Well-Posed Discrete Time Linear Systems and Their Feedbacks

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Abstract

In this paper we give definitions and basic results concerning various feedback and stability properties of the linear systems built around the following system of autonomous difference equations

$$\begin{cases} x_{j+1} &= Ax_j + Bu_j, \\ y_j &= Cx_j + Du_j, \quad j = 0, \pm 1 \pm 2 \dots \end{cases}$$

Here $u_j \in U$, $x_j \in H$ and $y_j \in Y$ (all of these are possibly infinite dimensional Hilbert spaces), and A, B, C and D are bounded linear operators.

In the companion paper [6] we develop a Riccati equation theory for a minimax control problem defined in this setting. This requires the introduction of several fundamental tools, such as feedback and stability notions which are developed here. We remark that we use stability notions (such as I/O-stability) that are weaker than the ordinary power stability; so our results, given here and in [6] hold for a larger class of linear systems than the usual one.

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1 Introduction

In the monograph [2] (Halanay, Ionescu, 1994) the authors state in the beginning of the preface:

> *After a short period of investigation it became clear to us that a systematic and coherent treatment of various subjects which are specific for the time-variant discrete systems is needed.*

Later in the same preface the authors remark:

> *Thus we have often found ourselves in something of a dilemma: on the other hand many facts should be known an on the other hand it is nearly impossible to give an adequate reference to all.*

The time-variant theory (where the operators in equations (1) of Section 2 may depend on the index j) given in [2] can naturally be applied to time-invariant problems as well. However, the fundamental stability notion used in [2] is exponential (also known as power) stability (see Definition 21). We note that the power stability is a very strong stability notion. The infinite dimensional power stable Riccati equation theory, analogous to the time-invariant theory given in [5] and [6] is now fairly well known — in many respects it resembles the classical finite dimensional theory. However, several new phenomena and problems appear when such Riccati equation theories are developed under weaker stability notions than the power stability.

For the infinite dimensional systems, it is possible to define many weaker stability notions than power stability. We remark that there are interesting infinite dimensional strongly stable (see Definition 21) systems that are not power stable. In this sense, the introduction of such weaker conditions is not only possible, but necessary. There also exist finite dimensional cases, when such a weaker stability notion reduces to the power stability (see [3, Theorem 2], [4, Theorem 5.24]). For this reason, the power stability is a very natural stability condition for finite dimensional systems. It is more difficult to say which stability notion (if any single) is equally natural for infinite dimensional systems.

The time-invariance and the notion of I/O-stability makes this work intersect with the operator theory and operator valued function theory (see e.g. [7], [8], [14] for monographs). However, the control theoretically interesting part of this theory is hidden rather deep in these works; so deep, in fact, that they are too seldom referred to in control theoretic publications. We conclude that, when it comes to the non-power stable infinite dimensional linear systems, the above citations from [2] are still valid. For this reason, instead of writing a long and confusing introductory section to [6] or resorting to an endless jungle of references to various works with incompatible notations, we have decided to write this companion paper. Our general style is closely reminiscent to the recent work by O. Staffans ([10], [12], [13], [11]) for the corresponding continuous time systems, which we gratefully acknowledge.

After these brief remarks, let us summarize the contents and general organization of this paper. Our basic notion is a data structure, associated to equations (1) of Section 2, called a *discrete time linear system (DLS).* In fact, we have two such data structures; one we call a DLS *in difference equation form* (Section 2), the other is a DLS *in I/O-form* (Section 3). Roughly, when we write the DLS in difference equation form, we work in the time domain formalism. Writing the DLS in I/O-form corresponds to working in the frequency domain formalism. It does not come as a surprise (Theorem 11 of Section 3) that these two notions are equivalent. As an application, we obtain the equivalent state feedback structure given for DLS in difference form (Section 4) and I/O-form (Section 5). Note that until now no stability assumptions have been made, if one does not regard the boundedness of the operators in (1) as a stability assumption (see Lemma 8).

In Section 6, the ℓ^2 - topologies of the input and output sequence spaces become important. Several stability notions for DLS's are introduced, such as H^2-stability and I/O-stability. The stability notions of Definitions 28, 34 are stated and studied in terms of the DLS in I/O-form. Some operator theoretic structure (the closed graph, the dense domain and the boundedness) for certain important operators are studied.

In Section 7 the notion called the compatibility condition is introduced. For DLS's satisfying the compatibility condition, the domain of the observability map is non-trivial. For example, I/O-stable DLS's are of this type. In Section 8, a stronger topology for the state space is introduced, using the results of Section 6 . With this new topology, we transform the original (I/O-stable) DLS into a modified system with the same I/O-map and algebraic properties, but with the additional property that the modified system is output stable (see Definition 34). By studying the modified DLS instead of the original, we can apply the Riccati equation theory of [6] for systems whose critical feedback operator could be unbounded in the original state space topology (see [6, Definition 7 and Theorem 40]. We conclude this paper in Section 9 where the stability theory of Section 6 is connected to the feedback theory of Sections 4 and 5.

The introduction of two different but equivalent forms of DLS's would first seem superfluous — even more so because of the fact that the I/O-stable (H^∞) systems (see Definition 28) we can use the transfer function representation (see [8, Theorem 1.15B]). However, operator theoretic study of these systems (as has been done in Sections 6, 7 and 8 of this paper) would become notationally very clumsy, if the basic operators are always stated as multiplications by transfer functions. We remark that in [8] the basic objects are unilateral shift operators together with Toeplitz operators, and the complex analysis results are presented more or less as an important application. From the control theoretic point of view, the interaction between controllability, observability and I/O-maps can be conveniently described in our formalism because these operators are the basic building blocks of the DLS in I/O-form. Also the generalizations to non-linear theories can be done easily with this notation.

These tools (DLS's in both I/O- and difference equation form) find their application in the minimax Riccati equation theory given in [5], [6]. The central concept there are certain inner-outer factorizations of the I/O-map, or equivalently, of the transfer function of the DLS. Here we follow the lines of [10], [16]. These factorizations are most naturally studied in the I/O-form notation. On the other hand, the minimax feedback law and Riccati equation are stated in the difference equation form. Note that in practical applications, the minimax state feedback should be given "in a state space form", as a static linear operator K^{crit} from the state space H into the input space U.

We use the following notations throughout the paper: $\mathbf{Z}$ is the set of integers. $\mathbf{Z}_+ := \{j \in \mathbf{Z} \mid j \geq 0\}$. $\mathbf{Z}_- := \{j \in \mathbf{Z} \mid j < 0\}$. $\mathbf{T}$ is the unit circle and $\mathbf{D}$ is the open unit disk of the complex plane. If H is a Hilbert space, then $\mathcal{L}(H)$ denotes the bounded linear operators in H. Elements of a Hilbert space are denoted by lower case letters; for example $u \in U$. The sequences in Hilbert spaces are denoted by $\tilde{u} = \{u_i\}_{i \in I} \subset U$, where I is the index set. Usually $I = \mathbf{Z}$ or $I = \mathbf{Z}_+$. Given a Hilbert space Z, the we define the sequence spaces

$$
\begin{aligned}
Seq(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} : z_i \in Z \quad \text{and} \quad \exists I \in \mathbf{Z} \quad \forall i \leq I : z_i = 0\big\} \\
Seq_+(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} : z_i \in Z \quad \text{and} \qquad \forall i < 0 : z_i = 0\big\} \\
Seq_-(Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \in Seq(Z) : z_i \in Z \quad \text{and} \qquad \forall i \geq 0 : z_i = 0\big\} \\
\ell^p(\mathbf{Z}; Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z : \sum_{i\in\mathbf{Z}} ||z_i||_Z^p < \infty\big\} \quad \text{for} \quad 1 \leq p < \infty \\
\ell^p(\mathbf{Z}_+; Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}_+} \subset Z : \sum_{i\in\mathbf{Z}_+} ||z_i||_Z^p < \infty\big\} \quad \text{for} \quad 1 \leq p < \infty \\
\ell^\infty(\mathbf{Z}; Z) &:= \big\{\{z_i\}_{i\in\mathbf{Z}} \subset Z : \sup_{i\in\mathbf{Z}} ||z_i||_Z < \infty\big\}
\end{aligned}
$$

Other notations are introduced when they are needed.

2 Algebraic structure of autonomous difference equation systems

In this section we introduce the basic algebraic notions associated to the system of difference equations (1). The notions of observability, controllability and I/O-maps are introduced, and their basic properties are given.

Let U, H and Y be Hilbert spaces. Consider the following system of difference equations:

$$
(1) \qquad \begin{cases} x_{j+1} &= Ax_j + Bu_j, \\ y_j &= Cx_j + Du_j, \quad j \in \mathbf{Z}, \end{cases}
$$

where $u_j \in U$, $x_j \in H$, $y_j \in Y$, and A, B, C, D are bounded linear operators between appropriate spaces. The index $j \in \mathbf{Z}$ is regarded as a time parameter.

The following formal definition sets up a data structure associated to equations (1). This data structure is called a discrete time linear system:

Definition 1. *Let U,Y,H be Hilbert spaces. Let the operators $A \in \mathcal{L}(H, H)$, $B \in \mathcal{L}(U, H)$, $C \in \mathcal{L}(H, Y)$, $D \in \mathcal{L}(U, Y)$.*

(i) The ordered quadruple $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ of operators A, B, C and D is called a discrete time linear system (DLS) in difference equation form.

(ii) The space U is called the input space, Y the output space and H the state space.

(iii) The mapping A is called the semi-group generator of the DLS ϕ. The operator B is called the control operator, the operator C is called the observation operator and the operator D is called the feedthrough operator of the DLS ϕ.

In the language of [15], the operators A, B, C, D are called the generating operators. For each sequence $\tilde{u} := \{u_j\}_{j\in\mathbf{Z}} \in Seq(U)$ there exists a sequence $\tilde{y} := \{y_j\}_{j\in\mathbf{Z}} \in Seq(Y)$ given by the equations (1). It can be easily seen that this mapping $Seq(U) \ni \tilde{u} \mapsto \tilde{y} \in Seq(Y)$ is well defined and linear. We call this mapping the I/O-(input-output)-map of the DLS.

Definition 2. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS.*

(i) The I/O-map $\mathcal{D}_\phi$ is the unique linear map $Seq(U) \ni \tilde{u} \mapsto \tilde{y} \in Seq(Y)$ associated to ϕ via (1).

(ii) The sequence $\tilde{u} := \{u_j\}_{j\in\mathbf{Z}}$ is called the input sequence of the DLS ϕ. The sequence $\{x_j\}_{j\in\mathbf{Z}}$ is a sequence of states, and $\tilde{y} := \{y_j\}_{j\in\mathbf{Z}}$ is an output sequence of the ϕ if it satisfies (1) for some input sequence $\tilde{u} \in Seq(U)$.

It is easy to calculate the formula for $\mathcal{D}_\phi$:

Proposition 3. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and let $\tilde{u}$ $Seq(U)$. The I/O-map is given by*

$$(\mathcal{D}_\phi \tilde{u})_j = \sum_{i=0}^{\infty} CA^i B u_{j-i-1} + D u_j \tag{2}$$

for $j \in \mathbf{Z}$.

Note that the sum (2) is actually finite, because we work in $Seq(U)$.

In order to study the time dynamics of the DLS, we define some projections and a shift operator on the time axis in the following manner:

Definition 4. *Let Z be a Hilbert space. Let $\tilde{z} \in Seq(Z)$. Then we define the following linear operators in $Seq(Z)$:*

(i) the interval projections for $j, k \in \mathbf{Z}$

$$\pi_{[j,k]}\tilde{z} := \{w_j\}; \quad w_i = z_i \quad \textit{for} \quad j \le i \le k, \quad 0 \quad \textit{otherwise};$$
$$\pi_j := \pi_{[j,j]},$$

(ii) the future and past projections

$$\pi_+ := \pi_{[1,\infty]}, \quad \pi_- := \pi_{[-\infty,-1]},$$

(iii) the composite projections

$$\bar{\pi}_+ := \pi_0 + \pi_+, \quad \bar{\pi}_- := \pi_0 + \pi_-,$$

(iv) the bilateral forward time shift τ and its (formal) adjoint, backward time shift τ^*

$$\tau\tilde{u} := \{w_j\} \quad \text{where} \quad w_j = u_{j-1},$$
$$\tau^*\tilde{u} := \{w_j\} \quad \text{where} \quad w_j = u_{j+1}.$$

We call τ^* to be the "adjoint" of τ rather than inverse for notational simplicity. At this stage, we do not yet have a Hilbert space structure on the sequence spaces that would make τ^* a true adjoint. This structure will appear later in this paper. By using the operator τ, we may give a formula for the I/O-map

$$\mathcal{D}_\phi\tilde{u} = D\tilde{u} + \sum_{i\geq 0} CA^iB\tau^{i+1}\tilde{u}. \tag{3}$$

The above converges pointwise: for all $k \in Z$, $\tilde{u} \in Seq(U)$, we have only finitely many non-zero terms in the sum $\pi_k\left(\sum_{i\geq 0} CA^iB\tau^{i+1}\tilde{u}\right)$, by causality. This notion of convergence gives the vector spaces $Seq(U)$, $Seq(Y)$ a topology of componentwise (pointwise) convergence.

In some subspaces of $Seq(U)$, the shift τ can be realized as a multiplication by a complex variable z. This gives us the transfer function representation for the I/O-map; the operator valued analytic transfer function being given by

$$\hat{\mathcal{D}}(z) = D + \sum_{i\geq 0} CA^iBz^{i+1}\tilde{u} \quad \text{for} \quad z \in \mathbf{C}, \tag{4}$$

where the power series converges in a neighbourhood of the origin. For example, this is true for all z satisfying $|z| < ||A||^{-1}$.

In addition to the I/O-map, we also define two other linear mappings — the controllability and observability maps.

Definition 5. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS.*

(i) The controllability map $\mathcal{B}_\phi : Seq(U) \to H$ is the linear mapping defined by

$$\mathcal{B}_\phi\tilde{u} := \sum_{i\geq 0} A^iBu_{-i-1} \tag{5}$$

for all $\tilde{u} \in Seq(U)$.

(ii) The observability map $\mathcal{C}_\phi : H \to Seq(Y)$ is the linear mapping defined by

$$(\mathcal{C}_\phi x_0)_j := \begin{cases} CA^j x_0, & \text{for} \quad j \geq 0, \\ 0, & \text{for} \quad j < 0, \end{cases} \tag{6}$$

for all $x_0 \in H$.

As we shall see in a moment, the controllability map brings data into the DLS. The state space H serves as a "memory" of the system. Finally, the observability map "reads the memory" and outputs its contents.

It is not always the case that we want to study the DLS with initial condition $x_J = 0$ very far in the past. In the *initial value setting*, we want to start at some specific time point (usually chosen to be $j = 0$) with a given nonzero initial state $x_0 \in H$. In fact, most of this work, as well as [6] and [5], are written in the initial value setting. The control sequences $\tilde{u}$ as well as the output sequences $\tilde{y}$ would then lie in the spaces $Seq_+(U)$ and $Seq_+(Y)$, respectively. The following notation will be used in the initial value setting:

Definition 6. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Let $x_0 \in H$ be an initial state at time $j = 0$, and $\tilde{u} \in Seq_+(U)$ be an input sequence.*

(i) The state of ϕ at time $j \geq 0$ is denoted by $x_j(x_0, \tilde{u})$, and it is defined by

$$x_j(x_0, \tilde{u}) := A^j x_0 + \sum_{i=0}^{j-1} A^i B u_{j-i} = A^j x_0 + \mathcal{B}_\phi \tau^{*j} \tilde{u}, \tag{7}$$

where τ is the time shift defined in Definition 4.

(ii) The output sequence $\tilde{y}(x_0, \tilde{u}) := \{y_j(x_0, \tilde{u})\}_{j \in \mathbf{Z}_+}$ of ϕ is defined by

$$y_j(x_0, \tilde{u}) := CA^j x_0 + \sum_{i=0}^{j-1} CA^i B u_{j-i} + D u_j = \pi_j(\mathcal{C}_\phi x_0 + \mathcal{D}_\phi \tilde{u}). \tag{8}$$

It is true that the mappings $\mathcal{D}_\phi$, $\mathcal{B}_\phi$ and $\mathcal{C}_\phi$ share an important property with our (notion of the) universe, namely causality. The following proposition collects the results how the I/O-map, controllability map and observability map interact with the time projections and shifts.

Lemma 7. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then*

(i) $\mathcal{D}_\phi$, $\mathcal{B}_\phi$ and $\mathcal{C}_\phi$ are causal; i.e. they satisfy

$$\pi_- \mathcal{D}_\phi \bar{\pi}_+ = 0, \quad \mathcal{B}_\phi \bar{\pi}_+ = 0, \quad \pi_- \mathcal{C}_\phi = 0,$$

(ii) $\mathcal{B}_\phi$ *satisfies*

$$\begin{aligned}\mathcal{B}_\phi \tau^* \tilde{u} &= (A\,\mathcal{B}_\phi + \mathcal{B}_\phi \tau^* \pi_0)\,\tilde{u} \\ &= A\,\mathcal{B}_\phi \tilde{u} + B u_0, \\ \mathcal{B}_\phi \tau^{*j} \tilde{u} &= A^j\,\mathcal{B}_\phi \tilde{u} + \sum_{i=0}^{j-1} A^i B u_{j-i-1},\end{aligned}$$

for all $j \geq 1,\ \tilde{u} \in Seq(U)$*,*

(iii) $\mathcal{C}_\phi$ *satisfies*

$$\bar{\pi}_+ \tau^* \mathcal{C}_\phi = \mathcal{C}_\phi A,$$

(iv) $\mathcal{D}_\phi$ *satisfies*

$$\bar{\pi}_+ \mathcal{D}_\phi - \mathcal{D}_\phi \bar{\pi}_+ = \bar{\pi}_+ \mathcal{D}_\phi \pi_- = \mathcal{C}_\phi \mathcal{B}_\phi, \quad \mathcal{D}_\phi \tau = \tau \mathcal{D}_\phi, \quad \mathcal{D}_\phi \tau^* = \tau^* \mathcal{D}_\phi.$$

Proof. Claim (i) is a direct consequence of Definition 5. The first part of claim (ii) will be proved by calculating for any $\tilde{u} \in Seq(U)$

$$\mathcal{B}_\phi \tau^* \tilde{u} = \sum_{i\geq 0} A^i B u_{-i} = A \sum_{i\geq 0} A^i B u_{-i-1} + B u_0 = A\mathcal{B}_\phi \tilde{u} + B u_0.$$

Quite easily we note that $Bu_0 = \mathcal{B}_\phi \tau^* \bar{\pi}_+ \tilde{u}$. This gives the first part of claim (ii). The latter part of claim (ii) follows from the first part by induction. The claim (iii) is an immediate conclusion of Definition 5.

The first equality of claim (iv) is trivial. For the proof of the second equality we proceed as follows ($j \geq 0$)

$$\begin{aligned}(\mathcal{D}_\phi \pi_- \tilde{u})_j &= \sum_{i=0}^{\infty} C A^i B (\pi_- \tilde{u})_{j-i-1} + D(\pi_- \tilde{u})_j \\ &= \sum_{i\geq j}^{\infty} C A^i B u_{j-i-1} = \sum_{i\geq 0}^{\infty} C A^{i+j} B u_{j-(i+j)-1} \\ &= C A^j \sum_{i\geq 0}^{\infty} C A^i B u_{-i-1} = (\mathcal{C}_\phi \mathcal{B}_\phi \tilde{u})_j.\end{aligned}$$

This proves the former part of claim (iv). The remaining part in claim (iv) is clear. This completes the proof the lemma. □.

We can speak about abstract linear, causal and shift invariant operators satisfying the conditions of Lemma 7. The following realization lemma characterizes the set of I/O-maps for DLS's in this larger set of linear, causal and shift invariant operators in $Seq(U)$.

Lemma 8. *Let $\mathcal{D}$ be a linear, causal, shift invariant operator from $Seq(U) \to Seq(Y)$. Then the following are equivalent*

(i) The unique representation for $\mathcal{D}$ as the componentwise convergent series

$$\mathcal{D} = \sum_{i \geq 0} T_i \, \tau^i \tag{9}$$

satisfies the growth bound $||T_i|| < C\, r^i$ for $\infty > C, r > 0$, where $T_i \in \mathcal{L}(U; Y)$ for all $i \geq 0$.

(ii) $\mathcal{D}$ is a I/O-map of a DLS.

Proof. The direction (ii) $\Rightarrow$ (i) follows trivially from formula (3) and the fact that the operators A, B, C, D are bounded.

The implication (i) $\Rightarrow$ (ii) requires a construction of a DLS whose I/O-maps equals $\mathcal{D}$. Let us first show that a linear, causal, shift invariant operator $\mathcal{D} : Seq(U) \to Seq(Y)$ can be always written in form

$$\mathcal{D} = \sum_{i \geq 0} T_i \, \tau^i, \tag{10}$$

where $T_i \in \mathcal{L}(U; Y)$. For $\tilde{u} \in Seq(U)$ satisfying $\pi_0 \tilde{u} = \tilde{u}$ we have

$$\mathcal{D}\pi_0\tilde{u} = \sum_{i \geq 0} \pi_i \mathcal{D} \pi_0 \tilde{u} = \sum_{i \geq 0} \tau^i \, (\tau^{*i} \pi_i \mathcal{D} \pi_0) \tilde{u} = \sum_{i \geq 0} T_i \tau^i \pi_0 \tilde{u}$$

where $T_i : U \to Y$ is given by $T_i := \tau^{*i} \pi_i \mathcal{D} \pi_0$. Uniqueness of this representation for inputs of type $\pi_0 \tilde{u}$ is clear. The boundedness of T_i's follows from the growth bound $||T_i|| < C\, r^i$. Shift invariance and linearity of $\mathcal{D}$ makes it possible to extend this for all $\tilde{u} \in Seq(U)$, by writing for $\tilde{u} = \sum_{j > J} \pi_j \tilde{u}$ in a unique way. So we have formula (9), in the sense of componentwise convergence.

To complete the proof, we must find bounded operators A, B, C and D in a Hilbert spaces U, Y and H such that

$$D = T_0, \quad C A^{i-1} B = T_i \quad \text{for} \quad i \geq 1.$$

The choice of D is clear. The input and output spaces U, Y are fixed by the choice of $\mathcal{D}$, the state space H will have to be constructed. We first define

$$\begin{aligned} A &:= \bar{\pi}_+ \tau^* : Seq_+(Y) \to Seq_+(Y) \\ B &:= [T_1 \, T_2 \, T_3 \cdots] : U \to Seq_+(Y) \\ C &:= \pi_0 : Seq_+(Y) \to Y \end{aligned}$$

and the define the state space H to be a certain Hilbert subspace of $Seq_+(Y)$ such that the operators become continuous.

Using the growth bound $||T_i|| < C\,r^i$, pick $r \leq \infty$ so large that $\sum_{i\geq 0} ||r^{-i}\,T_i||^2 < \infty$. An inner product can be defined in a subset of $Seq_+(Y)$ by

$$\langle \tilde{y}, \tilde{w} \rangle_H := \sum_{i\geq 0} r^{-2i} \, \langle y_i, w_i \rangle_Y .$$

Now the state space $H \subset Seq_+(Y)$ is the closure of the finite length sequences in this inner product. For $u \in U$, we have

$$\begin{aligned} ||Bu||_H^2 &= \sum_{i\geq 0} r^{-2i} \, \langle T_i u, T_i u \rangle_Y \\ &= \sum_{i\geq 0} ||r^{-i} T_i u||_Y^2 \leq ||u||_U \sum_{i\geq 0} ||r^{-i}\,T_i||^2. \end{aligned}$$

This proves the boundedness of B. To show the boundedness of A we calculate

$$\begin{aligned} \frac{||A\tilde{y}||_H^2}{||\tilde{y}||_H^2} &= \frac{\sum_{i\geq 1} r^{-2(i-1)} \, \langle y_i, y_i \rangle_Y}{\sum_{i\geq 0} r^{-2i} \, \langle y_i, y_i \rangle_Y} \\ &= r^2 \, \frac{\sum_{i\geq 1} r^{-2i} \, \langle y_i, y_i \rangle_Y}{\sum_{i\geq 0} r^{-2i} \, \langle y_i, y_i \rangle_Y} \leq r^2. \end{aligned}$$

Thus $||A||_H < r$. The boundedness of $C = \pi_0$ is trivial. This completes the proof. $\square$.

The number $\inf \{ r > 0 \,|\, \exists\, 0 < C < \infty : \, ||T_i|| \leq C\, r^i \}$ equals the spectral radius $\rho(A)$ of the semi-group generator A. The proof of Lemma 8 implies that given $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, for any $r > \rho(A)$, we can find another DLS ϕ' such that $\mathcal{D}_\phi = \mathcal{D}_{\phi'}$, whose semi-group generator A' satisfies $||A'|| \leq r$.

3 DLS in I/O-form

Let us review what was done in Section 2. In the definition of DLS, we associated to four bounded operators A, B, C and D a data structure $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$. To this ϕ, we associated three linear operators $\mathcal{B}_\phi$, $\mathcal{C}_\phi$, $\mathcal{D}_\phi$ satisfying the properties described in Lemma 7. In this section, we forget the operators B, C, D for a while and work only with operators A, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ that satisfy the properties of A, $\mathcal{B}_\phi$, $\mathcal{C}_\phi$ and $\mathcal{D}_\phi$ given by Lemma 7. We can, in fact, characterize the DLS starting from a set of operators satisfying the claims of Lemma 7. This will be the main result of this section.

Definition 9. *Let U, Y and H be Hilbert. Let $A \in \mathcal{L}(H)$. Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be linear operators of the following kind:*

(i) $\mathcal{B} : Seq_-(U) \to H$, $\mathcal{C} : H \to Seq_+(Y)$ and $\mathcal{D} : Seq(U) \to Seq(Y)$.

(ii) $\mathcal{D}$, $\mathcal{B}$ and $\mathcal{C}$ are causal in the sense of Lemma 7

$$\pi_- \mathcal{D} \bar{\pi}_+ = 0, \quad \mathcal{B} \bar{\pi}_+ = 0, \quad \pi_- \mathcal{C} = 0.$$

(iii) $\mathcal{B}$ *satisfies*

$$\mathcal{B}\tau^* = A\,\mathcal{B} + \mathcal{B}\tau^*\pi_0,$$
$$\mathcal{B}\pi_{-1} \in \mathcal{L}(U, H),$$

where U *is identified with* range (π_{-1}) *on* $Seq(U)$ *in the natural way.*

(iv) $\mathcal{C}$ *satisfies*

$$\bar{\pi}_+\tau^*\mathcal{C} = \mathcal{C}A,$$
$$\pi_0\mathcal{C} \in \mathcal{L}(H, Y),$$

where Y *is identified with* range (π_0) *on* $Seq(Y)$ *in the natural way.*

(v) $\mathcal{D}$ *satisfies*

$$\bar{\pi}_+\mathcal{D}\pi_- = \mathcal{C}\mathcal{B},$$
$$\mathcal{D}\tau = \tau\mathcal{D}, \quad \mathcal{D}\tau^* = \tau^*\mathcal{D},$$
$$\pi_0\mathcal{D}\pi_0 \in \mathcal{L}(U, Y),$$

where U, Y *are identified with* range (π_0) *in the natural way.*

Then the ordered quadruple

$$\Phi = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} \tag{11}$$

is called a discrete time linear system (DLS) in I/O-form. The operator A *is called the semi-group generator of* Φ*, and the family of the operators* $\{A^j\}_{j\geq 0}$ *is called the semi-group of* Φ*. The operator* $\mathcal{B}$ *is called the controllability map, the operator* $\mathcal{C}$ *is called the observability map and the operator* $\mathcal{D}$ *is called the I/O-map of* Φ*.*

We remind that the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ in Definition 1 is called a DLS in *difference equation form*. Lemma 7 associates to each DLS ϕ in difference equation form a DLS Φ in I/O-form. The converse of Lemma 7 is the following:

Lemma 10. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS in I/O-form. Then there are unique linear operators* $B \in \mathcal{L}(U, H)$*,* $C \in \mathcal{L}(H, Y)$*,* $D \in \mathcal{L}(U, Y)$ *such that* $\mathcal{B} = \mathcal{B}_\phi$*,* $\mathcal{C} = \mathcal{C}_\phi$*,* $\mathcal{D} = \mathcal{D}_\phi$ *for* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$*. The semi-group generator* ϕ *is the same operator* A *as the semi-group generator of* Φ*. Furthermore, we have*

(i) $B := \mathcal{B}\pi_{-1}$ *where* range (π_{-1}) *and* U *are identified in the natural way,*

(ii) $C := \pi_0\mathcal{C}$ *where* range (π_0) *and* Y *are identified in the natural way,*

(iii) $D := \pi_0\mathcal{D}\pi_0$*, where the range of the right* π_0 *is identified with* U*, and the range of the left* π_0 *is identified with* Y *in the natural way.*

Proof. We begin the proof by looking at the operator $\mathcal{B}$. Define $B := \mathcal{B}\pi_{-1}$. Then by assumption $B \in \mathcal{L}(U,H)$, and $\mathcal{B}\tau^* \tilde{u} = A\mathcal{B}\tilde{u} + Bu_0$ for all $\tilde{u} \in Seq(U)$. By induction just as in the proof of the last part of claim (ii) of Lemma 7, for all $j \geq 1$

$$\mathcal{B}\tau^{*j}\tilde{u} = A^j\,\mathcal{B}\tilde{u} + \sum_{i=0}^{j-1} A^i B u_{j-i-1}, \quad \tilde{u} = \{u_j\} \in Seq(U). \tag{12}$$

On the other hand, for each $\tilde{u} \in Seq(U)$ we have the finite sum representation

$$\pi_- \tilde{u} = \sum_{j<0} \pi_j \tilde{u}. \tag{13}$$

The linearity of $\mathcal{B}$ and formula (13) imply

$$\mathcal{B}\tilde{u} = \mathcal{B}\pi_-\tilde{u} = \sum_{j<0} \mathcal{B}\pi_j\tilde{u} = \sum_{j<0} \mathcal{B}\tau^{*|j|}\pi_0\tau^{|j|}\tilde{u}. \tag{14}$$

By formula (12) for all $j < 0$

$$\begin{aligned} \mathcal{B}\pi_j\tilde{u} &= \mathcal{B}\tau^{*|j|}\left(\pi_0\tau^{|j|}\tilde{u}\right) \\ &= A^{|j|}\mathcal{B}(\pi_0\tau^{|j|}\tilde{u}) + \sum_{i=0}^{j-1} A^i B(\pi_0\tau^{|j|}\tilde{u})_{|j|-i-1} = A^{|j|-1}Bu_{-|j|} = A^{-j-1}Bu_j. \end{aligned} \tag{15}$$

Formulae (14) and (15) together give

$$\mathcal{B}\tilde{u} = \sum_{j<0} A^{-j-1}Bu_j = \sum_{j\geq 1} A^{j-1}Bu_{-j} = \sum_{j\geq 0} A^j B u_{-j-1}.$$

This proves that $\mathcal{B} = \mathcal{B}_\phi$ for the DLS's of type $\phi = \left(\begin{smallmatrix} A & B \\ * & * \end{smallmatrix}\right)$ (here $*$ stands for an irrelevant entry).

To make a similar analysis for $\mathcal{C}$, we first define $C := \pi_0\mathcal{C}$. By assumption, $C \in \mathcal{L}(H,Y)$. For $x_0 \in H$, $j \geq 0$ a direct calculation gives

$$(\mathcal{C}x_0)_j = (\tau^{*j}\mathcal{C}x_0)_0 = \pi_0\tau^{*j}\mathcal{C}x_0.$$

But by assumption (iv) and definition of C

$$\pi_0\tau^{*j}\mathcal{C}x_0 = \pi_0\mathcal{C}A^j x_0 = CA^j x_0.$$

Thus $(\mathcal{C}x_0)_j = CA^j x_0$, and $\mathcal{C} = \mathcal{C}_\phi$ for all DLS's $\phi = \left(\begin{smallmatrix} A & * \\ C & * \end{smallmatrix}\right)$.

Our final task is to construct an operator a $D \in \mathcal{L}(U,Y)$ such that $\mathcal{D} = \mathcal{D}_\phi$ for the DLS $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where B, C are constructed as above. To start with, define $D := \pi_0\mathcal{D}\pi_0$. The theory of Section 2 gives us a the I/O-map $\mathcal{D}_\phi$ satisfying $\bar{\pi}_+\mathcal{D}_\phi\pi_- = \mathcal{C}_\phi\mathcal{B}_\phi = \mathcal{C}\mathcal{B}$, because in particular $\mathcal{B} = \mathcal{B}_\phi$, $\mathcal{C} = \mathcal{C}_\phi$ for $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ by what we have already proved above. Then we have

$$\begin{aligned} \mathcal{D}_\phi - \mathcal{D} = {} & \bar{\pi}_+(\mathcal{D}_\phi - \mathcal{D})\bar{\pi}_+ + \bar{\pi}_+(\mathcal{D}_\phi - \mathcal{D})\pi_- \\ & + \pi_-(\mathcal{D}_\phi - \mathcal{D})\bar{\pi}_+ + \pi_-(\mathcal{D}_\phi - \mathcal{D})\pi_-. \end{aligned}$$

Now, $\bar{\pi}_+(\mathcal{D}_\phi - \mathcal{D})\pi_- = \mathcal{CB} - \mathcal{CB} = 0$ by the construction of ϕ and assumption for $\mathcal{D}$. Furthermore, $\pi_-(\mathcal{D}_\phi - \mathcal{D})\bar{\pi}_+ = 0$ because both $\mathcal{D}_\phi$, $\mathcal{D}$ are causal. Thus the difference $\mathcal{D}_\phi - \mathcal{D}$ is a static shift invariant operator, and must equal the multiplication by operator $\pi_0(\mathcal{D}_\phi - \mathcal{D})\pi_0$. This operator, however, vanishes, because $\pi_0\mathcal{D}_\phi\pi_0 - \pi_0\mathcal{D}\pi_0 = D - D = 0$, by the choice of D. This proves the last part of the lemma. □

We now summarize an immediate conclusion of Lemmas 7 and 10.

Theorem 11. *There is one-to-one correspondence between DLS in difference equation form and DLS in I/O-form. To get the DLS given in difference equation form into the I/O-form, the formulae of Lemma 7 are used. To get the DLS given I/O-form into difference equation form, the formulae of Lemma 10 are used.*

If the DLS's $\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $\Phi = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ are equivalent in the sense of Theorem 11, we write $\phi = \Phi$.

We conclude this section by giving an important application of Theorem 11. Under certain conditions the I/O-map $\mathcal{D}_\phi$ can be inverted so that the inverse $\mathcal{D}_\phi$ is an I/O-map of another DLS, denoted by ϕ^{-1}. This situation is characterized by the following proposition:

Proposition 12. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then $\mathcal{D}_\phi : Seq(Y) \to Seq(U)$ is bijective and $\mathcal{D}_\phi^{-1}$ is an I/O-map of a DLS if and only if $D^{-1} \in \mathcal{L}(Y, U)$. When the equivalence holds then $\mathcal{D}_\phi^{-1} = \mathcal{D}_{\phi^{-1}}$ where*

$$\phi^{-1} = \begin{pmatrix} A - BD^{-1}C & BD^{-1} \\ -D^{-1}C & D^{-1} \end{pmatrix}.$$

Proof. We start with proving the "if" -part. Let ϕ be as stated above. Assume $\tilde{y} \in Seq(Y)$, $\tilde{u} \in Seq(U)$ satisfy $\tilde{y} = \mathcal{D}_\phi \tilde{u}$, such that D^{-1} is bounded. Then

$$\begin{aligned}
&\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ y_j = Cx_j + Du_j, \end{cases} \quad \text{for all} \quad j \geq 0, \\
\Leftrightarrow \quad &\begin{cases} x_{j+1} = Ax_j + Bu_j, \\ u_j = -D^{-1}Cx_j + D^{-1}y_j, \end{cases} \quad \text{for all} \quad j \geq 0, \\
\Leftrightarrow \quad &\begin{cases} x_{j+1} = (A - BD^{-1}C)x_j + BD^{-1}u_j, \\ u_j = -D^{-1}Cx_j + D^{-1}y_j, \end{cases} \quad \text{for all} \quad j \geq 0, \\
\Leftrightarrow \quad &\tilde{u} = \mathcal{D}_{\phi^{-1}}\tilde{y}
\end{aligned}$$

where the initial value is $x_J = 0$ for so large J that both $\tilde{u}$, $\tilde{y}$ have no nonzero components with index less than J. This in fact gives also equation (12).

To prove the "only if" part, assume that $\mathcal{D}_\phi^{-1}$ is an I/O-map of the DLS ϕ'. Then, because $\mathcal{I} = \mathcal{D}_\phi^{-1}\mathcal{D}_\phi = \mathcal{D}_\phi\mathcal{D}_\phi^{-1}$, we have $\pi_0 = \pi_0\mathcal{D}_\phi^{-1}\mathcal{D}_\phi\pi_0 = \pi_0\mathcal{D}_\phi^{-1}\pi_0\,\pi_0\mathcal{D}_\phi\pi_0$, by causality of both $\mathcal{D}_\phi^{-1}$ and $\mathcal{D}_\phi$. Now, $\pi_0\mathcal{D}_\phi\pi_0 = D$, and $I = D'D$, where $D' = \pi_0\mathcal{D}_\phi^{-1}\pi_0$. Similarly, $I = DD'$. It follows that D is a bounded bijection between Hilbert spaces U, Y. It thus has a bounded inverse $D^{-1} = D'$. □.

4 State feedback and closed loop DLS's in difference equation form

The basic tool in control theory to change some characteristics of a given system is to apply (state) feedback. In this section we study the feedbacks in difference equation form. In Section 5 we carry out the similar work for DLS's in I/O form. It will finally appear that these two feedback notions are equivalent (see Lemma 20). This is essentially a conclusion of Theorem 11.

In this section we introduce the notion of the *(state)* feedback pair, originally introduced in [10]. It comprises a pair of such bounded linear operators that can be "coupled" into a given DLS. These operators will serve as an "extra output" for the system; hence it can directly be used as a feedback signal for the original system. Because one of the operators in the feedback pair is allowed to read the whole state space, the question is actually about the *state* feedback.

Definition 13. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS with input space U, output space Y and state space H.*

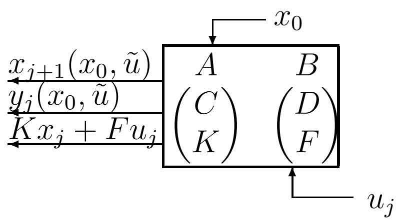
(i) The feedback pair (K, F) (in difference equation form) is an ordered pair of linear operators $K \in \mathcal{L}(H, U)$, $F \in \mathcal{L}(U, U)$ satisfying $(I - F)^{-1} \in \mathcal{L}(U, U)$.

(ii) Let (K, F) a feedback pair for ϕ as define above. Then

$$\phi^{ext} = (\phi, (K, F)) := \begin{pmatrix} A & B \\ \begin{bmatrix} C \\ K \end{bmatrix} & \begin{bmatrix} D \\ F \end{bmatrix} \end{pmatrix}$$

is called the extended DLS (in difference equation form) from ϕ with feedback pair (K, F). The input space of ϕ^{ext} is U, the output space is $Y \times U$ and the state space is H.

Following the language of [15], we can call the requirement $(I - F)^{-1} \in \mathcal{L}(U, U)$ admissibility of the feedback pair. By Proposition 12, this is quivalent with invertibility (in $Seq(U)$) of certain DLS. The following diagram illustrates the connections and signals for the extended system ϕ^{ext} with initial state $x_0 \in H$ at $j = 0$:



We remark that the extension of the DLS as defined in part (ii) of Definition (13) is nothing but the addition of an output to the original system ϕ. It is crucial that the output space we added is a copy of the input space U of the system. This makes it possible to couple the new output into the input . The feedback associated to pair (K, F) arises in a natural way by letting the input signal for ϕ be of form

$$u_j = v_j + (Kx_j + Fu_j), \quad \text{for} \quad j \geq 0, \tag{16}$$

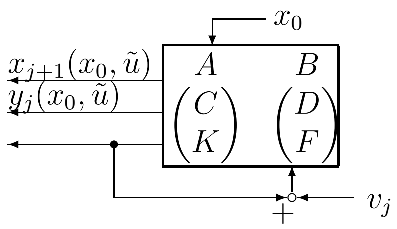
or equivalently

$$u_j = (I - F)^{-1}(Kx_j + v_j),$$

where $\tilde{v} := \{v_j\} \in Seq(U)$ is an external perturbation signal. This procedure is referred to as "closing the feedback loop". It is quite easy to show (in difference equation form) that closing the feedback loop given us another DLS that we may define formally as follows:

Definition 14. *Given an extended DLS ϕ^{ext} as defined above, the closed loop version $\phi^{ext}_{\diamond}$ is defined to be the DLS that is obtained from ϕ^{ext} by coupling to the input of ϕ to the signal $\tilde{u} = \{u_j\}$ given in equation (16). The perturbation signal $\tilde{v} = \{v_j\}$ is the input signal of the closed loop system.*

Definition 14 of the closed loop system $\phi^{ext}_{\diamond}$ i terms of the open loop DLS is diagrammatically as follows



As a result of a straightforward calculation, the formulae for closed loops systems are given below.

Lemma 15. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS and (K, F) a feedback pair for ϕ. The the closed loop extended system $\phi^{ext}_{\diamond}$ is given by*

$$\phi^{ext}_{\diamond} = (\phi, (K, F))_{\diamond} = \begin{pmatrix} A + B(I-F)^{-1}K & B(I-F)^{-1} \\ \begin{bmatrix} C \\ K \end{bmatrix} + \begin{bmatrix} D \\ F \end{bmatrix} (I-F)^{-1}K & \begin{bmatrix} D \\ F \end{bmatrix} (I-F)^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} A + B(I-F)^{-1}K & B(I-F)^{-1} \\ \begin{bmatrix} C + D(I-F)^{-1}K \\ (I-F)^{-1}K \end{bmatrix} & \begin{bmatrix} D(I-F)^{-1} \\ (I-F)^{-1} - I \end{bmatrix} \end{pmatrix}$$

The iterated feedbacks behave in an expected way. Given a DLS ϕ, we can define a product in the set of feedback pairs for ϕ by setting

$$(K_2, F_2)\,(K_1, F_1) := ((I - F_1)K_1 + K_2, F_1 + F_2 - F_2F_1)\,.$$

This gives the set of feedback pairs the structure of a non-commutative group, where the unit element is $(0,0)$. The iterated feedback is given by the formula

$$((\phi,(K_1,F_1)),(K_2,F_2)) = (\phi,(K_1,F_1)(K_2,F_2)).$$

The feedback pairs of form $(K,0)$ are an abelian subgroup of all the feedback pairs. In the litterature it is customary to use just these feedbacks, and set $F = 0$ in the closed loop formulae. In this paper we have chosen to have a nonvanishing F, because then the formulae of the closed loops systems in difference equation form will look like the corresponding formulae in the I/O-form, as introduced in Section 5. Also, to have a complete 1-to-1 correspondence between difference equation form feedbacks of this section, and the I/O-form feedbacks of the following section 5, we have to include $F \neq 0$, corresponding the feedthrough part $\pi_0 \mathcal{F} \pi_0$ of operator $\mathcal{F}$ in $[\mathcal{K},\mathcal{F}]$ in Definition 16. We remark that in [6], the critical feedback pairs are parameterized by the set of boundedly invertible operators in U; only one of these feedback pairs is of form $(K^{crit},0)$.

5 State feedback and closed loop DLS's in I/O-form

So far things have remained fairly simple, because we have worked with DLS's in difference equation form. Things get substantially more complicated when we study the analogous feedback structure with a DLS in I/O-form. We must make lengthy calculations to show that the closed loop "system" is indeed a DLS. As before, we start with a basic definition:

Definition 16. *Let* $\Phi = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS (in I/O-form) with input space* U*, output space* Y *and state space* H*. Then the feedback pair* $[\mathcal{K},\mathcal{F}]$ *(in I/O-form) for* Φ *is an ordered pair of linear operators* $\mathcal{K} : H \to Seq_+(U)$, $\mathcal{F} : Seq(U) \to Seq(U)$ *such that*

(i) $\Phi^{fb} = \left[\begin{smallmatrix} \mathcal{A}^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ *is a DLS (in I/O-form) with input space* U*, output space* U *and state space* H*.*

(ii) $(I - \mathcal{F})^{-1}$ *is an I/O-map of a DLS, mapping* $Seq(U) \to Seq(U)$*.*

The following proposition, together with Definition 13, will help us to understand the nature of condition (ii) of Definition 16. It is a direct consequence of Proposition 12.

Proposition 17. *Let* $\mathcal{F}$ *be an I/O-map of a DLS* Φ^{fb}*. Denote by* F *the bounded operator* $\pi_0 \mathcal{F} \pi_0$*, regarded as an operator in* U*, with the natural identification of spaces* range (π_0) *and* U*. Then* $(\mathcal{I} - \mathcal{F})^{-1}$ *is an I/O-map of a DLS if and only if* $I - F$ *has a bounded inverse in* U*. In that case,* $\pi_0 (\mathcal{I} - \mathcal{F})^{-1} \pi_0 = (I - F)^{-1}$*.*

Now that we have the feedback pairs, we have to associate a notion of feedback to them. We use the feedback pair $[\mathcal{K},\mathcal{F}]$ in roughly the same way as the feedback pair (K,F) in equation (16). The feedback is given by writing

$$\tilde{u} = \tilde{v} + (\mathcal{K}x_0 + \mathcal{F}\tilde{u}), \tag{17}$$

or equivalently

$$\tilde{u} = (I - \mathcal{F})^{-1}(\mathcal{K}x_0 + \tilde{v}),$$

where $\tilde{v} \in Seq(U)$ is an external perturbation signal. The introduction of the closed loop system is equivalent to Definition 14 with one exception: now it is not trivial to check that the system we obtain is in fact a DLS in I/O-form.

Definition 18. *Let* $\Phi = \begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}$ *be a DLS and* $[\mathcal{K}, \mathcal{F}]$ *a feedback pair of* Φ *as defined in Definition 16. Then*

(i) The DLS

$$\Phi^{ext} := \begin{pmatrix} A^j & \mathcal{B}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} \\ \mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D} \\ \mathcal{F} \end{bmatrix} \end{pmatrix}$$

is called the extended DLS (in I/O-form) from Φ *with feedback pair* $[\mathcal{K}, \mathcal{F}]$*. The input space of* Φ^{ext} *is* U*, the output space is* $Y \times U$ *and the state space is* H*. For brevity, we write* $\Phi^{ext} = [\Phi, [\mathcal{K}, \mathcal{F}]]$*.*

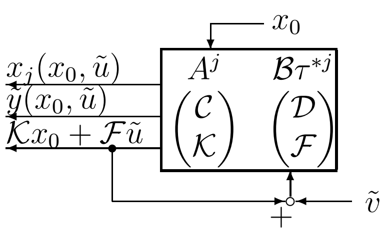
(ii) The closed loop extended system $\Phi_\diamond^{ext}$ *is the 6-tuple of operators*

$$\begin{aligned} \Phi_\diamond^{ext} &:= \begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(I - \mathcal{F})^{-1}\mathcal{K} & \mathcal{B}(I - \mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C} + \mathcal{D}(I - \mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I} - \mathcal{F})^{-1} \\ (\mathcal{I} - \mathcal{F})^{-1} - I \end{bmatrix} \end{bmatrix} \\ &=: \begin{bmatrix} A_\diamond(j) & \mathcal{B}_\diamond\tau^{*j} \\ \begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \end{bmatrix}. \end{aligned}$$

The input space of $\Phi_\diamond^{ext}$ *is* U*, the output space is* $Y \times U$ *and the state space is* H*. For brevity, we write* $\Phi_\diamond^{ext} = [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond$*.*

It is a matter of easy checking that $\Phi_\diamond^{ext}$ is well defined. This requires showing that the operator products appearing in $\Phi_\diamond^{ext}$ make sense for all j. However, note that at this stage we do not claim that $\Phi_\diamond^{ext}$ is a DLS, but just a 6-tuple of well defined linear operators between appropriate sequence spaces and the state space. The fact that $\Phi_\diamond^{ext}$ is a DLS will be proved in Lemma 19.

The following figure illustrates the feedback connection of the above defined closed loop extended system in terms of the open loop operators of Φ^{ext}.



Lemma 19. *The system $\Phi_\diamond^{ext}$ of Definition 18 is a DLS.*

Proof. First we note that it is sufficient to show that the system

$$\Phi_\diamond = \begin{bmatrix} A_\diamond(j) & \mathcal{B}_\diamond \tau^{*j} \\ \mathcal{C}_\diamond & \mathcal{D}_\diamond \end{bmatrix}$$

is a DLS. This is because the linear mappings A, $\mathcal{B}$, $\mathcal{K}$ and $\mathcal{F}$ form the DLS Φ^{fb} of Definition 16, and thus all the equations in Definition 9 are true with $\mathcal{C}$, $\mathcal{D}$ replaced by $\mathcal{K}$, $\mathcal{F}$, respectively. So it suffices to show that the operators $A_\diamond$, $\mathcal{B}_\diamond$, $C_\diamond$ and $\mathcal{D}_\diamond$ satisfy the conditions of Definition 9.

We start with establishing an important formula that will be used several times in the course of the proof. We have $\mathcal{K}\mathcal{B} = \bar{\pi}_+ \mathcal{F} \pi_-$ by property (v) of Definition 9 ($\mathcal{K}$ in place of $\mathcal{C}$). This and the causality of $\mathcal{F}$ implies the following identity

$$\begin{aligned} &(I-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}(I-\mathcal{F})^{-1} = (I-\mathcal{F})^{-1}\bar{\pi}_+\mathcal{F}\pi_-(I-\mathcal{F})^{-1} \\ &= \bar{\pi}_+(I-\mathcal{F})^{-1}\pi_-. \end{aligned} \tag{18}$$

Our proof begins with showing that $\{A^j + \mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K}\}_{j\geq 0}$ is a semi-group. We have

$$\begin{aligned} &(A + \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K})(A^j + \mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K}) \\ &= A^{j+1} + \overbrace{\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}A^j}^{(i)} \\ &+ \overbrace{A\mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}}^{(ii)} + \overbrace{\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K}}^{(iii)}. \end{aligned} \tag{19}$$

Now we study the enumerated parts of the previous equation separately. Term (i) satisfies

$$\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}A^j = \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}\bar{\pi}_+\tau^{*j}\mathcal{K}, \tag{20}$$

where property (iv) of Definition 9 ($\mathcal{K}$ in place of $\mathcal{C}$) and the causality of $(I-\mathcal{F})^{-1}$ has been used. Term (ii) satisfies

$$A\mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1} = \mathcal{B}\tau^{*(j+1)}(I-\mathcal{F})^{-1}\mathcal{K} - \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}\tau^{*j}\mathcal{K}, \tag{21}$$

where the property (iii) of Definition 9 has been used. The remaining term (iii) requires the most work. Now we have by the shift invariance of $(I-\mathcal{F})^{-1}$ and formula (18)

$$\begin{aligned} &\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K} = \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\bar{\pi}_+\mathcal{F}\pi_-(I-\mathcal{F})^{-1}\tau^{*j}\mathcal{K} \\ &= \mathcal{B}\tau^*\bar{\pi}_+(I-\mathcal{F})^{-1}\pi_-\tau^{*j}\mathcal{K} = \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}\pi_-\tau^{*j}\mathcal{K}, \end{aligned} \tag{22}$$

where the last equality follows immediately from the definition of $\mathcal{B}$. Now summing up formulae (20), (21) and (22) and combining that with formula (19), we obtain:

$$(A + \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K})(A^j + \mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K}) = (A^{j+1} + \mathcal{B}\tau^{*(j+1)}(I-\mathcal{F})^{-1}\mathcal{K}),$$

for all $j \geq 0$, or equivalently

$$(A^j + \mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K}) = \big(A + \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}\big)^j =: A_\diamond^j,$$

where $A_\diamond$ is the generator of the closed loop semi-group. This proves the claim about the semi-group.

In order to prove that $\mathcal{B}_\diamond := \mathcal{B}(I-\mathcal{F})^{-1}$ is a valid controllability map satisfying the conditions of Definition 9, we first check the causality of $\mathcal{B}_\diamond$. We have

$$\mathcal{B}_\diamond \bar{\pi}_+ = \mathcal{B}(I-\mathcal{F})^{-1}\bar{\pi}_+ = \mathcal{B}\bar{\pi}_+(I-\mathcal{F})^{-1}\bar{\pi}_+ = \mathcal{B}\pi_-\bar{\pi}_+(I-\mathcal{F})^{-1}\bar{\pi}_+ = 0,$$

where we have used the causality of $(I-\mathcal{F})^{-1}$ and $\mathcal{B}$. In order to see whether $\mathcal{B}_\diamond$ interacts correctly with the time shift τ^* and the semi-group generator $A_\diamond$, we have to show that $\mathcal{B}_\diamond\tau^* = A_\diamond\mathcal{B}_\diamond + \mathcal{B}_\diamond\tau^*\pi_0$. Let us proceed as follows:

$$\begin{aligned} \mathcal{B}_\diamond\tau^* &= \mathcal{B}(I-\mathcal{F})^{-1}\tau^* = \mathcal{B}\tau^*(I-\mathcal{F})^{-1} \\ &= A\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}. \end{aligned} \tag{23}$$

On the other hand, we have by the causality of $(I-\mathcal{F})^{-1}$ and equation (18)

$$\begin{aligned} &A_\diamond\mathcal{B}_\diamond + \mathcal{B}_\diamond\tau^*\pi_0 = (A + \mathcal{B}(I-\mathcal{F})^{-1}\tau^*\mathcal{K})(\mathcal{B}(I-\mathcal{F})^{-1}) + \mathcal{B}(I-\mathcal{F})^{-1}\tau^*\pi_0 \\ &= A\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{B}(I-\mathcal{F})^{-1}\tau^*\pi_0 \\ &= A\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{B}\tau^*\bar{\pi}_+(I-\mathcal{F})^{-1}\pi_- + \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\pi_0. \end{aligned}$$

The causality of $(I-\mathcal{F})^{-1}$ and the basic properties of $\mathcal{B}$ now allow us to continue

$$\begin{aligned} &= A\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}\pi_- + \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}\tau^*\pi_0 \\ &= A\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{B}\tau^*\pi_0(I-\mathcal{F})^{-1}. \end{aligned} \tag{24}$$

Now is it sufficient to compare the right sides of equations (23) and (24) to see that $\mathcal{B}_\diamond\tau^* = A_\diamond\mathcal{B}_\diamond + \mathcal{B}_\diamond\tau^*\pi_0$. This proves that $\mathcal{B}_\diamond$ is a valid controllability map for an DLS with semi-group generator $\mathcal{A}_\diamond$.

Next we check that the operator $\mathcal{C}_\diamond := \mathcal{C} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K}$ is a valid observability map satisfying the conditions of Definition 9. The causality of $\mathcal{C}_\diamond$ is quite clear. To establish $\mathcal{C}_\diamond A_\diamond = \bar{\pi}_+\tau^*\mathcal{C}_\diamond$, we calculate

$$\begin{aligned} \mathcal{C}_\diamond A_\diamond &= (\mathcal{C} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K})(A + \mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}) \\ &= \overbrace{\mathcal{C}A}^{(i)} + \overbrace{\mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K}A}^{(ii)} + \overbrace{\mathcal{C}\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}}^{(iii)} + \overbrace{\mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K}}^{(iv)}. \end{aligned} \tag{25}$$

The term (i) clearly equals $\bar{\pi}_+\tau^*\mathcal{C}$ by applying formula (iv) of Definition 9. Term (ii) can be seen to equal $\bar{\pi}_+\mathcal{D}(I-\mathcal{F})^{-1}\bar{\pi}_+\tau^*\mathcal{K}$ by applying condition (iv) of Definition 9, and noting that $\mathcal{D}$ is causal. In order to look at term (iii) we note that $\mathcal{C}\mathcal{B} = \bar{\pi}_+\mathcal{D}\pi_-$, by (v) of Definition 9. Then term (iii) takes form $\bar{\pi}_+\mathcal{D}\pi_-(I-\mathcal{F})^{-1}\tau^*\mathcal{K} = \bar{\pi}_+\mathcal{D}\pi_-(I-\mathcal{F})^{-1}\pi_-\tau^*\mathcal{K}$. The last term (iv) is again of the form of equation (18), and equals $\bar{\pi}_+\mathcal{D}\bar{\pi}_+(I-\mathcal{F})^{-1}\pi_-\tau^*\mathcal{K}$, where we have used the causality of $\mathcal{D}$, too.

Summing these formulae for all the terms (i) — (iv) of formula (25) gives the required identity $\mathcal{C}_\diamond A_\diamond = \bar{\pi}_+\tau^*\mathcal{C}_\diamond$.

So our final task is to check that the I/O-map candidate $\mathcal{D}_\diamond$ interacts correctly with the operators $A_\diamond$,$\mathcal{B}_\diamond$, $\mathcal{C}_\diamond$ and time shifts. Causality of $\mathcal{D}_\diamond$ is again no issue because its triviality, and so is the fact $\tau^*\mathcal{D}_\diamond = \mathcal{D}_\diamond\tau^*$. Our work lies in checking that $\bar{\pi}_+\mathcal{D}_\diamond\pi_- = \mathcal{C}_\diamond\mathcal{B}_\diamond$. The proof of this equality goes now in the familiar way by using equation (18) and causality of $\mathcal{D}$

$$\begin{aligned}\mathcal{C}_\diamond\mathcal{B}_\diamond &= (\mathcal{C} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K})\mathcal{B}(I-\mathcal{F})^{-1}\\ &= \mathcal{C}\mathcal{B}(I-\mathcal{F})^{-1} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K}\mathcal{B}(I-\mathcal{F})^{-1}\\ &= \bar{\pi}_+\mathcal{D}\pi_-(I-\mathcal{F})^{-1} + \mathcal{D}\bar{\pi}_+(I-\mathcal{F})^{-1}\pi_-\\ &= \bar{\pi}_+\mathcal{D}\pi_-(I-\mathcal{F})^{-1}\pi_- + \bar{\pi}_+\mathcal{D}\bar{\pi}_+(I-\mathcal{F})^{-1}\pi_-\\ &= \bar{\pi}_+\mathcal{D}(I-\mathcal{F})^{-1}\pi_- = \bar{\pi}_+\mathcal{D}_\diamond\pi_-.\end{aligned}$$

Now we have proved that the quadruple $\Phi_\diamond = \begin{bmatrix} A^j_\diamond & \mathcal{B}_\diamond\tau^{*j} \\ \mathcal{C}_\diamond & \mathcal{D}_\diamond \end{bmatrix}$ is a DLS. This completes the proof. □

A continuous time analogue to the previous theorem can be found in [15, Theorem 6.1].

We have found out that also the feedback in I/O-form gives a closed loop system, which still is a DLS $\Phi^{ext}_\diamond$, by Lemma 19. In Theorem 11 we stated that the DLS in difference form and I/O-form have 1-to-1 correspondence. Then the open loop system Φ has a representation ϕ in difference equation form, and so has the closed loop system $\Phi^{ext}_\diamond$ a representation ϕ', too. The final question is, whether ϕ' is equal to a closed loop system $(\phi, (K,F))_\diamond$ for some feedback pair (K,F)? And if so, then how how to relate the feedback pairs $[\mathcal{K},\mathcal{F}]$ and (K,F)? The following lemma answers these questions.

Lemma 20. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right] = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) = \phi$ *be the same DLS written in both I/O-form and difference equation form. Then the following are true:*

(i) There is 1-to-1 correspondence between the feedback pairs $[\mathcal{K},\mathcal{F}]$ *for* Φ*, and the feedback pairs* (K,F) *for* ϕ*. If* $[\mathcal{K},\mathcal{F}]$*,* (K,F) *correspond to each other in this sense, then*

$$\begin{bmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F}\end{bmatrix} = \begin{pmatrix} A & B \\ K & F\end{pmatrix}. \tag{26}$$

In this case, we write $[\mathcal{K},\mathcal{F}] = (K,F)$*.*

(ii) The feedback pairs satisfy $[\mathcal{K},\mathcal{F}] = (K,F)$ *if and only if the closed loop extended systems satisfy* $[\Phi,[\mathcal{K},\mathcal{F}]]_\diamond = (\phi,(K,F))_\diamond$*.*

Proof. The first claim (i) is a direct conclusion of Definition 16 and Theorem 11. To prove the claim (ii), we have to study when $[\Phi,[\mathcal{K},\mathcal{F}]]_\diamond = (\phi,(K,F))_\diamond$ or equivalently

$$\begin{aligned}&\begin{bmatrix} A^j + \mathcal{B}\tau^{*j}(I-\mathcal{F})^{-1}\mathcal{K} & \mathcal{B}(I-\mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix}\mathcal{C} + \mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}\end{bmatrix} & \begin{bmatrix}\mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ (\mathcal{I}-\mathcal{F})^{-1} - I\end{bmatrix}\end{bmatrix} \\ &= \begin{pmatrix} A + B(I-F)^{-1}K & B(I-F)^{-1} \\ \begin{bmatrix} C + D(I-F)^{-1}K \\ (I-F)^{-1}K\end{bmatrix} & \begin{bmatrix} D(I-F)^{-1} \\ (I-F)^{-1} - I\end{bmatrix}\end{pmatrix}\end{aligned} \tag{27}$$

under the assumption that $\Phi = \phi$.

Assume that $[\mathcal{K}, \mathcal{F}] = (K, F)$. We are to show that the equality hold in (27). We show only that the semi-group generators and observability maps in the right and left sides of formula (27) are equal; the other parts are left for the reader. For the semi-group generators we have

$$A + \mathcal{B}\tau^*(I - \mathcal{F})^{-1}\mathcal{K} = A + \mathcal{B}\pi_{-1}\tau^*\pi_0(I - \mathcal{F})^{-1}\bar{\pi}_+\mathcal{K} = A + \mathcal{B}\pi_{-1}\,\tau^*\,\pi_0(I - \mathcal{F})^{-1}\pi_0\,\mathcal{K},$$

where the latter equality if by the causality of $(I - \mathcal{F})^{-1}$ (see Definition 16). Now, by (26), $K = \pi_0\mathcal{K}$ and $\pi_0(I - \mathcal{F})^{-1}\pi_0 = (I - F)^{-1}$ where also Proposition 17 has been used. Also, because $\Phi = \phi$, $B = \mathcal{B}\pi_{-1}$. Thus

$$A + \mathcal{B}\tau^*(I - \mathcal{F})^{-1}\mathcal{K} = A + B(I - F)^{-1}K,$$

where natural identifications of appropriate intermediate spaces have been done.

To check that the observability maps in the right and left sides of formula (27) are equal, it suffices now to show that only the first components of them are equal. This is because we already proved the equality of the semi-group generators. So

$$\pi_0(\mathcal{C} + \mathcal{D}(I - \mathcal{F})^{-1}\mathcal{K}) = \pi_0\mathcal{C} + \pi_0\mathcal{D}\bar{\pi}_+(I - \mathcal{F})^{-1}\pi_+\mathcal{K} = \pi_0\mathcal{C} + \pi_0\mathcal{D}\pi_0(I - \mathcal{F})^{-1}\pi_0\mathcal{K},$$

where the second equality is by the causality of both $\mathcal{D}$ and $(I - \mathcal{F})^{-1}$. Now, $C = \pi_0\mathcal{C}$ and $D = \pi_0\mathcal{D}\pi_0$, because $\Phi = \phi$. Also $K = \pi_0\mathcal{K}$ and $\pi_0(I - \mathcal{F})^{-1}\pi_0 = (I - F)^{-1}$ as above, because $[\mathcal{K}, \mathcal{F}] = (K, F)$. It follows that

$$\pi_0(\mathcal{C} + \mathcal{D}(I - \mathcal{F})^{-1}\mathcal{K}) = C + D(I - F)^{-1}K,$$

again with natural identifications of appropriate spaces.

For the converse direction, assume that the equality holds in (27). Then in particular $\pi_0((\mathcal{I} - \mathcal{F})^{-1} - I)\pi_0 = (I - F)^{-1} - I)$ and $\pi_0(\mathcal{I} - \mathcal{F})^{-1}\pi_0 = (I - F)^{-1}$. Proposition 17 implies $\pi_0(\mathcal{I} - \mathcal{F})^{-1}\pi_0 = (I - \pi_0\mathcal{F}\pi_0)^{-1}$. But then $(I - \pi_0\mathcal{F}\pi_0)^{-1} = (I - F)^{-1}$ and thus $\pi_0\mathcal{F}\pi_0 = F$.

By using $\pi_0(\mathcal{I} - \mathcal{F})^{-1}\mathcal{K} = \pi_0(\mathcal{I} - \mathcal{F})^{-1}\pi_0\mathcal{K} = (I - \pi_0\mathcal{F}\pi_0)^{-1}\,\pi_0\mathcal{K} = (I - F)^{-1}K$, and the above proved identity $(I - \pi_0\mathcal{F}\pi_0)^{-1} = (I - F)^{-1}$ we have immediately $\pi_0\mathcal{K} = K$, because $(I - F)^{-1}$ has a bounded inverse, by Definition 13. Now we have $\pi_0\mathcal{F}\pi_0 = F$ and $\pi_0\mathcal{K} = K$, which implies (26), by Theorem 11. Thus $[\mathcal{K}, \mathcal{F}] = (K, F)$. □

We remark that if $[\mathcal{K}, \mathcal{F}] = (K, F)$, it is easy to find formulae connecting the operators $\mathcal{K}$, $\mathcal{F}$, K, F, by applying Theorem 11 to equation (26).

6 Stability notions of DLS's

In this section we give an inner product space structure to certain subspaces of the input and output sequence spaces $Seq(U)$, $Seq(Y)$ of a DLS. This makes it possible to "measure" things such as energies and costs of input and output sequences etc. which in turn requires topological rather than purely algebraic concepts. In particular, stability notions for DLS's are topological notions.

Two kinds of stability notions are considered here. In Definition 21 we start with the first kind of stability notions that only depend upon the structure of the semi-group generator A of Φ (see Definition 21). The second kind of stability notions depend upon the DLS in a more complicated manner (see Definition 34). We also study the conditions under which the topologized operators of an DLS are closed, densely defined and finally bounded.

Definition 21. *Let $A \in \mathcal{L}(H)$. Then*

(i) A is power (or exponentially) stable, if $\rho(A) < 1$,

(ii) A is strongly stable, if $A^j x_0 \to 0$ as $j \to \infty$,

(iii) A is power bounded, if $\sup_{j\geq 0} ||A^j||_H < \infty$.

The preceding semi-group stability notions as related to each other in the following way:

Proposition 22. *Let $A \in \mathcal{L}(H)$. Then, given the following enumeration of propositions:*

(i) $||A||_H < 1$,

(ii) A is power stable,

(iii) $||A^j x_0||_H < C(x_0)\delta_j$, where $C(x_0) < \infty$ in a set of second category in H, and $\sum_{j\geq 0} \delta_j < \infty$,

(iv) $||A^j x_0||_H < M\, ||x_0||_H \delta^j$ for a constant $M < \infty$ and $0 < \delta < 1$,

(v) A is strongly stable,

(vi) $\forall x_0 \in H : A^j x_0 \to \tilde{A} x_0 \in H$ for some operator $\tilde{A} \in \mathcal{L}(H)$,

(vii) A is power bounded,

(viii) $\rho(A) \leq 1$,

we have the following implications and equivalences:

$$(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Rightarrow (v) \Rightarrow (vi) \Rightarrow (vii) \Rightarrow (viii)$$

Proof. The first implication (i) $\Rightarrow$ (ii) is the inequality $\rho(A) \leq ||A||$. The equivalence (ii) $\Leftrightarrow$ (iv) is trivial, because

$$\rho(A) = \limsup_{j\to\infty} ||A^j||_{H\to H}^{\frac{1}{j}} \tag{28}$$

The implication (iii) $\Leftarrow$ (iv) is trivial. The other direction goes as follows. Define the bounded linear operators $T_k(z) := \sum_{j=0}^{k} (zA)^j$ on H for any $|z| \leq 1$. Then we have for each x_0 for which $C(x_0) < \infty$, $m \leq l$:

$$||(T_m(z) - T_l(z))x_0||_H \leq \sum_{j=m}^{l} ||A^j x_0||_H \leq C(x_0) \sum_{j=l}^{m} \delta_j$$

Because $\{\delta_j\}$ is absolutely summable, $\{T_j(z)x_0\}$ is Cauchy for all x_0 belonging to a set of second category. [9, Theorem 2.7(b)] implies now that the pointwise limit operator $T(z)x_0 := \lim_{j\to\infty} T_j(z)x_0$ is bounded. It is easy to check that $T(z)(I - zA) = (I - zA)T(z) = I$ and thus $\frac{1}{z} \notin \sigma(A)$. Because $|z| \leq 1$ was arbitrary, we have $\sigma(A) \subset \mathbf{D}$ and $\rho(A) < 1$. This completes the proof of the equivalence part of this proposition.

The implications (iv) $\Rightarrow$ (v) $\Rightarrow$ (vi) are trivial. The implication (vi) $\Rightarrow$ (vii) is an immediate consequence of Banach-Steinhaus Theorem [9, Theorem 2.5], and the last implication (vii) $\Rightarrow$ (viii) follows again from formula (28). This completes the proof of the proposition. □

The stability of the semi-group generator is reflected to the I/O-properties of ϕ in the following manner.

Proposition 23. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS.*

(i) If A is power stable, then $\mathcal{D}$ is bounded from $\ell^p(\mathbf{Z};U) \cap Seq(U)$ into $\ell^p(\mathbf{Z};Y) \cap Seq(Y)$ for all $1 \leq p < \infty$.

(ii) If A is power bounded, then $\mathcal{D}$ is bounded from $\ell^1(\mathbf{Z};U) \cap Seq(U)$ into $\ell^\infty(\mathbf{Z};Y)$.

Proof. We prove first claim (i). We use the characterization (iv) of Proposition 22 for the power stability. For $\tilde{u} \in \ell^p(\mathbf{Z};U) \cap Seq(U)$, $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ we write

$$\begin{aligned} ||\mathcal{D}\tilde{u}||^p_{\ell^p(\mathbf{Z};Y)} &= \sum_j || \sum_{i \geq 0} CA^i B u_{j-i-1} ||^p_Y \\ &\leq ||C||^p_{H \to Y} \sum_j \big(\sum_{i \geq 0} ||A^i B u_{j-i-1}||_H\big)^p \\ &\leq ||C||^p_{H \to Y} M^p \sum_j \big(\sum_{i \geq 0} \delta^i \, ||B u_{j-i-1}||_H\big)^p \end{aligned}$$

$$
\begin{aligned}
&\le ||B||^p_{U\to H}||C||^p_{H\to Y}M^p \sum_j (\sum_{i\ge 0} \delta^{\frac{i}{q}}\, \delta^{\frac{i}{p}} ||u_{j-i-1}||_U)^p \\
&\le ||B||^p_{U\to H}||C||^p_{H\to Y}M^p \sum_j ([\sum_{i\ge 0} \delta^{\frac{i}{q}q}]^{\frac{1}{q}}\, [\sum_{i\ge 0} \delta^{\frac{i}{p}p} ||u_{j-i-1}||^p_U]^{\frac{1}{p}})^p \\
&\le ||B||^p_{U\to H}||C||^p_{H\to Y}M^p \left(\frac{1}{1-\delta}\right)^{\frac{p}{q}} \sum_j \sum_{i\ge 0} \delta^i ||u_{j-i-1}||^p_U \\
&\le ||B||^p_{U\to H}||C||^p_{H\to Y}M^p \left(\frac{1}{1-\delta}\right)^{\frac{p}{q}} \sum_{i\ge 0} \left(\delta^i \sum_j ||u_{j-i-1}||^p_U\right) \\
&= ||B||^p_{U\to H}||C||^p_{H\to Y}M^p \left(\frac{1}{1-\delta}\right)^{\frac{p}{q}} \sum_{i\ge 0} \left(\delta^i ||\tilde{u}||^p_{\ell^p(\mathbf{Z};U)}\right) \\
&= ||B||^p_{U\to H}||C||^p_{H\to Y}M^p \left(\frac{1}{1-\delta}\right)^{p} ||\tilde{u}||^p_{\ell^p(\mathbf{Z};U)},
\end{aligned}
$$

where we have used the Hölder inequality and the Theorem of Fubini. The case $p = 1$ is even easier. This completes the proof of claim (i) (see also [4, Theorem 4.33]).

To prove claim (ii), we calculate by Proposition 3 for any $\tilde{u} = \{u_j\} \in \ell^1(\mathbf{Z}; U) \cap Seq(U)$:

$$
\begin{aligned}
&||(\mathcal{D}_\phi \tilde{u})_j||_Y \le ||D||_{U\to Y}\, ||u_j||_U + ||C||_{H\to Y}\, ||B||_{U\to H} \sup_{j\ge 0} ||A^j||_H \sum_{j\ge 1} ||u_{j-i-1}||_U \\
&\le (||D||_{U\to Y} + ||C||_{H\to Y}\, ||B||_{U\to H} \sup_{j\ge 0} ||A^j||_H)\, ||\tilde{u}||_{\ell^1(\mathbf{Z};U)}.
\end{aligned}
$$

Taking supremum over $j \in \mathbf{Z}$ completes the proof. □

Definition 5 of the controllability and observability maps is purely algebraic without explicit topological connection; the Hilbert spaces U, H and Y were used only as vector spaces, and the only topology we used in spaces $Seq(U)$, $Seq(Y)$ was the very weak topology of componentwise convergence. What we need, is a stronger inner product topology for the input and output sequences.

In this section, we take the inputs and outputs for the DLS from the inner product spaces $\ell^2(\mathbf{Z}; U) \cap Seq(U)$ $(\ell^2(\mathbf{Z}; Y) \cap Seq(Y))$. The projections π_+, π_-, π_0, $\bar{\pi}_+$, $\bar{\pi}_-$, $\pi_{[j,k]}$ and the shift τ are restricted to these spaces. Then all these projections become orthogonal projections and their operator norms are 1. The bilateral shift τ becomes a unitary operation, thus of norm 1. The following definition gives us the restricted sets that will be domains of linear operators:

Definition 24. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be a DLS. The we define the following sets:*

$$\mathrm{dom}\,(\underline{\mathcal{B}}) := Seq_-(U), \tag{29}$$

$$\mathrm{dom}\,(\underline{\mathcal{C}}) := \{x_0 \in H \,|\, \mathcal{C}x_0 \in \ell^2(\mathbf{Z}_+; Y)\}, \tag{30}$$

$$\mathrm{dom}\,(\underline{\mathcal{D}}) := \{\tilde{u} \in \ell^2(\mathbf{Z}; U) \cap Seq(U) \,|\, \mathcal{D}\tilde{u} \in \ell^2(\mathbf{Z}; Y) \cap Seq(U)\}, \tag{31}$$

(32) $$\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+}) := \{\tilde{u} \in \ell^2(\mathbf{Z}_+;U) \,|\, \mathcal{D}\bar{\pi}_+\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)\},$$

(33) $$\mathrm{dom}\,(\underline{\mathcal{D}\pi_0}) := \{\tilde{u} \in \mathrm{range}\,(\pi_0) \,|\, \mathcal{D}\pi_0\tilde{u} \in \ell^2(\mathbf{Z}_+;Y)\},$$

(34) $$\mathrm{dom}\,(\underline{\mathcal{D}\pi_j}) := \tau^j \mathrm{dom}\,(\underline{\mathcal{D}\pi_0}) \quad \textit{for all} \quad j \in \mathbf{Z} \setminus \{0\},$$

where $\ell^2(\mathbf{Z}_+;Y)$ *and* $\bar{\pi}_+\ell^2(\mathbf{Z};Y)$ *are identified in (32).*

It easy to see that the sets of Definition 24 are vector spaces. We throughout use the ℓ^2-topology on $\mathrm{dom}\,(\underline{\mathcal{B}})$, $\mathrm{dom}\,(\underline{\mathcal{D}})$ and $\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})$. The set $\mathrm{dom}\,(\underline{\mathcal{D}\pi_j})$ has the topology of U. This gives all these spaces an inner product space structure. In $\mathrm{dom}\,(\underline{\mathcal{C}})$ we use use the topology of H, but later we introduce a stronger (inner product) topology there.

Moreover, the vector space $\mathrm{dom}\,(\underline{\mathcal{B}})$ is dense in $\ell^2(\mathbf{Z}_-;U)$. The other spaces $\mathrm{dom}\,(\underline{\mathcal{C}})$, $\mathrm{dom}\,(\underline{\mathcal{D}})$, $\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})$, $\mathrm{dom}\,(\underline{\mathcal{D}\pi_j})$ need not be dense, and for systems unbounded enough they even might be empty. One should also notice that the kernels of $\underline{\mathcal{C}}$ and $\underline{\mathcal{D}}$ are always in their respective domains. If there is nothing else, then we say that the domains in question are trivial. The following definition is to be expected.

Definition 25. *Let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be the algebraic controllability, observability and I/O-map as defined in Definition 5. Then the corresponding topological maps are defined by*

$$\begin{aligned}
\underline{\mathcal{B}} &:= \mathcal{B}|_{\mathrm{dom}(\underline{\mathcal{B}})}, \\
\underline{\mathcal{C}} &:= \mathcal{C}|_{\mathrm{dom}(\underline{\mathcal{C}})}, \\
\underline{\mathcal{D}} &:= \mathcal{D}|_{\mathrm{dom}(\underline{\mathcal{D}})}, \\
\underline{\mathcal{D}\bar{\pi}_+} &:= \mathcal{D}\bar{\pi}_+|_{\mathrm{dom}(\underline{\mathcal{D}\bar{\pi}_+})}, \\
\underline{\mathcal{D}\pi_j} &:= \mathcal{D}\pi_j|_{\mathrm{dom}(\underline{\mathcal{D}\pi_j})} \quad \textit{for all} \quad j \in \mathbf{Z}.
\end{aligned}$$

$\underline{\mathcal{B}}$ is the topological controllability map, $\underline{\mathcal{C}}$ is the topological observability map and $\underline{\mathcal{D}}$ is the topological I/O-map of Φ. $\underline{\mathcal{D}\bar{\pi}_+}$ is the causal Toeplitz operator of $\underline{\mathcal{D}}$. The operators $\underline{\mathcal{D}\bar{\pi}_j}$ are the impulse response operator of Φ. The sets $\mathrm{dom}\,(\underline{\mathcal{B}})$, $\mathrm{dom}\,(\underline{\mathcal{C}})$, $\mathrm{dom}\,(\underline{\mathcal{D}})$, $\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})$ *and* $\mathrm{dom}\,(\underline{\mathcal{D}\pi_j})$ *are called the domains of the respective operators.*

The basic relations between the sets $\mathrm{dom}\,(\underline{\mathcal{D}})$, $\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})$ and $\mathrm{dom}\,(\underline{\mathcal{D}\pi_0})$ are the following:

Proposition 26. *Let $\mathcal{D}$ be an I/O-map of the DLS Φ. Then*

(i) $\mathrm{dom}\,(\underline{\mathcal{D}}) = \cup_{j\geq 0}\tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})$,

(ii) $\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+}) \subset \bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}})$,

(iii) $\mathrm{dom}\,(\underline{\mathcal{D}\pi_j}) \subset \pi_j\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})$ *and* $\mathrm{dom}\,(\underline{\mathcal{D}\pi_j} \subset \mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+}))$ *for all* $j \in \mathbf{Z}_+$,

(iv) $\overline{\mathrm{dom}\,(\underline{\mathcal{D}\pi_0})} = U \quad \Rightarrow \quad \overline{\mathrm{dom}\,(\underline{\mathcal{D}\bar{\pi}_+})} = \ell^2(\mathbf{Z}_+;U) \quad \Rightarrow \quad \overline{\mathrm{dom}\,(\underline{\mathcal{D}})} = \ell^2(\mathbf{Z};U)$.

Proof. To prove claim (i), let $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}})$. Then $\tau^j \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ for some $j \in \mathbf{Z}$, and $\mathcal{D}\tilde{u} \in \ell^2(\mathbf{Z}_+; Y)$. Thus $\mathcal{D}\tau^j\tilde{u} \in \ell^2(\mathbf{Z}_+; Y)$ by causality and shift invariance. This implies that $\tau^j\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and $\tilde{u} \in \tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \cup_{j\geq 0}\tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$.

Conversely, if $\tilde{u} \in \cup_{j\geq 0}\tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$, then $\tilde{u} = \tau^{*j}\tilde{w}$ for some $j \in \mathbf{Z}$ and $\tilde{w} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$. Thus $\tau^j\tilde{u} = \tilde{w} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and so $\mathcal{D}\tau^j\tilde{u} = \tau^j\mathcal{D}\tilde{u} \in \ell^2(\mathbf{Z}_+; Y)$. It follows that $\mathcal{D}\tilde{u} \in \ell^2(\mathbf{Z}; Y)$, whence $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}})$ because $u_{-j-k} = 0$ for all $k \geq 0$.

To prove claim (ii), we use claim (i) and calculate

$$\begin{aligned}
&\bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}}) = \bar{\pi}_+ \cup_{j\geq 0} \tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \\
&= \big(\cup_{j\geq 1}\bar{\pi}_+\tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\big) \cup \bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \big(\cup_{j\geq 1}\bar{\pi}_+\tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\big) \cup \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\,.
\end{aligned}$$

But then (ii) immediately follows.

The proof of claim (iii) is trivial. The first part of claim (iv) is proved as follows: Let $\tilde{u} \in \ell^2(\mathbf{Z}_+; U)$, $\epsilon > 0$ be arbitrary. Take $j \in \mathbf{Z}_+$ so large that

$$||\pi_{[0,j]}\tilde{u} - \tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \leq \epsilon/2. \tag{35}$$

For each i satisfying $0 \leq i \leq j$, take $\tilde{u}_{i,j} \in \mathrm{dom}\,(\underline{\mathcal{D}}\pi_0) \subset U$ such that

$$||\tau^i\tilde{u}_{i,j} - \pi_i\tilde{u}||_U \leq \epsilon/(2\sqrt{j+1}). \tag{36}$$

This is possible because $\overline{\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)} = U$ is assumed, and range (π_j) is identified with U. Then define

$$\tilde{u}_j = \sum_{i=0}^{j} \tau^i\tilde{u}_{i,j} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\,,$$

where the inclusion follows because we have a finite sum of shifted $\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)$ -terms, each in $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$, by claim (iii) of this proposition. Now estimate

$$||\tilde{u}_j - \tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \leq ||\tilde{u}_j - \pi_{[0,j]}\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} + ||\pi_{[0,j]}\tilde{u} - \tilde{u}||_{\ell^2(\mathbf{Z}_+;U)}.$$

The latter term on the right is under $\epsilon/2$, by equation (35). The first term is estimated above by

$$||\tilde{u}_j - \tilde{u}||^2_{\ell^2(\mathbf{Z}_+;U)} = \sum_{i=0}^{j} ||\tau^i\tilde{u}_{i,j} - \pi_i\tilde{u}_j||^2_U \leq (j+1)\,\epsilon^2/(4(j+1)) = (\epsilon/2)^2,$$

by equation (36). So $||\tilde{u}_j - \tilde{u}||_{\ell^2(\mathbf{Z}_+;U)} \leq \epsilon$, where $\tilde{u}_j \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$. The claim follows because ϵ was arbitrary. The second part of claim (iv) is proved analogously. □

The bad news is that, given Φ, various topological pathologies can occur. Most of this section handles the cases when we have good news. We start with showing that three of the operators in Definition 25 are closed. In particular the closed graph property of $\underline{\mathcal{C}}$ will be used in Section 8.

Lemma 27. *The operators $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}\bar{\pi}_+$ and $\underline{\mathcal{D}}\pi_j$, $j \in \mathbf{Z}$ are closed.*

Proof. We prove only the claim for the Toeplitz operator $\underline{\mathcal{D}}\bar{\pi}_+$. The proofs of the other two claims are analogous.

Let $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \ni \tilde{u}_j \to \tilde{u} \in \ell^2(\mathbf{Z}_+; U)$ be a convergent sequence in H such that

$$\mathcal{D}\bar{\pi}_+\tilde{u}_j \to \tilde{y} \in \ell^2(\mathbf{Z}_+; Y)$$

in the norm of $\ell^2(\mathbf{Z}_+; Y)$. We show that $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and then $\mathcal{D}\bar{\pi}_+\tilde{u} = \underline{\mathcal{D}}\bar{\pi}_+\tilde{u} = \tilde{y}$, which proves the closed graph property for $\underline{\mathcal{D}}\bar{\pi}_+$.

For each $k \geq 0$ we have

$$\pi_k\mathcal{D}\bar{\pi}_+\tilde{u}_j \to \pi_k\tilde{y} \quad \text{as} \quad j \to \infty \tag{37}$$

in the norm of Y (with $\mathrm{range}\,(\pi_k)$ and Y identified), because $\mathcal{D}\bar{\pi}_+\tilde{u}_j \to \tilde{y}$, by assumption. On the other hand,

$$\pi_k\mathcal{D}\bar{\pi}_+\tilde{u}_j = \pi_k\mathcal{D}\pi_{[0,k]}\tilde{u}_j \to \pi_k\mathcal{D}\bar{\pi}_+\tilde{u} \tag{38}$$

in the norm of Y, because $\pi_k\mathcal{D}\pi_{[0,k]}$ is a bounded operator on $\ell^2(\mathbf{Z}_+; U)$ as a finite sum of bounded operators; see the discussion following formula (3).

Now equations (37) and (38) imply, by the uniqueness of the limit in Y, that $\pi_k\mathcal{D}\bar{\pi}_+\tilde{u} = \pi_k\tilde{y}$ for all $k \geq 0$, or equivalently $\mathcal{D}\bar{\pi}_+\tilde{u} = \tilde{y}$ for the algebraic I/O-map. But then, because $\tilde{y} \in \ell^2(\mathbf{Z}_+; Y)$, we have $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}})$ and $\tilde{y} = \underline{\mathcal{D}}\bar{\pi}_+\tilde{u}$. This completes the proof of the lemma. □

Continuity properties of the operators $\underline{\mathcal{B}}$, $\underline{\mathcal{C}}$, $\underline{\mathcal{D}}$, $\underline{\mathcal{D}}\bar{\pi}_+$, $\underline{\mathcal{D}}\pi_0$ are stability properties of the DLS. The following definition gives stability notions associated to the I/O-map $\underline{\mathcal{D}}$.

Definition 28. *The* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS.*

- *(i)* Φ *is I/O-stable if* $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \ell^2(\mathbf{Z}_+; U)$.
- *(ii)* Φ *is (strongly)* H^2 *(Hardy -2) -stable if* $\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0) = U$.

Quite trivially (i) implies (ii). If (ii) holds, we also say that Φ has a (strong) L^2 impulse response.

An I/O -stable $\underline{\mathcal{D}}\bar{\pi}_+$ is a bounded linear operator from $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \ell^2(\mathbf{Z}; U)$ into $\ell^2(\mathbf{Z}; Y)$, because a closed operator with complete (closed) domain is bounded, by the Closed Graph Theorem (see [9, Theorem 2.15]). The operator norm of $\underline{\mathcal{D}}\bar{\pi}_+$ given by

$$||\underline{\mathcal{D}}\bar{\pi}_+||_{\mathrm{dom}(\underline{\mathcal{D}}\bar{\pi}_+)\to\ell^2(\mathbf{Z}_+;Y)} := \sup_{\tilde{u}\in\mathrm{dom}(\underline{\mathcal{D}}\bar{\pi}_+),\, ||\tilde{u}||_{\ell^2(\mathbf{Z}_+;U)}=1} ||\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)}$$

can be called the energy gain of Φ. The transfer function (see equation (4)) of a DLS Φ is in the Hardy space $H^\infty(U; Y)$ if and only if Φ is I/O-stable. These are objects around which the linear H^∞ control theory is built.

The analogous treatment can be given for H^2-stable DLS; now all the one step inputs give a finite energy output. In other word, an H^2-stable DLS has a bounded impulse response operator. The basic properties of H^2 -stable DLS's are given in the following lemma.

Lemma 29. *Let Φ be a DLS. Enumerate the statements as follows:*

(i) $\sum_{j\geq 0} ||CA^jB||^2 < \infty$,

(ii) $\sum_{j\geq 0} ||CA^jBu_0||^2 < \infty$ *for all* $u_0 \in U$, *or equivalently,* Φ *is (strongly)* H^2 *-stable,*

(iii) $\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0) = U$ *and* $||\underline{\mathcal{D}}\pi_0||_{U\to\ell^2\mathbf{Z}_+;Y} < \infty$,

(iv) $\ell^1(\mathbf{Z}_+;U) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ *and* $\underline{\mathcal{D}}\bar{\pi}_+ \in \mathcal{L}(\ell^1(\mathbf{Z}_+;U), \ell^2(\mathbf{Z}_+;Y))$,

(v) $\underline{\mathcal{D}}\bar{\pi}_+$ *is a densely defined closed operator on* $\ell^2(\mathbf{Z}_+;U)$.

Then (i) $\Rightarrow$ *(ii)* $\Rightarrow$ *(iii)* $\Rightarrow$ *(iv)* $\Rightarrow$ *(v).*

Proof. The first implication (i) $\Rightarrow$ (ii is trivial.

The implication (ii) $\Rightarrow$ (iii) is proved as follows. Assume (ii). Equation (3) upon input $\tilde{u} = \pi_0\tilde{u}$, we see that $\sum_{j\geq 0} ||CA^jBu_0||^2 < \infty$ implies $u_0 = \pi_0\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)$. Because this is the case for all $u_0 \in U$, $\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0) = U$. $\underline{\mathcal{D}}\pi_0$ is then closed (Lemma 27) with complete domain and consequently bounded by the Closed Graph Theorem (see [9, Theorem 2.15]). Now claim (iii) follows.

To prove the implication (iii) $\Rightarrow$ (iv), we note that any $\tilde{u} \in \ell^1(\mathbf{Z}_+;U)$ can be written as $\tilde{u} = \sum_{j\geq 0} \pi_j\tilde{u}$, where the sum converges in the norm of $\ell^1(\mathbf{Z}_+;U)$. Thus

$$\begin{aligned} &||\underline{\mathcal{D}}\bar{\pi}_+\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \leq \sum_{j\geq 0} ||\underline{\mathcal{D}}\pi_j\tilde{u}||_{\ell^2(\mathbf{Z}_+;Y)} \\ &\leq \sum_{j\geq 0} ||\underline{\mathcal{D}}\pi_j||_{U\to\ell^2(\mathbf{Z}_+;Y)}\,||\pi_j\tilde{u}||_U = ||\underline{\mathcal{D}}\pi_0||_{U\to\ell^2(\mathbf{Z}_+;Y)} \sum_{j\geq 0} ||\pi_j\tilde{u}||_U, \end{aligned}$$

where the last equality is by the shift invariance of $\underline{\mathcal{D}}$. Now $\sum_{j\geq 0} ||\pi_j\tilde{u}||_U =: ||\tilde{u}||_{\ell^1(\mathbf{Z}_+;U)}$ and the claim follows.

The final implication (iv) $\Rightarrow$ (v) is trivial because $\ell^1(\mathbf{Z}_+;U)$ is dense in $\ell^2(\mathbf{Z}_+;Y)$. Also part (iv) of Proposition 26 could be used. □

The condition (i) of Lemma 29 can be called the *uniform* H^2-stability. For the uniformly H^2-stable DLS's, the the power series coefficients of the operator valued transfer function $\mathcal{D}(z)$ are square summable; i.e. $\mathcal{D}(z)$ lies in the uniform $H^2(U;Y)$. Claim (iv) of Proposition 26 gives a sufficient condition for $\underline{\mathcal{D}}\bar{\pi}_+$ to be a densely defined closed operator. Sometimes this condition is also necessary, as indicated in the following lemma:

Lemma 30. *Assume that* $\overline{\pi_0 \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)} = \overline{\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)}$. *Then* $\overline{\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)} = U$ *if and only if* $\overline{\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)} = \ell^2(\mathbf{Z}_+; U)$.

Proof. The "only if" part is given by claim (iv) of Proposition 26. The "if" part is proved by

$$\begin{aligned} U &\supset \overline{\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)} = \overline{\pi_0 \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)} \\ &\supset \pi_0 \overline{\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)} = \pi_0 \ell^2(\mathbf{Z}_+; U) = U, \end{aligned}$$

where the second inclusion is by the continuity of π_0. This completes the proof. □

The nature of the condition $\overline{\pi_0 \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)} = \overline{\mathrm{dom}\,(\underline{\mathcal{D}}\pi_0)}$ is at the first sight quite obscure. The stronger condition $\pi_j \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \mathrm{dom}\,(\underline{\mathcal{D}}\pi_j)$, for all $j \in \mathbf{Z}_+$, has a deep operator theoretic meaning, given in Lemma 33. From a control theoretic point of view, it means that given an arbitrary $\bar{\pi}_+ \tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ giving a finite energy output, also all its components $\pi_j \tilde{u}$ alone give a finite energy output. Before giving Lemma 33 we have to introduce a preliminary definition and proposition.

Definition 31. *The operator* $\tau\bar{\pi}_+ \in \ell^2(\mathbf{Z}_+; U)$ *is called the unilateral shift. If a subspace* $A \subset \ell^2(\mathbf{Z}_+; U)$ *satisfies*

$$\begin{aligned} (\tau\bar{\pi}_+)\, A &\subset A, \\ (\tau\bar{\pi}_+)^*\, A &\subset A, \end{aligned}$$

then we say that the subspace A *reduces the unilateral shift.*

The unilateral shift is a fundamental object in the operator and function theory. The *closed* subspaces of $\ell^2(\mathbf{Z}_+; U)$ reducing the unilateral shift are of form $\ell^2(\mathbf{Z}_+; U')$, where U' is a closed Hilbert subspace of U (see [8, Corollary 5.2]). From the control theoretic point of view, the case when $\overline{\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)}$ is of this form is important because then its orthogonal complement (the unstable part) can be projected away by a static (time independent) projection.

Proposition 32. *The following inclusions hold:*

(i) $(\tau\bar{\pi}_+)\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ *for all* $j \in \mathbf{Z}_+$,

(ii) $(\tau\bar{\pi}_+)^*\bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}}) \subset \bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}})$ *for all* $j \in \mathbf{Z}_+$,

Proof. To prove claim (i), we take $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$. Then $\mathcal{D}\bar{\pi}_+\tilde{u} \in \ell^2(\mathbf{Z}_+; Y)$. Now

$$\mathcal{D}(\tau\bar{\pi}_+)\,\bar{\pi}_+\tilde{u} = \mathcal{D}\tau\bar{\pi}_+\tilde{u} = \tau(\mathcal{D}\bar{\pi}_+\tilde{u}) \in \ell^2(\mathbf{Z}_+; Y).$$

Thus $(\tau\bar{\pi}_+)\,\bar{\pi}_+\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and (i) follows.

To prove claim (ii), let $\tilde{u} \in \bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}}) = \bar{\pi}_+ \cup_{j\geq 0} \tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$. Then

$$\begin{aligned} &(\tau\bar{\pi}_+)^*\,\tilde{u} \in \bar{\pi}_+\tau^* \cup_{j\geq 0} \tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \\ &= \bar{\pi}_+ \cup_{j\geq 1} \tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \bar{\pi}_+ \cup_{j\geq 0} \tau^{*j}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \bar{\pi}_+\mathrm{dom}\,(\underline{\mathcal{D}}), \end{aligned}$$

giving (ii). □

Lemma 33. *The following are equivalent:*

(i) $\pi_j \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \mathrm{dom}\,(\underline{\mathcal{D}}\pi_j)$,

(ii) $\pi_{[j,\infty]} \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\pi_j)$,

(iii) $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = \bar{\pi}_+ \mathrm{dom}\,(\underline{\mathcal{D}})$,

(iv) $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ *reduces the unilateral shift.*

Proof. To prove (i) $\Rightarrow$ (ii), let $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and $j \in \mathbf{Z}_+$ be arbitrary. Then, by (i) and the shift invariance of $\underline{\mathcal{D}}$, for all i, $0 \le i \le j$

$$\pi_i \tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_j) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\,.$$

It follows that $\pi_{[0,j]}\tilde{u} = \sum_{i=0}^{j} \pi_i \tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ because $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ is a vector space. Because $\tilde{u}$, j were arbitrary, claim (ii) follows.

The implication (ii) $\Rightarrow$ (iii) is proved by writing

$$\bar{\pi}_+ \mathrm{dom}\,(\underline{\mathcal{D}}) = \left(\cup_{j\ge 1} (\bar{\pi}_+ \tau^*)^j \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\right) \cup \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$$

as in the proof of claim (ii) of Proposition 26. By using this we obtain the following equivalences with condition (iii)

$$\begin{aligned} &\bar{\pi}_+ \mathrm{dom}\,(\underline{\mathcal{D}}) = \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \\ \Leftrightarrow &\cup_{j\ge 1} (\bar{\pi}_+ \tau^*)^j \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \\ \Leftrightarrow &(\bar{\pi}_+ \tau^*)^j \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \quad \text{for all} \quad j \in \mathbf{Z}_+. \end{aligned}$$

Now, if (ii) holds, the for arbitrary $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$, $j \in \mathbf{Z}_+$ we have $\pi_{[j,\infty]}\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$. The shift invariance and causality of $\mathcal{D}$ imply $\ell^2(\mathbf{Z}_+; Y) \ni \tau^{*j}\mathcal{D}\pi_{[j,\infty]}\tilde{u} = \mathcal{D}\tau^{*j}\pi_{[j,\infty]}\tilde{u} = \mathcal{D}\bar{\pi}_+\tau^{*j}\tilde{u}$. Then $\bar{\pi}_+\tau^{*j}\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ and claim (iii) follows.

The implication (iii) $\Rightarrow$ (iv) is a trivial consequence of Proposition 32. Finally, to obtain (iv) $\Rightarrow$ (i), assume (iv). We have $(\bar{\pi}_+\tau^*)\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$. Because $(\tau\bar{\pi}_+)\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ by Proposition 32,

$$\pi_{[1,\infty]}\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) = (\tau\bar{\pi}_+)(\bar{\pi}_+\tau^*)\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+) \subset \mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)\,.$$

Because $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ is a vector space, (i) follows. This completes the proof. □

We remark that if $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ reduces the unilateral shift, then so does its closure $\overline{\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)}$, because the unilateral shift $\tau\bar{\pi}_+$ is a bounded operator. So, if the equivalent conditions of Lemma 33 hold, then $\overline{\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)} = \ell^2(\mathbf{Z}_+; U')$, where U' is a closed Hilbert subspace of U. Furthermore, if $\mathrm{dom}\,(\underline{\mathcal{D}}\bar{\pi}_+)$ is closed and shift invariant, then then input space U can be restricted to a Hilbert subspace U', so that the restricted system becomes a I/O-stable DLS on space $\ell^2(\mathbf{Z}_+; U')$. We shall not develop this any further here.

Further stability notions are given below. They are used in Section 9 where stabilities of the closed loop systems are studied.

Definition 34. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *be a DLS.*

(i) If $\underline{\mathcal{B}} \in \mathcal{L}(\mathrm{dom}\,(\underline{\mathcal{B}})\,, H)$*, then* Φ *is input stable.*

(ii) If $\mathrm{dom}\,(\underline{\mathcal{C}}) = H$*, then* Φ *is output stable.*

(iii) If all the above holds, and Φ *is in addition both I/O-stable and* A *is power bounded, then* Φ *is stable.*

(iv) If Φ *is stable, and in addition the semi-group generator* A *of* Φ *is strongly stable, the* Φ *is strongly stable.*

The following lemmas have simple proofs, and are now omitted.

Lemma 35. *Let* Φ *be a DLS with a power stable semi-group generator. Then* Φ *is strongly stable.*

Lemma 36. *Let* Φ *be an I/O-stable DLS. Then its I/O-map* $\underline{\mathcal{D}}$ *has a continuous extension to the whole of* $\ell^2(\mathbf{Z} : U)$*, denoted also by* $\underline{\mathcal{D}}$*. The extension is shift invariant: i.e.* $\underline{\mathcal{D}}\tau = \tau\underline{\mathcal{D}}$*.*

7 Stability and the structure of the state space

In this section we study how the controllability and observability maps of the DLS interact. It is easy to construct a DLS that reads input into the state space by the controllability map $\mathcal{B}$, processes the information in the state space with the semi-group generator A, but this information always remains in a subspace of the state space H that is never "seen" by the observability map $\mathcal{C}$. It is easy to think that this kind of DLS's are in some sense trivial and uninteresting. In this section we show that if we require some stability from Φ, this does not happen.

Definition 37. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS. We say that* ϕ *satisfies the compatibility condition for controllability and observability maps, if*

$$BU \subset \mathrm{dom}\left(\underline{\mathcal{C}}_\phi\right)$$

The following proposition shows us, when the domain of the topological observability map consists of full trajectories. The role of the compatibility condition of Definition 37 is seen to be crucial.

Lemma 38. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS. Then*

(i) $A\,\mathrm{dom}\left(\underline{\mathcal{C}}_\phi\right) \subset \mathrm{dom}\left(\underline{\mathcal{C}}_\phi\right)$

(ii) If, in addition, ϕ satisfies the compatibility condition of Definition 37, then

$$x_j(x_0, \tilde{u}) \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right), \quad \forall j \in \mathbf{Z}_+, \quad \forall \tilde{u} \in Seq_+(U)$$

whenever $x_0 \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$.

Proof. In order to prove (i), we calculate:

$$\begin{aligned} x_0 \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right) \quad &\Leftrightarrow \quad \mathcal{C}_\phi x_0 \in \ell^2(\mathbf{Z}_+; Y) \subset Seq_+(Y) \\ &\Rightarrow \quad \mathcal{C}_\phi A x_0 = \bar{\pi}_+ \tau^* \mathcal{C}_\phi x_0 \in Seq_+(Y) \end{aligned}$$

by part (iii) of Lemma 7. But now, because $\underline{\mathcal{C}}_\phi x_0 \in \ell^2(\mathbf{Z}_+; Y)$ and $\bar{\pi}_+$, τ^* are of norm 1 in $\ell^2(\mathbf{Z}; Y)$, it follows that $\mathcal{C}_\phi A x_0 \in \ell^2(\mathbf{Z}_+; Y)$; i.e. $Ax_0 \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$ for the domain of the topological observability map. This proves the first claim of the lemma.

So as to prove the remaining part (ii), use the inductive reasoning as follows. By assumption, $x_0 \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$. Assume that it has already been proved that $x_j(x_0, \tilde{u}) \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$. Now $x_{j+1}(x_0, \tilde{u}) = Ax_j(x_0, \tilde{u}) + Bu_j$, where $Ax_j(x_0, \tilde{u}) \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$ by the first part of this lemma. $Bu_j \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$ by Definition 37. But then $x_{j+1}(x_0, \tilde{u}) \in \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$, because $\operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$ is a vector space. This proves the remaining part of the lemma. □

The following lemma gives us a characterization of those DLS's in terms of operators $\underline{\mathcal{B}}$ and $\underline{\mathcal{C}}$ that satisfy the compatibility condition. It is an immediate consequence of Lemma 38.

Lemma 39. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. The ϕ satisfies the compatibility condition of Definition 37 if and only if*

$$\operatorname{range}\left(\underline{\mathcal{B}}_\phi\right) \subset \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right) \tag{39}$$

Sufficient conditions for the compatibility condition are the following:

Lemma 40. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS. Then the following conditions are sufficient to guarantee that ϕ satisfies the compatibility condition of Definition 37:*

(i) ϕ is output stable,

(ii) ϕ is (strongly) H^2-stable,

(iii) ϕ is I/O-stable.

Proof. Claim (i) is trivially proved, once we note that for output stable DLSs ϕ, $\operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right) = H$. Because $BU \subset H$, we must have $BU \subset \operatorname{dom}\left(\underline{\mathcal{C}}_\phi\right)$ This proves claim (i).

In order to prove claim (ii) let $\tilde{u} \in \ell^2(\mathbf{Z}; U)$ satisfying $\pi_{-1}\tilde{u} = \tilde{u}$ be arbitrary. Then

$$\mathcal{D}_\phi \pi_{-1}\tilde{u} = \mathcal{D}_\phi \tilde{u} = \pi_{-1}\mathcal{D}_\phi \pi_{-1}\tilde{u} + \mathcal{C}\mathcal{B}\pi_{-1}\tilde{u} = D\pi_{-1}\tilde{u} + \mathcal{C}Bu, \tag{40}$$

where $u = \pi_{-1}\tilde{u} \in U$ with obvious identification of the spaces. Trivially $D\pi_{-1}\tilde{u} \in \ell^2(\mathbf{Z}; Y)$. $\mathcal{D}_\phi \pi_{-1}\tilde{u} \in \ell^2(\mathbf{Z}; Y)$ because $\mathcal{D}$ is shift invariant and ϕ is H^2 -stable. It thus follows from equation (40) that $\mathcal{C}Bu \in \ell^2(\mathbf{Z}_+; Y)$ and thus $Bu \in \operatorname{dom}(\underline{\mathcal{C}})$. Because u is arbitrary, (ii) follows. Claim (iii) follows because I/O-stability implies H^2-stability. □

8 Graph topology of the state space

In this section we study the structure of the state space of an I/O-stable DLS more carefully. We give a stronger inner product topology for the state space of original DLS, without affecting the properties of the topological I/O-map $\underline{\mathcal{D}}$. The resulting system will have essentially the same algebraic structure, but in addition it is output stable in the new state space topology. The closed graph property of $\underline{\mathcal{C}}$ is the key in the construction of the new inner product topology (see [1, Chapter 2]). The new complete state space will be the vector space $\operatorname{dom}(\underline{\mathcal{C}})$ and the new topology will be called the graph topology of the observability map.

The introduction of the stronger topology for the state space enables us to study I/O-stable but not output stable systems as if they were output stable. For example, the critical feedback operator in [6, Definition 7] given by

$$\mathcal{K}^{crit} := -(\bar{\pi}_+\underline{\mathcal{D}}^* J\underline{\mathcal{D}}\bar{\pi}_+)^{-1}\bar{\pi}_+\underline{\mathcal{D}}^* J\underline{\mathcal{C}}$$

could be unbounded in the topology of H, if $\underline{\mathcal{C}}$ is unbounded. However, in the stronger topology $\mathcal{K}^{crit}$ becomes bounded, if the Toeplitz operator $(\bar{\pi}_+\underline{\mathcal{D}}^* J\underline{\mathcal{D}}\bar{\pi}_+)^{-1}$ and I/O-map $\underline{\mathcal{D}}$ are bounded.

In the rest of this section we assume that $\overline{\operatorname{dom}(\underline{\mathcal{C}})} = H$, i.e. $\underline{\mathcal{C}}$ is a densely defined closed operator. To avoid trivialities, we must have $\operatorname{dom}(\underline{\mathcal{C}}) \neq 0$. This is true when Φ is I/O-stable, by part (iii) of Lemma 40. For I/O-stable DLS's, then we can restrict the state space to $\overline{\operatorname{dom}(\underline{\mathcal{C}})}$, if we are only interested in ℓ^2 -inputs for the system.

We start with the necessary technical tools: Definition 41 and Lemmas 42, 43 and 44.

Definition 41. *Let Φ be a DLS non-trivial* $\operatorname{dom}(\underline{\mathcal{C}})$.

(i) Define the inner product $\langle , \rangle_E$ in $\operatorname{dom}(\underline{\mathcal{C}})$ *by*

$$\langle x, y\rangle_E := \langle x, y\rangle_H + \langle \underline{\mathcal{C}}x, \underline{\mathcal{C}}y\rangle_{\ell^2(\mathbf{Z}_+;Y)}.$$

(ii) By E denote the inner product space $\operatorname{dom}(\underline{\mathcal{C}})$ *equipped with the inner product $\langle , \rangle_E$.*

(iii) The topology of $\operatorname{dom}(\underline{\mathcal{C}})$ *given by $\langle , \rangle_E$ is called the graph topology of the observability map.*

It is easy to check that $\langle , \rangle_E$ is indeed an inner product in $\operatorname{dom}(\underline{\mathcal{C}})$. $\langle , \rangle_E$ coincides with $\langle , \rangle_H$ in the kernel of $\underline{\mathcal{C}}$. This is a closed subspace of H, because $\underline{\mathcal{C}}$ is closed.

The following to consequences of the closed graph property of $\underline{\mathcal{C}}$ are basic.

Lemma 42. *Introduce the same notations as in Definition 41. Then*

(i) E *is Hilbert,*

(ii) $\underline{\mathcal{C}} \in \mathcal{L}(E; \ell^2(\mathbf{Z}_+; Y)$ *and* $||\underline{\mathcal{C}}||_{E \to \ell^2(\mathbf{Z}_+;Y)} \leq 1$,

(iii) $C := \pi_0 \underline{\mathcal{C}} \in \mathcal{L}(E; Y)$ *and* $||C||_{E \to Y} \leq 1$.

Proof. In order to show claim (i), it is sufficient to show that E is complete. For this end, let $\{x_j\} \subset E = \mathrm{dom}\,(\underline{\mathcal{C}})$ be a Cauchy sequence in the topology of E. Because the norm of E majorizes the norm of H, it follows that $\{x_j\}$ is also Cauchy in the topology of H, and similarly so is the sequence $\{\underline{\mathcal{C}} x_j\}$. It follows that the sequence $\{x_j\}$ has a limit $x \in H$ and $\{\underline{\mathcal{C}} x_j\}$ has a limit $\tilde{y} \in \ell^2(\mathbf{Z}_+; Y)$, by the completeness of H and $\ell^2(\mathbf{Z}_+; Y)$.

Because $\underline{\mathcal{C}}$ is closed by Lemma 27, it follows that $x \in \mathrm{dom}\,(\underline{\mathcal{C}}) = E$ and $y = \underline{\mathcal{C}} x$. Now we can write

$$||x_j - x||_E^2 = ||x_j - x||_H^2 + ||\underline{\mathcal{C}} x_j - \underline{\mathcal{C}} x||_{\ell^2(\mathbf{Z}_+;Y)}^2 \to 0$$

as $j \to \infty$. Thus the sequence $\{x_j\}$ has a limit x in E, and E is complete.

The proof of claim (ii) is a direct consequence of the definition of norm $|| \cdot ||_E$. Claim (iii) follows from claim (ii), because π_0 is of norm 1. The proof of this lemma is now complete. □

Under no stronger assumptions, we can also say several essential things about the operator A as a linear mapping on E.

Lemma 43. *Introduce the same notations as in Definition 41. Then the following claims are true:*

(i) A *maps* E *linearly into itself.*

(ii) $A|_E \in \mathcal{L}(E)$

(iii) $A|_E$ *is a power bounded linear operator in* $\mathcal{L}(E)$ *if* A *is power bounded in* $\mathcal{L}(H)$. *Furthermore,*

$$sup_{j>0} ||(A|_E)^j||_E \leq \max\,(sup_{j>0} ||A^j||_H, 1).$$

(iv) For all $x_0 \in E$ *we have*

$$||A^j x_0||_H \to 0 \Rightarrow ||A^j x_0||_E \to 0.$$

In particular, if A *is strongly stable, then* $A|_E$ *is strongly stable.*

Proof. In order to prove claim (i), we refer at claim (i) of Lemma 38, and note that $E = \mathrm{dom}\,(\underline{\mathcal{C}})$ as the algebraic vector space. Claims (ii) and (iii) follow immediately from the calculation

$$\begin{aligned}
\frac{||A^j x_0||_E^2}{||x_0||_E^2} &= \frac{||A^j x_0||_H^2 + ||\underline{\mathcal{C}} A^j x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2}{||x_0||_H^2 + ||\underline{\mathcal{C}} x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2} \\
&\leq \frac{||A^j x_0||_H^2 + ||\bar{\pi}_+ \tau^{*j} \underline{\mathcal{C}} x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2}{||x_0||_H^2 + ||\underline{\mathcal{C}} x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2} \leq \frac{||A^j||_{H\to H}^2\, ||x_0||_H^2 + ||\underline{\mathcal{C}} x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2}{||x_0||_H^2 + ||\underline{\mathcal{C}} x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2} \\
&\leq \max\,(||A^j||_{H\to H}, 1).
\end{aligned}$$

The proof of claim (iv) requires the following computation for $x_0 \in E$

$$\begin{aligned}
||A^j x_0||_E^2 &= ||A^j x_0||_H^2 + ||\underline{\mathcal{C}} A^j x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2 \\
&= ||A^j x_0||_H^2 + ||\bar{\pi}_+ \tau^{*j} \underline{\mathcal{C}} x_0||_{\ell^2(\mathbf{Z}_+;Y)}^2.
\end{aligned}$$

Now the first part of the right hand side approaches zero by assumption. The second part approaches zero, because $\underline{\mathcal{C}} x_0 \in \ell^2(\mathbf{Z}_+; Y)$ by the definition of $\mathrm{dom}\,(\underline{\mathcal{C}})$ and the identity of Parseval. This completes the proof of the lemma. □

One could regard claim (iv) of the previous lemma as a partial complementary result to the obvious implication

$$||x_j||_E \to 0 \Rightarrow ||x_j||_H \to 0 \tag{41}$$

for all sequences $\{x_j\} \in E$. The topology of E is in general genuinely stronger that that inherited from H, and the full converse to formula (41) is in general not true. The topologies of E and H coincide if and only if $\underline{\mathcal{C}}$ is bounded (output stable). If this is not the case, then E is a set of first category in H. For I/O-stable DLS's we can say even more about the relation of Φ and the graph topology.

Lemma 44. *Let Φ be an H^2-stable DLS. Then the following claims are true:*

(i) $||\underline{\mathcal{C}} B||_{U\to\ell^2(\mathbf{Z}_+;Y)} \leq ||\underline{\mathcal{D}}\pi_0||_{U\to\ell^2(\mathbf{Z}_+;Y)} < \infty$

(ii) $B \in \mathcal{L}(U; E)$ *and*

$$||B||_{U\to E}^2 \leq ||B||_{U\to H}^2 + ||\underline{\mathcal{D}}\pi_0||_{\mathrm{dom}(\underline{\mathcal{D}}\pi_0)\to\ell^2(\mathbf{Z}_+;Y)}^2 < \infty.$$

(iii) If, in addition, Φ is I/O-stable and input stable, then we have $\underline{\mathcal{B}} \in \mathcal{L}(\mathrm{dom}\,(\underline{\mathcal{B}})\,; E)$ and

$$||\underline{\mathcal{B}}||_{\mathrm{dom}(\underline{\mathcal{B}})\to E}^2 \leq ||\underline{\mathcal{B}}||_{\mathrm{dom}(\underline{\mathcal{B}})\to H}^2 + ||\underline{\mathcal{D}}||_{\mathrm{dom}(\underline{\mathcal{D}})\to\ell^2(\mathbf{Z};Y)}^2 < \infty.$$

Proof. Claim (i) follows immediately from Definition 28 and formula (3).

Claim (ii) is a trivial consequence of the first part and the definition of the norm of E, if we just remember that $B \in \mathcal{L}(U;H)$ by the definition of a DLS.

In order to prove the claim (iii), we use the input stability of Φ as follows:

$$\begin{aligned}
\frac{||\underline{\mathcal{B}}\tilde{u}||_E^2}{||\tilde{u}||^2_{\mathrm{dom}(\underline{\mathcal{B}})}} &= \frac{||\underline{\mathcal{B}}\tilde{u}||_H^2 + ||\underline{\mathcal{C}}\underline{\mathcal{B}}\tilde{u}||^2_{\ell^2(\mathbf{Z}_-;U)}}{||\tilde{u}||^2_{\mathrm{dom}(\underline{\mathcal{B}})}} \\
&\leq \frac{||\underline{\mathcal{B}}\tilde{u}||_H^2 + ||\bar{\pi}_+\underline{\mathcal{D}}\pi_-\tilde{u}||^2_{\ell^2(\mathbf{Z}_-;Y)}}{||\tilde{u}||^2_{\mathrm{dom}(\underline{\mathcal{B}}))}} \\
&\leq ||\underline{\mathcal{B}}||^2_{\mathrm{dom}(\underline{\mathcal{B}})\to H} + ||\underline{\mathcal{D}}||^2_{\mathrm{dom}(\underline{\mathcal{D}})\to \ell^2(\mathbf{Z};Y)} < \infty,
\end{aligned}$$

for all nonzero $\tilde{u} \in \mathrm{dom}\,(\underline{\mathcal{B}}) \subset \mathrm{dom}\,(\underline{\mathcal{D}})$. This completes the proof of the lemma.☐

The following definition of the modified DLS $\phi^{(m)}$ comes no longer as a surprise. The main properties of the modified DLS are given in Theorem 46.

Definition 45. *Let $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ be a DLS satisfying the compatibility condition. Then the ordered quadruple of linear operators*

$$\phi^{(m)} := \begin{pmatrix} A|E & B \\ C|E & D \end{pmatrix},$$

is called the modified DLS $\phi^{(m)}$ associated to ϕ, where the linear operators are defined by

- *(i) E is the space* $\mathrm{dom}\,(\underline{\mathcal{C}})$ *as topologized in Definition (41).*
- *(ii) $A|_E$ is the restriction of A into E,*
- *(iii) $C|_E$ is the restriction of C into E.*

The space U is called the input space, Y the output space and E the state space of system $\phi^{(m)}$.

The DLS $\phi^{(m)}$ is well defined; this amounts to first showing that E is A-invariant and $\mathrm{range}\,(B) \subset E$. From claim (i) of Lemma 38 we know that the algebraic vector space $E = \mathrm{dom}\,(\underline{\mathcal{C}})$ is A-invariant. The requirement $\mathrm{range}\,(B) \subset E$ follows immediately from Definition (37). The boundedness of the operators appearing in the formula for $\phi^{(m)}$ is now a nontrivial fact. $A|_E \in \mathcal{L}(E)$, by Lemma 43. Lemma 44 implies that $B \in \mathcal{L}(U;E)$. Finally, Lemma 42 gives us $C \in \mathcal{L}(E;Y)$. Thus Definition 45, indeed, defines a DLS.

The basic properties of the modified I/O-stable DLS are given by the following theorem:

Theorem 46. *Let* $\phi = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *be a DLS satisfying the compatibility condition of Definition 37. Then*

(i) $\phi^{(m)}$ *is an output stable DLS satisfying* $\mathcal{D}_{\phi^{(m)}} = \mathcal{D}_{\phi}$ *in* $Seq(U)$*. In particular,* $\phi^{(m)}$ *is* H^2*-stable if and only if* ϕ *is.* $\phi^{(m)}$ *is I/O-stable if and only if* ϕ *is.*

(ii) The semi-group generator of $\phi^{(m)}$ *is power bounded if* Φ *is. The semi-group generator of* $\phi^{(m)}$ *is strongly stable if* Φ *is.*

(iii) Assume, moreover, that ϕ *is I/O-stable. Then* $\phi^{(m)}$ *is input stable if* Φ *is.*

(iv) $\phi^{(m)}$ *is stable if* Φ *is input stable and I/O-stable with a power bounded semi-group generator. It is strongly stable if in addition the semigroup generator of* Φ *is strongly stable.*

Proof. Let us start with proving claim (i). The output stability is the continuity of $\underline{\mathcal{C}}$ in the norm of E, implied by claim (ii) of Lemma 42. We prove that $\mathcal{D}_{\phi^{(m)}} = \mathcal{D}_{\phi}$ in $Seq(U)$. The formula

$$\mathcal{D}\pi_j \tilde{u} = D\pi_j \tilde{u} + \tau^{j+1} \mathcal{C} B u_j$$

is a consequence of formula (3) for all $\tilde{u} = \{u_i\} \in Seq(U)$. Then, because $D = D_{\phi^{(m)}}$ and $\mathcal{C}_{\phi^{(m)}} B_{\phi^{(m)}} = \mathcal{C}B$ as linear operators from U to $Seq(Y)$, it follows that $\mathcal{D}\pi_j = \mathcal{D}_{\phi^{(m)}}\pi_j$ as mappings from $Seq(U)$ to $Seq(Y)$, for all j. Thus, $\mathcal{D}$ and $\mathcal{D}_{\phi^{(m)}}$ coincide on $Seq(U)$ by linearity. Then the domains dom $(\underline{\mathcal{D}}\pi_0)$ and dom $(\underline{\mathcal{D}}\bar{\pi}_+)$ coincide, by their definitions. Both the claims about H^2-stability and I/O-stability follow, completing the proof of claim (i).

Claim (ii) follows directly from claims (iii) and (iv) of Lemma 43. To show claim (iii), we refer to claim (iii) of Lemma 44. The last claim (iv) is an immediate conclusion of the previous parts of this theorem, combined with claim (iv) of Lemma 43 for the strong stability part. This completes the proof. □

9 Stability of closed loop systems

In this final section we study the feedbacks of DLS's with the additional requirement that the topology of the input and output sequence spaces plays a significant role. We restrict the notion of feedback pair as presented in Section 5 to take these additional requirements into consideration. We study both I/O-stable and stable systems, and how the open loop stability is preserved in the closed loop system.

Definition 47. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ an I/O-stable DLS. Then the pair $[\mathcal{K}, \mathcal{F}]$ of linear operators is called I/O-stable feedback pair for system Φ, if*

(i) the DLS $\Phi^{fb} := \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{K} & \mathcal{F} \end{smallmatrix}\right]$ is I/O-stable with input space U, state space H and output space U,

(ii) we have the inclusion

$$\mathrm{dom}\,(\underline{\mathcal{C}}) \subset \{x_0 \in H \mid \mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+; U)\} =: \mathrm{dom}\,(\mathcal{K})\,,$$

(iii) $(\mathcal{I} - \mathcal{F})^{-1}$ is an I/O-map of an I/O-stable DLS, mapping $\ell^2(\mathbf{Z}_+; U) \to \ell^2(\mathbf{Z}_+; U)$.

Instead of (iii) we can say that $\mathcal{I} - \mathcal{F}$ is outer with bounded inverse (see [6, Definition 18]) An even stronger notion of stable feedback pair is defined as follows:

Definition 48. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ a stable DLS and let $[\mathcal{K}, \mathcal{F}]$ be an I/O-stable feedback pair for Φ. Then $[\mathcal{K}, \mathcal{F}]$ is called stable feedback pair for Φ, if in addition $\mathcal{K} \in \mathcal{L}(H, \ell^2(\mathbf{Z}_+; U))$.*

In the following Theorem 49 we apply an I/O-stable feedback pair onto an I/O-stable DLS. In Theorem 51 we do the same thing with a stable DLS and a stable feedback pair. The proof are rather elementary because they contain just restatements of the theory presented in previous sections.

Theorem 49. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ an I/O-stable DLS. Let $[\mathcal{K}, \mathcal{F}]$ be a I/O-stable feedback pair for Φ. Then*

(i) the extended DLS $\Phi^{ext} = [\Phi, [\mathcal{K}, \mathcal{F}]]$ is I/O-stable,

(ii) the closed loop extended DLS $\Phi^{ext}_\diamond = [\Phi, [\mathcal{K}, \mathcal{F}]]_\diamond$ is an I/O-stable DLS with input space U, state space H and output space $Y \times U$,

(iii) if we denote the operators in $\Phi^{ext}_\diamond$ by

$$\Phi^{ext}_\diamond = \begin{bmatrix} A^j_\diamond & \mathcal{B}_\diamond \tau^{*j} \\ \begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \end{bmatrix},$$

then we have for the domain of the observability map of $\Phi^{ext}_\diamond$

$$\mathrm{dom}\,(\underline{\mathcal{C}}) \subset \mathrm{dom}\left(\underline{\begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix}}\right). \tag{42}$$

Proof. The claim (i) is quite trivial. In order to prove claim (ii), we recall the formula for the closed loop extended DLS from Definition (18)

$$\Phi_\diamond^{ext} = \begin{bmatrix} (A+\mathcal{B}\tau^*(I-\mathcal{F})^{-1}\mathcal{K})^j & \mathcal{B}(I-\mathcal{F})^{-1}\tau^{*j} \\ \begin{bmatrix} \mathcal{C}+\mathcal{D}(I-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \end{bmatrix} & \begin{bmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1} \\ (\mathcal{I}-\mathcal{F})^{-1}-I \end{bmatrix} \end{bmatrix} =: \begin{bmatrix} A_\diamond^j & \mathcal{B}_\diamond\tau^{*j} \\ \begin{bmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{bmatrix} & \begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \end{bmatrix} \tag{43}$$

Let us study the I/O-map of $\Phi_\diamond^{ext}$. We have

$$\begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \bar\pi_+ = \begin{bmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\bar\pi_+ \\ ((\mathcal{I}-\mathcal{F})^{-1}-\mathcal{I})\,\bar\pi_+ \end{bmatrix} = \begin{bmatrix} \mathcal{D}\bar\pi_+(\mathcal{I}-\mathcal{F})^{-1}\bar\pi_+ \\ \bar\pi_+\,((\mathcal{I}-\mathcal{F})^{-1}-\mathcal{I})\,\bar\pi_+ \end{bmatrix}$$

by the causality of $\mathcal{D}$ and $(\mathcal{I}-\mathcal{F})^{-1}$. Now it is easy to see that $\operatorname{dom}\left(\begin{bmatrix} \mathcal{D}_\diamond \\ \mathcal{F}_\diamond \end{bmatrix} \bar\pi_+\right) = \ell^2(\mathbf{Z}_+;U)$ by the boundedness of both $\mathcal{D}$ and $(\mathcal{I}-\mathcal{F})^{-1}$ in $\ell^2(\mathbf{Z}_+;U)$, the former by I/O-stability of Φ and the latter by Definition 47. Thus the I/O-stability of $\Phi_\diamond^{ext}$ follows.

To establish the inclusion (42) we first recall that $\operatorname{dom}(\underline{\mathcal{C}}) \subset \{x_0 \in H \mid \mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+;U)\}$, by requirement (ii) of Definition 47. Furthermore, for $x_0 \in \operatorname{dom}(\underline{\mathcal{C}})$ we have

$$(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}x_0 = \bar\pi_+(\mathcal{I}-\mathcal{F})^{-1}\bar\pi_+\mathcal{K}x_0 \in \ell^2(\mathbf{Z}_+;U)$$

This immediately gives formula (42), because $\mathcal{D}\bar\pi_+$ is bounded from $\ell^2(\mathbf{Z}_+;U)$ into $\ell^2(\mathbf{Z}_+;Y)$, by the I/O-stability of Φ. This completes the proof the theorem. □

The next theorem shows us, how the stability of the semi-group is preserved under the closing of the feedback loop. The role of input stability should be carefully noted.

Theorem 50. *Let* $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ *an I/O-stable and input stable DLS. Let* $[\mathcal{K},\mathcal{F}]$ *be a stable feedback pair for* Φ. *By* A $(A_\diamond)$ *denote the semi-group generator of* Φ ($\Phi_\diamond^{ext}$, *respectively). Then*

(i) $A_\diamond$ *is strongly stable if* A *is,*

(ii) $A_\diamond$ *is power bounded if and only if* A *is.*

Proof. We start with proving the claim (i). Let $x_0 \in H$ be arbitrary. Then

$$(A_\diamond^j - A^j)x_0 = \underline{\mathcal{B}}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}x_0.$$

We estimate the right hand side of the previous equation and see that it gets small if j is increased. We have

$$\begin{aligned} &\|\underline{\mathcal{B}}\tau^{*j}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}x_0\|_H \\ &< \|\underline{\mathcal{B}}\tau^{*j}\pi_{[0,J]}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}x_0\|_H + \|\underline{\mathcal{B}}\tau^{*j}\pi_{[J+1,\infty]}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}x_0\|_H. \end{aligned} \tag{44}$$

The second term on the right of equation (44) gets small by increasing J for any fixed x_0, because $(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}x_0$ is in $\ell^2(\mathbf{Z}_+;U)$ and $\underline{\mathcal{B}}$ is bounded. Also the first term gets small, as shown by the following inequality (implied by claim (ii) of Lemma 7) for any $\tilde{u} \in \ell^2(\mathbf{Z}_+;U)$ and $j > J$

$$
\begin{aligned}
||\underline{\mathcal{B}}\tau^{*j}\pi_{[0,J]}\tilde{u}||_H &\leq ||A^j\underline{\mathcal{B}}\pi_{[0,J]}\tilde{u}||_H + ||\sum_{i=0}^{j-1} A^i B(\pi_{[0,J]}\tilde{u})_{j-i-1}||_H \\
&= ||A^{j-J-1}\left(\sum_{i=0}^{J} A^i B u_{J-i}\right)||_H.
\end{aligned}
$$

This proves that $A_\diamond$ is strongly stable if A is, thus establishing claim (i). In order to prove claim (ii), we calculate

$$
\begin{aligned}
&\big|\, ||A_\diamond^j||_{H\to H} - ||A^j||_{H\to H} \big| \leq ||A_\diamond^j - A^j||_{H\to H} \\
&\leq ||\underline{\mathcal{B}}||_{\ell^2(\mathbf{Z}_-;U)\to H)}\, ||(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}||_{H\to\ell^2(\mathbf{Z}_-;U)} < \infty
\end{aligned}
$$

Thus either both $A_\diamond$ and A are power bounded, or neither are. This completes the proof of the theorem. □

The following theorem gives results, what happens if the feedback pair $[\mathcal{K},\mathcal{F}]$ is not only I/O-stable, but stable.

Theorem 51. *Let $\Phi = \left[\begin{smallmatrix} A^j & \mathcal{B}\tau^{*j} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right]$ be an I/O-stable DLS, and $J \in \mathcal{L}(Y)$ be self adjoint. Assume that the feedback pair $[\mathcal{K},\mathcal{F}]$ is stable.Then the following is true:*

(i) $\Phi_\diamond^{ext}$ is input stable if and only if Φ is.

(ii) $\Phi_\diamond^{ext}$ is output stable if and only if Φ is.

(iii) $\Phi_\diamond^{ext}$ is stable if and only if Φ is.

(iv) $\Phi_\diamond^{ext}$ is strongly stable if Φ is.

Proof. Claim (i) trivial, because $(\mathcal{I}-\mathcal{F})^{-1}$ is both bounded and coercive in $\ell^2(\mathbf{Z}_-;U)\cap Seq(U)$, and $\mathcal{B}_\diamond = \mathcal{B}(\mathcal{I}-\mathcal{F})^{-1}$. In order to prove claim (ii), we first note that $(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K}$ is bounded in H if and only if the I/O-stable feedback pair $[\mathcal{K},\mathcal{F}]$ is in fact stable ($\mathcal{K}$ is bounded). Now the claimed equivalence is trivial because the observability map of $\Phi_\diamond^{ext}$ is

$$
\begin{pmatrix} \mathcal{C}_\diamond \\ \mathcal{K}_\diamond \end{pmatrix} = \begin{pmatrix} \mathcal{C} \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{D}(\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \\ (\mathcal{I}-\mathcal{F})^{-1}\mathcal{K} \end{pmatrix}.
$$

The proof of claims (iii) and (iv) is a composition of the claims (ii) and (i) of this theorem and claim (ii) of Theorem 50. This completes the proof. □

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