^*P_1B + B^*(P_2 - P_1)B = \Lambda_{P_2},\end{aligned}$$

and

$$\begin{aligned}(62)\qquad \Lambda_{P_2}\tilde{K}_{\Delta P} = \tilde{\Lambda}_{\Delta P}\tilde{K}_{\Delta P} &= -I^* \cdot \Lambda_{P_1} \cdot (-K_{P_1}) - B^*\Delta P A\\ &= -D^*JC - B^*P_1A - B^*(P_2 - P_1)A = \Lambda_{P_2}K_{P_2},\end{aligned}$$

or $\tilde{K}_{\Delta P} = K_{P_2}$. But this gives for the spectral minimax node

$$(\phi_{P_1}, \Lambda_{P_1})_{\Delta P} \equiv \left(\begin{pmatrix} A & B \\ -\tilde{K}_{\Delta P} & I \end{pmatrix}, \tilde{\Lambda}_{\Delta P}\right) \equiv \left(\begin{pmatrix} A & B \\ -K_{P_2} & I \end{pmatrix}, \Lambda_{P_2}\right),$$

and equality (58) follows. It remains to consider the minimax node $(\phi_{P_1}, \Lambda_{P_1})^{\Delta P}$. By part (iii) of Definition 57, we have

$$(\phi_{P_1}, J)^{P_2} \equiv \left(\begin{pmatrix} \tilde{A}_{\Delta P} & B \\ \tilde{C}_{\Delta P} & I \end{pmatrix}, \tilde{\Lambda}_{\Delta P} \right)$$

where $\tilde{\Lambda}_{\Delta P} = \Lambda_{P_2}$ as above,

$$\tilde{A}_{\Delta P} = A + B\tilde{K}_{\Delta P} = A + BK_{P_2} = A + BK_{P_2} = A_{P_2},$$

and

$$\tilde{C}_{\Delta P} = -K_{P_1} + \tilde{K}_{\Delta P} = -K_{P_1} + K_{P_2}.$$

This proves the final claim (59). $\square$

The following "commutation" result will be important in applications:

Corollary 60. *Let (ϕ, J) be a minimax node, and $P_1, P_2 \in \mathcal{L}(H)$ self-adjoint. Then*

$$\left((\phi_{P_1})^{P_2 - P_1}, \Lambda_{P_1}\right) \equiv \left((\phi^{P_2})_{P_1}, \Lambda_{P_1}\right).$$

Proof. This is an immediate consequence of formulae (56) and (59) of Proposition 59. $\square$

Now we have introduced the notion of a minimax node, and defined two algebraic operations on such nodes: transition to inner and spectral minimax nodes. In the following definition, a discrete time algebraic Riccati equation (DARE) is associated to each minimax node in the familiar form, see [26, Definition 18].

Definition 61. *Let $(\phi, J) \equiv \left(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), J\right)$ be a minimax node. Then the following system of operator equations*

$$\begin{cases} A^*PA - P + C^*JC = K_P^*\Lambda_P K_P \\ \Lambda_P = D^*JD + B^*PB \\ \Lambda_P K_P = -D^*JC - B^*PA \end{cases} \tag{63}$$

is called the discrete time algebraic Riccati equation (DARE) and denoted by $Ric(\phi, J)$. The linear operators are required to satisfy $\Lambda_P, \Lambda_P^{-1} \in \mathcal{L}(U)$ and $K_P \in \mathcal{L}(H; U)$. Here P is a unknown self-adjoint operator to be solved. If $P \in \mathcal{L}(H)$ satisfies (63), we write $P \in Ric(\phi, J)$.

As before, we use the same symbol $Ric(\phi, J)$ both for the solution set of a DARE, and the DARE itself. This should not cause confusion. When we write expressions such as

$$P \in Ric(\phi, J), \quad Ric(\phi, J) = Ric(\phi, J), \quad Ric(\phi, J) \subset Ric(\phi, J),$$

the symbol $Ric(\phi, J)$ denotes the solution set. Clearly, different minimax nodes can give the same DARE because the DARE depends on the operators C^*JC, D^*JC, and D^*JD, but not directly on C, D, or J. When two DAREs $Ric(\phi_1, J_1)$ and $Ric(\phi_2, J_2)$ equal in this way, we write $Ric(\phi_1, J_1) \doteq Ric(\phi_2, J_2)$. We have

$$(\phi_1, J_1) \equiv (\phi_2, J_2) \Rightarrow Ric(\phi_1, J_1) \doteq Ric(\phi_2, J_2) \Rightarrow Ric(\phi_1, J_1) = Ric(\phi_2, J_2),$$

and none of the implications is an equivalence. In particular, the equality $Ric(\phi, J) = Ric(\phi, J)$ does not imply that the two Riccati equations were same, and even less that the two minimax nodes were the same. If $(\phi_1, J_1) \equiv (\phi_2, J_2)$, then we write $Ric(\phi_1, J_1) \equiv Ric(\phi_2, J_2)$.

The inner and spectral minimax nodes of an original minimax node (ϕ, J) give rise to new DAREs: namely the inner and spectral DAREs, centered at the self-adjoint operator $P \in \mathcal{L}(U)$. In order to obtain something interesting, we must now require that in fact $P \in Ric(\phi, J)$.

Definition 62. *Let $(\phi, J) \equiv (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), J)$ be a minimax node. Let $P \in Ric(\phi, J)$ be arbitrary. Let ϕ_P and ϕ^P as given in Definition 58, and by Λ_P, K_P denote the indicator and feedback operators of P, respectively.*

(i) The DARE $Ric(\phi, J)_P :\equiv Ric(\phi_P, \Lambda_P)$

$$\begin{cases} A^*\tilde{P}A - \tilde{P} + K_P^*\Lambda_P K_P = \tilde{K}_{\tilde{P}}^*\tilde{\Lambda}_{\tilde{P}}\tilde{K}_{\tilde{P}} \\ \tilde{\Lambda}_{\tilde{P}} = \Lambda_P + B^*\tilde{P}B \\ \tilde{\Lambda}_{\tilde{P}}\tilde{K}_{\tilde{P}} = \Lambda_P K_P - B^*\tilde{P}A \end{cases} \tag{64}$$

is the spectral (ϕ, J)-DARE, centered at $P \in Ric(\phi, J)$. Here $\tilde{P}$ is an unknown self-adjoint operator to be solved.

(ii) The DARE $Ric(\phi, J)^P :\equiv Ric(\phi^P, J)$

$$\begin{cases} A_P^*\tilde{P}A_P - \tilde{P} + C_P^*JC_P = \tilde{K}_{\tilde{P}}^*\Lambda_{\tilde{P}}\tilde{K}_{\tilde{P}} \\ \Lambda_{\tilde{P}} = D^*JD + B^*\tilde{P}B \\ \Lambda_{\tilde{P}}\tilde{K}_{\tilde{P}} = -D^*JC_P - B^*\tilde{P}A_P, \end{cases} \tag{65}$$

is the inner (ϕ, J)-DARE, centered at $P \in Ric(\phi, J)$. Here $\tilde{P}$ is an unknown self-adjoint operator to be solved, and $A_P := A + BK_P$, $C_P := C + DK_P$.

We start with discussing the spectral Riccati equation $Ric(\phi, J)_P$. The following proposition is basic, and serves as a prerequisite for Lemma 64.

Proposition 63. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$. Then $Ric(\phi, J)_P$ can be written in the equivalent form*

$$\begin{cases} A^*\tilde{P}A - \tilde{P} + K_P^*\Lambda_P K_P = K_{P+\tilde{P}}^*\Lambda_{P+P}K_{P+\tilde{P}} \\ \Lambda_{P+\tilde{P}} = D^*JD + B^*(P+\tilde{P})B \\ \Lambda_{P+\tilde{P}}K_{P+\tilde{P}} = -D^*JC - B^*(P+\tilde{P})A. \end{cases}$$

Proof. By equation (61), $\tilde{\Lambda}_{\tilde{P}} = \Lambda_{P+\tilde{P}}$, and by equation (62), $\tilde{K}_{\tilde{P}} = K_{P+\tilde{P}}$. $\square$

Lemma 64. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$ and $\tilde{P}$ be a bounded self-adjoint operator. Then the following are equivalent*

(i) $P + \tilde{P} \in Ric(\phi, J)$,

(ii) $\tilde{P} \in Ric(\phi, J)_P$.

Proof. Assume claim (i). Because both $P, (P + \tilde{P}) \in Ric(\phi, J)$, we have by Proposition 63

$$A^*(P + \tilde{P})A - (P + \tilde{P}) + C^*JC = K^*_{P+\tilde{P}}\Lambda_{P+\tilde{P}}K_{P+\tilde{P}},$$
$$A^*PA - P + C^*JC = K^*_P\Lambda_P K_P.$$

Here Λ_Q and K_Q denote the indicator and the feedback operator of the self-adjoint operator Q, relative to the original minimax node (ϕ, J). Subtracting these two Riccati equations we obtain

$$A^*\tilde{P}A - \tilde{P} + K^*_P\Lambda_P K_P = K^*_{P+\tilde{P}}\Lambda_{P+\tilde{P}}K_{P+\tilde{P}}.$$

But now, by Proposition 63, $\tilde{P} \in Ric(\phi, J)_P$, and claim (ii) follows.

For the converse direction, assume claim (ii). Let $P \in Ric(\phi, J)$, $P \in Ric(\phi_P, \Lambda_P) = Ric(\phi, J)_P$ be arbitrary. By adding the DAREs $Ric(\phi, J)$ and $Ric(\phi, J)_P$ we obtain

$$A^*(P + \tilde{P})A - (P + \tilde{P}) + C^*JC = K^*_{P+\tilde{P}}\Lambda_{P+P}K_{P+\tilde{P}}$$

where Proposition 63 has been used again. Thus claim (i) immediately follows. $\square$

The remaining part of this section is devoted to the study of the inner Riccati equation $Ric(\phi, J)^P$. Given any $P \in Ric(\phi, J)$, the relation between the solution sets of $Ric(\phi, J)^P$ and $Ric(\phi, J)$ appears to be very simple.

Lemma 65. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$ be arbitrary. Then the following are equivalent:*

(i) $\tilde{P} \in Ric(\phi, J)^P$,

(ii) $\tilde{P} \in Ric(\phi, J)$.

Proof. We prove the direction (i) $\Rightarrow$ (ii); the proof of the other direction is obtained by reading this proof in the reverse direction. Let $\tilde{P} \in Ric(\phi, J)^P$. Then the left hand side of the first equation in (65) takes the form

$$\begin{aligned} &A^*_P\tilde{P}A_P - \tilde{P} + C^*_PJC_P \\ &= A^*\tilde{P}A - \tilde{P} + C^*JC - K^*_P\Lambda_{\tilde{P}}K_{\tilde{P}} - K^*_{\tilde{P}}\Lambda_{\tilde{P}}K_P + K^*_P\Lambda_{\tilde{P}}K_P. \end{aligned} \tag{66}$$

Here Λ_Q and K_Q denote the indicator and the feedback operator of the self-adjoint operator Q, relative to the original minimax node (ϕ, J). By equation (60), $\tilde{K}_{\tilde{P}} = K_{\tilde{P}} - K_P$ and the right hand side of the first equation in (65) becomes

$$\tilde{K}_{\tilde{P}}^* \Lambda_{\tilde{P}} \tilde{K}_{\tilde{P}} = K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_{\tilde{P}} - K_P^* \Lambda_{\tilde{P}} K_{\tilde{P}} - K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_P + K_P^* \Lambda_{\tilde{P}} K_P.$$

This, together with equation (66) gives

$$A^* \tilde{P} A - \tilde{P} + C^* J C = K_{\tilde{P}}^* \Lambda_{\tilde{P}} K_{\tilde{P}}.$$

Thus $\tilde{P} \in Ric(\phi, J)$. This completes the proof. $\square$

As an immediate corollary, we can put $Ric(\phi, J)^P$ in a different form

Proposition 66. *Let (ϕ, J) be a minimax node. Let $P \in Ric(\phi, J)$. Then $Ric(\phi, J)^P$ can be written in the equivalent form*

$$\begin{cases} A_P^* \tilde{P} A_P - \tilde{P} + C_P^* J C_P = (K_{\tilde{P}} - K_P)^* \Lambda_{\tilde{P}} (K_{\tilde{P}} - K_P) \\ \Lambda_{\tilde{P}} = D^* J D + B^* \tilde{P} B \\ \Lambda_{\tilde{P}} K_{\tilde{P}} = -D^* J C - B^* \tilde{P} A, \quad \Lambda_P K_P = -D^* J C - B^* P A. \end{cases}$$

Proof. This is because $\tilde{K}_{\tilde{P}} = K_{\tilde{P}} - K_P$, by equation (60). $\square$

The results of Lemmas 64 and 65 can be given in a short form

$$\begin{aligned} Ric(\phi, J) &= P + Ric(\phi, J)_P = P + Ric(\phi_P, \Lambda_P), \\ Ric(\phi, J) &= Ric(\phi, J)^P = Ric(\phi^P, J) \end{aligned} \tag{67}$$

for all $P \in Ric(\phi, J)$. It now follows that the iterated transitions to inner and spectral DAREs satisfy the following rules of calculation.

Corollary 67. *Let $(\phi, J) \equiv (\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), J)$ be a minimax node. Let $P_1, P_2 \in Ric(\phi, J)$, and $\Delta P := P_2 - P_1 \in Ric(\phi, J)_{P_1}$. Then*

$$Ric(\phi^{P_1}, J)_{P_2} \equiv Ric(\begin{pmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{pmatrix}, \Lambda_{P_2}) = Ric(\phi, J) - P_2, \tag{68}$$

$$Ric(\phi^{P_1}, J)^{P_2} = Ric(\phi, J), \tag{69}$$

$$Ric(\phi_{P_1}, \Lambda_{P_1})_{\Delta P} = Ric(\phi, J) - P_2, \tag{70}$$

$$Ric(\phi_{P_1}, \Lambda_{P_1})^{\Delta P} \equiv Ric(\begin{pmatrix} A & B \\ K_{P_2} - K_{P_1} & I \end{pmatrix}, \Lambda_{P_2}) = Ric(\phi, J) - P_1. \tag{71}$$

We remark that the DLS $\phi_{P_2,P_1} := \left(\begin{smallmatrix} A_{P_1} & B \\ K_{P_1} - K_{P_2} & I \end{smallmatrix}\right)$ is familiar from [26, Proposition 56].

10 Liapunov equation theory

The operator equation

$$A^*PA - P + C^*JC = 0, \tag{72}$$

is called the discrete time Liapunov equation or the (symmetric) Stein equation. As with the Riccati equation, the operators are as follows: the operator $A \in \mathcal{L}(H)$ is the semigroup generator, $C \in \mathcal{L}(H, Y)$ is the output operator, and the self-adjoint operator $J \in \mathcal{L}(Y)$ is the cost operator. The solution P is required to be self-adjoint. It is clear that the observability and controllability Gramians $\mathcal{C}^*\mathcal{C}$ and $\mathcal{B}\mathcal{B}^*$ of a DLS are solutions of Liapunov equations, see e.g. [56, p. 71].

A fairly complete Liapunov equation theory is given e.g. in [15] and [56] for the case when A, C and J are matrices, and $J > 0$. It is well known that the matrix Liapunov equation has a unique solution for any self-adjoint matrix C^*JC if and only if $\sigma(A) \cap \left(\overline{\sigma(A)}\right)^{-1} = \emptyset$, see [15, Theorem 5.2.3]. When this spectral separation holds, the solution P can be expressed as a Cauchy integral, see [15, Theorem 5.2.4]. When we do not have the spectral separation, the Cauchy integral cannot be defined because an integration contour cannot be drawn such that $\sigma(A)$ and $\left(\overline{\sigma(A)}\right)^{-1}$ lie on the "opposite sides" of the contour. The Cauchy integral solution makes perfect sense even for some operator Liapunov equations, provided that the required spectral separation exists. Even if we produced the dimension free variants of these results, the spectral separation would be too restrictive a condition to be useful for non-power stable but nevertheless strongly stable semigroup generators A. If $\sigma(A) \subset \overline{\mathbf{D}}$, then the spectral separation forces $\sigma(A) \subset \mathbf{D}$, and so A is power stable.

In the present work, our main interest is not in finding solutions for Liapunov equations. Quite conversely, we are given a nonnegative solution P of the Liapunov equation (72), with $J \geq 0$. Our task is to show that the output stability of an associated observability map $\mathcal{C}_{\phi'} := \{J^{\frac{1}{2}}CA^j\}_{j\geq 0}$ follows, see Lemma 74. Then, an expression can be found for the minimal nonnegative solution P_0 of (72), and the other solutions are parameterized by their residual cost operators $L_{A,P} := \mathrm{s}-\lim_{j\to\infty} A^{*j}PA^j$, see Corollary 71. Recall that the residual cost operator is defined as a strong limit $L_{A,P} := \mathrm{s}-\lim_{j\to\infty} A^{*j}PA^j$, see [26, Definition 21].

We now briefly discuss the connection of the Liapunov equation to stability questions. The Liapunov equation is connected to the Liapunov stability theory of DLSs, see [17] for an exposition of the matrix case. For another view into this, suppose $Q \geq 0$ and $P > 0$ satisfies $A^*PA - P + Q = 0$. Then by writing for $x \neq 0$,

$$||Ax||_P^2 - ||x||_P^2 := \left\langle P^{\frac{1}{2}}Ax, P^{\frac{1}{2}}Ax\right\rangle - \left\langle P^{\frac{1}{2}}x, P^{\frac{1}{2}}x\right\rangle = -\langle Qx, x\rangle \leq 0, \tag{73}$$

we see that such solution P defines an inner product topology such that the operator A becomes a contraction. Because P is bounded, we have

$||x||_P \leq ||P|| \cdot ||x||$, which implies that the $||.||_P$-topology is generally weaker that the original. Clearly the topologies coincide if P has a bounded inverse. This gives some functional analytic meaning for the Liapunov stability theory of linear systems.

Another instance where a Liapunov equation arises is connected to DARE and given in the following proposition. Its proof is a straightforward calculation, and clearly connected to the inner Riccati equation $Ric(\phi, J)^P$ of Definition 62 and Lemma 65.

Proposition 68. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Then $P \in Ric(\phi, J)$ if and only if*

$$A_P^* P A_P - P + C_P^* J C_P = 0, \tag{74}$$

where $A_P := A + BK_P$ and $C_P := C + DK_P$. Furthermore, $D^* J C_P + B^* P A_P = 0$.

By solving the Liapunov equation (74), the operator $P \in Ric(\phi, J)$ can be recovered from the operators Λ_P and K_P, provided that the solution of the Liapunov equation is unique or we know the residual cost operator $L_{A_P,P}$ apriori. Unfortunately, it is difficult to check (for uniqueness of P) the spectral separation $\sigma(A_P) \cap \left(\overline{\sigma(A_P)}\right)^{-1} = \emptyset$ for solutions $P \in Ric(\phi, J)$ of interest. By iteration, the following algebraic triviality is shown.

Proposition 69. *Assume that $A \in \mathcal{L}(H)$, $C \in \mathcal{L}(H, Y)$ and $J \in \mathcal{L}(Y)$. Assume that a possibly unbounded linear map $P : H \supset \mathrm{dom}(P) \to H$, $A\,\mathrm{dom}(P) \subset \mathrm{dom}(P)$, satisfies the Liapunov equation $A^*PA - P + C^*JC = 0$. Then*

$$Px = \sum_{j=0}^{n-1} A^{*j} C^* J C A^j x + A^{*n} P A^n x, \quad \textit{for all} \quad x \in \mathrm{dom}(P), n \geq 1.$$

We start to study solutions P of the Liapunov equation (72) for which the residual cost operator $L_{A,P}$ exists. The fact that the mapping $P \mapsto A^*PA - P$ is bounded and linear, gives the background for the following proposition:

Proposition 70. *Assume that the linear mappings $A \in \mathcal{L}(H)$, $C \in \mathcal{L}(H, Y)$ and $J \in \mathcal{L}(Y)$ self-adjoint. Then the following are equivalent:*

(i) There is a solution P_0 of the Liapunov equation such that the residual cost operator vanishes: $L_{A,P_0} = 0$.

(ii) There is at least one solution $\tilde{P}$ of the Liapunov equation such that the residual cost operator $L_{A,\tilde{P}} \in \mathcal{L}(H)$ exists.

(iii) The Liapunov equation has at least one solution, and for all solutions P, the residual cost operator $L_{A,P} \in \mathcal{L}(H)$ exists.

If, in addition, $J \geq 0$, then we have a third equivalent condition *(iv) The DLS* $\phi' := \left(\begin{smallmatrix} A & * \\ J^{\frac{1}{2}}C & * \end{smallmatrix} \right)$ *is output stable.*

Proof. The implication (i) $\Rightarrow$ (ii) is trivial. To prove the implication (ii) $\Rightarrow$ (iii), note that by Proposition 69 $\sum_{j=0}^{n-1} A^{*j}C^*JCA^j x = \tilde{P}x - A^{*n}\tilde{P}A^n x$ for all $x \in H$. Thus $\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j = \tilde{P} - L_{A,\tilde{P}}$ exists if (ii) holds. Now, for all solutions P of the Liapunov equation the strong limit $L_{A,P} = \mathrm{s}-\lim_{n\to\infty} A^{*n}PA^n$ exists, because the limit on the right hand side for the following equation exists

$$A^{*n}PA^n x = Px - \sum_{j=0}^{n-1} A^{*j}C^*JCA^j x$$

for all $x \in H$.

To prove the implication (iii) $\Rightarrow$ (i), assume $\tilde{P}$ is a solution such that $L_{A,\tilde{P}} \in \mathcal{L}(H)$ exists. It follows that the strong limit operator $P_0 := \mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j$ exists and equals $\tilde{P} - L_{A,\tilde{P}} \in \mathcal{L}(H)$. We show that P_0 is a solution of the Liapunov equation such that $L_{A,P_0} = 0$. Let $x_1, x_2 \in H$ be arbitrary. Then

$$\begin{aligned}(75)\qquad \langle x_1, (A^*P_0A - P_0)x_2\rangle_H &= \left\langle Ax_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)Ax_2 \right\rangle_H \\ &- \left\langle x_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)x_2 \right\rangle_H .\end{aligned}$$

Now the latter part on the right hand side of equation (75) takes the form

$$\begin{aligned}&\left\langle x_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)x_2 \right\rangle_H = \left\langle x_1, \lim_{n\to\infty}(\sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_2) \right\rangle_H \\ &= \lim_{n\to\infty}\left\langle x_1, (\sum_{j=0}^{n-1} A^{*j}C^*JCA^j x_2) \right\rangle_H = \lim_{n\to\infty}\sum_{j=0}^{n-1}\left\langle x_1, A^{*j}C^*JCA^j x_2\right\rangle_H \\ &= \sum_{j=0}^{\infty}\left\langle x_1, A^{*j}C^*JCA^j x_2\right\rangle_H,\end{aligned}$$

where the second equality holds because $\langle x_1, \cdot\rangle_H$ is a continuous linear functional for each $x_1 \in H$. Similarly,

$$\left\langle Ax_1, (\mathrm{s}-\lim_{n\to\infty}\sum_{j=0}^{n-1} A^{*j}C^*JCA^j)Ax_2 \right\rangle_H = \left\langle x_1, \sum_{j=0}^{\infty} A^{*(j+1)}C^*JCA^{(j+1)}x_2 \right\rangle_H .$$

Subtracting these two limits, together with equation (75), gives $\langle x_1, (A^*P_0A - P_0)x_2\rangle_H = -\langle x_1, C^*JCx_2\rangle_H$. Because x_1 and x_2 are arbitrary,

P_0 solves the Liapunov equation. To show that $L_{A,P_0} = \mathrm{s} - \lim_{n\to\infty} A^{*n} P_0 A^n = 0$, we note that for each $x_1 \in H$, $n \in \mathbf{N}$

$$\begin{aligned}
||A^{*n} P_0 A^n x_1|| &= ||P_0 x_1 - \sum_{j=0}^{n-1} A^{*j} C^* J C A^j x_1|| \\
&= || \lim_{m\to\infty} \sum_{j=0}^{m} A^{*j} C^* J C A^j x_1 - \sum_{j=0}^{n-1} A^{*j} C^* J C A^j x_1|| \\
&= || \sum_{j=n}^{\infty} A^{*j} C^* J C A^j x_2|| \to 0,
\end{aligned}$$

as a tail of a convergent series.

We complete the proof by studying the additional part (iv). Assume that both (ii) and (iii) hold, P is a solution of the Liapunov equation such that $L_{A,P}$ exists, and $J \geq 0$. Then both the bounded operators $J^{\frac{1}{2}}$ and $\mathrm{s} - \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j = P - L_{A,P}$ exist. We calculate for any $x \in H$

$$\begin{aligned}
&||P - L_P|| \cdot ||x||^2 \geq |\langle x, (P - L_P)x\rangle_H| \\
&= \left| \left\langle x, \left(\mathrm{s} - \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j \right) x \right\rangle_H \right| = \left| \left\langle x, lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j x \right\rangle_H \right| \\
&= \left| \lim_{n\to\infty} \left\langle x, \sum_{j=0}^{n-1} (A^{*j} C^* J C A^j x) \right\rangle_H \right| = \lim_{n\to\infty} \sum_{j=0}^{n-1} \left\langle J^{\frac{1}{2}} C A^j x, J^{\frac{1}{2}} C A^j x \right\rangle_H \\
&= ||\{J^{\frac{1}{2}} C A^j x\}_{j\geq 0}||^2_{\ell^2(\mathbf{Z}_+;Y)} = ||\mathcal{C}_{\phi'} x||^2_{\ell^2(\mathbf{Z}_+;Y)},
\end{aligned}$$

where the third equality holds because $\langle x, \cdot \rangle_H$ is a continuous linear functional for each $x \in H$.

It follows that the observability map $\mathcal{C}_{\phi'}$ of the DLS ϕ' maps all of a (complete) Hilbert space H into $\ell^2(\mathbf{Z}_+;Y)$. However, the observability map of a DLS is a closed operator (see [26, Lemma 3]) and now the domain $\mathrm{dom}(\mathcal{C}_{\phi'}) = H$ is complete. The Closed Graph Theorem implies the boundedness of $\mathcal{C}_{\phi'}$; i.e. the output stability of ϕ'. So claim *(iv)* follows. The implication *(iv)* $\Rightarrow$ (i) follows because the output stability of ϕ' implies the strong convergence of the sum $\mathrm{s} - \lim_{n\to\infty} \sum_{j=0}^{n-1} A^{*j} C^* J C A^j$, thus defining the solution P_0 of the Liapunov equation. This completes the proof. $\square$

Compare the above proof to the proof of [26, Proposition 43]. An immediate consequence is the following:

Proposition 71. *If there is a solution P of the Liapunov equation (72) such that the residual cost operator $L_{A,P} \in \mathcal{L}(H)$ exists, then there is a solution P_0 such that $L_{A,P_0} = 0$. Such P_0 is unique, and given by $P_0 x_0 = \sum_{j=0}^{\infty} (A^{*j} C^* J C A^j x_0)$ for all $x_0 \in H$. All other bounded solutions P of the Liapunov equation satisfy*

$$P = P_0 + L_{A,P}, \quad L_{A,P} = \mathrm{s} - \lim_{j\to\infty} A^{*j} P A^j.$$

If A is strongly stable, then P_0 is the unique solution of the Liapunov equation.

Proof. The existence of P_0 is the matter of the implication (ii) $\Rightarrow$ (i) of Proposition 70. The formula for P_0 is found in the proof of implication (iii) $\Rightarrow$ (i) of Proposition 70. The parameterization of all the solutions is a direct consequence of Proposition 69. Claim about the uniqueness of P_0 is proved by noting that for two solutions $P_1, P_2 \in \mathcal{L}(H)$ we have

$$A^{*j}(P_1 - P_2)A^j = P_1 - P_2$$

for all $j > 0$. If both $\mathrm{s} - \lim_{j\to\infty} A^{*j} P_1 A^j = 0$ and $\mathrm{s} - \lim_{j\to\infty} A^{*j} P_2 A^j = 0$, then the left hand side converges to zero pointwise in H, as j grows. The right hand side does not even depend on j. Thus $P_1 = P_2$. The claim involving the strongly stable semigroup is trivial. $\square$

As discussed in the beginning of this section, a fair amount of stability results for DLSs can be given with the aid of the Liapunov equation. The following result is [56, Lemma 21.6], stating that an unstable eigenvector of the semigroup is undetectable.

Proposition 72. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS, and $J \geq 0$ a cost operator. Let $P \in Ric(\phi, J)$, $P \geq 0$ be arbitrary. Assume that $Ax = \lambda x$ for $|\lambda| \geq 1$. Then $J^{\frac{1}{2}}Cx = 0$.*

Proof. If $Ax = \lambda x$, the Liapunov equation takes the form

$$(|\lambda|^2 - 1)\langle Px, x\rangle + \left\langle J^{\frac{1}{2}}Cx, J^{\frac{1}{2}}Cx \right\rangle = 0. \tag{76}$$

Now, if $|\lambda|^2 - 1 \geq 0$, then $(|\lambda|^2 - 1)\langle Px, x\rangle \geq 0$ because $P \geq 0$. Because $J \geq 0$, equation (76) implies that $J^{\frac{1}{2}}Cx = 0$, and the claim is proved. $\square$

Unfortunately this is too weak to be useful for our purposes. Clearly, this approach is restricted to the cases when the eigenvectors of the semigroup generator A span (the interesting part of) the state space. However, the case when A is a diagonalizable matrix or a Riesz spectral operator is covered, see [3, p. 37]. In order to obtain a more general theory for the operator Riccati equation, a stronger infinite-dimensional Liapunov equation theory is required. In Lemma 74, an essential analogue of Proposition 72 is proved for DLSs with much more complicated semigroups. We start with a result known as the Vigier's theorem in [30, Theorem 4.1.1].

Proposition 73. *Let $\{T_j\}_{j\geq 0} \subset \mathcal{L}(H)$ be a sequence of nonnegative self-adjoint operators such that*

$$0 \leq \langle x, T_j x\rangle \leq \langle x, T_{j-1}x\rangle, \quad j > 0.$$

Then there is a nonnegative self-adjoint operator $T \in \mathcal{L}(H)$ such that $0 \leq T \leq T_j$ for all $j \geq 0$, and

$$\langle x, Tx\rangle = \lim_{j\to\infty} \langle x, T_j x\rangle.$$

Proof. Define $a_j(x,y) := \langle x, T_j y \rangle_H$, for all $j \geq 0$. It is easy to see that $a_j(x,y)$ is a bounded conjugate symmetric sesquilinear form on $H \times H$. Now, because $\{\langle x, T_j x \rangle\}_{j \geq 0}$ is a nonincreasing sequence of nonnegative real numbers, the limit exists for all $x \in H$. The polarization identity

$$\begin{aligned} &4a_j(x,y) = 4 \cdot \langle x, T_j y \rangle \\ &= \langle x+y, T_j(x+y) \rangle - \langle x-y, T_j(x-y) \rangle + \\ &i \langle x+iy, T_j(x+iy) \rangle - i \langle x-iy, T_j(x-iy) \rangle . \end{aligned}$$

implies that the limit $a(x,y) := \lim_{j \to \infty} a_j(x,y)$ exists, for all $x, y \in H$. It remains to show that $a(x,y)$ is a bounded conjugate symmetric sesquilinear form on $H \times H$.

The linearity in the first argument x and the conjugate linearity in the second argument y is a trivial consequence of the limit process, because this is true for each $a_j(x,y)$ by the properties of the inner product. The same is true about the conjugate symmetricity of $a(x,y)$. To show the boundedness, we see that

$$|a(x,y)| = \lim_{j \to \infty} |a_j(x,y)| = \lim_{j \to \infty} | \langle x, T_j y \rangle | \leq \lim_{j \to \infty} ||T_j|| \, ||x|| \, ||y||.$$

Now, the family $\{T_j\}_{j \geq 0}$ is uniformly bounded by $||T_0||$, because the norms $||T_j||$ are in fact a nonincreasing sequence

$$||T_j|| = \sup_{||x||=1} \langle x, T_j x \rangle \leq \sup_{||x||=1} \langle x, T_{j-1} x \rangle = ||T_{j-1}||,$$

where we have used the assumption that $0 \leq \langle x, T_j x \rangle \leq \langle x, T_{j-1} x \rangle$, for all $x \in H$. As a bounded sesquilinear form, $a(x,y)$ can be written in form $a(x,y) = \langle x, Ty \rangle$, for a unique operator $T \in \mathcal{L}(H)$ (see [35, Theorem 12.8]). T is self-adjoint because $\langle x, Ty \rangle = a(x,y) = \overline{a(y,x)} = \overline{\langle y, Tx \rangle} = \overline{\langle T^* y, x \rangle} = \langle x, T^* y \rangle$. Because the nonnegativity of T is trivial, T satisfies the claims of this proposition. □

By claim (ii) of Proposition 70, we saw that if the Liapunov equation has one solution $\tilde{P}$ such that the residual cost operator $L_{A,P}$ exists, then a number of nice results followed. Now we use Proposition 73 to give an existence of such $L_{A,P}$ for a given nonnegative solution P.

Lemma 74. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be DLS, and $J \geq 0$ a self-adjoint cost operator. Assume that the Liapunov equation*

$$A^* P A - P + C^* J C = 0,$$

has a nonnegative solution $P \in \mathcal{L}(H)$. Then

(i) The DLS $\phi' := \begin{pmatrix} A & * \\ J^{\frac{1}{2}} C & * \end{pmatrix}$ is output stable, and the residual cost operator $L_{A,P} := \mathrm{s} - \lim_{j \to \infty} A^{*j} P A^j$ exists.

(ii) The operator P_0 is the minimal nonnegative solution of the Liapunov equation (72), where $P_0 := \mathcal{C}_{\phi'}^*\mathcal{C}_{\phi'}$, and $L_{A,P_0} = 0$.

The assumption $J \geq 0$ can be replaced by the assumption $C^*JC \geq 0$, if ϕ' is replaced by $\left(\begin{smallmatrix} A & * \\ (C^*JC)^{\frac{1}{2}} & * \end{smallmatrix} \right)$.

Proof. Let $P \geq 0$ be the nonnegative solution whose existence is assumed. By Proposition 69, we have for all $x \in H$ and $n \geq 1$

$$\langle x, Px \rangle - \sum_{j=0}^{n-1} ||J^{\frac{1}{2}}CA^jx||^2 = \langle x, A^{*n}PA^nx \rangle \, ,$$

because $J \geq 0$ by assumption. Define $T_n := A^{*n}PA^n$. It immediately follows that $\langle x, T_nx \rangle$ is a nonincreasing sequence of nonnegative real numbers, because $P \geq 0$. We can apply Proposition 73, and obtain the largest lower bound operator T, such that $0 \leq T \leq A^{*n}PA^n$ for all $n \geq 0$. We proceed show that $T = \mathrm{s} - \lim_{n\to\infty} A^{*n}PA^n =: L_{A,P}$. We have, because $\langle x, Tx \rangle = \lim_{n\to\infty} \langle x, A^{*n}PA^nx \rangle$ for all $x \in H$:

$$0 = \lim_{n\to\infty} \langle x, (A^{*n}PA^n - T)x \rangle = \lim_{n\to\infty} ||(A^{*n}PA^n - T)^{\frac{1}{2}}x||^2.$$

So $(A^{*n}PA^n - T)^{\frac{1}{2}} \to 0$ in the strong operator topology, and $\{(A^{*n}PA^n - T)^{\frac{1}{2}}\}_{n\geq 0}$ is thus a uniformly bounded family, by the Banach–Steinhaus theorem. It follows that $(A^{*n}PA^n - T)x \to 0$ for all $x \in H$, and so we have $T = L_{A,P}$ which, in particular, exists. We conclude that the equivalent conditions of Proposition 70 hold. Furthermore, because $J \geq 0$, ϕ' is output stable.

The proof of the second claim (ii) goes as follows. Because ϕ' is output stable, it follows from Proposition 71 that $P_0 = \mathcal{C}_{\phi'}^*\mathcal{C}_{\phi'}$ is a bounded solution of the Liapunov equation, satisfying $L_{A,P_0} = 0$. It is nonnegative because $J \geq 0$. To show that P_0 is minimal nonnegative, let $P_1 \in \mathcal{L}(H)$ is another nonnegative solution of the Liapunov equation. Then the strong limit L_{A,P_1} exists, by Proposition 70, and because $P_1 \geq 0$, it follows that $L_{P_1} \geq 0$. By Proposition 71, $P_1 = P_0 + L_{A,P_1} \geq P_0$. So P_0 is a minimal nonnegative solution of the Liapunov equation. The final comment follows by replacing C by $(C^*JC)^{\frac{1}{2}}$, and J by I. The proof is now complete. □

We now consider the special case when the Liapunov equation is connected to DARE $Ric(\phi, J)$ for $J \geq 0$, and its nonnegative solution $P \in Ric(\phi, J)$, as in Proposition 68. By applying Lemma 74 with A_P in place for A and C_P in place for C, we get an important results that is used several times in Section 11.

Corollary 75. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS, and $J \geq 0$ a cost operator. Let $P \in Ric(\phi, J)$ such that $P \geq 0$. Then the DLS $\phi' := \left(\begin{smallmatrix} A_P & * \\ J^{\frac{1}{2}}C_P & * \end{smallmatrix} \right)$ is output* *stable, and the (closed loop) residual cost operator* $L_{A_P,P}$ *exists. Furthermore,* $P_0 := \mathcal{C}_{\phi'}^* \mathcal{C}_{\phi'}$ *is a minimal nonnegative solution of the Liapunov equation*

$$A_P^* \tilde{P} A_P - \tilde{P} + C_P^* J C_P = 0,$$

where $A_P := A + BK_P$, $C_P := C + DK_P$, *and* $\tilde{P}$ *is the operator to be solved. Also* $L_{A_P,P_0} = 0$.

We conclude that not bad instabilities of A_P are seen through the operator C_P, as a dimension independent analogy to Proposition 72. We remark that P_0 does not necessarily solve the DARE $Ric(\phi, J)$. Under stronger conditions, it is shown in Lemma 100 that $L_{A_P,P} = 0$ and then $P = P_0$, by Proposition 71.

We complete this section by considering a case when the Liapunov equation technique is applicable to a nonnegative solution of DARE $Ric(\phi, J)$, even if the cost operator J could be indefinite. In Corollary 75, the closed loop residual condition of P was considered. A conclusion about the open loop residual cost operator $L_{A,P}$ is considered in the following.

Corollary 76. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ *be a DLS, and* $J \in \mathcal{L}(Y)$ *a self-adjoint cost operator. Let* $P \in Ric(\phi, J)$ *such that* $P \geq A^*PA \geq 0$. *Then* $P \in Ric_{00}(\phi, J)$.

Proof. Because $P \in Ric(\phi, J)$, we have the Liapunov equation

$$A^*PA - P + \begin{bmatrix} C^* & K_P^* \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & -\Lambda_P \end{bmatrix} \begin{bmatrix} C \\ K_P \end{bmatrix} = 0.$$

Now $P \geq A^*PA$ if and only if $\left[\begin{smallmatrix} C^* & K_P^* \end{smallmatrix} \right] \left[\begin{smallmatrix} J & 0 \\ 0 & -\Lambda_P \end{smallmatrix} \right] \left[\begin{smallmatrix} C \\ K_P \end{smallmatrix} \right] \geq 0$. Now claim (i) of Lemma 74 (in its modified form for the indefinite cost operator) shows that the residual cost operator $L_{A,P}$ exists. □

Note that the condition $P \geq A^*PA \geq 0$ implies that $\ker(P)$ is A-invariant, and the orthogonal complement $\ker(P)^\perp$ is A^*-invariant but not necessarily A-invariant. For this reason, we have to introduce the compression of the semigroup generator.

Definition 77. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ *be a DLS and* J *self-adjoint. Let* $P \in Ric(\phi, J)$. *Define the closed subspace* $H^P := \ker(P)^\perp \subset H$, *the orthogonal projection* Π_P *onto* H^P, *and the compression of the semigroup* $A^P := \Pi_P A | H^P \in \mathcal{L}(H^P)$.

A nonnegative solution $P \in Ric(\phi, J)$ induces an inner product space structure into $H^P := \ker(P)^\perp$. Everything goes in the same way as discussed in connection with equation (73) for the Liapunov equations, with the exception that now the (generally nontrivial) null space of P must be divided away. It is easy to see that $P \geq A^*PA \geq 0$ is equivalent to

$$||A^P x||_P := ||P^{\frac{1}{2}} A x|| \leq ||P^{\frac{1}{2}} x|| =: ||x||_P \quad \text{for all} \quad x \in H^P. \tag{77}$$

In this case, we say that the compression A^P is a $||\cdot||_P$-contraction. If $P^{\max} \in Ric(\phi, J)$ was nonnegative and injective, then $H^{P^{\max}} = H$ but the norm $||\cdot||_{P^{\max}}$ could give weaker topology that the original norm of H. More generally, H^P need not be complete, when equipped with the norm $||\cdot||_P$.

Proposition 78. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and J self-adjoint. Let $P \in Ric(\phi, J)$, $P \geq 0$ such that the compression $A^P = \Pi_P A|H^P$ is a $||\cdot||_P$-contraction, where the objects are given in Definition 77. Then the following holds*

(i) $P \in Ric_{00}(\phi, J)$. If, in addition, ϕ is output stable and $\Lambda_P > 0$, then ϕ_P is output stable.

(ii) Assume, in addition, that ϕ is output stable and I/O stable, the input operator B is Hilbert–Schmidt, and the input space U is separable.

If $P \in Ric_{uw}(\phi, J)$ and $\Lambda_P > 0$, then $P \in ric_{00}(\phi, J) \cap ric_{uw}(\phi, J)$. If $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$, then

$$\{P \in Ric_{uw}(\phi, J) \mid P \geq 0, \Lambda_P > 0\} \subset ric_0(\phi, J). \tag{78}$$

Proof. The first part of claim (i) is Corollary 76. The rest follows from [26, claim (i) of Proposition 43]. Claim (ii) follows from [26, Corollary 47 and equation (35)]. □

The reader is instructed to compare [26, equations (34) and (35)], and equation (78). They all characterize subsets $ric_0(\phi, J)$, where J can be indefinite but the indicators Λ_P must be positive.

The P-contractivity condition $P \geq A^*PA \geq 0$ can be given a game theoretic interpretation. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be output stable and I/O stable, and let $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ be a regular critical solution which is assumed nonnegative. If the cost operator J is indefinite, the special case of the minimax cost optimization problem, associated to (ϕ, J), can be seen as a (full information, state feedback) minimax game, where the minimizing and maximizing players are given an initial state x_0 and their task to do the best they can. Some additional information structure of the game itself must be imposed; e.g. the input space U must be divided into two parts, and one player must not have access to the other players input space, but we now disregard all the details. Now, each noncritical solution $P \in ric_0(\phi, J)$ is associated to a strategy where both players have, in a rough sense, made an agreement that the game is played (i.e. the cost is measured by P) only inside the restricted state space H^P.

Let now $P \in ric_0(\phi, J)$ be such that $P \geq A^*PA \geq 0$. Now the open loop trajectories $x_j = A^j x_0$ (with zero input from both players) are nonnegative and nonincreasing, in the sense of the cost functional $\langle x_j, Px_j\rangle$. Thus, the maximizing player "loses money" if he does not do anything, but the future game always has a nonnegative cost if the feedback loop is closed (by the maximizing player) at some later moment. In fact, the maximizing player wins the game also in the open loop, and the final cost at infinite future is $\lim_{j\to\infty} \langle A^{*j}PA^j x_0, x_0\rangle = \langle L_{A,P}x_0, x_0\rangle \geq 0$, because $P \geq 0$ is assumed.

11 Factorization of the I/O-map

In this section we study the natural partial ordering of the solution set of the H^∞DARE, induced by the cone of nonnegative self-adjoint operators. We work under the assumption that the cost operator $J \geq 0$, and the equivalent conditions of [26, Theorem 27] hold. In this case, we have a nonnegative regular critical solution $P_0^{\mathrm{crit}} = (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$.

In [26, Theorem 27], we have indicated that the critical solution $P_0^{\mathrm{crit}} \in ric_0(\Phi, J)$ gives a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization of the I/O-map. The (generally noncritical) solutions $P \in ric_{uw}(\Phi, J)$ induce other factorizations of the Popov operator $\mathcal{D}^* J \mathcal{D} = \mathcal{D}_{\phi_P}^* \Lambda_P \mathcal{D}_{\phi_P}$ with I/O stable $\mathcal{D}_{\phi_P}$, see [26, Theorem 50]. However, these do not necessarily lead to a factorization of the I/O-map $\mathcal{D}$ as a composition of two I/O stable operators, in the same way as the spectral factorization leads to the $(J, \Lambda_{P_0^{\mathrm{crit}}})$ inner-outer factorization of $\mathcal{D}$. The task of this section is to describe which solutions P actually do give a factorization of the I/O-map $\mathcal{D}$ into compositions of I/O stable I/O-maps.

Consider the following. Let $P \in ric(\Phi, J)$, where Φ is output stable and I/O stable. The operator pair $(K_P, 0)$ is a perfectly valid state feedback pair for Φ in the sense of [19, Definition 13]. However, if P is not a critical solution, this feedback pair is not I/O stable in the sense of [19, Definition 47]. This means that even if the open loop DLS, extended with the feedback pair $(K_P, 0) = [-\mathcal{C}_{\phi_P}, \mathcal{I} - \mathcal{D}_{\phi_P}]$

$$(\Phi, (K_P, 0)) = \left(\begin{matrix} A & B \\ \begin{bmatrix} C \\ K_P \end{bmatrix} & \begin{bmatrix} D \\ 0 \end{bmatrix} \end{matrix} \right) = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} \\ -\mathcal{C}_{\phi_P} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{I} - \mathcal{D}_{\phi_P} \end{bmatrix} \end{bmatrix},$$

is output stable and I/O stable, the closed loop extended system

$$\begin{aligned} (79) \qquad & (\Phi, (K_P, 0))_\diamond \\ & = \left(\begin{matrix} A_P & B \\ \begin{bmatrix} C_P \\ K_P \end{bmatrix} & \begin{bmatrix} D \\ 0 \end{bmatrix} \end{matrix} \right) = \begin{bmatrix} A^j - \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j}\mathcal{C}_{\phi_P} & \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} - \mathcal{D}\mathcal{D}_{\phi_P}^{-1}\mathcal{C}_{\phi_P} \\ -\mathcal{D}_{\phi_P}^{-1}\mathcal{C}_{\phi_P} \end{bmatrix} & \begin{bmatrix} \mathcal{D}\mathcal{D}_{\phi_P}^{-1} \\ \mathcal{D}_{\phi_P}^{-1} - \mathcal{I} \end{bmatrix} \end{bmatrix} \end{aligned}$$

need not be, where $A_P = A + BK_P$ and $C_P = C + DK_P$. This is the bad news. However, if $P \geq 0$, together with proper technical assumptions, it follows that the upper two rows of the closed loops DLS (79) give an I/O stable DLS. Furthermore, this partial DLS is exactly $\phi^P = \left(\begin{smallmatrix} A_P & B \\ C_P & D \end{smallmatrix} \right)$; the inner DLS (of Φ and J) of Definition 58, centered at P. Note that $\mathcal{D}_{\phi^P} := \mathcal{D}\mathcal{D}_{\phi_P}^{-1}$ for the I/O-map of ϕ^P, and this algebraic fact does not depend on the stability properties of the systems, apart from the boundedness of the static operators A, B, C, D, and K_P.

Let us review some analogous results of the matrix theory when all the spaces U, H and Y of the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ are finite dimensional. If the pair (A, B) is stabilizable, $J \geq 0$ and $D^* J D$ coercive, there is a unique maximal positive solution $P^{\max}$ of the Riccati equation such that the closed loop spectrum $\sigma(A_{P^{\max}}) \subset \overline{\mathbf{D}}$, see [15, Corollary 12.1.2]. If $J = I$, $D^* D = I$, $D^* C = 0$

and (C, A) detectable, then the power stability $\sigma(A_{P^{\max}}) \subset \mathbf{D}$ follows, see [15, Corollary 13.5.3]. Such $P^{\max}$ is called the (power) stabilizing solution of $Ric(\Phi, J)$. If the open loop semigroup generator A is power stable and (A, B) is controllable, then $P^{\max}$ clearly equals the unique critical solution (which is defined only for DAREs associated to I/O stable DLSs) in the sense of [26, Theorem 27]. Indeed, the semigroup generators of both $\phi_{P^{\max}}$ and $\phi_{P^{\max}}^{-1}$ are power stable, by the formulae given in claim (ii) of Proposition 55.

To obtain a matrix H^∞DARE example, let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be a DLS whose spaces U, H and Y are finite dimensional, and the semigroup generator A is power stable; $\sigma(A) \subset \mathbf{D}$. We take $J = I$ to be the cost operator, and assume that the transfer function $\mathcal{D}_\phi(z)$ has no zeroes on the unit circle $\mathbf{T}$. By the assumed finite dimensionality of all the spaces, the last condition can always be achieved, if necessary, by a small perturbation of the DLS ϕ. Then the Popov operator $\mathcal{D}^*\mathcal{D}$ is coercive, and the nonnegative regular critical solution $P_0^{crit} = \left(\mathcal{C}_\phi^{\mathrm{crit}}\right)^* \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists, by [26, Corollary 32]. It follows that $A_{P_0^{crit}}$ is power stable, by [21, claim (i) of Theorem 50] and the finite dimensionality of the state space H. If there was another power stabilizing solution P^{stab}, it would also be a critical solution in $ric_0(\phi, J)$. Thus, if ϕ, in addition, is controllable range$(\mathcal{B}_\phi) = H$, then P_0^{crit} is the unique power stabilizing solution of H^∞DARE $ric(\phi, J)$, see [26, claim (i) of Corollary 30]. In fact, P_0^{crit} is the maximal nonnegative solution in $Ric(\phi, J)$, by Corollary 94 and the fact that the power stability of A implies the equality of solution sets $Ric(\phi, J) = ric_0(\phi, J)$. It is easy to see by a numerical example, using the matrix DARE theory given in [15, Corollary 12.1.2], that it is possible (and even a generic case) that DARE $Ric(\phi, J)$ has long increasing chains of self-adjoint solutions. By using Lemma 64, we can, if necessary, replace $Ric(\phi, J)$ by its spectral DARE $Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ for $\tilde{P}$ "small". So there exists a H^∞DARE $ric(\phi, J)$ (with a power stable semigroup generator) that has an arbitrarity long increasing chain of nonnegative solutions, if $\dim H$ is increased sufficiently. We conclude that the power stabilizing solution P_0^{crit} need not be the only nonnegative H^∞ solution of a (matrix) H^∞DARE. For the other nonmaximal $P \in Ric(\phi, J)$, $P_0^{\mathrm{crit}} \geq P \geq 0$, the inner DLS

$$\phi^P := \begin{pmatrix} A_P & B \\ C_P & D \end{pmatrix} = \begin{bmatrix} A^j - \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j}\mathcal{C}_{\phi_P} & \mathcal{B}\mathcal{D}_{\phi_P}^{-1}\tau^{*j} \\ \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} & \mathcal{D}_{\phi^P} \end{bmatrix}, \tag{80}$$

is nevertheless I/O stable by the following Lemma 79 and the assumption that $J = I$ has a bounded inverse. However, the closed loop semigroup generators A_P are not power stable. In this sense, all the nonnegative solutions of the Riccati equation are I/O-stabilizing, but only the maximal nonnegative P_0^{crit} gives a power stable semigroup generator in the closed loop, under the indicated additional assumptions.

This phenomenon can be viewed from two directions. The first "state space" view is that the DLS ϕ^P is I/O stable because the unstable part of A_P is not "seen" through the output operator C_P of ϕ^P. The second view is the input/output view; that a kind of zero-pole-cancellation process is involved when the feedback loop is closed. In the language of the transfer functions $\mathcal{D}_{\phi^P}(z) = \mathcal{D}(z)\mathcal{D}_{\phi_P}(z)^{-1}$, some of the zeroes of $\mathcal{D}(z)$ get canceled by the poles of $\mathcal{D}_{\phi_P}(z)^{-1}$, at least in the cases when the transfer functions are complex-valued ($U = Y = \mathbf{C}$). We remark that the condition $\dim H < \infty$ amounts to the fact that the inner factors of both $\mathcal{D}(z)$ and $\mathcal{D}_{\phi_P}(z)$ are finite Blaschke products, and the zero–pole cancellation idea makes perfect sense. We remark that using a nonnegative but nonmaximal solution $P \in Ric(\Phi, J)$ for feedback control leads to a partial stabilization of the (unstable) open loop DLS, see [4] and the references therein.

In the following lemma we show that if $P \geq 0$, then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is an I/O-map from $\ell^1(\mathbf{Z}_+; U)$ into $\ell^2(\mathbf{Z}_+; Y)$; i.e. the transfer function $\mathcal{D}_{\phi^P}(z) \in \mathrm{sH}^2(\mathbf{D}; \mathcal{L}(U;Y))$. Step by step, we finally conclude that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O stable under stronger assumptions. If J has a bounded inverse, the same conclusions clearly hold for the I/O-map $\mathcal{D}_{\phi^P}$, too.

Lemma 79. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau{}^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists. Let $P \in ric(\Phi, J)$, such that $P \geq 0$. By ϕ_P and ϕ^P denote the spectral and inner DLS of Definition 58, both centered at P.*

Then the following holds:

(i) We have

$$\mathcal{D} = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}, \tag{81}$$

where ϕ_P is I/O stable and output stable. The DLS $J^{\frac{1}{2}}\phi^P$ is output stable, and the impulse response operator $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\bar{\pi}_0$ is bounded. The Toeplitz operator $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\bar{\pi}_+ : \ell^1(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$ is bounded, and $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\bar{\pi}_+ : \ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$ is a densely defined closed operator.

(ii) The transfer function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)$ is analytic in the whole unit disk $\mathbf{D}$. For each $u_0 \in U$, the analytic function $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)u_0 \in H^2(\mathbf{D}; Y)$. We can write

$$J^{\frac{1}{2}}\mathcal{D}(z) = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)\mathcal{D}_{\phi_P}(z) \quad \text{for all} \quad z \in \mathbf{D}. \tag{82}$$

If, in addition, $P \in ric_{uw}(\Phi, J)$, then

$$J^{\frac{1}{2}}\mathcal{N}(z) = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)\mathcal{N}_P(z) \quad \text{for all} \quad z \in \mathbf{D}, \tag{83}$$

where $\mathcal{N}$, ($\mathcal{N}_P$) are the $(J, \Lambda_{P_0^{\mathrm{crit}}})$- inner, ($(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner) factors of $\mathcal{D}$, ($\mathcal{D}_{\phi_P}$, respectively).

Assume, in addition, that the input operator B of Φ is Hilbert–Schmidt, and both the spaces U and Y are separable. Then:

(iii) Then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U;Y))$. The boundary trace function

$$J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta}) := \mathrm{s} - \lim_{z \to e^{i\theta}} J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z)$$

exists as a nontangential strong limit, a.e. (modulo Lebesque measure of $\mathbf{T}$) on $e^{i\theta} \in \mathbf{T}$.

(iv) *For $P \in ric_{uw}(\Phi, J)$, the boundary trace $\Lambda_P^{-\frac{1}{2}} \mathcal{N}_P(e^{i\theta}) \Lambda_{P_0^{\text{crit}}}^{\frac{1}{2}}$ is unitary a.e. $e^{i\theta} \in \mathbf{T}$. In particular, $\mathcal{N}_P(e^{i\theta})$ has a bounded inverse a.e. $e^{i\theta} \in \mathbf{T}$, and the nontangential strong limit $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(e^{i\theta})$ satisfies*

$$J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(e^{i\theta}) = J^{\frac{1}{2}} \mathcal{N}(e^{i\theta}) \mathcal{N}_P(e^{i\theta})^{-1} \quad \text{a.e. on} \quad e^{i\theta} \in \mathbf{T}, \tag{84}$$

Furthermore, $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(z) \Lambda_P^{-\frac{1}{2}} \in H^\infty(\mathbf{D}; \mathcal{L}(U;Y))$, and is inner from the left. The I/O-map $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}$ is (I, Λ_P)-inner (but $\mathcal{D}_{\phi^P}$ need not be I/O stable if J is not coercive).

We remark that the function $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(e^{i\theta})$ means the boundary trace of $(J^{\frac{1}{2}} \mathcal{D}_{\phi^P})(z)$. As an analytic transfer function $\mathcal{D}_{\phi^P}(z)$, $P \geq 0$ makes perfect sense for $z \in \mathbf{D}$, but it need not be of bounded type.

Proof. Claim (i) is proved as follows. The equality (81) of the I/O-maps is given by formula (79), in form $\mathcal{D}_{\phi^P} = \mathcal{D} \mathcal{D}_{\phi_P}^{-1}$. We see that the $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}$ is the I/O-map of DLS

$$\phi'' = \begin{pmatrix} A_P & B \\ J^{\frac{1}{2}} C_P & J^{\frac{1}{2}} D \end{pmatrix},$$

which is output stable, by Corollary 75 and the assumption $P \geq 0$. Also the (closed loop) residual cost operator $L_{A_P,P}$ exists, but this is not needed here.

But then, if $H \ni x = Bu_0$, with $u_0 \in U$, we have $J^{\frac{1}{2}} \mathcal{D}_{\phi^P} \pi_0 u_0 = J^{\frac{1}{2}} D \pi_0 u_0 + \tau \mathcal{C}_{\phi''} B u_0 = D \pi_0 u_0 + \tau \mathcal{C}_{\phi''} x \in \ell^2(\mathbf{Z}_+; Y)$ because $\mathrm{dom}(\mathcal{C}_{\phi''}) = H$, by the output stability of ϕ''.

$\mathcal{D}_{\phi^P} \pi_0 : U = \mathrm{range}(\pi_0) \to \ell^2(\mathbf{Z}_+; U)$, i.e. $\mathrm{dom}(\mathcal{D}_{\phi^P} \pi_0) = U$ is complete, see [21, Definition 24]). Because the impulse response operator $\mathcal{D}_{\phi^P} \pi_0$ is closed by [21, Lemma 27], it follows from the Closed Graph Theorem that $\mathcal{D}_{\phi^P} \pi_0$ is bounded. It immediately follows that $J^{\frac{1}{2}} \mathcal{D}_{\phi^P} \in \mathcal{L}(\ell^1(\mathbf{Z}_+; U), \ell^2(\mathbf{Z}_+; U))$ by the triangle inequality, and the shift invariance of $\mathcal{D}_{\phi^P}$. The Toeplitz operator $\mathcal{D}_{\phi^P} \bar{\pi}_+$ is thus densely defined on $\ell^2(\mathbf{Z}_+; U)$ and closed, by [21, Lemma 27]. This completes the proof of claim (i).

Consider now claim (ii). $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(z)$ is analytic in the whole of $\mathbf{D}$ by [26, Proposition 11] because it is a transfer function of an output stable system ϕ''. Also $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(z) \in sH^2(\mathbf{D}; Y)$, by [26, Definition 10 and Proposition 11].

Because $\mathcal{D}_{\phi^P} = \mathcal{D} \mathcal{D}_{\phi_P}^{-1}$, then also $\mathcal{D}_{\phi^P} \mathcal{D}_{\phi_P} = \mathcal{D}$ on $Seq(U)$. For the transfer functions, we have $\mathcal{D}_{\phi^P}(z) \mathcal{D}_{\phi_P}(z) = (\mathcal{D}_{\phi^P} \mathcal{D}_{\phi_P})(z) = \mathcal{D}(z)$ for all $z \in N_0$, by [26, Corollary 8]. Here N_0 is a nonempty open neighborhood of the origin. In fact, $\mathcal{D}(z), \mathcal{D}_{\phi_P}(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U;Y))$, by [26, Proposition 9] and the assumed I/O stability of Φ and ϕ_P. As indicated above, also $J^{\frac{1}{2}} \mathcal{D}_{\phi^P}(z)$ is analytic in $\mathbf{D}$. By using a basic analytic continuation technique we conclude that $\mathcal{D}_{\phi^P}(z) \mathcal{D}_{\phi_P}(z) = \mathcal{D}(z)$ for all $z \in \mathbf{D}$, which is equation (82).

To prove equation (83), proceed as follows. Because the existence of the regular critical solution $P_0^{\text{crit}} \in ric_0(\Phi, J)$ is assumed, the equivalent conditions of [26, Theorem 27] hold, we can write $\mathcal{D} = \mathcal{N}\mathcal{X}$, where $\mathcal{X}$ is outer with a bounded inverse, and $\mathcal{N}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. Furthermore, because $P \in ric_{uw}(\Phi, J)$ we can also write $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$, by [26, Proposition 55]. By Corollary 8, $\mathcal{D}(z) = \mathcal{N}(z)\mathcal{X}(z)$ and $\mathcal{D}_{\phi_P}(z) = \mathcal{N}_P(z)\mathcal{X}(z)$, for all $z \in \mathbf{D}$. Because $\mathcal{X}$ is outer with a bounded inverse, i.e. $\mathcal{X}^{-1} \in \mathcal{L}(\ell^2(\mathbf{Z}; U))$, both $\mathcal{X}$ and $\mathcal{X}^{-1}$ are I/O-maps of I/O stable systems. It follows from Corollary 8 that the transfer function $\mathcal{X}(z) \in \mathcal{L}(U)$ has a bounded inverse for all $z \in \mathbf{D}$. Now equation (83) follows.

We proceed to prove claim (iii). The Hilbert–Schmidt property of the input operator B admits us to apply [26, Corollary 42] to the output stable DLS ϕ'', defined above. It follows that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U;Y))$, and this is a function of bounded type. The existence of the nontangential strong limit $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})$ is from [33, Theorem 4.6A], as discussed in [26, Section 2.5].

It remains to prove the final claim (iv). We first note that because $J \geq 0$, then $\Lambda_P > 0$ for all $P \geq 0$. This makes is possible to define the normalized operators $\mathcal{N}^\circ := J^{\frac{1}{2}}\mathcal{N}\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$ and $\mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$. Then both $\mathcal{N}^\circ$ and $\mathcal{N}_P^\circ$ are inner from the left (i.e. (I, I)-inner). We have

$$\mathcal{N}^\circ = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\Lambda_P^{-\frac{1}{2}} \cdot \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}} = \mathcal{M}_P^\circ\mathcal{N}_P^\circ,$$

where $\mathcal{M}_P^\circ := J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\Lambda_P^{-\frac{1}{2}}$. For the corresponding transfer functions and their nontangential limits, we can write

$$\mathcal{N}^\circ(e^{i\theta}) = \mathcal{M}_P^\circ(e^{i\theta})\mathcal{N}_P^\circ(e^{i\theta}), \tag{85}$$

a.e. $e^{i\theta} \in \mathbf{T}$. This is legal because all the transfer functions are of bounded type in the sense of [26, Definition 12] and the discussion associated to it.

The inner from the left transfer function $\mathcal{N}_P^\circ(z) \in H^\infty(\mathbf{D}; \mathcal{L}(U))$ is in fact inner from both sides, see [26, Definition 33]. To see this, we use [26, Proposition 34] in a trivial way, with $z_0 = 0$. Now $\mathcal{N}_P^\circ(0) = \Lambda_P^{\frac{1}{2}}\mathcal{N}_P(0)\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}} = \Lambda_P^{\frac{1}{2}}\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$ by the realization given for $\mathcal{N}_P$ in [26, Proposition 55]. But $\mathcal{N}_P^\circ(0)$ is now boundedly invertible, and $\mathcal{N}_P^\circ$ is inner, by [26, Proposition 34]. So $\mathcal{N}_P^\circ(e^{i\theta}) \in \mathcal{L}(U)$ is a unitary operator for a.e. $e^{i\theta} \in \mathbf{T}$. Applying this on equation (85) gives

$$\mathcal{N}^\circ(e^{i\theta})\mathcal{N}_P^\circ(e^{i\theta})^* = \mathcal{M}_P^\circ(e^{i\theta})$$

a.e. $e^{i\theta} \in \mathbf{T}$. Because $\mathcal{N}_P^\circ(e^{i\theta})^*$ is unitary and $\mathcal{N}^\circ(e^{i\theta})$ is an isometry, it follows that $\mathcal{M}_P^\circ(e^{i\theta})$ is an isometry a.e. $e^{i\theta} \in \mathbf{T}$. But now $\mathcal{M}_P^\circ(e^{i\theta}) \in L^\infty(\mathcal{L}(U;Y)) \cap H^2(\mathbf{T}; \mathcal{L}(U;Y))$, and by [26, Lemma 36], $\mathcal{M}_P^\circ(e^{i\theta}) \in H^\infty(\mathbf{T}; \mathcal{L}(U;Y))$ is inner from the left. This completes the proof. □

The following normalization, presented in the proof of Lemma 79, will be used throughout the rest of this paper. By [26, Corollary 54], it makes sense even for indefinite solutions P.

Corollary 80. *Make the same assumptions as in claim (iii) of Lemma 79. By $P_0^{\rm crit} \in ric_0(\Phi, J)$ denote the regular critical solution. Let $P \in ric_{uw}(\Phi, J)$, $P \geq 0$ be arbitrary. Denote*

$$\begin{aligned}
&\mathcal{D}^\circ := J^{\frac{1}{2}}\mathcal{D}, \quad \mathcal{D}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{D}_{\phi_P},\\
&\mathcal{M}_P^\circ := J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\Lambda_P^{-\frac{1}{2}}, \quad \mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\rm crit}}^{-\frac{1}{2}}, \quad \mathcal{X}^\circ = \Lambda_{P_0^{\rm crit}}^{\frac{1}{2}}\mathcal{X}.
\end{aligned}$$

Then

$$\mathcal{D}^\circ = \mathcal{M}_P^\circ\mathcal{D}_P^\circ = \mathcal{M}_P^\circ\mathcal{N}_P^\circ\mathcal{X}_P^\circ, \tag{86}$$

where $\mathcal{M}_P^\circ : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; Y)$ is inner from the left, $\mathcal{N}_P^\circ : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is two-sided inner, and $\mathcal{X}^\circ : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is outer with a bounded inverse.

The following Theorem is a variation of Lemma 79. Now, a solution $P \in Ric(\Phi, J)$, $P \geq 0$ gives a factorization of a H^∞-transfer function, such that both the factors are in H^∞. However, the solution is not in $ric_{uw}(\phi, P)$ by an explicit assumption, and ϕ_P is not *a priori* required to be output stable or I/O stable as has been required in Lemma 79.

Theorem 81. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that both the spaces U and Y are separable. Assume that the input operator $B \in \mathcal{L}(U; H)$ of Φ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\rm crit} := (\mathcal{C}^{\rm crit})^* J \mathcal{C}^{\rm crit} \in ric_0(\Phi, J)$ exists. Let $P \in Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J)$, $P \geq 0$.*

Then both the DLSs ϕ_P and $J^{\frac{1}{2}}\phi^P$ are output stable and I/O stable. Furthermore, we have the factorization $J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{D}_{\phi_P} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P} \cdot \mathcal{N}_P \cdot \mathcal{X}$ where all factors are I/O stable. Here $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is (I, Λ_P)-inner, $\mathcal{N}_P$ is $(\Lambda_P, \Lambda_{P_0^{\rm crit}})$-inner, and $\mathcal{X}$ is outer with a bounded inverse.

Proof. Because $J \geq 0$ and $P \geq 0$, it follows that $D^*JD + B^*PB = \Lambda_P \geq 0$, and then $\Lambda_P > 0$ because the indicator has a bounded inverse, by definition. Because $P \in Ric_{00}(\Phi, J)$, the residual cost operator $L_{A,P}$ exists and [26, Proposition 43] implies that ϕ_P is output stable. Because $P \in Ric_{uw}(\Phi, J)$, [26, Corollary 47] implies that ϕ_P is I/O stable. Now $P \in ric_{uw}(\Phi, J)$ as in [26, equation (35)], and we can apply all claims of Lemma 79. In particular, this gives the output stability and I/O stability of the normalized inner DLS $J^{\frac{1}{2}}\phi^P$. The proof is now complete. $\square$

If A is strongly stable, then $Ric(\Phi, J) = Ric_0(\Phi, J) = Ric_{00}(\Phi, J) = Ric_{uw}(\Phi, J)$. But now $Ric(\Phi, J) = Ric_{00}(\Phi, J) \cap Ric_{uw}(\Phi, J)$, and all non-negative solutions $P \in Ric(\Phi, J)$ give a factorization of Theorem 81. The following lemma is more general than Lemma 79, and it refers to something we might call "generalized factorizations" of an unstable $\mathcal{D}$. Now the spectral DLS ϕ_P need not be I/O stable.

Lemma 82. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be output stable and* $J \geq 0$. *Let* $P \in Ric_{00}(\Phi, J)$, $P \geq 0$. *Then the following holds:*

(i) *The I/O-maps satisfy* $\mathcal{D} = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}$ *on* $Seq(U)$, *and both* ϕ_P *and* $J^{\frac{1}{2}}\phi^P$ *are output stable.*

(ii) *Assume, in addition, that the input operator* B *is Hilbert–Schmidt, and both* U *and* Y *are separable. Then we have the factorization*

$$J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}, \tag{87}$$

where $J^{\frac{1}{2}}\mathcal{D}(z), J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U; Y))$ *and* $\mathcal{D}_{\phi_P}(z) \in H^2(\mathbf{D}; \mathcal{L}(U))$.

Proof. As before, $\Lambda_P > 0$ for any nonnegative solution. [26, Proposition 43] implies that ϕ_P is output stable. Corollary 75 implies that $J^{\frac{1}{2}}\phi^P$ is output stable. This proves claim (i) because the (algebraic) factorization of the well-posed I/O-maps of DLSs does not require any kind of stability. Claim (ii) is a consequence of [26, Corollary 42]. $\square$

In particular, Lemma 82 gives H^2 factorizations to H^∞ transfer functions. Note that the existence of a critical regular solution $P_0^{\text{crit}} \in ric_0(\phi, J)$ is not required. Under stronger asssumptions, such generalized factorizations easily become ordinary H^∞ factorizations, by Theorem 81. We complete this section by showing that the finite increasing chains of solutions $P_i \in ric_{uw}(\Phi, J)$ behave expectedly.

Theorem 83. *Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint cost operator. Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}_\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS. Assume that the input operator* $B \in \mathcal{L}(U; H)$ *is Hilbert–Schmidt, and both the spaces* U *and* Y *are separable. Assume that the regular critical solution* $P_0^{\text{crit}} = \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ *exists.*

Let $P_i \in ric_{uw}(\Phi, J)$, $i = 1, \ldots, n+1$ *be a sequence of solutions such that* $P_i \leq P_{i+1}$ *and* $\Lambda_{P_i} > 0$ *for all* $i = 1, \ldots, n$. *Denote by* $\mathcal{D}_{\phi_{P_i}} = \mathcal{N}_{P_i}\mathcal{X}$ *the* $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$ *-inner-outer factorization of* $\mathcal{D}_{\phi_{P_i}}$ *where* $\mathcal{X} = \mathcal{D}_{P_0^{\text{crit}}}$ *and* $\mathcal{N}_{P_i} := \mathcal{D}_{\phi_{P_i}}\mathcal{X}^{-1}$ *Then the following holds:*

(i) *Then there is a sequence of causal shift-invariant operators* $\mathcal{N}_{P_i,P_{i+1}} := \mathcal{D}_{\phi_{P_i}}\mathcal{D}_{\phi_{P_{i+1}}}^{-1}$ *on* $Seq(U)$ *such that*

$$\mathcal{N}_{P_i} = \mathcal{N}_{P_i,P_{i+1}}\mathcal{N}_{P_{i+1}} \quad \text{for all} \quad i = 1, \ldots, n. \tag{88}$$

The operator $\mathcal{N}_{P_i,P_{i+1}}$ *is the I/O-map of the I/O stable DLS*

$$\phi_{P_i,P_{i+1}} = \begin{pmatrix} A_{P_{i+1}} & B \\ K_{P_{i+1}} - K_{P_i} & I \end{pmatrix}. \tag{89}$$

Furthermore, each $\mathcal{N}_{P_i,P_{i+1}}$ *is* $(\Lambda_{P_i}, \Lambda_{P_{i+1}})$*-inner.*

(ii) We have the factorization

$$\mathcal{N}_{P_1} = \left(\prod_{i=1}^{n} \mathcal{N}_{P_i,P_{i+1}} \right) \mathcal{N}_{P_{n+1}}, \tag{90}$$

where the elements with increasing i enter the product from the left. If, in addition, $J \geq 0$ and $P_{n+1} = P_0^{\rm crit}$, then

$$J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}} \left(\prod_{i=1}^{n} \mathcal{N}_{P_i,P_{i+1}} \right) \mathcal{X}, \tag{91}$$

where $J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}$ is I/O stable and (I, Λ_{P_1}) -inner, and $\mathcal{X} = \mathcal{D}_{\phi_{P_0^{\rm crit}}}$ is outer with a bounded inverse.

Proof. In order to prove claim (i), note that $(\Lambda_P, \Lambda_{P_0^{\rm crit}})$-inner-outer factorization $\mathcal{D}_{\phi_{P_i}} = \mathcal{N}_{P_i}\mathcal{X}$ exists for all i, by [26, Proposition 55]. Because the feed-through operator of all spectral DLSs is identity, we can speak about the inverse $\mathcal{D}_{\phi_{P_i}}^{-1}$ as a causal shift-invariant operator on $Seq(U)$, see [26, Proposition 2]. Because the outer factor (with a bounded inverse) is common for all $\mathcal{D}_{\phi_{P_i}}$, we see that equation (88) holds.

Fix the arbitrary two consecutive elements $P_i \leq P_{i+1}$ in the sequence $\{P_i\}$, define $\Delta P_i := P_{i+1} - P_i \geq 0$. Then $\Delta P_i \in Ric(\phi_{P_i}, \Lambda_{P_i})$, by Lemma 64. Now, $Ric(\phi_{P_i}, \Lambda_{P_i})$ is a H^∞DARE with a nonnegative cost operator Λ_{P_i}, but we do not know whether ΔP_i is a its H^∞ solution. To see that this is the case, we must consider the spectral DLS $(\phi_{P_i})_{\Delta P_i}$, centered at the solution ΔP_i and relative to the cost operator $\Lambda_{P_i} > 0$ of the spectral DARE. We have for the minimax nodes

$$(\phi_{P_i}, \Lambda_{P_i})_{\Delta P_i} \equiv (\phi_{P_i + \Delta P_i}, \Lambda_{P_i + \Delta P_i}) \equiv \left(\phi_{P_{i+1}}, \Lambda_{P_{i+1}}\right), \tag{92}$$

see equation (58) of Proposition 59. So, the spectral DLS $(\phi_{P_i})_{\Delta P_i}$ of ΔP_i equals $\phi_{P_{i+1}}$ which is an I/O stable and output stable DLS because $P_{i+1} \in ric(\Phi, J)$, by assumption. We conclude that $\Delta P \in ric(\phi_{P_i}, \Lambda_{P_i})$. The indicator $\tilde{\Lambda}_{\Delta P}$ of $\Delta P \in ric(\phi_{P_i}, \Lambda_{P_i})$ equals $\Lambda_{P_{i+1}}$, by equation (92).

Trivially range$(\mathcal{B})$ = range$(\mathcal{B}_{\phi_{P_i}})$ because $\mathcal{B} = \mathcal{B}_{\phi_{P_i}}$. Because both P_i and P_{i+1} satisfy the ultra weak residual cost condition with the same semigroup generator A, so does $\Delta P_i = P_{i+1} - P_i$, and we have $\Delta P_i \in ric_{uw}(\phi_{P_i}, \Lambda_{P_i})$.

Now we have reached the situation described in Lemma 79. We see that the operator $\mathcal{N}_{P_i,P_{i+1}} := \mathcal{D}_{\phi_{P_i}}\mathcal{D}_{\phi_{P_{i+1}}}^{-1} = \mathcal{D}_{\phi_{P_i}}\mathcal{D}_{(\phi_{P_i})_{\Delta P_i}}^{-1}$ actually plays the part of the operator $\mathcal{D}_{\phi^P}$ in Lemma 79, when the DLS Φ is replaced by ϕ_{P_i}, the cost operator J is replaced by Λ_{P_i}, the solution P is replaced by ΔP_i, the spectral DLS ϕ_P is replaced by $(\phi_{P_i})_{\Delta P_i} = \phi_{P_{i+1}}$ and the indicator Λ_P is replaced by $\Lambda_{P_{i+1}}$.

Because the input operator B of ϕ_{P_i} is Hilbert–Schmidt, we conclude that $\mathcal{N}_{P_i,P_{i+1}}$ is I/O stable and $(\Lambda_{P_i}, \Lambda_{P_{i+1}})$-inner, by claim (iv) of Lemma 79, and the fact that Λ_{P_i} (used as the cost operator) has a bounded inverse.

Realization (89) is valid because $\mathcal{N}_{P_i,P_{i+1}} = \mathcal{N}_{P_i}\mathcal{N}_{P_{i+1}}^{-1}$, by equation (89) and [26, claim (iii) of Proposition 56]. This completes the proof of claim (i).

The factorization in (90) is clearly obtained by applying the first part of this theorem n times. The second factorization (91) is obtained by first factorizing $J^{\frac{1}{2}}\mathcal{D} = J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}\mathcal{D}_{\phi_{P_1}}$, where $J^{\frac{1}{2}}\mathcal{D}_{\phi^{P_1}}$ is I/O stable and (I, Λ_{P_1}) -inner, by claim (iv) of Lemma 79. This is the only place where we have used the nonnegativity of J. Then the $(\Lambda_{P_1}, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}_{P_1}$ of $\mathcal{D}_{\phi_{P_1}}$ is factorized as in (90), noting that the last factor $\mathcal{N}_{P_{n+1}} = \mathcal{I}$ because $P_{n+1} = P_0^{\mathrm{crit}}$, by claim (ii) of [26, Proposition 55]. After multiplying from the right by the common outer factor $\mathcal{X}$ of $\mathcal{D}$ and $\mathcal{D}_{\phi_{P_1}}$, the claim follows. $\square$

By [26, Lemma 53], it is sufficient to require $\Lambda_P > 0$ only for one solution $P \in ric_{uw}(\Phi, J)$ that need not be an element of the chain $\{P_i\}$. Clearly, the order of the operator products in claim (ii) is significant, if $\dim U > 1$. The transfer function $\mathcal{N}_{P_i,P_{i+1}}(z)$ can be normalized to $\mathcal{N}^{\circ}_{P_i,P_{i+1}}(z) := \Lambda_{P_i}^{\frac{1}{2}}\mathcal{N}_{P_i,P_{i+1}}(z)\Lambda_{P_{i+1}}^{-\frac{1}{2}}$ which is inner from both sides. The zero evaluation $\mathcal{N}^{\circ}_{P_i,P_{i+1}}(0) = \Lambda_{P_i}^{\frac{1}{2}}\Lambda_{P_{i+1}}^{-\frac{1}{2}}$ satisfies the spectral condition $\sigma(\Lambda_{P_i}^{\frac{1}{2}}\Lambda_{P_{i+1}}^{-\frac{1}{2}}) \subset (0,1)$, as an immediate consequence of the fact that $\Lambda_{P_{i+1}} \geq \Lambda_{P_i}$. However, $\Lambda_{P_i}^{\frac{1}{2}}\Lambda_{P_{i+1}}^{-\frac{1}{2}}$ is generally not normal and, in particular, self-adjoint. In Theorem 83, we have considered only finite increasing chains of solutions. To cover the case of the (countably) infinite chains, one would be lead to consider a limit process, not totally different from the one involved in the study of the Blaschke–Potapov representations for the (matrix-valued) bounded analytic functions. Several applications, references and historical remarks about the Blaschke–Potapov factorizations can be found in the survey article [12, p. 28] by Yu. P. Ginzburg and L. V. Shevchuk.

12 I/O stability of the inner DLS

In this section, we consider converse results to those given in Section 11. Roughly, we show that for $P \in ric_{uw}(\phi, J)$, the I/O stability of ϕ^P implies $P \geq 0$. The nonnegativity of the cost operator $J \geq 0$ is assumed in the main results.

We start by considering solutions $P \in ric(\phi, J)$ such that ϕ^P is I/O stable. Out of such solutions, those that have (J, Λ_P)-inner I/O-maps satisfy the minimax condition of Definition 84, by Proposition 86. In particular, all solutions in $ric_{uw}(\phi, J)$ with an I/O stable inner DLS ϕ^P are of this kind, by Proposition 85. In Propositions 87 and 88, the minimax condition of P is connected to an associated Liapunov equation and the DARE $ric(\Phi, J)$. The main result of this section is Lemma 89, which is a partial converse Lemma 79. An equivalence result is finally given in Theorem 90, under stronger assumptions.

Definition 84. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, and $J \in \mathcal{L}(Y)$ a cost operator. Let $P \in ric(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable. We say that P satisfies the minimax condition if*

$$\bar{\pi}_+ \mathcal{D}^*_{\phi^P} J \mathcal{C}_{\phi^P} = 0, \tag{93}$$

where $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P} \mathcal{C}_{\phi_P}$ it the observability map of inner DLS ϕ^P.

The regular critical solution $P_0^{\text{crit}} := (\mathcal{C}^{\text{crit}})^* J \mathcal{C}^{\text{crit}}$ (as discussed in connection with [26, Theorem 27]) always satisfies the minimax condition. This is because in this case $\mathcal{D} = \mathcal{N}\mathcal{X}$ (where $\mathcal{N} = \mathcal{D}_{\phi^{P_0^{\text{crit}}}}$ and $\mathcal{X} = \mathcal{D}_{\phi_{P_0^{\text{crit}}}}$) is the $(J, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, and $\mathcal{C}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}^{\text{crit}}$ is the critical (closed loop) observability map. By [19, Lemma 4], $\bar{\pi}_+ \mathcal{D}^* J \mathcal{C}^{\text{crit}} = \bar{\pi}_+ \mathcal{X}^* \bar{\pi}_+ \mathcal{N}^* J \mathcal{C}^{\text{crit}} = 0$, and the minimax condition holds.

In fact, the orthogonality of $\overline{\text{range}(\mathcal{D}\bar{\pi}_+)} = \overline{\text{range}(\mathcal{N}\bar{\pi}_+)}$ and the range of the desired closed loop observability map $\mathcal{C}_{\phi^{P_0^{\text{crit}}}} = \mathcal{C}^{\text{crit}}$ can be used to find the critical P_0^{crit} without explicitly solving the DARE, see [19, Section 3]. For a noncritical P, however, one should *a priori* know the (range of the) partial inner factor $\mathcal{D}_{\phi^P}\bar{\pi}_+$ of $\mathcal{D}\bar{\pi}_+$ associated to the yet unknown P, before the correct minimax formulation could be written in the first place.

We proceed to show that quite many interesting solutions $P \in ric(\Phi, J)$ (such that $\mathcal{D}_{\phi^P}$ is I/O stable) satisfy the minimax condition. This will be used as a technical tool to obtain Lemma 89, a rough converse of Lemma 79.

Proposition 85. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that the spaces U and Y are separable. Let $J \geq 0$ be a cost operator. Assume that the regular critical solution $P_0^{\text{crit}} = \left(\mathcal{C}^{\text{crit}}\right)^* J \mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ exists. Let $P \in ric_{uw}(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable. Then $\mathcal{D}^*_{\phi^P} J \mathcal{D}_{\phi^P} = \Lambda_P$; i.e. the I/O-map $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner.*

Proof. We have the familiar factorization of the I/O-maps $\mathcal{D} = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}$. Because P_0^{crit} exists, the conditions of [26, Theorem 27] hold, and we can factorize $\mathcal{D} = \mathcal{N}\mathcal{X}$, $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$, where $\mathcal{N}$, $(\mathcal{N}_P)$ is $(J, \Lambda_{P_0^{\text{crit}}})$-inner, $((\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner, respectively). Here we have used the residual cost assumption $P \in ric_{uw}(\Phi, J)$ and [26, claim (i) of Theorem 50]. The operator $\mathcal{X}$ is a common outer factor with a bounded inverse; for details, see [26, Proposition 55]. This gives us the factorization

$$\mathcal{N} = \mathcal{D}_{\phi^P}\mathcal{N}_P \tag{94}$$

where all the factors I/O stable, the I/O-map $\mathcal{D}_{\phi^P}$ by our explicit assumption.

Consider the factor $\mathcal{N}_P$ more carefully. By [26, Corollary 54], $\Lambda_P > 0$ for all $P \in ric_{uw}(\Phi, J)$, because the conditions of [26, Theorem 27] hold and $J \geq 0$ implies that $P_0^{\text{crit}} \geq 0$ and $\Lambda_{P_0^{\text{crit}}} > 0$. So we can normalize $\mathcal{N}_P^\circ := \Lambda_P^{\frac{1}{2}}\mathcal{N}_P\Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}$ which is (I, I)-inner, and its transfer function $\mathcal{N}_P^\circ(z)$ is inner from the left. Because $\mathcal{N}_P^\circ(0) = \Lambda_P^{\frac{1}{2}}\Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}$ has a bounded inverse, [26, Proposition 34] implies that $\mathcal{N}_P^\circ(z)$ is inner from both sides, and its boundary trace $\mathcal{N}_P^\circ(e^{i\theta})$ takes unitary values a.e. $e^{i\theta} \in \mathbf{T}$. We remark that here the separability of U is used.

Because also Y is separable, equation (94) implies for the boundary traces

$$\mathcal{D}_{\phi^P}(e^{i\theta}) = \mathcal{N}(e^{i\theta})\mathcal{N}_P(e^{i\theta})^{-1}$$

a.e. $e^{i\theta} \in \mathbf{T}$, as in the proof of claim (iv) of Lemma 79. But now for almost all $e^{i\theta} \in \mathbf{T}$

$$J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})\Lambda_P^{-\frac{1}{2}} = \mathcal{N}^\circ(e^{i\theta})\mathcal{N}_P^\circ(e^{i\theta})^*,$$

where $\mathcal{N}^\circ(e^{i\theta}) := J^{\frac{1}{2}}\mathcal{N}(e^{i\theta})\Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}$ is isometric a.e. $e^{i\theta} \in \mathbf{T}$. It follows that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}(e^{i\theta})\Lambda_P^{-\frac{1}{2}}$ is isometric a.e. $e^{i\theta} \in \mathbf{T}$, and thus $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner. This completes the proof of the proposition. □

Proposition 86. *Let* $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS. Assume that the regular critical solution* $P_0^{\text{crit}} = \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ *exists. Let* $P \in ric(\Phi, J)$ *such that the inner DLS* ϕ^P *is I/O stable and its I/O-map is* (J, Λ_P)*-inner. If* $\overline{\text{range}(\mathcal{B})} = H$, *then* P *satisfies the minimax condition; i.e.* $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_{\phi^P} = 0$.

Proof. Let $\tilde{u} \in Seq_-(U)$ be arbitrary. Because $\mathcal{D}_{\phi^P}^* J\mathcal{D}_{\phi^P} = \Lambda_P$ and $\bar{\pi}_+\mathcal{D}_{\phi^P}^*\pi_- = 0$, we have

$$\bar{\pi}_+\mathcal{D}_{\phi^P}^* J(\bar{\pi}_+\mathcal{D}_{\phi^P}\pi_-\tilde{u}) = \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{D}_{\phi^P}\pi_-\tilde{u} = \bar{\pi}_+\Lambda_P\pi_-\tilde{u} = 0.$$

because $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner. Define $x = \mathcal{B}_{\phi^P}\pi_-\tilde{u}$. Now $\mathcal{C}_{\phi^P}x = \mathcal{C}_{\phi^P}\mathcal{B}_{\phi^P}\pi_-\tilde{u} = \bar{\pi}_+\mathcal{D}_{\phi^P}\pi_-\tilde{u}$, it follows that $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_{\phi^P}x = 0$. Because $\tilde{u} \in Seq_-(U)$ is arbitrary, we have $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_{\phi^P}x = 0$ for all $x \in \text{range}(\mathcal{B}_{\phi^P})$.

It remains to show that $\overline{\mathrm{range}(\mathcal{B}_{\phi^P})} = H$. Because $\mathcal{B}_{\phi^P} = \mathcal{B}\mathcal{D}_{\phi_P}^{-1}$, we show that
$\mathrm{range}(\mathcal{B}\mathcal{D}_{\phi_P}^{-1}) = \mathrm{range}(\mathcal{B})$. To see this, let $x \in \mathrm{range}(\mathcal{B})$ be arbitrary. Then $x = \mathcal{B}\pi_-\tilde{u}$ for some $\tilde{u} \in Seq_-(U)$. Define $\tilde{w} = \mathcal{D}_{\phi_P}\tilde{u} \in Seq(U)$. Then $\pi_-\tilde{w} \in Seq_-(U)$ has only finitely many nonzero components, and $\mathcal{B}\mathcal{D}_{\phi_P}^{-1}\pi_-\tilde{w} = \mathcal{B}\pi_-\mathcal{D}_{\phi_P}^{-1}\pi_-\tilde{w} = \mathcal{B}\pi_-\mathcal{D}_{\phi_P}^{-1}\tilde{w} = \mathcal{B}\pi_-\mathcal{D}_{\phi_P}^{-1}\mathcal{D}_{\phi_P}\pi_-\tilde{u} = \mathcal{B}\pi_-\tilde{u}$, where we have used the causality of $\mathcal{D}_{\phi_P}^{-1}$. This proves the inclusion $\mathrm{range}(\mathcal{B}) \subset \mathrm{range}(\mathcal{B}\mathcal{D}_{\phi_P}^{-1})$. The other inclusion follows similarly by interchanging the causal shift-invariant operators $\mathcal{D}_{\phi_P}^{-1}$, $\mathcal{D}_{\phi_P}$ on $Seq(U)$, and noting that nothing in the proof depends upon the boundedness of neither of these operators. We have now proved that a feedback does not change the reachable subspace.

Because $\mathrm{range}(\mathcal{B}_{\phi^P}) = \mathrm{range}(\mathcal{B})$ and $\overline{\mathrm{range}(\mathcal{B})} = H$, it follows that $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_P = 0$, provided $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_P$ is bounded. Now $\mathcal{D}_{\phi^P}^*$ is bounded because $\mathcal{D}_{\phi^P}$ is assumed to be. Also $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$ is bounded because both Φ and ϕ_P are assumed to be output stable. The proof is now complete. □

Proposition 87. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS, and J be a cost operator. Let $P \in ric(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable, and its I/O-map is (J, Λ_P)-inner. Then the following are equivalent:*

(i) P satisfies the minimax condition; i.e. $\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_{\phi^P} = 0$.

(ii) $\mathcal{C}_{\phi_P} = \Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}$

(iii) $-K_P = \Lambda_P^{-1} \cdot \pi_0\mathcal{D}_{\phi^P}^* J\mathcal{C}$, *with the identification of spaces* $\mathrm{range}(\pi_0)$ *and* U.

Proof. Proof of the equivalence (i) ⇔ (ii) is the following equivalence:

$$\begin{aligned} &\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}_{\phi^P} = \bar{\pi}_+\mathcal{D}_{\phi^P}^* J(\mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}) = 0 \\ \Leftrightarrow\quad &\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C} = (\bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{D}_{\phi^P}\bar{\pi}_+)\mathcal{C}_{\phi_P} = \Lambda_P \cdot \mathcal{C}_{\phi_P}. \end{aligned}$$

Because $\mathcal{C}_{\phi_P} = \{-K_P A^j\}_{j\geq 0}$ by Definition 58, the implication (ii) ⇒ (iii) is immediate. For the converse direction, we have to show that $\Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}$ is an observability map of a DLS whose semigroup generator is A — we already know that the first component $-K_P$ is correct if (iii) holds. It remains to prove

$$\left(\Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}\right) A = \bar{\pi}_+\tau^* \left(\Lambda_P \cdot \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}\right).$$

But this is the case:

$$\begin{aligned} &\left(\Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}_{\phi^P}^* J\mathcal{C}\right) A = \Lambda_P^{-1} \cdot \bar{\pi}_+\mathcal{D}_{\phi^P}^* \bar{\pi}_+\tau^* J\mathcal{C} \\ &= \bar{\pi}_+\tau^* \left(\Lambda_P^{-1} \cdot \mathcal{D}_{\phi^P}^* \pi_+ J\mathcal{C}\right) = \bar{\pi}_+\tau^* \left(\Lambda_P \cdot \mathcal{D}_{\phi^P}^* \bar{\pi}_+ J\mathcal{C}\right), \end{aligned}$$

where the last equality follows because $\pi_+\mathcal{D}_{\phi^P}^* \pi_0 = 0$, by the anti-causality of $\mathcal{D}_{\phi^P}^*$. This completes the proof. □

In claim (ii) of the following proposition, the minimax condition is connected to a Liapunov equation that is almost the Riccati equation.

Proposition 88. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be an I/O stable and output stable DLS, and J be a cost operator. Let* $P \in ric(\Phi, J)$ *such that* $\mathcal{D}_{\phi^P}$ *is I/O stable and* (J, Λ_P)*-inner. Define* $P_0 := \mathcal{C}^*_{\phi^P} J \mathcal{C}_{\phi^P} \in \mathcal{L}(H)$*. Then*

(i) P_0 *satisfies the Liapunov equation*

$$\begin{aligned} A^*P_0A - P_0 + C^*JC & \\ = -K_P^*\Lambda_P K_P + K_P^*\Lambda_P \left(-\Lambda_P^{-1} \cdot \pi_0 \mathcal{D}^*_{\phi^P} J\mathcal{C}\right) & \\ + \left(-\Lambda_P^{-1} \cdot \pi_0 \mathcal{D}^*_{\phi^P} J\mathcal{C}\right)^* \Lambda_P K_P, & \end{aligned} \tag{95}$$

and the residual cost operator satisfies $L_{A,P_0} = 0$*.*

(ii) Assume, in addition, P satisfies the minimax condition $\bar{\pi}_+ \mathcal{D}^*_{\phi^P} J \mathcal{C}_{\phi^P} = 0$*. Then* P_0 *satisfies the Liapunov equation*

$$A^*P_0A - P_0 + C^*JC = K_P^*\Lambda_P K_P. \tag{96}$$

Furthermore, $A^*(P - P_0)A = P - P_0$*, and if* $P \in ric_{00}(\Phi, J)$*, then* $P - P_0 = L_{A,P}$*. If* $P \in ric_0(\Phi, J)$ *then* $P = P_0$*.*

Proof. We first remark that is ϕ^P is output stable because $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$, and all the operators $\mathcal{C}$, $\mathcal{D}_{\phi^P}$, $\mathcal{C}_{\phi_P}$ are assumed to be bounded. So $\mathcal{C}^*_{\phi^P}$ makes sense, and P_0 is well defined. The proof of claim (i) is the following technical calculation. Because $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}$, we obtain

$$\begin{aligned} P_0 &:= \mathcal{C}^*J\mathcal{C} - \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} - \mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C} + \mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} \\ &= \mathcal{C}^*J\mathcal{C} - \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} - \mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C} + \mathcal{C}_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P}, \end{aligned}$$

where the latter equality is because $\mathcal{D}_{\phi^P}$ is assumed to be (J, Λ_P)-inner. But then

$$\begin{aligned} & A^*P_0A - P_0 + C^*JC \\ &= \overbrace{\left(A^*\mathcal{C}^*J\mathcal{C}A - \mathcal{C}^*J\mathcal{C} + \mathcal{C}^*J\mathcal{C}\right)}^{(i)} + \overbrace{\left(-A^*\mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}A + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}\right)}^{(ii)} \\ &+ \overbrace{\left(-A^*\mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C}A + \mathcal{C}^*_{\phi_P}\mathcal{D}^*_{\phi^P}J\mathcal{C}\right)}^{(iii)} + \overbrace{\left(A^*\mathcal{C}_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P}A - \mathcal{C}_{\phi_P}\Lambda_P\mathcal{C}_{\phi_P}\right)}^{(iv)}. \end{aligned}$$

Part *(i)* vanishes trivially. Parts *(ii)* and *(iii)* are adjoints of each other, and because A is the semigroup generator of both ϕ and ϕ_P, we have

$$\begin{aligned} & -A^*\mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P}A + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} = -\mathcal{C}^*J\pi_+(\tau\mathcal{D}_{\phi^P}\tau^*)\pi_+\mathcal{C}_{\phi_P} + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} \\ &= -\mathcal{C}^*J\pi_+\mathcal{D}_{\phi^P}\pi_+\mathcal{C}_{\phi_P} + \mathcal{C}^*J\mathcal{D}_{\phi^P}\mathcal{C}_{\phi_P} = \mathcal{C}^*J(\bar{\pi}_+\mathcal{D}_{\phi^P}\bar{\pi}_+ - \pi_+\mathcal{D}_{\phi^P}\pi_+)\mathcal{C}_{\phi_P} \\ &= \mathcal{C}^*J\mathcal{D}_{\phi^P}\pi_0 \cdot \pi_0\mathcal{C}_{\phi_P}, \end{aligned}$$

where the last equality is by the causality of $\mathcal{D}_{\phi^P}$. But $\pi_0 \mathcal{C}_{\phi_P} = -K_P$ with the natural identification of the spaces U and range(π_0). So part *(ii)* equals $-\mathcal{C}^* J \mathcal{D}_{\phi^P} \pi_0 \cdot K_P$, and part *(iii)* equals $-K_P \cdot^* \pi_0 \mathcal{D}^*_{\phi^P} J\mathcal{C}$. A similar calculation as required for part *(i)* shows that part *(iv)* equals $-K_P^* \Lambda_P K_P$. Collecting out results together, we have (95).

Because both $\mathcal{C}$ and $\mathcal{C}_{\phi_P}$ are bounded by assumptions, and A is the semigroup generator of both Φ and ϕ_P, trivially $\mathcal{C}A^j = \bar{\pi}_+ \tau^j \mathcal{C} \to 0$ and $\mathcal{C}_{\phi_P} A^j = \bar{\pi}_+ \tau^j \mathcal{C}_{\phi_P} \to 0$ in the strong operator topology. Because $\mathcal{C}_{\phi^P} = \mathcal{C} - \mathcal{D}_{\phi^P} \mathcal{C}_{\phi_P}$ where $\mathcal{D}_{\phi^P}$ is bounded, it follows that $\mathcal{C}_{\phi^P} A^j \to 0$ in the strong operator topology. By the Banach–Steinhaus Theorem, the family of operators $\{\mathcal{C}_{\phi^P} A^j\}_{j \geq 0}$ is uniformly bounded, and so is the family of their adjoints. It now follows that for all $x \in H$

$$||A^{*j} P_0 A^j x|| \leq \sup_{j \geq 0} ||A^{*j} \mathcal{C}^*_{\phi^P} J|| \cdot ||\mathcal{C}_{\phi^P} A^j x|| \to 0$$

as $j \to \infty$. This completes the proof of claim (i).

In order to prove claim (ii), we use the equivalence of (i) and (iii) in Proposition 87; now P is, in addition, assumed to satisfy the minimax condition. Replacing $-\Lambda_P^{-1} \cdot \pi_0 \mathcal{D}^*_{\phi^P} J\mathcal{C}$ by K_P in (95) gives (96). Note that the Riccati equation solution P, by definition, satisfies the Liapunov equation (96) with P in place of P_0, and then $A^*(P - P_0)A = P - P_0$. This completes the proof. $\square$

In the following Lemma, the main result of this section, we give a partial converse result to Lemma 79.

Lemma 89. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS, such that the spaces U and Y are separable. Assume that $\overline{\text{range}(\mathcal{B})} = H$. Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, $J \geq 0$. Assume that the regular critical solution $P_0^{\text{crit}} = \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ exists.*

If $P \in ric_0(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable, then $P \geq 0$.

Proof. Let $P \in ric_0(\Phi, J)$ such that the inner DLS ϕ^P is I/O stable. By Proposition 85, $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner because $P \in ric_{uw}(\Phi, J) \subset ric_0(\Phi, J)$. By Proposition 86, P satisfies the minimax condition $\bar{\pi}_+ \mathcal{D}^*_{\phi^P} J\mathcal{C}_{\phi^P} = 0$. Define $P_0 := \mathcal{C}^*_{\phi^P} J\mathcal{C}_{\phi^P}$ as in Proposition 88. Because $J \geq 0$, then $P_0 \geq 0$. Because $P \in ric_0(\Phi, J)$, it follows that $P = P_0$ by claim (ii) of Proposition 88. Thus $P \geq 0$, and the proof is complete. $\square$

The following theorem states that the exactly those state feedback laws that associated to nonnegative solutions of DARE, are I/O-stabilizing. We could also say that such solutions partially stabilize the closed loop semigroup generator A_P, and hide the unstable part of A_P to the unobservable (undetectable) subspace.

Theorem 90. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that the spaces U and Y are separable. Assume that $\overline{\text{range}(\mathcal{B})} =$*

H, and the input operator $B \in \mathcal{L}(U;Y)$ is Hilbert–Schmidt. Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator, $J \geq 0$. Assume that the regular critical solution $P_0^{\mathrm{crit}} = \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists. Let $P \in ric_0(\Phi, J)$ be arbitrary.

Then $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O stable if and only if $P \geq 0$.

Proof. If $P \geq 0$, then claim (iv) of Lemma 79 implies that $J^{\frac{1}{2}}\mathcal{D}_{\phi^P}$ is I/O stable. The converse direction is an application of Lemma 89. However, we first have to "absorb" the cost operator J into the DLS Φ by replacing the feed-through operator D by $J^{\frac{1}{2}}D$, and the output operator C by $J^{\frac{1}{2}}C$. Call this modified DLS ϕ'. Finally replace the cost operator J by I. Clearly the assumptions of Φ and ϕ' correspond to each other one-to-one, the DARE remains unchanged, and Lemma 89 implies that $P \geq 0$. $\square$

13 Partial ordering and factorization

Assume that Φ is an output stable and I/O stable DLS, and the cost operator J is nonnegative. Furthermore, assume that the regular critical solution $P_0^{\text{crit}} \in ric_0(\Phi, J)$ exists. In this section, we consider the partial ordering of the solution set $ric_0(\Phi, J)$ as self-adjoint operators. Recall that for $P \in ric_0(\Phi, J)$, the closed ranges $\text{range}(\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+) \subset \ell^2(\mathbf{Z}_+; U)$ of the Toeplitz operators $\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+$ are shift-invariant, see Lemma 91 and Corollary 92. Here $\widetilde{\mathcal{D}}_{\phi_P}$ denotes the adjoint of the I/O-map $\mathcal{D}_{\phi_P}$ of the spectral DLS ϕ_P. Inclusions of the subspaces $\text{range}(\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+)$ are considered in Lemma 93. In Corollary 94, the maximality property of the regular critical solution $P_0^{\text{crit}} = \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ is proved. The order-preserving equivalence

$$ric_0(\Phi, J) \ni P \mapsto \text{range}(\widetilde{\mathcal{N}}_P\bar{\pi}_+) \subset \ell^2(\mathbf{Z}_+; U)$$

is considered in Theorem 95. Here $\widetilde{\mathcal{N}}_P$ denotes the adjoint I/O-map of $\mathcal{N}_P$, the $(\Lambda_P, \Lambda_{P_0^{\text{crit}}})$-inner factor of $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$.

We start with reminding some classical results. The Beurling–Lax–Halmos Theorem on the shift-invariant subspaces is the following:

Lemma 91. *Let U be a separable Hilbert space. The following are equivalent*

(i) H_1 *be a shift-invariant subspace of* $\ell^2(\mathbf{Z}_+; U)$,

(ii) $H_1 = \text{range}(\Theta\bar{\pi}_+) = \Theta\ell^2(\mathbf{Z}_+; U')$, *where* $U' \subset U$ *is a Hilbert subspace, and*
$\Theta : \ell^2(\mathbf{Z}; U') \to \ell^2(\mathbf{Z}; U)$ *is a causal, shift-invariant and bounded operator, which is inner from the left.*

Furthermore, if $\text{range}(\Theta_1\bar{\pi}_+) = \text{range}(\Theta_2\bar{\pi}_+)$ *then there is a unitary (static) operator* $V \in \mathcal{L}(U)$ *such that* $\Theta_1 = \Theta_2 V$.

For proofs, see e.g. [32, Lecture 9, Corollary 9] or [8, Chapter IX, Theorem 2.1]. We can get rid of indexing over the subspaces $U' \subset U$ if we modify the definition of the inner (from the left) operator. This convention is taken in [33], where the inner operators are defined to be such that $\Theta(e^{i\theta})$ is a partial isometry, a.e. $e^{i\theta} \in \mathbf{T}$. Actually this indexing is only over all the cardinalities of the subspaces U, because two Hilbert subspaces of the same dimension can be unitarily identified. For the following corollary, see e.g. [32, Lecture I, Corollary 8]:

Corollary 92. *Let* Θ_1, Θ_2 *be inner from both sides. Then* $\text{range}(\Theta_2\bar{\pi}_+) \subset \text{range}(\Theta_1\bar{\pi}_+)$ *if and only if there is an inner operator* Θ_3 *such that* $\Theta_2 = \Theta_1\Theta_3$.

We now consider the inclusions of the shift-invariant subspaces range$(\widetilde{\mathcal{D}}_{\phi_P}\bar{\pi}_+)$. Under the J-coercivity assumption $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for some $\epsilon > 0$, these subspaces are closed, see [26, Proposition 38].

Lemma 93. *Let $J \in \mathcal{L}(Y)$ be a cost operator, and $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Assume that $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for some $\epsilon > 0$.*

Let $P_1, P_2 \in ric_{uw}(\Phi, J)$ such that $P_1 \leq P_2$. Then range$(\widetilde{\mathcal{D}}_{\phi_{P_1}}\bar{\pi}_+) \subset$ range$(\widetilde{\mathcal{D}}_{\phi_{P_2}}\bar{\pi}_+)$.

Proof. We begin the proof by centering the problem at the smaller of the solutions P_1. Define $\Delta P := P_2 - P_1 \geq 0$. Then we have $P_2 = P_1 + \Delta P$ where $\Delta P \in Ric(\phi_{P_1}, \Lambda_{P_1})$, by Lemma 64. The spectral DARE $Ric(\phi_{P_1}, \Lambda_{P_1})$ is a H^∞DARE because $P_1 \in ric(\Phi, J)$, by assumption. Also $0 \in ric_0(\phi_{P_1}, \Lambda_{P_1})$ is a trivial solution, corresponding to the solution of the original DARE P_1 itself. By [26, Corollary 54], both the indicators satisfy $\Lambda_{P_1} > 0$ and $\Lambda_{P_2} > 0$.

Note that we have not written $\Delta P \in ric(\phi_{P_1}, \Lambda_{P_1})$ because we do not know *a priori* the output stability and I/O stability of the spectral DLS $(\phi_{P_1})_{\Delta P}$. However, a computation with the minimax nodes reveals that the spectral DLS $(\phi_{P_1})_{\Delta P}$ is a spectral DLS associated to the original Φ and J

$$\left((\phi_{P_1})_{\Delta P}, \tilde{\Lambda}_{\Delta P}\right) \equiv (\phi_{P_1}, \Lambda_{P_1})_{\Delta P} \equiv (\phi_{P_1+\Delta P}, \Lambda_{P_1+\Delta P}) \equiv (\phi_{P_2}, \Lambda_{P_2}), \tag{97}$$

see equation (58) of Proposition 59. Because $P_2 \in ric(\Phi, J)$ by assumption, it follows that the spectral DLS $(\phi_{P_1})_{\Delta P}$ is output stable and I/O stable. Thus $\Delta P \in ric(\phi_{P_1}, \Lambda_{P_1})$. For all $x_0 \in$ range$(\mathcal{B})$, we have

$$\left\langle \Delta P A^j x_0, A^j x_0 \right\rangle = \left\langle P_2 A^j x_0, A^j x_0 \right\rangle - \left\langle P_1 A^j x_0, A^j x_0 \right\rangle \to 0 \tag{98}$$

as $j \to \infty$, because both P_1 and P_2 are assumed to satisfy the ultra weak residual cost condition of Definition 21. Because the DLSs Φ and ϕ_{P_1} have the common controllability map, we have range$(\mathcal{B})$ = range$(\mathcal{B}_{\phi_P})$, and then equation (98) implies that $\Delta P \in ric_{uw}(\phi_{P_1}, \Lambda_{P_1})$. From equation (97) we also see that $\Delta P \in ric_{uw}(\phi_{P_1}, \Lambda_{P_1})$ has a positive indicator $\tilde{\Lambda}_{\Delta P} = \Lambda_{P_2} > 0$.

Now we want to apply claim (iii) of Lemma 79 with $(\phi_{P_1}, \Lambda_{P_1})$ in place for (Φ, J), and $\Delta P \in ric_{uw}(\phi_{P_1}, \Lambda_{P_1})$ in place of $P \in ric_{uw}(\Phi, J)$. We have to check that the DLS ϕ_{P_1}, cost operator Λ_{P_1} and solution ΔP satisfy the additional conditions. Firstly, the equivalent conditions of [26, Theorem 27] hold for the pair $(\phi_{P_1}, \Lambda_{P_1})$ because they hold for (Φ, J), by the coercivity assumption $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ and [26, Corollary 54]. For details see [26, Proposition 55] and the discussion following it. We conclude that there is a regular critical solution $\tilde{P}_0^{\rm crit} \in ric_0(\phi_{P_1}, \Lambda_{P_1})$.

The input operator B is common for both Φ and ϕ_{P_1}, and so the Hilbert–Schmidt assumption holds for ϕ_{P_1}. The same is true for the separability of the Hilbert space U, which is the input and the output space of ϕ_{P_1}. Now claim (iii) of Lemma 79 gives

$$\mathcal{D}_{\phi_{P_1}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{D}_{(\phi_{P_1})_{\Delta P}}, \tag{99}$$

where $(\phi_{P_1})^{\Delta P}$ is the inner DLS, and $(\phi_{P_1})_{\Delta P}$ is the spectral DLS of ϕ_{P_1}, centered at ΔP. Both $(\phi_{P_1})^{\Delta P}$ and $(\phi_{P_1})_{\Delta P}$ are output stable and I/O stable; the former by claim (iii) of Lemma 79, and the latter because $\Delta P \in ric_{uw}(\phi_P, \Lambda_P)$. It also follows from Lemma 79 that the I/O-map $\mathcal{D}_{(\phi_{P_1})^{\Delta P}}$ is in fact $(\Lambda_{P_1}, \Lambda_{P_2})$-inner, because $\tilde{\Lambda}_{\Delta P} = \Lambda_{P_2}$ is the indicator of $\Delta P \in ric(\phi_{P_1}, \Lambda_{P_1})$, as discussed above. Note that because the nonnegative cost operator Λ_{P_1} has a bounded inverse, we do not need to include the square root of it into equation (99), as has been done in Lemma 79 for possibly noncoercive cost operator J.

It follows from equation (97) that $\mathcal{D}_{(\phi_{P_1})_{\Delta P}} = \mathcal{D}_{\phi_{P_2}}$. By [26, Corollary 32], the regular critical solution $P_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}\right)^* J\mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$ exists because $\bar{\pi}_+\mathcal{D}^* J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ is assumed for some $\epsilon > 0$. We now obtain from equation (99)

$$\mathcal{N}_{P_1}\mathcal{X} = \mathcal{D}_{\phi_{P_1}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{D}_{\phi_{P_2}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{N}_{P_2}\mathcal{X}, \tag{100}$$

where $\mathcal{D}_{\phi_{P_1}} = \mathcal{N}_{P_1}\mathcal{X}$ ($\mathcal{D}_{\phi_{P_2}} = \mathcal{N}_{P_2}\mathcal{X}$) are $(\Lambda_{P_1}, \Lambda_{P_0^{\mathrm{crit}}})$ ($(\Lambda_{P_2}, \Lambda_{P_0^{\mathrm{crit}}})$)-inner-outer factorizations, respectively. The outer factor $\mathcal{X}$ (having a bounded inverse) is common for both the I/O-maps $\mathcal{D}_{\phi_{P_1}}$ and $\mathcal{D}_{\phi_{P_2}}$, see [26, Proposition 55]. As noted earlier, $\mathcal{D}_{(\phi_{P_1})^{\Delta P}}$ is bounded and $(\Lambda_{P_1}, \Lambda_{P_2})$-inner.

Divide the outer factor away from (100), to obtain $\mathcal{N}_{P_1} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \cdot \mathcal{N}_{P_2}$. Normalize, as in Corollary 80 $\mathcal{N}_{P_1}^\circ = \mathcal{M}_{P_1,\Delta P}^\circ\mathcal{N}_{P_2}^\circ$, where $\mathcal{N}_{P_1}^\circ := \Lambda_{P_1}^{\frac{1}{2}}\mathcal{N}_{P_1}\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$, $\mathcal{N}_{P_2}^\circ := \Lambda_{P_2}^{\frac{1}{2}}\mathcal{N}_{P_2}\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}$ are two-sided inner mappings $\ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$, and $\mathcal{M}_{P_1,\Delta P}^\circ := \Lambda_{P_1}^{\frac{1}{2}} \cdot \mathcal{D}_{(\phi_{P_1})^{\Delta P}} \cdot \Lambda_{P_2}^{-\frac{1}{2}} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ is inner from the left. Note that the static part of $\mathcal{D}_{(\phi_{P_1})^{\Delta P}}$ equals the identity $I \in \mathcal{L}(U)$. Because both $\Lambda_{P_1}^{\frac{1}{2}}$ and $\Lambda_{P_2}^{\frac{1}{2}}$ are boundedly invertible, it follows that $\mathcal{M}_{P_1,\Delta P}^\circ$ is inner from both sides, by [26, Proposition 34]. By using the adjoint I/O-maps, we change the order of factors

$$\widetilde{\mathcal{N}}_{P_1}^\circ = \widetilde{\mathcal{N}}_{P_2}^\circ\widetilde{\mathcal{M}}_{P_1,\Delta P}^\circ,$$

where all the factors are inner from the both sides. Now Corollary 92 implies that

$$\operatorname{range}(\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+) = \operatorname{range}(\widetilde{\mathcal{N}}_{P_1}^\circ\bar{\pi}_+) \subset \operatorname{range}(\widetilde{\mathcal{N}}_{P_2}^\circ\bar{\pi}_+) = \operatorname{range}(\Lambda_{P_0^{\mathrm{crit}}}^{-\frac{1}{2}}\widetilde{\mathcal{N}}_{P_2}\bar{\pi}_+). \tag{101}$$

By considering the outer transfer functions as in [26, claim (ii) of Proposition 37], it is easy to see that $\widetilde{\mathcal{X}}$ is outer with a bounded inverse if and only if $\mathcal{X}$ is outer with a bounded inverse. In particular, $\widetilde{\mathcal{X}}\Lambda_{P_0^{\mathrm{crit}}}^{\frac{1}{2}}$ is outer with a bounded inverse, and the Toeplitz operator $\tilde{\mathcal{X}}\Lambda_{P_0^{\mathrm{crit}}}^{\frac{1}{2}}\bar{\pi}_+$ is a bounded bijection on $\ell^2(\mathbf{Z}_+; U)$. Thus the inclusion of ranges in (101) remains valid if we multiply the operators from the left by $\tilde{\mathcal{X}}\Lambda_{P_0^{\mathrm{crit}}}^{\frac{1}{2}}\bar{\pi}_+$. Now the claim follows. $\square$

The following corollary is somewhat analogous to [15, Theorem 13.5.2].

Corollary 94. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O stable and output stable DLS. Assume that the input space U and the output space Y are separable Hilbert spaces, and the input operator $B \in \mathcal{L}(U;H)$ of Φ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\text{crit}} := \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ exists.*

(i) Let $P_0 \in ric_0(\Phi, J)$ be such that $P_1^{\text{crit}} \leq P_0$ where $P_1^{\text{crit}} \in ric_0(\Phi, J)$ is any regular critical solution. Then P_0 is a regular critical solution.

(ii) If, in addition, $\overline{\text{range}(\mathcal{B})} = H$, then the unique critical solution $P_0^{\text{crit}} := (\mathcal{C}^{\text{crit}})^* J\mathcal{C}^{\text{crit}}$ is maximal in the set $ric_0(\Phi, J)$.

Proof. By Lemma 93, equation (101) gives for the ranges of the adjoined operators, because $P_0 \geq P_1^{\text{crit}}$

$$\ell^2(\mathbf{Z}_+; U) = \text{range}(\tilde{\mathcal{D}}_{\phi_{P_1^{\text{crit}}}} \bar{\pi}_+) \subset \text{range}(\tilde{\mathcal{D}}_{\phi_{P_0}} \bar{\pi}_+) \subset \ell^2(\mathbf{Z}_+; U),$$

and immediately $\text{range}(\tilde{\mathcal{D}}_{\phi_{P_0}} \bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U)$. By $\mathcal{D}_{\phi_{P_0}} = \mathcal{N}_{P_0}\mathcal{X}$ denote the $(\Lambda_{P_0}, \Lambda_{P_0^{\text{crit}}})$-inner-outer factorization, and normalize the inner part as before: $\widetilde{\mathcal{N}}_{P_0}^{\circ} = \Lambda_{P_0}^{\frac{1}{2}} \widetilde{\mathcal{N}}_{P_0} \Lambda_{P_0^{\text{crit}}}^{-\frac{1}{2}}$. Then $\text{range}(\widetilde{\mathcal{N}}_{P_0} \bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U)$, as in the last part of the proof of Lemma 93. Now the uniqueness part of Lemma 91 shows that $\tilde{\mathcal{N}}_{P_0}^{\circ}$ is a static unitary constant operator $V \in \mathcal{L}(U)$. By cancelling the normalization, we obtain $\mathcal{D}_{\phi_{P_0}} = \Lambda_{P_0}^{-\frac{1}{2}} V^* \Lambda_{P_0^{\text{crit}}}^{\frac{1}{2}} \mathcal{X}$. Because the static part of both $\mathcal{D}_{\phi_{P_0}}$ and $\mathcal{X}$ is the identity operator $I \in \mathcal{L}(U)$, it follows that $\Lambda_{P_0}^{-\frac{1}{2}} V^* \Lambda_{P_0^{\text{crit}}}^{\frac{1}{2}} = I$ and hence $\mathcal{D}_{\phi_{P_0}} = \mathcal{X}$. Because $P_0 \in ric_0(\phi, J)$, it is a regular critical solution, and the first claim (i) is verified. Under the approximate controllability $\overline{\text{range}(\mathcal{B})} = H$, an application of [26, claim (i) of Corollary 30] proves the remaining claim. □

We remark that the solution $P_0^{\text{crit}} := (\mathcal{C}^{\text{crit}})^* J\mathcal{C}^{\text{crit}}$ is not generally maximal in the full solution set $Ric(\Phi, J)$. A plenty of examples about this are provided by Lemma 101 in Section 14. Even if $\overline{\text{range}(\mathcal{B})} = H$ is assumed, we do not yet know whether P_0^{crit} is the largest element of $ric_0(\Phi, J)$ — there could be a solution $P \in ric_0(\Phi, J)$ that is not comparable to P_0^{crit}. However, this is not the case, as shown in Theorem 96. This result is based on the following equivalence of the two order relations.

Theorem 95. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that $\overline{\text{range}(\mathcal{B})} = H$. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Assume that the regular critical solution $P_0^{\text{crit}} := \left(\mathcal{C}^{\text{crit}}\right)^* J\mathcal{C}^{\text{crit}} \in ric_0(\Phi, J)$ exists.*

For $P_1, P_2 \in ric_0(\Phi, J)$, the following are equivalent

(i) $P_1 \leq P_2$.

(ii) $\mathrm{range}(\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+) \subset \mathrm{range}(\widetilde{\mathcal{N}}_{P_2}\bar{\pi}_+)$*, where* $\mathcal{N}_P$ *is the* $(\Lambda_P, \Lambda_{P_0^{\mathrm{crit}}})$*-inner factor of* $\mathcal{D}_{\phi_P}$*.*

In other words, the mapping

$$ric_0(\Phi, J) \ni P \mapsto \mathrm{range}(\widetilde{\mathcal{N}}_P\bar{\pi}_+) \subset \ell^2(\mathbf{Z}_+; U)$$

is order-preserving from the POSET $ric_0(\Phi, J)$ *(ordered by the natural partial ordering of self-adjoint operators) into the sub-POSET* $\{\mathrm{range}(\widetilde{\mathcal{N}}_P\bar{\pi}_+)\}_{P \in ric_0(\Phi,J)}$ *of the shift-invariant subspaces of* $\ell^2(\mathbf{Z}_+; U)$ *(ordered by the inclusion of subspaces).*

Proof. The implication (i) $\Rightarrow$ (ii) is Lemma 93. We just remark that if $J \geq 0$, the existence of the regular critical solution P_0^{crit} is equivalent to $\bar{\pi}_+\mathcal{D}^*J\mathcal{D}\bar{\pi}_+ \geq \epsilon\bar{\pi}_+$ for $\epsilon > 0$, see [26, Theorem 27 and Corollary 31]. For the converse direction (ii) $\Rightarrow$ (i), note that $\mathrm{range}(\widetilde{\mathcal{N}}_{P_1}\bar{\pi}_+) \subset \mathrm{range}(\widetilde{\mathcal{N}}_{P_2}\bar{\pi}_+)$ is equivalent to $\mathrm{range}(\widetilde{\mathcal{N}}_{P_1}^\circ\bar{\pi}_+) \subset \mathrm{range}(\widetilde{\mathcal{N}}_{P_2}^\circ\bar{\pi}_+)$, where the normalization is as in Corollary 80. This normalization is possible because both the indicators Λ_{P_1}, Λ_{P_2} and $\Lambda_{P_0^{\mathrm{crit}}}$ are positive, by [26, Corollary 54]. By Corollary 92, there is an inner (from both sides) operator Θ such that $\widetilde{\mathcal{N}}_{P_2}^\circ\Theta = \widetilde{\mathcal{N}}_{P_1}^\circ$, or equivalently

$$\mathcal{D}_{\phi_{P_1}} = \Lambda_{P_1}^{-\frac{1}{2}}\widetilde{\Theta}\Lambda_{P_2}^{\frac{1}{2}} \cdot \mathcal{D}_{\phi_{P_2}}, \tag{102}$$

because we can factorize $\mathcal{D}_{\phi_P} = \mathcal{N}_P\mathcal{X}$ for $P \in ric_{uw}(\phi, J)$, by [26, Proposition 55].

Now we continue as in proof of Lemma 93, and center the problem around the smaller solution P_1. As in the proof of Lemma 93, we have the solution $\Delta P := P_2 - P_1 \in ric(\phi_{P_1}, \Lambda_{P_1})$ whose nonnegativity is to be shown. We have $(\phi_{P_1})_{\Delta P} = \phi_{P_2}$ and

$$\mathcal{D}_{\phi_{P_1}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{D}_{(\phi_{P_1})_{\Delta P}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}\mathcal{D}_{\phi_{P_2}}, \tag{103}$$

as in the proof of Lemma 93.

We have to check that ϕ_{P_1}, Λ_{P_1} and ΔP satisfy the assumptions of Lemma 89. Firstly, the separable U is the input space and the output space of the output stable and I/O stable DLS ϕ_{P_1}. Also $\overline{\mathrm{range}(\mathcal{B}_{\phi_{P_1}})} = H$, because $\mathcal{B}_{\phi_{P_1}} = \mathcal{B}$. The indicator Λ_{P_1}, serving as the cost operator, is nonnegative as already has been discussed. The H^∞DARE $ric(\phi_{P_1}, \Lambda_{P_1})$ has a regular critical solution because the original H^∞DARE $ric(\Phi, J)$ has, see [26, Theorem 27 and claim (i) of Proposition 55]. Because $\Delta P = P_2 - P_1$ and $P_1, P_2 \in ric_0(\Phi, J)$ by assumption, the residual cost operator $L_{A,\Delta P}$ exists. Furthermore, $L_{A,\Delta P} = L_{A,P_2} - L_{A,P_1} = 0$, and it follows that $\Delta P \in ric_0(\phi_{P_1}, \Lambda_{P_1})$ because A is the common semigroup generator of all the DLSs Φ, ϕ_{P_1} and $(\phi_{P_1})_{\Delta P}$. Now we see that the assumptions of Lemma 89 are satisfied.

By comparing (102) and (103), we see that the inner DLS $(\phi_{P_1})^{\Delta P}$ is I/O stable. Compare, for example, the transfer functions in a small neighbourhood of the origin, to convince yourself that $\Lambda_{P_1}^{-\frac{1}{2}}\widetilde{\Theta}\Lambda_{P_2}^{\frac{1}{2}} = \mathcal{D}_{(\phi_{P_1})^{\Delta P}}$. Also [26, claim (ii) of Proposition 38] can be used, to see that the I/O-map $\mathcal{D}_{\phi_{P_2}}$ has a bounded, shift-invariant but generally noncausal inverse in $\ell^2(\mathbf{Z};U)$. By Lemma 89, $\Delta P \geq 0$ and the proof is completed. $\square$

We proceced to give an order-theoretic characterization of the set of nonnegative regular H^∞ solutions of the H^∞DARE $ric(\phi, J)$. Under approximate controllability, these are exactly those that give H^∞ factorizations in Lemma 79, see [26, Corollary 44].

Theorem 96. *Let $J \geq 0$ be a cost operator. Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an I/O stable and output stable DLS, such that $\overline{\mathrm{range}(\mathcal{B})} = H$. Assume that the input space U and the output space Y are separable, and the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt. Assume that there is a (unique) regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}^{\mathrm{crit}})^* J \mathcal{C}^{\mathrm{crit}} \in ric_0(\Phi, J)$. Then*

$$\{P \in ric_0(\Phi, J) \quad | \quad P \geq 0\} = \{P \in Ric(\Phi, J) \quad | \quad 0 \leq P \leq P_0^{\mathrm{crit}}\}.$$

Proof. The inclusion $\supset$ has already been established in [26, claim (ii) of Corollary 48]. For the converse inclusion, let a nonnegative $P \in ric_0(\Phi, J)$ be arbitrary. Because $\widetilde{\mathcal{N}}_{P_0^{\mathrm{crit}}} = \mathcal{I}$, it follows that the range of the Toeplitz operator $\widetilde{\mathcal{N}}_{P_0^{\mathrm{crit}}} \bar{\pi}_+$ is all of $\ell^2(\mathbf{Z}_+;U)$. In particular, $\mathrm{range}(\widetilde{\mathcal{N}}_P \bar{\pi}_+) \subset \mathrm{range}(\widetilde{\mathcal{N}}_{P_0^{\mathrm{crit}}} \bar{\pi}_+)$, and it follows that $P \leq P_0^{\mathrm{crit}}$, by Theorem 95. The proof is complete. $\square$

14 H^∞ solutions of the inner and spectral DAREs

We start with a motivation of the contents of this section. For simplicity, assume for a while that the nonnegative cost operator J is boundedly invertible. In claim (iv) Lemma 79, we introduce the factorization of the I/O-map as a composition of two I/O stable I/O-maps

$$\mathcal{D}_\phi = \mathcal{D}_{\phi^{\tilde P}} \cdot \mathcal{D}_{\phi_{\tilde P}}, \tag{104}$$

for any nonnegative $\tilde P \in ric_0(\phi, J)$. As a conclusion of the same lemma, it follows that the inner DLS $\phi^{\tilde P}$ is output stable and I/O stable. The technical assumptions of Lemma 79, such as the separability of the Hilbert spaces and the Hilbert–Schmidt compactness of the common input operator $B \in \mathcal{L}(U;H)$, are inherited from ϕ by $\phi^{\tilde P}$. This makes it possible to apply claim (iv) of Lemma 79 to inner DLS ϕ^P and the associated inner H^∞DARE $ric(\phi^P, J)$. In this way, the $(J, \Lambda_{\tilde P})$-inner factor $\mathcal{D}_{\phi^{\tilde P}}$ can be further factorized by the nonnegative solutions $P \in ric_0(\phi^{\tilde P}, J)$. A similar consideration can be given for the right factor $\mathcal{D}_{\phi_{\tilde P}}$, which is the I/O-map of the spectral DLS ϕ_P, and a stable spectral factor of the Popov operator $\mathcal{D}_\phi^* J \mathcal{D}_\phi$, too. The nonnegative solutions $P \in ric_0(\phi_{\tilde P}, \Lambda_{\tilde P})$ of the spectral DARE factorize $\mathcal{D}_{\phi_{\tilde P}}$ into I/O stable factors.

Because of the possibility of a recursive factorization of factors in equation (104), we conclude that both the solutions sets

$$\begin{aligned}
&ric_0(\phi_{\tilde P}, \Lambda_{\tilde P}), \quad \text{for all} \quad \tilde P \in ric_0(\phi, J),\\
&\left\{P \in ric_0(\phi^{\tilde P}, J) \quad | \quad P \geq 0\right\}, \quad \text{for all} \quad \tilde P \in ric_0(\phi, J), \quad \tilde P \geq 0
\end{aligned}$$

are quite interesting. So it is desirable to characterize them in terms of the original data, namely the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, the cost operator J, and the solution sets $Ric(\phi, J)$ and $ric_0(\phi, J)$ of the original DARE. This is the subject of the present section.

We start with considering the spectral DARE, as it is quite easy. In fact, the result on the spectral DLSs has already been used in the proof of Theorem 95.

Lemma 97. *Let $J \geq 0$ a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS. Assume that the input operator $B \in \mathcal{L}(U;H)$ is Hilbert–Schmidt and the input space U is separable. Let $\tilde P \in ric_0(\phi, J)$ be arbitrary.*

Then the following are equivalent:

(i) $\Delta P \in ric_0(\phi_{\tilde P}, \Lambda_{\tilde P})$,

(ii) $\tilde P + \Delta P \in ric_0(\phi, J)$.

Proof. To prove the implication (i) $\Rightarrow$ (ii), let $\Delta P \in ric_0(\phi_{\tilde P}, \Lambda_{\tilde P})$ be arbitrary. Then, because A is the semigroup generator of both ϕ and $\phi_{\tilde P}$, it follows that the residual cost operator $L_{A,\tilde P + \Delta P}$ exists and satisfies $L_{A,\tilde P+\Delta P} = L_{A,\tilde P} + L_{A,,\Delta P} = 0$. By Lemma 64, $\tilde P + \Delta P \in Ric_0(\phi, J)$.

Because $J \geq 0$, it follows that $P_0^{\text{crit}} = \left(\mathcal{C}_\phi^{\text{crit}}\right)^* J\mathcal{C}_\phi^{\text{crit}} \geq 0$ and also $\Lambda_{P_0^{\text{crit}}} > 0$. By [26, Theorem 27 and Lemma 53], it follows that $\Lambda_{\tilde{P}} > 0$ because $\tilde{P} \in ric_0(\phi, J)$. The spectral H^∞DARE $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ has a regular critical solution $\tilde{P}_0^{\text{crit}} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ because $P_0^{\text{crit}} \in ric_0(\phi, J)$ is assumed to exist, see [26, Proposition 55]. Because the cost operator of DARE $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is nonnegative, the indicator $\tilde{\Lambda}_{\tilde{P}_0^{\text{crit}}}$ is nonnegative and the same is true for the indicator $\tilde{\Lambda}_{\Delta P}$, by [26, Lemma 53] and the assumption $\Delta P \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. Now, by equation (58) of Proposition 59, $\Lambda_{\tilde{P}+\Delta P} = \tilde{\Lambda}_{\Delta P} > 0$.

Now we have concluded that $\tilde{P} + \Delta P \in Ric_0(\phi, J)$, and its indicator is positive. It follows that $\tilde{P} + \Delta P \in ric_0(\phi, J)$, by [26, Corollary 47]. This completes the proof of the first implication.

To prove the other direction (ii) $\Rightarrow$ (i), assume that $P_2 := \tilde{P} + \Delta P \in ric_0(\phi, J)$. Then $\Delta P = P_2 - \tilde{P} \in Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ by Lemma 64, and also $L_{A,\Delta P} = 0$. Thus $\Delta P \in Ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ because the same A is the semigroup generator of all spectral DLSs. The indicator $\tilde{\Lambda}_\Delta$ of $\Delta P \in Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ satisfies $\tilde{\Lambda}_\Delta = \Lambda_{P_2}$, by equation (58) of Proposition 59. But the latter is positive because $P_2 \in ric_0(\phi, J)$, by the same argument that is presented in the first part of the proof for $\Lambda_{\tilde{P}}$.

We have proved that $\Delta P \in Ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$, and its indicator $\tilde{\Lambda}_{\Delta P}$ is positive. Now, because the Hilbert–Schmidt class input operator B and the separable input space U is common for all spectral DLSs, an application of [26, Corollary 47] completes the proof. □

A similar results can be given for other residual cost conditions introduced in Definition 21. The case of the ultra weak residual cost condition has been considered in the proof of Lemma 93. We proceed to characterize a regular critical solution of the spectral DARE.

Corollary 98. *Make the same assumption as in Lemma 97. By* $P_0^{\text{crit}} := (\mathcal{C}_\phi^{\text{crit}})^* J\mathcal{C}_\phi^{\text{crit}} \in ric_0(\phi, J)$ *denote the regular critical solution.*

Then $P_0^{\text{crit}} - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ *is a regular critical solution. If, in addition,* $\overline{\text{range}(\mathcal{B}_\phi)} = H$, *then it is the unique regular critical solution.*

Proof. By Lemma 97, we see that $\Delta P := P_0^{\text{crit}} - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. By equation (58) of Proposition 59, we have for $(\phi_{\tilde{P}})_{P_0^{\text{crit}}-\tilde{P}} = \phi_{P_0^{\text{crit}}}$, whose I/O-map is the outer factor $\mathcal{X}$ of $\mathcal{D}_\phi$, by the definition of the critical solution P_0^{crit}. It follows that $P_0^{\text{crit}} - \tilde{P} \in ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is a regular critical solution of the spectral H^∞DARE $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. If $\overline{\text{range}(\mathcal{B}_\phi)} = H$, then also $\overline{\text{range}(\mathcal{B}_{\phi_{\tilde{P}}})} = H$ because the controllability maps of ϕ and $\phi_{\tilde{P}}$ coincide. The uniqueness of the regular critical solution of $ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ follows from [26, Corollary 30]. □

The spectral DLS and DARE can be used to show that the solution set $ric_0(\phi, J)$ is order-convex:

Lemma 99. *Let $J \geq 0$ be a cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an output stable and I/O stable DLS. Assume that the input space U is separable, and the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt. By $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ denote the regular critical solution.*

Then $ric_0(\phi, J)$ is order-convex in the following sense: if $\tilde{P} \in ric_0(\phi, J)$ is such that $\tilde{P} \leq P_0^{\mathrm{crit}}$, then all $P \in Ric(\phi, J)$ such that $\tilde{P} \leq P \leq P_0^{\mathrm{crit}}$ satisfy $P \in ric_0(\phi, J)$.

Proof. Because $\tilde{P} \leq P \leq P_0^{\mathrm{crit}}$, then $0 \leq P - \tilde{P} \leq P_0^{\mathrm{crit}} - \tilde{P}$. By Lemma 64, $P - \tilde{P} \in Ric(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. By Corollary 98, $P_0^{\mathrm{crit}} - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$ is a regular critical solution. By [26, claim (ii) of Corollary 48], $P - \tilde{P} \in ric_0(\phi_{\tilde{P}}, \Lambda_{\tilde{P}})$. The proof is now complete. □

Now we have dealt with the spectral DLSs and DAREs. We proceed to study the regular H^∞ solutions for the inner H^∞DARE $ric(\phi^{\tilde{P}}, J)$, centered at $\tilde{P} \geq 0$. We need to assume that the nonnegative cost operator J has a bounded inverse. By Lemma 79, this quarantees that $\phi^{\tilde{P}}$ is output stable and I/O stable, when questions about H^∞ solutions become meaningful.

Lemma 100. *Let $J > 0$ a boundedly invertible cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$. Assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space U and the output space Y of ϕ are separable. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Let $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, be arbitrary.*

Then the inner DLS $\phi^{\tilde{P}}$ is output stable and I/O stable. The inner DARE $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE. Furthermore, $\tilde{P}$ is the unique regular critical solution of its own inner DARE $ric_0(\phi^{\tilde{P}}, J)$. In particular, $L_{A_{\tilde{P}}, \tilde{P}} = 0$.

Proof. Let $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, be arbitrary. By claim (iv) of Lemma 79, $\phi^{\tilde{P}}$ is output stable and I/O stable, because $J > 0$ has a bounded inverse. Thus $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE, and it makes sense to ask about the regular H^∞ solutions $P \in ric_0(\phi^{\tilde{P}}, J)$.

By claim (iv) of Lemma 79, $\mathcal{D}_{\phi^{\tilde{P}}}$ is $(J, \Lambda_{\tilde{P}})$-inner. Because $\tilde{P} \geq 0$ and $J \geq 0$, it follows that $\Lambda_{\tilde{P}} > \epsilon I$ for some $\epsilon > 0$. Thus the Popov operator satisfies $\mathcal{D}_{\phi^{\tilde{P}}}^* J \mathcal{D}_{\phi^{\tilde{P}}} = \Lambda_P \cdot \mathcal{I} \geq \epsilon \mathcal{I}$, and by [26, Corollary 54], there is a regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi^{\tilde{P}}, J)$. It follows from the approximate controllability assumption $\overline{\mathrm{range}(\mathcal{B})} = H$ of ϕ that the inner DLS $\phi^{\tilde{P}}$ is approximately controllable, too, because $\mathrm{range}(\mathcal{B}_{\phi^{\tilde{P}}}) = \mathrm{range}(\mathcal{B}_\phi)$ as in the proof of Proposition 86. Now [26, claim (i) of Corollary 30] implies that $\tilde{P}_0^{\mathrm{crit}}$ is the unique regular critical solution of H^∞DARE $ric(\phi^{\tilde{P}}, J)$. Furthermore, $\tilde{P}_0^{\mathrm{crit}}$ is nonnegative, because $J > 0$. Expectedly, the outer factor of $\mathcal{D}_{\phi^{\tilde{P}}}$ is the static identity operator $\mathcal{I}$, which equals the I/O-map $(\phi^{\tilde{P}})_{\tilde{P}_0^{\mathrm{crit}}}$ of the corresponding spectral DLS (associated to pair $(\phi^{\tilde{P}}, J)$).

Let $P \in Ric(\phi^{\tilde{P}}, J) = Ric(\phi, J)$, $P \geq 0$, be arbitrary. Then the spectral DLS $(\phi^{\tilde{P}})_P$ can be put into form

$$(105) \qquad \left((\phi^{\tilde{P}})_P, \tilde{\Lambda}_P\right) :\equiv \left(\phi^{\tilde{P}}, J\right)_P \equiv \left(\begin{pmatrix} A_{\tilde{P}} & B \\ K_{\tilde{P}} - K_P & I \end{pmatrix}, \Lambda_P\right),$$

see equation (56) of Proposition 59. Here $A_{\tilde{P}} := A + BK_{\tilde{P}}$, $\Lambda_Q = D^*JD + B^*QB$, and $\Lambda_Q K_Q = -D^*JC - B^*QA$ for $Q = \tilde{P}, P$ are the closed loop semigroup generator, indicator and feedback operator, relative to the original DLS ϕ and the cost operator J.

By setting $P = \tilde{P}$ in equation (105), we get

$$(\phi^{\tilde{P}})_{\tilde{P}} = \begin{pmatrix} A_{\tilde{P}} & B \\ -\tilde{K}_{\tilde{P}} & I \end{pmatrix} = \begin{pmatrix} A_{\tilde{P}} & B \\ 0 & I \end{pmatrix},$$

and the feedback operator $\tilde{K}_{\tilde{P}}$, associated to pair $(\phi^{\tilde{P}}, J)$, satisfies $\tilde{K}_{\tilde{P}} = 0$.

However, the same is true for the unique regular critical solution $\tilde{P}_0^{\mathrm{crit}} \in ric_0(\phi^{\tilde{P}}, J)$ if $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$. It follows that $\mathrm{range}(\mathcal{B}_{\phi^{\tilde{P}}}) = \mathrm{range}(\mathcal{B}_\phi)$ as in the proof of Proposition 86. But now assumption $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$ implies $\overline{\mathrm{range}(\mathcal{B}_{\phi^{\tilde{P}}})} = H$. Furthermore, because the controllability maps of a DLS and any of its spectral DLSs are equal, the approximate controllability $\overline{\mathrm{range}(\mathcal{B}_{(\phi^{\tilde{P}})_P})} = H$ follows for all $P \in Ric(\phi, J)$. Now, for $P = \tilde{P}_0^{\mathrm{crit}}$ equation (105) gives

$$(\phi^{\tilde{P}})_{\tilde{P}_0^{\mathrm{crit}}} = \begin{pmatrix} A_{\tilde{P}} & B \\ -\tilde{K}_{\tilde{P}_0^{\mathrm{crit}}} & I \end{pmatrix} = \begin{pmatrix} A_{\tilde{P}} & B \\ K_{\tilde{P}} - K_{\tilde{P}_0^{\mathrm{crit}}} & I \end{pmatrix}.$$

By the definition of the critical solution, the I/O-map of the spectral DLS $(\phi^{\tilde{P}})_{\tilde{P}_0^{\mathrm{crit}}}$ is the outer factor of $\mathcal{D}_{\phi^{\tilde{P}}}$. But this is the static identity operator $\mathcal{I}$, as discussed above. Thus $\tilde{K}_{\tilde{P}_0^{\mathrm{crit}}} | \mathrm{range}(\mathcal{B}_{(\phi^{\tilde{P}})_P}) = 0$, and by the approximate controllability assumption, it follows that $\tilde{K}_{\tilde{P}_0^{\mathrm{crit}}} = 0$.

By the definition of the inner DARE $ric(\phi^{\tilde{P}}, J)$, the following Liapunov equations are satisfied

$$\begin{aligned} A_{\tilde{P}}^* \tilde{P} A_{\tilde{P}} - \tilde{P} + C_{\tilde{P}}^* J C_{\tilde{P}} &= \tilde{K}_{\tilde{P}}^* \tilde{\Lambda}_{\tilde{P}} \tilde{K}_{\tilde{P}} = 0, \\ A_{\tilde{P}}^* \tilde{P}_0^{\mathrm{crit}} A_{\tilde{P}} - \tilde{P}_0^{\mathrm{crit}} + C_{\tilde{P}}^* J C_{\tilde{P}} &= \tilde{K}_{\tilde{P}_0^{\mathrm{crit}}}^* \tilde{\Lambda}_{\tilde{P}_0^{\mathrm{crit}}} \tilde{K}_{\tilde{P}_0^{\mathrm{crit}}} = 0. \end{aligned}$$

But now $\tilde{P}_0^{\mathrm{crit}} - \tilde{P} = A_{\tilde{P}}^* (\tilde{P}_0^{\mathrm{crit}} - \tilde{P}) A_{\tilde{P}}$ and by iterating

$$\tilde{P}_0^{\mathrm{crit}} - \tilde{P} - A_{\tilde{P}}^{*j} \tilde{P}_0^{\mathrm{crit}} A_{\tilde{P}}^j = -A_{\tilde{P}}^{*j} \tilde{P} A_{\tilde{P}}^j.$$

Because $\tilde{P}_0^{\mathrm{crit}}$ is the regular critical solution of $ric(\phi^{\tilde{P}}, J)$, it follows that $A_{\tilde{P}}^{*j} \tilde{P}_0^{\mathrm{crit}} A_{\tilde{P}}^j$ converges strongly to zero as $j \to \infty$. But then $L_{A_{\tilde{P}}, \tilde{P}} := \mathrm{s} - \lim_{j \to \infty} A_{\tilde{P}}^{*j} \tilde{P} A_{\tilde{P}}^j$ exists, and

$$(106) \qquad \tilde{P}_0^{\mathrm{crit}} - \tilde{P} = -L_{A_{\tilde{P}}, \tilde{P}}.$$

A similar kind of calculation can be carried out with the open loop operators. Because $\tilde{K}_{\tilde{P}_0^{\rm crit}} = 0$ as shown above, and by formula (105), $\tilde{K}_{\tilde{P}_0^{\rm crit}} = K_{\tilde{P}} - K_{\tilde{P}_0^{\rm crit}}$, it follows that $K_{\tilde{P}} = K_{\tilde{P}_0^{\rm crit}}$. For the indicators we have $\Lambda_{\tilde{P}} = \Lambda_{\tilde{P}_0^{\rm crit}}$, too. To see this equality, consider first the solution $\tilde{P} \in ric_0(\phi, J)$. The I/O-map of its inner DLS $\phi^{\tilde{P}}$ is $(J, \Lambda_{\tilde{P}})$-inner, as has already been mentioned. The critical solution $\tilde{P}_0^{\rm crit} \in ric_0(\phi^{\tilde{P}}, J)$ gives the $(J, \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}})$-inner-outer factorization

$$\mathcal{D}_{\phi^{\tilde{P}}} = \mathcal{D}_{(\phi^{\tilde{P}})^{\tilde{P}_0^{\rm crit}}} \cdot \mathcal{D}_{(\phi^{\tilde{P}})_{\tilde{P}_0^{\rm crit}}} = \mathcal{D}_{(\phi^{\tilde{P}})^{\tilde{P}_0^{\rm crit}}} \cdot \mathcal{I} = \mathcal{D}_{(\phi^{\tilde{P}})^{\tilde{P}_0^{\rm crit}}}$$

by claim (iv) of Lemma 79, and the uniqueness of the (J, S)-inner-outer factorizations of an I/O-map if the feed-through part of the outer factor is normalized to identity, see [19, Proposition 21]. We conclude that $\mathcal{D}_{\phi^{\tilde{P}}}$ is $(J, \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}})$-inner. So, $\mathcal{D}_{\phi^{\tilde{P}}}$ is simultaneously both $(J, \Lambda_{\tilde{P}})$-inner and $(J, \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}})$-inner. This implies that $\Lambda_{\tilde{P}_0^{\rm crit}} = \tilde{\Lambda}_{\tilde{P}_0^{\rm crit}} = \Lambda_{\tilde{P}}$ because the indicator of a solution is not changed under transition to any inner DARE.

Because $K_{\tilde{P}} = K_{\tilde{P}_0^{\rm crit}}$ and $\Lambda_{\tilde{P}_0^{\rm crit}} = \Lambda_{\tilde{P}}$ holds, the open loop DARE $Ric(\phi, J)$ gives us the equality

$$A^* \tilde{P}_0^{\rm crit} A - \tilde{P}_0^{\rm crit} = A^* \tilde{P} A - \tilde{P},$$

because both the operator $\tilde{P}_0^{\rm crit}$ and $\tilde{P}$ are solutions of the original DARE $Ric(\phi, J)$, and the right hand sides of the DARE at these solutions coincide.

Thus $\tilde{P}_0^{\rm crit} - \tilde{P} = A^*(\tilde{P}_0^{\rm crit} - \tilde{P})A$ and in the same way as proving equation (106) we obtain

$$\tilde{P}_0^{\rm crit} - \tilde{P} = L_{A, \tilde{P}_0^{\rm crit}} - L_{A, \tilde{P}} = L_{A, \tilde{P}_0^{\rm crit}}. \tag{107}$$

Here the strong limit exists and equality holds because $L_{A,\tilde{P}} = 0$, by assumption $P \in ric_0(\phi, J)$.

Comparing equations (106) and (107), we see that $-L_{A_{\tilde{P}}, \tilde{P}} = L_{A, \tilde{P}_0^{\rm crit}}$. Both the residual cost operators are nonnegative, as strong limits of sequences of nonnegative operators. It immediately follows that $L_{A_{\tilde{P}}, \tilde{P}} = L_{A, \tilde{P}_0^{\rm crit}} = 0$. Thus $\tilde{P} \in ric_0(\phi^{\tilde{P}}, J)$ is the critical regular solution of its own inner DARE. This completes the proof. $\square$

In the following Lemma 101 we characterize the regular H^∞ solutions of the inner DARE $Ric(\phi^{\tilde{P}}, J)$ for nonnegative $\tilde{P} \in ric_0(\phi, J)$. As in Lemma 100, we have to be a little careful to see that $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE. For this reason, we assume again that the cost operator $J > 0$ has a bounded inverse. It is important that the particular case when $\tilde{P} = P_0^{\rm crit} = (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit}$ can be solved for general $J \geq 0$, see Theorem 105.

Lemma 101. *Let $J > 0$ a boundedly invertible cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$. Assume that the input operator $B \in \mathcal{L}(U; H)$ is Hilbert–Schmidt, and the input space*

U and the output space Y of ϕ are separable. Assume that the regular critical solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit} \in ric_0(\phi, J)$ exists.

Then for all $\tilde{P} \in ric_0(\phi, J)$, $\tilde{P} \geq 0$, the DLS $\phi^{\tilde{P}}$ is output stable and I/O stable. Furthermore, we have the following equality of the solution sets of H^∞DAREs

$$\left\{ P \in ric_0(\phi, J) \quad | \quad P \leq \tilde{P} \right\} = ric_0(\phi^{\tilde{P}}, J).$$

Proof. The output stability and I/O stability of $\phi^{\tilde{P}}$ follow from Lemma 79 and the assumption that J has a bounded nonnegative inverse. We conclude that the inner DARE $Ric(\phi^{\tilde{P}}, J)$ is a H^∞DARE, and the claim about the solution sets $ric_0(\phi, J)$ and $ric_0(\phi^{\tilde{P}}, J)$ is meaningful. We proceed to prove the equality of the solution sets. Fix $\tilde{P} \in ric_0(\phi, J)$ such that $\tilde{P} \geq 0$.

To prove inclusion "$\subset$", let $P \in ric_0(\phi, J)$ be arbitrary, such that $P \leq \tilde{P}$. By Lemma 97, $\Delta P := \tilde{P} - P \in ric_0(\phi_P, \Lambda_P)$ and we can consider the inner DARE of $ric(\phi_P, \Lambda_P)$, centered at $\Delta P \geq 0$. Because the input operator B of ϕ_P is Hilbert–Schmidt, the input space U is separable, the cost operator $\Lambda_P > 0$ is boundedly invertible, and the H^∞ solution $\Delta P \in ric_0(\phi_P, \Lambda_P)$ is nonnegative, Lemma 100 implies that ΔP is the unique regular critical solution of its own inner DARE $ric((\phi_P)^{\Delta P}, \Lambda_P)$.

By Corollary 60, the minimax nodes have the "commutation" relation

$$((\phi_P)^{\Delta P}, \Lambda_P) \equiv \left((\phi^{\tilde{P}})_P, \Lambda_P \right). \tag{108}$$

Because the semigroup generator of $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$, equalling that of $\phi^{\tilde{P}}$, is $A_{\tilde{P}}$, it follows

$$0 = L_{A_{\tilde{P}}, \Delta P} = L_{A_{\tilde{P}}, (\tilde{P} - P)} = L_{A_{\tilde{P}}, \tilde{P}} - L_{A_{\tilde{P}}, P} = -L_{A_{\tilde{P}}, P},$$

where the first equality is because $\Delta P \in ric_0((\phi_P)^{\Delta P}, \Lambda_P)$ as the unique regular critical solution, and the last follows from the last claim of Lemma 100. This implies the existence of $L_{A_{\tilde{P}}, P}$ as a strong limit and also $L_{A_{\tilde{P}}, P} = 0$. Because $A_{\tilde{P}}$ is also the semigroup generator of $\phi^{\tilde{P}}$, it remains to prove that $P \in ric(\phi^{\tilde{P}}, J)$.

By identity (108), we conclude that $(\phi^{\tilde{P}})_P$ is output stable and I/O stable, because this DLS equals $(\phi_P)^{\Delta P}$, which is I/O stable and output stable by claim (iv) of Lemma 79 and the fact that $\Delta P \in ric_0(\phi_P, \Lambda_P)$ is nonnegative, as discussed earlier.

Here we have used the fact that the cost operator Λ_P of DARE $ric(\phi_P, \Lambda_P)$ is nonnegative with a bounded inverse, by Lemma 53, because $P \in ric_0(\phi, J)$ and the regular critical solution $P_0^{\rm crit} := (\mathcal{C}_\phi^{\rm crit})^* J \mathcal{C}_\phi^{\rm crit} \geq 0$ surely has a positive indicator, by the nonnegativity of J. This completes the first part of the proof.

For the converse inclusion "$\supset$", let $P \in ric_0(\phi^{\tilde{P}}, J)$ be arbitrary, and define $\Delta P = \tilde{P} - P$. Now our task is to show that ϕ_P is output stable and I/O stable, and $\tilde{P} \geq P$. To clarify things, we first write the observability map of $\phi^{\tilde{P}}$ in I/O-form, by using formula (79), with ϕ_P in place of ϕ, ΔP in place of P, and so on. Recall that this formula does not require any stability properties of any of the DLSs involved (apart from the boundedness of the generating operators), because is solely based on the equivalence of DLSs (and their feedbacks) in I/O-form and difference equation form, presented in the sense of [21, Lemma 19 and Lemma 20]. We obtain

$$\mathcal{C}_{(\phi^{\tilde{P}})_P} = \mathcal{C}_{(\phi_P)^{\Delta P}} = \mathcal{C}_{\phi_P} - \mathcal{D}_{\phi_P}\mathcal{D}^{-1}_{(\phi_P)_{\Delta P}}\mathcal{C}_{(\phi_P)_{\Delta P}},$$

where the first equality is because $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$, by equation (108). Furthermore, $\mathcal{D}_{\phi_P}\mathcal{D}^{-1}_{(\phi_P)_{\Delta P}} = \mathcal{D}_{(\phi_P)^{\Delta P}}$, as causal, shift invariant operators in the sequence space $Seq(U)$, by formulae (79) and (80). But now $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$ implies that $\mathcal{D}_{(\phi_P)^{\Delta P}} = \mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}$ in $Seq(U)$. Because $(\phi_P)_{\Delta P} = \phi_{\tilde{P}}$ by equation (58) of Proposition 59, we get

$$\mathcal{C}_{\phi_P} = \mathcal{C}_{(\phi^{\tilde{P}})_P} + \left(\mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}\right)\cdot \mathcal{C}_{\phi_{\tilde{P}}}. \tag{109}$$

Because $P \in ric_0(\phi^{\tilde{P}}, J)$ by assumption, both $\mathcal{C}_{(\phi^{\tilde{P}})_P} : H \to \ell^2(\mathbf{Z}_+; U)$ and $\mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P} : \ell^2(\mathbf{Z}; U) \to \ell^2(\mathbf{Z}; U)$ are bounded. Similarly $\mathcal{C}_{\phi_{\tilde{P}}} : H \to \ell^2(\mathbf{Z}_+; U)$ is bounded because $\tilde{P} \in ric_0(\phi, J)$, by assumption. We now conclude that ϕ_P is output stable, because all the operators in equation (109) are bounded between the corresponding (dense subspaces of the) Hilbert spaces H, $\ell^2(\mathbf{Z}_+; U)$, and $\ell^2(\mathbf{Z}; U)$.

We proceed to show the I/O stability of ϕ_P. As above, $\mathcal{D}_{\phi_P}\mathcal{D}^{-1}_{(\phi_P)_{\Delta P}} = \mathcal{D}_{(\phi_P)^{\Delta P}} = \mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}$ in $Seq(U)$. Also, $\mathcal{D}_{(\phi_P)_{\Delta P}} = \mathcal{D}_{\phi_{\tilde{P}}}$ because $(\phi_P)_{\Delta P} = \phi_{\tilde{P}}$. Because the feed-through operator of the spectral DLS $\phi_{\tilde{P}}$ is always the invertible identity operator, it follows from Proposition 1 that $\mathcal{D}_{\phi_{\tilde{P}}}$ is a causal bijection in $Seq(U)$. It follows that $\mathcal{D}_{\phi_P} = \mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}\mathcal{D}_{\phi_{\tilde{P}}}$ in $Seq(U)$. From assumptions $\tilde{P} \in ric_0(\phi, J)$ and $P \in ric_0(\phi^{\tilde{P}}, J)$ it follows that both $\mathcal{D}_{\phi_{\tilde{P}}}$ and $\mathcal{D}_{\left(\phi^{\tilde{P}}\right)_P}$ are bounded in $\ell^2(\mathbf{Z}; U)$, and so is $\mathcal{D}_{\phi_P}$. We have now proved that $P \in ric(\phi, J)$, and thus $Ric(\phi_P, \Lambda_P)$ is a H^∞DARE.

Because $P \in ric_0(\phi^{\tilde{P}}, J)$, it follows from claim (iii) of Lemma 79 that the I/O-map of the inner DLS $\left(\phi^{\tilde{P}}\right)^P$ is I/O stable and $(J, \tilde{\Lambda}_P)$-inner. The indicator $\tilde{\Lambda}_P$ of P, as a solution of the inner DARE $Ric(\phi^{\tilde{P}}, J)$, equals the indicator Λ_P of P, as a solution of the original DARE $Ric(\phi, J)$. Because $\left(\phi^{\tilde{P}}\right)^P = \phi^P$ by equation (57) of Proposition 59, it follows that $\mathcal{D}_{\phi^P}$ is (J, Λ_P)-inner.

Thus $\mathcal{D}_\phi = \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}$ where both the factors are bounded. For the Popov operator we get

$$\mathcal{D}_\phi^* J \mathcal{D}_\phi = \left(\mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P}\right)^* J \mathcal{D}_{\phi^P}\mathcal{D}_{\phi_P} = \mathcal{D}^*_{\phi_P}\cdot \mathcal{D}^*_{\phi^P} J \mathcal{D}_{\phi^P}\cdot \mathcal{D}_{\phi_P} = \mathcal{D}^*_{\phi_P}\Lambda_P\mathcal{D}_{\phi_P}.$$

Because we already know that $P \in ric(\phi, J)$, it follows that the residual cost operator in I/O-form satisfies $\mathcal{L}_{\phi,P} = 0$, by [26, claim (ii) of Lemma 52]. Because $\overline{\text{range}(\mathcal{B}_\phi)} = H$ is assumed, it follows that $L_{A,P} = 0$, by [26, claim (iii) of Lemma 52]. We have now shown that $P \in ric_0(\phi, J)$.

Because $\tilde{P}, P \in ric_0(\phi, J)$, Lemma 97 implies that that $\Delta P := \tilde{P} - P \in ric_0(\phi_P, \Lambda_P)$. Because $(\phi_P)^{\Delta P} = (\phi^{\tilde{P}})_P$ and $P \in ric_0(\phi^{\tilde{P}}, J)$, it follows that the inner DLS $(\phi_P)^{\Delta P}$ at solution ΔP is I/O stable. Because the DLS ϕ_P, the cost operator Λ_P, and the solution $\Delta P \in ric_0(\phi_P, \Lambda_P)$ satisfy the conditions of Theorem 96, it follows that $\Delta P \geq 0$ and thus $\tilde{P} \geq P$. This completes the proof. $\square$

15 Reduction of H^∞DARE to an inner DARE

In this section, we consider the H^∞DARE $ric(\phi, J)$ that has a regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$, where

$$\mathcal{C}_\phi^{\mathrm{crit}} := (\mathcal{I} - \bar{\pi}_+ \mathcal{D}_\phi (\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J) \mathcal{C}_\phi. \tag{110}$$

In essence, we show under technical assumptions that $ric(\phi, J)$ and $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ are practically equivalent, as H^∞DAREs. Many of these results hold for general cost operator J; the nonnegativity assumption $J \geq 0$ is required only when the sets $ric_0(\phi, J)$ and $ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$ of regular H^∞ solutions are related to each other.

Suppose we are interested in the H^∞ solutions of H^∞DARE $ric(\phi, J)$. If we know *some* solution $\tilde{P} \in ric(\phi, J)$, we can study the (possibly non-H^∞) inner DARE $Ric(\phi^{\tilde{P}}, J)$ in place of the original $ric(\phi, J)$. Furthermore, under the conditions of claim (iv) Lemma 79, if we can find a *nonnegative* solution $\tilde{P} \in ric_{uw}(\phi, J)$ for $J \geq 0$, then the inner DARE $Ric(\phi^{\tilde{P}}, J)$ is essentially the H^∞DARE $ric(J^{\frac{1}{2}}\phi^{\tilde{P}}, I)$, with an $(I, \Lambda_{\tilde{P}})$-inner I/O-map $J^{\frac{1}{2}}\mathcal{D}_{\phi^{\tilde{P}}}$. If, in addition, the nonnegative cost operator J has a bounded inverse, then $Ric(\phi^{\tilde{P}}, J)$ itself is a H^∞DARE. We remark that an inner DLS $\phi^{\tilde{P}}$ is generally not observable (i.e. $\ker(\mathcal{C}_{\phi^{\tilde{P}}}) \neq \{0\}$), and the semigroup generator A_P is generally not even power bounded.

In Lemmas 100 and 101 we have considered the solution set $ric_0(\phi^{\tilde{P}}, J)$ for $\tilde{P} \geq 0$ and boundedly invertible $J > 0$. In this section, we give stronger results in the particular case $\tilde{P} = P_0^{\mathrm{crit}}$, where P_0^{crit} is given by (110). The I/O-map $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$ is now the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of the I/O-map $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$, and there is no need to assume a bounded inverse for J to make $\phi^{P_0^{\mathrm{crit}}}$ output stable and I/O stable. The outer factor $\mathcal{X}$ of the I/O-map $\mathcal{D}_\phi$ is not very important from the Riccati equation point of view, as implied by Theorem 105, the main result of this section. An important application of these results is in the last section of [25].

We start by answering the uniqueness questions associated to various critical operators.

Proposition 102. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an output stable and I/O stable DLS, and $J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Assume that the regular critical solution $P_0^{\mathrm{crit}} := (\mathcal{C}_\phi^{\mathrm{crit}})^* J \mathcal{C}_\phi^{\mathrm{crit}} \in ric_0(\phi, J)$ exists.*

Then

(i) the critical indicators satisfy $\Lambda_{P^{\mathrm{crit}}} = \Lambda_{P_0^{\mathrm{crit}}}$ for all critical $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$,

(ii) If $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$, then the critical feedback operators satisfy $K_{P^{\mathrm{crit}}} = K_{P_0^{\mathrm{crit}}}$ for all critical $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$. Furthermore, the closed loop operators $A_{P^{\mathrm{crit}}} = A_{P_0^{\mathrm{crit}}}$ and $C_{P^{\mathrm{crit}}} = C_{P_0^{\mathrm{crit}}}$, where critical $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$ is arbitrary. P_0^{crit} is the unique critical solution in the set $ric_{00}(\phi, J)$.

(iii) If $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$*, and the open loop semigroup* A *is strongly stable, then there is only one critical solution* $P^{\mathrm{crit}} \in Ric_{uw}(\phi, J)$*, and it equals* P_0^{crit}*.*

We conclude that if $\overline{\mathrm{range}(\mathcal{B})} = H$, it makes sense to speak about the critical (closed loop) feedback operator K^{crit}, the critical semigroup A^{crit} and critical output operator C^{crit}, because these are now independent of the choice of the critical solution. In our earlier work [20, Definitions 7 and 10], we defined the objects K^{crit}, A^{crit} and C^{crit} differently. We proceed to show that under approximate controllability $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$, both these definitions coincide. This makes it possible to write the inner DLS $\phi^{P_0^{\mathrm{crit}}} = \left(\begin{smallmatrix} A_{P_0^{\mathrm{crit}}} & B \\ C_{P_0^{\mathrm{crit}}} & D \end{smallmatrix}\right)$ in I/O-form, without explicit reference to the solution P_0^{crit}.

Proposition 103. *Let* $J \in \mathcal{L}(Y)$ *be a self-adjoint cost operator. Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be an output stable and I/O stable DLS, such that* $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$*. Assume that there exists a regular critical solution* $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$*.*

Define the critical (closed loop) feedback operator

$$\mathcal{K}^{\mathrm{crit}} := -(\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{C}_\phi$$

and the critical (closed loop) observability map $\mathcal{C}_\phi^{\mathrm{crit}} := \mathcal{C}_\phi + \mathcal{D}_\phi \mathcal{K}^{\mathrm{crit}}$*. By* $\mathcal{X}$ *and* $\mathcal{N}$ *denote the* $(J, \Lambda_{P_0^{\mathrm{crit}}})$*-inner and outer factors in the* $(J, \Lambda_{P_0^{\mathrm{crit}}})$*-inner-outer factorization* $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$*.*

Then

(i) $K_{P_0^{\mathrm{crit}}} = K^{\mathrm{crit}}$*, where* $K^{\mathrm{crit}} := \pi_0 \mathcal{K}^{\mathrm{crit}}$ *with the natural identification of spaces* $\mathrm{range}(\pi_0)$ *and* U*,*

(ii) the observability map of the spectral DLS satisfies $\mathcal{C}_{\phi_{P_0^{\mathrm{crit}}}} = \mathcal{X}\mathcal{K}^{\mathrm{crit}}$*,*

(iii) $A_{P_0^{\mathrm{crit}}} := A + BK_{P_0^{\mathrm{crit}}} = A^{\mathrm{crit}}$*, where* $A^{\mathrm{crit}} := A + \mathcal{B}_\phi \tau^* \mathcal{K}^{\mathrm{crit}}$*,*

(iv) $C_{P_0^{\mathrm{crit}}} := C + DK_{P_0^{\mathrm{crit}}} = C^{\mathrm{crit}}$*, where* $C^{\mathrm{crit}} := \pi_0 \mathcal{C}_\phi^{\mathrm{crit}}$ *with the natural identification of spaces* $\mathrm{range}(\pi_0)$ *and* Y*.*

(v) In particular, the inner DLS $\phi^{P_0^{\mathrm{crit}}}$ *is given in I/O-form by the critical (closed loop) DLS*

$$\Phi^{P_0^{\mathrm{crit}}} = \begin{bmatrix} (A^{\mathrm{crit}})^j & \mathcal{B}_\phi \mathcal{X}^{-1} \tau^{*j} \\ \mathcal{C}_\phi^{\mathrm{crit}} & \mathcal{N} \end{bmatrix}. \tag{111}$$

Proof. Let $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ be the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, where the outer part $\mathcal{X}$ has a bounded inverse, and the feed-through operator is normalized $\pi_0 \mathcal{X} \pi_0 = I$. The existence of such factorization follows from the assumption that the critical solution P_0^{crit} exists, by [26, Theorem 27]. It also follows that the Popov operator $\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+$ has a bounded inverse, and it follows that all the operators $\mathcal{K}^{\mathrm{crit}}$, K^{crit}, A^{crit}, $\mathcal{C}_\phi^{\mathrm{crit}}$ and C^{crit} are well defined.

Then, as in the proof of [19, Lemma 25], it follows that the outer factor $\mathcal{X}$ has the realization, written in I/O-form

$$\Phi_{\mathcal{X}} = \begin{bmatrix} A^j & \mathcal{B}_\phi \tau^{*j} \\ -\mathcal{K} & \mathcal{X} \end{bmatrix}, \tag{112}$$

where $\mathcal{K} := -\Lambda_{P_0^{\text{crit}}}^{-1} \mathcal{N}^* J \mathcal{C}_\phi$. On the other hand, the critical (closed loop) feedback operator $\mathcal{K}^{\text{crit}} := -(\bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{D}_\phi \bar{\pi}_+)^{-1} \bar{\pi}_+ \mathcal{D}_\phi^* J \mathcal{C}_\phi$ can be written in form $\mathcal{K}^{\text{crit}} = \mathcal{X}^{-1} \cdot \mathcal{K}$, by [19, Lemma 22]. We have now enough information to translate the DLS $\Phi_{\mathcal{X}}$ in formula (112) into difference equation form; we have

$$\phi_{\mathcal{X}} = \begin{pmatrix} A & B \\ -K^{\text{crit}} & I \end{pmatrix}, \quad K^{\text{crit}} := \pi_0 \mathcal{K}^{\text{crit}}, \tag{113}$$

because $\pi_0 \mathcal{X} \pi_0 = I$ implies that $\pi_0 \mathcal{X}^{-1} \pi_0 = I$, and then $\pi_0 \mathcal{K} = \pi_0 \mathcal{K}^{\text{crit}}$. Note that we have identified the spaces range(π_0) and U in the natural way.

Now, because $P_0^{\text{crit}} \in ric_0(\phi, J)$ is a critical solution, the outer factor $\mathcal{X}$ can be expressed also as the I/O-map of the spectral DLS $\phi_{P_0^{\text{crit}}} = \left(\begin{smallmatrix} A & B \\ -K_{P_0^{\text{crit}}} & I \end{smallmatrix} \right)$. Because the controllability maps of $\phi_{P_0^{\text{crit}}}$ and $\phi_{\mathcal{X}}$ coincide with $\mathcal{B}_\phi$, we conclude that $K^{\text{crit}} | \text{range}(\mathcal{B}_\phi) = K_{P_0^{\text{crit}}} | \text{range}(\mathcal{B}_\phi)$. By approximate controllability, $K^{\text{crit}} = K_{P_0^{\text{crit}}}$, because both the operators are bounded. This proves now claim (i), and claim (ii) immediately follows because $\mathcal{K} = \mathcal{C}_{\phi_{P_0^{\text{crit}}}}$ and $\mathcal{K}^{\text{crit}} = \mathcal{X}^{-1} \cdot \mathcal{K}$, as discussed above.

Claims (iii), (iv) and (v) are consequences of [21, Lemma 20], where it is shown that the state feedback structures of DLSs in I/O-form and difference equation form are equivalent. More precisely, the pairs $[\mathcal{K}, \mathcal{I} - \mathcal{X}]$ and $(K^{\text{crit}}, 0)$ are corresponding state feedback pairs for the (open loop) DLS ϕ in I/O-form and difference equation form, respectively. It follows that the closed loop DLSs $[\phi, [\mathcal{K}, \mathcal{I} - \mathcal{X}]]_\diamond$ in I/O-form and $\left(\phi, \left(K^{\text{crit}}, 0\right)\right)_\diamond$ in difference equation form are equal, by [21, Lemma 20]. But these equal $\Phi^{P_0^{\text{crit}}}$ and $\phi^{P_0^{\text{crit}}}$, extended by the equal feedback pairs. $\square$

Now we have tools to find out how the continuity properties of ϕ are inherited by the inner DLS $\phi^{P_0^{\text{crit}}}$.

Proposition 104. *$J \in \mathcal{L}(Y)$ a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$ be an output stable and I/O stable DLS. Assume that $\overline{\text{range}(\mathcal{B}_\phi)} = H$, and the (unique) regular critical solution $P_0^{\text{crit}} \in ric_0(\phi, J)$ exists. Then*

- *(i) $\phi^{P_0^{\text{crit}}}$ is output stable and I/O stable. The I/O-map of $\phi^{P_0^{\text{crit}}}$ is the $(J, \Lambda_{P_0^{\text{crit}}})$-inner factor $\mathcal{N}$ of $\mathcal{D}_\phi = \mathcal{N} \mathcal{X}$. Furthermore, ϕ is input stable if and only if $\phi^{P_0^{\text{crit}}}$ is.*
- *(ii) We have $\overline{\text{range}(\mathcal{B}_{\phi^{P_0^{\text{crit}}}})} = H$. If ϕ is input stable, then $\mathcal{B}_\phi \ell^2(\mathbf{Z}_-; U) = H$ if and only if $\mathcal{B}_{\phi^{P_0^{\text{crit}}}} \ell^2(\mathbf{Z}_-; U) = H$.*

Proof. In claim (i), the output stability and I/O stability of $\phi^{P_0^{\mathrm{crit}}}$ follows directly from equation (111) in Proposition 103. More precisely, the observability map $\mathcal{C}_\phi^{\mathrm{crit}}$ is bounded because all operators in (110) are bounded by our explicit assumptions; in particular, the inverse of the Popov operator $\bar\pi_+\mathcal{D}_\phi^* J\mathcal{D}_\phi\bar\pi_+$ is bounded because P_0^{crit} exists, see [26, Theorem 27]. Also the I/O-map of $\phi^{P_0^{\mathrm{crit}}}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of $\mathcal{D}_\phi$, by equation (111).

To complete the proof, we first show that show that the bounded, anti-causal Toeplitz operator $\pi_-\mathcal{X}^{-1}\pi_- : \ell^2(\mathbf{Z}_-;U) \to \ell^2(\mathbf{Z}_-;U)$ with a causal symbol $\mathcal{X}^{-1}$ is a bijection in this space. Let us start with the surjectivity. Let $\pi_-\tilde u \in \ell^2(\mathbf{Z}_-;U)$ be arbitrary. Because $\mathcal{X}$ is outer with a bounded inverse, it follows that $\mathcal{X}^{-1} : \ell^2(\mathbf{Z};U) \to \ell^2(\mathbf{Z};U)$ is a bounded, shift-invariant and causal bijection. Thus there is a $\tilde v \in \ell^2(\mathbf{Z};U)$ such that $\pi_-\tilde u = \mathcal{X}^{-1}\tilde v$. But now

$$\pi_-\tilde u = \mathcal{X}^{-1}\pi_-\tilde v + \mathcal{X}^{-1}\bar\pi_+\tilde v = \pi_-\mathcal{X}^{-1}\pi_-\tilde v + \pi_-\mathcal{X}^{-1}\bar\pi_+\tilde v.$$

The causality of $\mathcal{X}^{-1}$ implies that $\pi_-\mathcal{X}^{-1}\bar\pi_+\tilde v = 0$ and so $\pi_-\tilde u = \pi_-\mathcal{X}^{-1}\pi_- \cdot \pi_-\tilde v$. The surjectivity of $\pi_-\mathcal{X}^{-1}\pi_-$ follows because $\pi_-\tilde v \in \ell^2(\mathbf{Z}_-;U)$.

We show the injectivity of $\pi_-\mathcal{X}^{-1}\pi_-$. Assume $\pi_-\tilde v \in \ell^2(\mathbf{Z}_-;U)$ is such that $\pi_-\mathcal{X}^{-1}\pi_-\tilde v = 0$. Then

$$0 = \mathcal{X}\pi_-\mathcal{X}^{-1}\pi_-\tilde v = \mathcal{X}\mathcal{X}^{-1}\pi_-\tilde v - \mathcal{X}\bar\pi_+\mathcal{X}^{-1}\pi_-\tilde v = \pi_-\tilde v - \mathcal{X}\bar\pi_+\mathcal{X}^{-1}\pi_-\tilde v,$$

or equivalently $\pi_-\tilde v = \mathcal{X}\bar\pi_+\mathcal{X}^{-1}\pi_-\tilde v = \pi_-\mathcal{X}\bar\pi_+\mathcal{X}^{-1}\pi_-\tilde v$. The causality of $\mathcal{X}$ implies that $\pi_-\mathcal{X}\bar\pi_+ = 0$, and so $\pi_-\tilde v = 0$. We conclude that the Toeplitz operator $\pi_-\mathcal{X}^{-1}\pi_-$ in injective, and thus a bounded bijection. It then follows from the Open Mapping Theorem, that $\pi_-\mathcal{X}^{-1}\pi_-$ has a bounded inverse in $\ell^2(\mathbf{Z}_-;U)$. Because $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{B}_\phi\mathcal{X}^{-1} = \mathcal{B}_\phi \cdot \pi_-\mathcal{X}^{-1}\pi_-$ by equation (111) in Proposition 103, the equivalence of the input stabilities of ϕ and $\phi^{P_0^{\mathrm{crit}}}$ follows.

It remains to consider claims (ii) about the range of $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}}$. Again, we have $\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{B}_\phi\pi_- \cdot \pi_-\mathcal{X}^{-1}\pi_-$. As a causal operator, $\pi_-\mathcal{X}^{-1}$ maps the domain of any controllability map (consisting of the sequences $Seq_-(U) \subset \ell^2(\mathbf{Z}_-;U)$ that have only finitely many nonzero components) onto itself. This implies that $\mathrm{range}(\mathcal{B}_\phi) = \mathrm{range}(\mathcal{B}_{\phi^{P_0^{\mathrm{crit}}}})$, and the approximate controllability claim follows. The (infinite time) exact controllability claim follows because the Toeplitz operator $\pi_-\mathcal{X}^{-1}\pi_-$ is boundedly invertible. The proof is now complete. □

Now that we have related the DLSs ϕ and $\phi^{P_0^{\mathrm{crit}}}$, we proceed to consider the inner DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ and give the main result of this section. The significance of the following theorem is that the structure of a H^∞DARE does not essentially depend on the outer factor of $\mathcal{D}_\phi$ if the cost operator J is nonnegative. It is then possible, under proper technical assumptions, to replace an original H^∞DARE $ric(\phi, J)$ by the inner H^∞DARE $ric(\phi^{P_0^{\mathrm{crit}}}, J)$ that has a $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner I/O-map. This result has an application in the final section of [25].

Theorem 105. *Let $J \in \mathcal{L}(Y)$ be a self-adjoint cost operator. Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be an output stable and I/O stable DLS, such that $\overline{\mathrm{range}(\mathcal{B}_\phi)} = H$. Assume that the regular critical solution $P_0^{\mathrm{crit}} \in ric_0(\phi, J)$ exists. Then the following holds:*

(i) The inner DARE $Ric(\phi^{P_0^{\mathrm{crit}}}, J)$ is a H^∞DARE. The full solution sets satisfy $Ric(\phi, J) = Ric(\phi^{P_0^{\mathrm{crit}}}, J)$. The I/O-map $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$ is the $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner factor $\mathcal{N}$ of $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$.

(ii) The unique regular critical solution $\tilde{P}_0^{\mathrm{crit}} := \left(\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}}\right)^* J \mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}} \in ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$ satisfies $\tilde{P}_0^{\mathrm{crit}} = P_0^{\mathrm{crit}}$.

(iii) Assume, in addition, the input space U and output space Y are separable, the input operator B is Hilbert–Schmidt, and $J \geq 0$. Then

$$ric_0(\phi, J) = ric_0(\phi^{P_0^{\mathrm{crit}}}, J). \tag{114}$$

Proof. By claim (i) of Proposition 104, $\phi^{P_0^{\mathrm{crit}}}$ is output stable and I/O stable. It follows that $Ric(\phi^{P_0^{\mathrm{crit}}}, J)$ is a H^∞DARE. By claim (v) of Proposition 103, the I/O-map of $\phi^{P_0^{\mathrm{crit}}}$ is $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner. The full solution sets satisfy $Ric(\phi, J) = Ric(\phi^{P_0^{\mathrm{crit}}}, J)$, by Lemma 65.

We prove claim (ii) by calculating an expression for the critical (closed loop) observability map $\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}}$ for the inner DLS $\phi^{P_0^{\mathrm{crit}}}$ and the cost operator J. Clearly, $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{N} = \mathcal{N}\mathcal{I}$ is the unique $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, where $\mathcal{I}$ is the unique outer factor whose feed-through operator is the identity of U. By [19, claim (iii) of Lemma 22], we obtain

$$\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{C}_{\phi^{P_0^{\mathrm{crit}}}} - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}_{\phi^{P_0^{\mathrm{crit}}}}. \tag{115}$$

By claim (v) of Proposition 103, $\mathcal{C}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{C}^{\mathrm{crit}}_\phi$, and again, by [19, claim (iii) of Lemma 22]

$$\mathcal{C}^{\mathrm{crit}}_\phi = \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}_\phi, \tag{116}$$

because $\mathcal{D}_\phi = \mathcal{N}\mathcal{X}$ is the unique $(J, \Lambda_{P_0^{\mathrm{crit}}})$-inner-outer factorization, where $\mathcal{X}$ is the unique outer factor whose feed-through operator is the identity of U. By combining equations (115) and (116), we obtain

$$\begin{aligned}
\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}} &= \left(\mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}_\phi\right) \\
&- \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \left(\mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}_\phi\right) \\
&= \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J \mathcal{C}_\phi \\
&+ \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+ \cdot \left(\mathcal{N}^* J \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\right) \cdot \bar{\pi}_+\mathcal{N}^* J \mathcal{C}_\phi.
\end{aligned}$$

Because $\mathcal{N}^* J \mathcal{N} = \Lambda_{P_0^{\mathrm{crit}}}$, the last two terms on the right hand side cancel each other, and it follows

$$\mathcal{C}^{\mathrm{crit}}_{\phi^{P_0^{\mathrm{crit}}}} = \mathcal{C}_\phi - \mathcal{N}\Lambda^{-1}_{P_0^{\mathrm{crit}}}\bar{\pi}_+\mathcal{N}^* J\mathcal{C}_\phi = \mathcal{C}^{\mathrm{crit}}_\phi,$$

where the last equality is by [19, claim (iii) of Lemma 22]. Now claim (ii) is verified.

We prove now the inclusion "$\subset$" of claim (iii). In fact, the inclusion "$\subset$" of Lemma 101 is almost what we need, if we set $\tilde{P} = P_0^{\mathrm{crit}} \in ric_0(\phi, J)$. In the proof of this lemma, the bounded inverse of the cost operator $J > 0$ was only needed to show that $\phi^{\tilde{P}}$ is output stable and I/O stable. In the special case when $\tilde{P} = P_0^{\mathrm{crit}}$, we know by Proposition 104 that $\phi^{\tilde{P}}$ is output stable and I/O stable, even if $J \geq 0$ is not boundedly invertible. We now conclude that

$$\left\{P \in ric_0(\phi, J), \quad | \quad P \leq P_0^{\mathrm{crit}}\right\} \subset ric_0(\phi^{P_0^{\mathrm{crit}}}, J).$$

as in the proof of Lemma 101. By Theorem 96, P_0^{crit} is the largest element of the set $ric_0(\phi, J)$, and $P \leq P_0^{\mathrm{crit}}$ need not be explicitly written. The claimed inclusion now follows.

The proof of the converse inclusion "$\supset$" is identical to that given in Lemma 101 for $\tilde{P} = P_0^{\mathrm{crit}}$. We remark that the invertibility of the cost operator J is never used in the proof of this converse inclusion "$\supset$". The proof is now complete. $\square$

The statement on Theorem 105 is in a perfect harmony with the following intuitive observation of this paper: finding solutions for the H^∞ Riccati equation $ric(\phi, J)$ is related to moving in the lattice of the inner factors of $\mathcal{D}_\phi$. We remark that the input operator $B \in \mathcal{L}(U : H)$ is required to be Hilbert–Schmidt and the cost operator J nonnegative only in claim (iii) of Theorem 105. All the other results in this section hold for arbitrary B and self-adjoint J.

Under the assumptions of claim (iii) of Theorem 105, it is enough to be able to solve (numerically) H^∞DAREs with an inner I/O-map. To transform ϕ into $\phi^{P_0^{\mathrm{crit}}}$, we need not directly solve the original DARE $ric(\phi, J)$; the regular critical solution P_0^{crit} can be computed from $\mathcal{C}^{\mathrm{crit}}_\phi$ by using formula (110). We remark that in this process, the most requiring thing is to calculate the inverse of the (Toeplitz) Popov operator $\bar{\pi}_+\mathcal{D}^*_\phi J\mathcal{D}_\phi\bar{\pi}_+$. At least when U is finite dimensional, and there is some smoothness in the Popov function $e^{i\theta} \mapsto \mathcal{D}_\phi(e^{i\theta})^* J\mathcal{D}_\phi(e^{i\theta})$, we can efficiently solve the required Toeplitz systems of equations iteratively, see [23], [18], and [28]. We conclude that we have some hope in this direction, even from the numerical analysis point of view.

So as to the numerical solution of the resulting H^∞DARE with an inner I/O-map, things seem to be wide open. It is not even clear what a nice solver would have to do, in order to be nice. Particularly interesting would be algorithms that would not require the dimensionality of the state space, and would not reduce the computation into some type of generalized eigenvalue problem. Such a solver could possibly be an iterative process, formulated for infinite dimensional objects and without any discretization. State space isomorphism techniques could be helpful, so that convenient (minimal) realizations of $\mathcal{D}_{\phi^{P_0^{\mathrm{crit}}}}$ could be used instead. Some additional functionality would have to be required, to enable such solver to move in the solution set of DARE and to find a particular solution of interest. It is not clear, how the natural lattice operations of the set $ric_0(\phi^{P_0^{\mathrm{crit}}}, J)$ can be realized, without replacing them by intersections and spans of subspaces. These problems we leave open for the future research.

References

[1] S. Bittanti, A. J. Laub, and J. C. Willems (Eds.). *The Riccati equation.* Springer Verlag, 1991.

[2] F. M. Callier, L. Dumortier, and J. Winkin. On the nonnegative self adjoint solutions of the operator Riccati equation for infinite dimensional systems. *Integral equations and operator theory*, 22:162–195, 1995.

[3] R. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics.* Springer Verlag, New York, Berlin, 1995.

[4] L. Dumortier. *Partially stabilizing linear-quadratic optimal control for stabilizable semigroup systems.* PhD thesis, Facultes universitaires Notre-Dame de la Paix, 1998.

[5] N. Dunford and J. Schwarz. *Linear Operators; Part II: Spectral Theory.* Interscience Publishers, Inc. (J. Wiley & Sons), New York, London, 1963.

[6] P. L. Duren. *Theory of H^2 spaces*, volume 38 of *Pure and applied mathematics.* Academic press, 1970.

[7] L. Finesso and G. Picci. A characterization of minimal spectral factors. *IEEE Transactions on Automatic Control*, AC-27:122–127, 1982.

[8] C. Foias and A. E. Frazho. *The commutant lifting approach to interpolation problems*, volume 44 of *Operator Theory: Advances and applications.* Birkhäuser Verlag, Basel, Boston, Berlin, 1990.

[9] P. A. Fuhrmann. *Linear systems and operators in Hilbert space.* McGraw-Hill, Inc., 1981.

[10] P. A. Fuhrmann. On the characterization and parameterization of minimal spectral factors. *Journal of Mathematical Systems, Estimation, and Control*, 5(4):383–444, 1995.

[11] P. A. Fuhrmann and J. Hoffmann. Factorization theory for stable discrete-time inner functions. *Journal of Matematical Systems, Estimation, and Control*, 7(4):383–400, 1997.

[12] I. Gohberg and L. A. Sachnovich (Eds.). *Matrix and Operator Valued Functions*, volume 72 of *Operator Theory Advances and Applications.* Birkhäuser, 1994.

[13] J. W. Helton. A spectral factorization approach to the distributed stable regulator problem; the algebraic Riccati equation. *SIAM Journal of Control and Optimization*, 14:639–661, 1976.

[14] E. Hille and R. S. Phillips. *Functional analysis and semi-groups.* American Mathematical Society, 1957.

[15] P. Lancaster and L. Rodman. *Algebraic Riccati equations.* Clarendon press, Oxford, 1995.

[16] H. Langer, A. C. M. Ran, and D. Temme. Nonnegative solutions fo algebraic Riccati equations. *Linear algebra and applications*, 261:317-352, 1997.

[17] J. P. LaSalle. *The stability and Control of Discrete Processes.* Springer Verlag, 1986.

[18] J. Malinen. On the Properties for Iteration of a Compact Operator with Unstructured Perturbation. *Helsinki University of Technology Institute of Mathematics Research Reports*, A360, 1996.

[19] J. Malinen. Minimax Control of Distributed Discrete Time Systems through Spectral Factorization. *Proceedings of EEC97, Brussels, Belgium*, 1997.

[20] J. Malinen. Nonstandard Discrete Time Cost Optimization Problem: The Spectral Factorization Approach. *Helsinki University of Technology Institute of Mathematics Research Reports*, A385, 1997.

[21] J. Malinen. Well-posed Discrete Time Linear Systems and Their Feedbacks. *Helsinki University of Technology Institute of Mathematics Research Reports*, A384, 1997.

[22] J. Malinen. Discrete Time Riccati Equations and Invariant Subspaces of Linear Operators. *Conference Proceedings of MMAR98*, 1998.

[23] J. Malinen. Properties of Iteration of Toeplitz Operators with Toeplitz Preconditioners. *BIT Numerical Mathematics*, 38(2), June 1998.

[24] J. Malinen. Solutions of the Riccati equation for H^∞ discrete time systems. *Conference Proceedings of MTNS98*, 1998.

[25] J. Malinen. Discrete Time Riccati Equations and Invariant Subspaces of Linear Operators. *Helsinki University of Technology Institute of Mathematics Research Reports*, A407, 1999.

[26] J. Malinen. Riccati Equations for H^∞ Discrete Time Systems: Part I. *Helsinki University of Technology Institute of Mathematics Research Reports*, A405, 1999.

[27] J. Malinen. Riccati Equations for H^∞ Discrete Time Systems: Part II. *Helsinki University of Technology Institute of Mathematics Research Reports*, A406, 1999.

[28] J. Malinen. Toeplitz Preconditioning of Toeplitz Matrices — an Operator Theoretic Approach. *Helsinki University of Technology Institute of Mathematics Research Reports*, A404, 1999.

[29] K. Mikkola. On the stable H^2 and H^∞ infinite-dimensional regulator problems and their algebraic Riccati equations. *Helsinki University of Technology, Institute of mathematics, Research Report A383*, 1997.

[30] G. J. Murphy. *C^*-algebras and operator theory.* Academic Press, 1990.

[31] O. Nevanlinna. On the growth of the resolvent operators for power bounded operators. *Linear operators, Banach center publications*, 1997.

[32] N. K. Nikolskii. *Treatise on the Shift Operator*, volume 273 of *Grundlehren der mathematischen Wissenschaften.* Springer Verlag, 1986.

[33] M. Rosenblum and J. Rovnyak. *Hardy classes and operator theory.* Oxford university press, New York, 1985.

[34] W. Rudin. *Real and Complex Analysis.* McGraw-Hill Book Company, New York, 3 edition, 1986.

[35] W. Rudin. *Functional Analysis.* McGraw-Hill Book Company, New York, TMH edition, 1990.

[36] O. J. Staffans. Quadratic optimal control of stable abstract linear systems. *To appear in Proceedings of the IFIP Conference Modelling and Optimization of Distributed Parameter Systems with Applications to Engineering, Warshaw, Poland*, July 17-21 1995.

[37] O. J. Staffans. Quadratic optimal control of stable systems through spectral factorization. *Mathematics of Control, Signals and Systems*, 8:167–197, 1995.

[38] O. J. Staffans. On the discrete and continuous time infinite dimensional algebraic Riccati equations. *Systems and Control Letters*, 29:131–138, 1996.

[39] O. J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Transactions of American Mathematical Society*, 349:3679–3715, 1997.

[40] O. J. Staffans. Coprime factorizations and well-posed linear systems. *SIAM Journal on Control and Optimization*, 36:1268–1292, 1998.

[41] O. J. Staffans. Quadratic optimal control of well-posed linear systems. *SIAM Journal of Control and Optimization*, 1998.

[42] O. J. Staffans. Feedback representations of critical controls for well-posed linear systems. *International Journal of Robust and Nonlinear Control*, 1999.

[43] O. J. Staffans. On the distributed stable full information H^∞ minimax problem. *International Journal of Robust and Nonlinear Control*, 1999.

[44] O. J. Staffans. Quadratic optimal control through coprime and spectral factorizations. *European Journal of Control*, 1999.

[45] O. J. Staffans. *Well-Posed Linear Systems.* In preparation, 1999.

[46] B. Sz.-Nagy and C. Foias. *Harmonic Analysis of Operators on Hilbert space.* North-Holland Publishing Company, Amsterdam, London, 1970.

[47] E. G. F. Thomas. Vector-valued integration with applications to the operator-values H^∞-space. *IMA Journal of mathematical control and information*, (14):109–136, 1997.

[48] G. Weiss. Weakly ℓ^p-stable linear operators are power stable. *International Journal of systems science*, 20(11):2323–2328, 1989.

[49] G. Weiss. Representations of shift-invariant operators on L^2 by H^∞ tranfer functions: An elementary proof, a generalization to L^p, and a counterexample for L^∞. *Mathematics of Control, Signals, and Systems*, 4:193–203, 1991.

[50] G. Weiss. Regular linear systems with feedback. *Mathematics of Control, Signals, and Systems*, 7:23–57, 1994.

[51] G. Weiss. Transfer functions of regular linear systems, Part I: Characterizations of regularity. *Transactions of American Mathematical Society*, 342(2):827–854, 1994.

[52] M. Weiss. *Riccati equation in Hilbert space: A Popov function approach.* PhD. Thesis. The University of Groningen, 1994.

[53] M. Weiss. Riccati equation theory for Pritchard-Salamon systems: a Popov function approach. *IMA Journal of Mathematics, Control and Information*, 1996.

[54] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems. *Mathematics of control, signals and systems*, pages 287–330, 1997.

[55] H. K. Wimmer. The set of positive semidefinite solutions of the algebraic Riccati equation of discrete-time optimal control. *IEEE Transactions on automatic control*, 41(5):660–671, 1996.

[56] K. Zhou, J. C. Doyle, and K. Glover. *Robust and optimal control.* Prentice Hall, 1996.

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