ON A TAUBERIAN CONDITION FOR BOUNDED LINEAR OPERATORS

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Abstract

We study the relation between the growth of sequences $\|T^n\|$ and $\|(n+1)(I-T)T^n\|$ for operators $T \in \mathcal{L}(X)$ satisfying variants of the Ritt resolvent condition $\|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda-1|}$ in various subsets of $\{|\lambda| > 1\}$.

1. Introduction

Let $T \in \mathcal{L}(X)$; a bounded linear operator on a (complex) Banach space X. It was R. K. Ritt himself who first studied the *Ritt resolvent condition*

$$\|(\lambda - T)^{-1}\| \leq \frac{C}{|\lambda - 1|} \tag{1.1}$$

for $|\lambda| > 1$. He proved that if T satisfies (1.1) for $|\lambda| > 1$, then $\lim_{n\to\infty} \|T^n/n\| = 0$, see [17]. Clearly (1.1) implies that $\sigma(T) \subset \mathbb{D} \cup \{1\}$ but, in fact, even $\sigma(T) \subset K_\delta^c \cap (\mathbb{D} \cup \{1\})$ for some $\delta > 0$, where

$$K_\delta := \{\lambda = 1 + re^{i\theta} : r > 0 \text{ and } |\theta| < \frac{\pi}{2} + \delta\}; \tag{1.2}$$

see O. Nevanlinna [13, Theorem 4.5.4] and Yu. Lyubich [9].

An operator $T \in \mathcal{L}(X)$ is *power bounded* if $\sup_{n\geq 1} \|T^n\| < \infty$. Y. Katznelson and L. Tzafriri proved in 1986 that for a power bounded T, we have $\sigma(T) \subset \mathbb{D} \cup \{1\}$ if and only if $\lim_{n\to\infty} \|(I-T)T^n\| = 0$, see [8]. Related to this, J. Zemánek asked in 1992 whether (1.1) implies $\lim_{n\to\infty} \|(I-T)T^n\| = 0$, too. This was answered in positive by O. Nevanlinna, and he also noted that if (1.1) holds in the larger set $K_\delta \cup \mathbb{D}^c$ for some $\delta > 0$, then T is power bounded, see [13, Theorem 4.5.4], [14] and [21].

It was observed in 1998 independently by Yu. Lyubich [9], B. Nagy and J. Zemánek

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[12], and O. Nevanlinna that if (1.1) holds for all $|\lambda| > 1$, then the same estimate holds for all $\lambda \in K_\delta \cup \mathbb{D}^c$ for some $\delta > 0$ (with another possibly larger constant C_δ in place for C). Hence, if T satisfies (1.1) for all $|\lambda| > 1$, then T is power bounded. The upper bound $\sup_{n\geq 1} \|T^n\| \leq (eC^2)/2$ was given by N. Borovykh, D. Drissi and M. N. Spijker, see [1]. A tighter estimate $\sup_{n\geq 1} \|T^n\| \leq C^2$ was shown by O. El-Fallah and T. Ransford in [3].

Much of these developments culminate in the following fundamental result connecting power boundedness, the Ritt resolvent condition and the *Tauberian condition* (1.3):

Proposition 1.1. *The following are equivalent:*

(i) *T satisfies (1.1) for all $|\lambda| > 1$,*

(ii) *$\sigma(T) \subset \mathbb{D} \cup \{1\}$, and there exists $\delta > 0$ and $C = C(\delta)$ such that (1.1) holds for all $\lambda \in K_\delta$, and*

(iii) *T is power bounded, and it satisfies the Tauberian condition*

$$\sup_{n\geq 1} (n+1)\|(I-T)T^n\| \leq M \tag{1.3}$$

for some $M < \infty$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has already been discussed above. The implication (ii) $\Rightarrow$ (iii) is given in [13, Theorem 4.5.4]. That (iii) implies (i) was reported in [14, Theorem 2.1]. The proof relies on the theory of analytic semigroups, and it follows closely [15, Theorem 5.2]; note that the restrictive assumption $0 \in \rho(A)$ can be removed from [15, Theorem 5.2] by a more careful analysis. The equivalence (i) $\Leftrightarrow$ (iii) is explicitly given in [12, p. 147]. ■

The conditions of Proposition 1.1 can be combined in another way. We shall prove the following Tauberian theorem and discuss its consequences:

Theorem 1.2. *Assume that $T \in \mathcal{L}(X)$ satisfies Tauberian condition (1.3), and*

$$\|(\lambda-1)(\lambda-T)^{-1}\| \leq C \tag{1.4}$$

for all $\lambda \in (1, 1+\epsilon)$ for some $\epsilon > 0$. Then T is power bounded with the estimates

$$\|T^n\| \leq 2 + C\|T\| + 2M \quad \text{and} \quad \limsup_{n\to\infty} \|T^n\| \leq 2 + C\|T\| + (1+1/e)\,M. \tag{1.5}$$

The proof of this theorem is given in Section 2 below.

2. Equivalent conditions under the Tauberian condition

Let us remind the results of the classical Tauberian theorem in the scalar case. Let $\{a_n\}$ be a complex sequence and $s_n = a_0 + a_1 + ... + a_n$ for $n \geq 0$. A. Tauber proved in 1897 that if

(i) $\lim_{n\to\infty}(n+1)a_n = 0$, and
(ii) $\lim_{r\to 1_-} f(r) = s$, where $f(r) = \sum_0^\infty a_n r^n$ for $0 < r < 1$,

then $\lim_{n\to\infty} s_n = s$ follows. J. E. Littlewood showed in 1910 that condition (i) can in fact be replaced by the weaker Tauberian condition $\sup_n n|a_n| < \infty$ but the proof with this modification becomes considerable harder. Good references to this and other related Tauberian theorems are [16] and [22].

If we take $a_n = (I-T)T^n$, we see that the weaker Tauberian condition $\sup_n n|a_n| < \infty$ corresponds to assumption (1.3), and the partial sums are simply $s_n = I - T^{n+1}$. We remark that the stronger Tauberian condition (i) above is too restrictive in operator context because a power-bounded $T \neq I$ with $\sigma(T) = \{1\}$ satisfies $\liminf_{n\to\infty}(n+1)\|(I-T)T^n\| \geq 1/e$; see [4], [7], and [10] for various proofs.

In this paper, we are not interested in the *limit behaviour* of $\{s_n\}$ (in other words, the ergodicity of T) but in the *boundedness* of this sequence when (1.3) holds. For this reason, we may adapt the original simple approach by Tauber to prove Theorem 1.2, without having to deal with the more complicated technique by Littlewood; see also [16, Remark 2 on p. 67]. For a related ergodicity result, see [18, Theorem III-1 on p. 150].

PROOF OF THEOREM 1.2. Define

$$s_n := \sum_{j=0}^{n-1}(I-T)T^j = 1 - T^n, \quad f_n(r) := \sum_{j=0}^{n-1}(I-T)T^j r^j \quad \text{and}$$
$$f(r) := \sum_{j=0}^{\infty}(I-T)T^j r^j = I + T(r-1)(1-rT)^{-1}.$$

Then for all $r \in (0,1)$ and $n \geq 0$, we have

$$\|s_n\| \leq \|s_n - f_n(r)\| + \|f_n(r) - f(r)\| + \|f(r)\|. \tag{2.1}$$

By condition (1.4), the last term on the right hand side of (2.1) is bounded by $1 + C\|T\|$ uniformly for all $r \in (0,1)$. For the second term, we have

$$\begin{aligned}\|f_n(r) - f(r)\| &= \|\textstyle\sum_{j\geq n}(I-T)T^j r^j\| \leq \sum_{j\geq n}\frac{M}{j+1}r^j \\ &= \textstyle\frac{M}{n+1}\sum_{j\geq n}\frac{n+1}{j+1}r^j \leq \frac{M}{n+1}r^n(1-r)^{-1}\end{aligned}$$

by (1.3). From now on, we choose $r_n := 1 - 1/n$ in (2.1). Then

$$\frac{M}{n+1}r_n^n\,(1-r_n)^{-1} = \frac{M}{n+1}\left(1-\frac{1}{n}\right)^n n \quad \begin{cases}\to M/e \text{ as } n\to\infty;\\ \leq M \text{ for all } n \geq 1.\end{cases}$$

So the second term on the right hand side of (2.1) is bounded with this choice of $r = r_n$.

The first term of the right side of inequality (2.1) (when choosing $r = r_n$) we have

$$s_n - f_n(r_n) = \sum_{j=0}^{n-1} (I-T)T^j(1-r_n^j).$$

By the mean value theorem, there exists $r_0^j \in [r_n, 1)$ for any $j > 0$, such that we can estimate

$$1 - r_n^j = j r_0^{j-1}(1-r_n) \leq j(1-r_n) = \frac{j}{n}.$$

This together with (1.3) yields

$$\|s_n - f_n(r_n)\| \leq \sum_{j=0}^{n-1} \frac{j}{n} \|(I-T)T^j\| \leq \sum_{j=0}^{n-1} \frac{j}{n} \frac{M}{j+1} \leq M \frac{1}{n} \sum_{j=0}^{n-1} 1 = M.$$

So the sequence $\{s_n\}_{n \geq 0}$ is uniformly bounded from above, which is equivalent to the power boundedness of T. It is also clear from this argument that the constants in (1.5) are as claimed. ■

If $T \in \mathcal{L}(X)$ satisfies the Tauberian condition (1.3), then a number of conditions are equivalent. The following theorem is analogous to [14, Theorem 2.1], except that now (1.3) is a standing assumption instead of power boundedness. We remark that condition (iv) constitutes a slight improvement to Theorem 1.2.

Theorem 2.1. *Assume that $T \in \mathcal{L}(X)$ satisfies the Tauberian condition* (1.3). *Then the following are equivalent:*

(i) *T is power bounded,*

(ii) *there exists $0 < \delta \leq 1 \leq C < \infty$ such that T satisfies the Ritt resolvent condition* (1.1) *for all $\lambda \in K'_\delta := \{\lambda = 1 + re^{i\theta} | r > 0, |\theta| < \frac{\pi}{2} + \delta\}$,*

(iii) *there exists $C_K < \infty$ such that T satisfies the iterated Kreiss resolvent condition*

$$\|(\lambda - T)^{-k}\| \leq \frac{C_K}{(|\lambda| - 1)^k} \quad \text{for all} \quad |\lambda| > 1 \quad \text{and} \quad k \in \mathbb{N},$$

(iv) *for some $k \in \mathbb{N}$ there exists $0 < \eta_k \leq 1 \leq C_k < \infty$ such that T satisfies the kth order resolvent condition*

$$\|(\lambda - 1)^k (\lambda - T)^{-k}\| \leq C_k \quad \text{for all} \quad \lambda \in (1, 1 + \eta_k),$$

(v) *there exists $C_{HY} < \infty$ such that $A = T - I$ satisfies the Hille–Yosida resolvent condition*

$$\|(\lambda - 1)^k (\lambda - T)^{-k}\| \leq C_{HY} \quad \text{for all} \quad \lambda > 1 \quad \text{and} \quad k \in \mathbb{N},$$

(vi) *$A = T - I$ generates an uniformly bounded, norm continuous, analytic semigroup $t \mapsto e^{At}$ of linear operators,*

(vii) *the operators* $M_n := \frac{1}{n+1}\sum_{j=0}^{n} T^j$ *are uniformly bounded, and*
(viii) *there exists* $C_{UA} < \infty$ *such that* T *is uniformly Abel bounded, i.e.,*

$$\|(\lambda - 1)\sum_{k=0}^{n} \lambda^{-k-1} T^k\| < C_{UA} \quad \textit{for all} \quad n \in \mathbb{N} \quad \textit{and } \lambda > 1.$$

PROOF. Claims (i) and (ii) are equivalent by Proposition 1.1 and an extension result that can be found, e.g., in [12]. By estimating the Neumann series it follows that (i) ⇒ (iii). It is trivial that (iii) ⇒ (iv).

We argue next that if condition (iv) holds with some $k \in \mathbb{N}$, then it holds with $k = 1$, too. Using the identity $(\lambda - T)^{-k-1} = \sum_{j=0}^{\infty} \binom{n+j}{n} T^j \lambda^{-n-j-1}$ for $|\lambda| > 1$ we get from (1.3) the estimate

$$\|(I - T)(\lambda - T)^{-k-1}\| \leq \frac{C}{k}\left(\frac{1}{(\lambda - 1)^k} - \frac{1}{(\lambda)^k}\right) \text{ for } \lambda > 1,$$

see [14, Theorem 2.1]. From this it follows for all $k \geq 1$ that

$$\|(\lambda - 1)^k(\lambda - T)^{-k} - (\lambda - 1)^{k+1}(\lambda - T)^{-k-1}\| \leq \frac{C}{\lambda} \text{ for all } \lambda > 1.$$

Having shown this, it follows directly from Theorem 1.2 that (iv) ⇒ (i) because (1.4) is only used near point 1 in the proof of Theorem 1.2.

Assume (i). The semigroup generated by $T - I$ is norm continuous since $T \in \mathcal{L}(X)$, and it is uniformly bounded since $\|e^{tT}\| \leq \sum_{j \geq 0} \frac{\|T^j\| t^j}{j!} \leq \sup_{j \geq 0} \|T^j\| \cdot e^t$ for all $t \geq 0$. Moreover, it is not difficult to see that (1.3) implies $\|Ae^{tA}\| \leq Mt^{-1}(1 - e^{-t})$ where $A := T - I$. This implies that e^{tA} is analytic, by a slight generalization of [15, Theorem 5.2]. We conclude that (i) ⇒ (vi). We have (vi) ⇒ (v) the classical theorem of E. Hille and K. Yosida on strongly continuous semigroups; see, e.g., [6]. The implication (v) ⇒ (iv) is trivial. We have now shown that all the conditions (i) – (vi) are equivalent.

It is trivial that (i) ⇒ (vii). The implication (vii) ⇒ (viii) is given in [5, Theorem 2]. That (viii) ⇒ (iv) with $k = 1$ is trivial, and the proof is now complete. ■

A number of remarks are now in order. That (iii) with $k = 1$ implies (i) was first proved by using a Cauchy integration argument, see [20]. It is shown in [11, Theorem 3.1], conditions (vii) and (viii) are equivalent even without assuming (1.3). Likewise, condition (vii) implies always (iv) (with $k = 1$) by [14, Theorem 4.2].

If T satisfies the following weaker form of Tauberian condition

$$\sup_{n \geq 0} \sqrt{n+1}\|(I - T)T^n\| < \infty,$$

then (i) ⇔ (v) follows from [16, Theorem III.5 on p. 68]. However, then the bounded semigroup generated by $I - T$ need not be analytic as can be seen by applying [14, Theorem 2.3] on a contraction T with $\sigma(T) = \overline{\mathbb{D}}$.

Recall from the ergodicity theory the identity $(n+1)(M_n - M_{n-1}) = T^n - M_{n-1}$ where M_n is defined as in claim (vii). Thus claims (i) and (vii) are equivalent if $\limsup_{n\to\infty}(n+1)\|M_n - M_{n-1}\| < \infty$. Note that (1.3) implies

$$\limsup_{n\to\infty}(n+1)\|M_n - M_{n-1}\| < \infty.$$

All this was pointed out in [19, Proposition 6.1].

If the Banach space X is reflexive, condition (i) implies that the operator sequence M_n in claim (vii) converges in strong operator topology to a bounded projection without assuming (1.3); see, e.g., [2, Corollary III.5.4]. Then (vii) holds by the Banach–Steinhaus theorem.

We remark that the Tauberian condition (1.3) implies $\|T^n\| = \mathcal{O}(\ln n)$, and by [7, Theorem 3.3], the growth can really be there for an operator in a Banach space. Condition (1.3) "almost" implies condition (iv) of Theorem 2.1, too. Indeed, as $(1-r)(I-rT)^{-1} = I - r(I-T)(I-rT)^{-1}$ for all $|r| < 1$, we obtain the estimate

$$\|(1-r)(I-rT)^{-1}\| \le 1 + M\sum_{j\ge 0}\frac{|r|^{j+1}}{j+1} = 1 + M\ln\frac{1}{1-|r|}$$

for all $0 \le |r| < 1$. Setting $r = 1/\lambda$ for $\lambda > 1$ gives now

$$\|(\lambda-1)(\lambda-T)^{-1}\| \le 1 + M\ln\frac{\lambda}{\lambda-1}.$$

Hence $\|(\lambda-T)^{-1}\| = \mathcal{O}\left((\lambda-1)\ln(\lambda-1)\right)$ as $\lambda \to 1+$. Again, the logarithmic term can really be present on the right hand side, as can be seen by studying more carefully the example given in [7, Theorem 3.3].

Finally, the Tauberian condition (1.3) "almost" implies condition (vi) of Theorem 2.1. Indeed, as $\|Ae^{tA}\| \le Mt^{-1}(1-e^{-t})$ where $A := T - I$, and the function $t \mapsto t^{-1}(1-e^{-t})$ is decreasing for $t \ge 0$, it follows that

$$\|e^{tA}\| \le 1 + \int_0^t \|Ae^{tA}\|\,dt \le 1 + M + M(1-e^{-1})\ln t \quad \text{for all} \quad t \ge 1.$$

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