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ScienceDirect
J. Differential Equations 231 (2006) 290–312
Journal of Differential Equations
www.elsevier.com/locate/jde

Conservative boundary control systems

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Received 13 February 2006; revised 8 May 2006

Available online 5 July 2006

Abstract

We study continuous time linear dynamical systems of boundary control/observation type, satisfying a Green–Lagrange identity. Particular attention is paid to systems which have a well-defined dynamics both in the forward and the backward time directions. As we change the direction of time we also interchange inputs and outputs. We show that such a boundary control/observation system gives rise to a continuous time Livšic–Brodskiĭ (system) node with strictly unbounded control and observation operators. The converse is also true. We illustrate the theory by a classical example, namely, the wave equation describing the reflecting mirror.
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MSC: 47A48; 47N70; 93B28; 93C25

Keywords: Conservative system; Boundary control; Time–flow invertibility; Operator node; Cayley transform

1. Introduction

In this paper, we give simple necessary and sufficient conditions for the (scattering) conservativity of linear boundary control/observation systems described by differential equations of form

$$u(t) = Gz(t), \quad \dot{z}(t) = Lz(t), \quad y(t) = Kz(t), \quad t \in \mathbb{R}^+ = [0, \infty),$$
$$z(0) = z_0. \tag{1.1}$$

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1 O. Staffans gratefully acknowledges the financial support from the Academy of Finland under grant 203991.

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doi:10.1016/j.jde.2006.05.012

These conditions are stated in terms of data given; namely the (unbounded) operators G, L, and K. In a typical application L is a partial differential operator, and G and K are boundary trace operators.

We shall assume throughout that the operators G, L, and K in (1.1) give rise to a *boundary node* of the following type.

Definition 1.1. A triple $\Xi := (G, L, K)$ is a *boundary node* on the Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ if the following conditions are satisfied:

(i) G, L, and K are linear operators with the same domain $\mathcal{Z} \subset \mathcal{X}$;
(ii) $\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix}\right]$ is a closed linear operator mapping $\mathcal{Z}$ into $\mathcal{U} \times \mathcal{X} \times \mathcal{Y}$;
(iii) G is surjective and $\mathcal{N}(G)$ is dense in $\mathcal{X}$;
(iv) The operator $L|\mathcal{N}(G)$ (interpreted as an operator in X with domain $\mathcal{N}(G)$) has a nonempty resolvent set.

This boundary node is *internally well-posed* (in the forward time direction) if, in addition,

(v) $L|\mathcal{N}(G)$ generates a C_0 semigroup.

We call $\mathcal{U}$ the input space, $\mathcal{X}$ the state space, $\mathcal{Y}$ the output space, $\mathcal{Z}$ the *solution space*, G the *input boundary operator*, L the *interior operator*, and K the *output boundary operator*.

If $\Xi = (G, L, K)$ is internally well-posed, then (1.1) has a unique solution for sufficiently smooth input functions u and initial states z_0 compatible with $u(0)$. More precisely, as we show in Lemma 2.6, for all $z_0 \in \mathcal{X}$ and $u \in C^2(\mathbb{R}^+; \mathcal{U})$ with $Gz_0 = u(0)$ the first, second and fourth of Eqs. (1.1) have a unique solution $z \in C^1(\mathbb{R}^+; \mathcal{X}) \cap C(\mathbb{R}^+; \mathcal{Z})$,[2] and hence we can define $y \in C(\mathbb{R}^+; \mathcal{Y})$ by the third equation in (1.1). In the rest of this article, when we say "a smooth solution of (1.1) on $\mathbb{R}^+$" we mean a solution with the above properties.

Definition 1.2. A boundary node Ξ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is *energy preserving* if it is internally well-posed and all smooth solutions of (1.1) on $\mathbb{R}^+$ satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2_{\mathcal{X}} + \|y(t)\|^2_{\mathcal{Y}} = \|u(t)\|^2_{\mathcal{U}}, \quad t \in \mathbb{R}^+. \tag{1.2}$$

As we show in Proposition 4.2, this identity is equivalent to the *Green–Lagrange identity*

$$2\,\mathrm{Re}\langle z, Lz\rangle_{\mathcal{X}} + \|Kz\|^2_{\mathcal{Y}} = \|Gz\|^2_{\mathcal{U}}, \quad z \in \mathcal{Z} = \mathcal{D}om\left(\begin{bmatrix} G \\ L \\ K \end{bmatrix}\right). \tag{1.3}$$

Many boundary nodes defined by PDEs are *time–flow invertible*, i.e., they have the property that they remain boundary nodes if we reverse the direction of time and interchange the roles of K and G.

[2] Here we use the graph norm of $\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix}\right]$ in $\mathcal{Z}$, see (2.9).

Definition 1.3. A boundary node $\Xi = (G, L, K)$ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is *time–flow invertible* if the triple $\Xi^{\leftarrow} := (K, -L, G)$ is a boundary node on $(\mathcal{Y}, \mathcal{X}, \mathcal{U})$. We call $\Xi^{\leftarrow}$ the *time–flow inverse* of Ξ.

Definition 1.4. A boundary node $\Xi = (G, L, K)$ is *conservative* if it is time–flow invertible and both Ξ itself and the time–flow inverse $\Xi^{\leftarrow}$ are energy preserving.

The following theorem is the first of our main results.

Theorem 1.5. *Let $\Xi := (G, L, K)$ be a boundary node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. Then Ξ is conservative if and only if the following three additional conditions hold*:

(i) *K is surjective and $\mathcal{N}(K)$ is dense in $\mathcal{X}$,*
(ii) *$\rho(L|\mathcal{N}(G)) \cap \overline{\mathbb{C}^+} \neq \emptyset$,*
(iii) *$\rho(-L|\mathcal{N}(K)) \cap \overline{\mathbb{C}^+} \neq \emptyset$,*
(iv) *the Green–Lagrange identity* (1.3) *holds.*

As shown by the first author in [24, Theorem 5] using [26, Theorem 4.4], it is possible to replace condition (iv) of Theorem 1.5 by two slightly weaker conditions (with $\mathcal{Z} = \mathcal{D}om\left(\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix}\right]\right)$):

(iv′) $2\,\mathrm{Re}\langle x, Lx\rangle_{\mathcal{X}} + \|Kx\|^2_{\mathcal{Y}} = 0$ for all $x \in \mathcal{N}(G)$,
(v′) $\langle z, Lx\rangle_{\mathcal{X}} + \langle Lz, x\rangle_{\mathcal{X}} = \langle Gz, Gx\rangle_{\mathcal{U}}$ for all $z \in \mathcal{Z}$ and $x \in \mathcal{N}(K)$.

However, in practice it does not appear to be easier to check conditions (iv′) and (v′) than to check the full Green–Lagrange identity (1.3).

The proof of Theorem 1.5 is based on the notion of a system node. In this work we do not just study internally well-posed boundary nodes for their own sake, but we interpret them as system nodes in a natural way. (See Definition 2.1 for the exact definition.) This opens up the possibility of applying existing results for system nodes (e.g., on feedback, generalized solutions; see [38]) to internally well-posed boundary nodes. For example, in the conservative case it is possible to use the theory of well-posed linear systems to replace the class of smooth solutions of (1.1) by solutions where u and y belong locally to L^2 and z is continuous in the space $\mathcal{X}$.

A system node is a special case of an *operator node*. By this we mean a closed densely defined linear operator $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \supset \mathcal{D}om(S) \to \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with certain additional properties. In the case of a system node it generates a dynamical system through the equations

$$\begin{bmatrix} \dot{z}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \qquad z(0) = z_0. \tag{1.4}$$

Here $u \in C^2(\mathbb{R}^+; \mathcal{U})$ and $\left[\begin{smallmatrix} z_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}om(S)$, and Eq. (1.4) has a unique solution $z \in C^1(\mathbb{R}^+; \mathcal{X})$ and output function $y \in C(\mathbb{R}^+; \mathcal{Y})$ (see Lemma 2.2). If $\dim \mathcal{X} < \infty$, then S can always be written as $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, where A, B, C, and D are bounded linear operators between the appropriate spaces, and (1.4) takes the familiar form

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \\ y(t) &= Cz(t) + Du(t), \quad t \in \mathbb{R}^+, \\ z(0) &= z_0. \end{aligned} \tag{1.5}$$

Given a boundary node Ξ it is possible to construct a unique operator node S with the property that, formally, the solutions of (1.1) coincide with those of (1.4). For an internally well-posed boundary node Ξ this correspondence is not only formal but actually valid for all smooth solutions of (1.1), and S is then a system node. We give a complete description of those operator/system nodes S that arise in this way from some boundary node Ξ. We say that these operator/system nodes are of *boundary control type*

The main result in Section 3 is the following theorem.

Theorem 1.6. *A boundary node Ξ is time–flow invertible in the sense of Definition* 1.3 *if and only if the corresponding operator node S* (*see Theorems* 2.3 *and* 2.4) *is time–flow invertible in the usual operator node sense* (*see Definition* 3.1).

We call an operator node S *energy preserving* if it is a system node and the smooth solutions of (1.4) satisfy (1.2). Clearly, if S arises from an internally well-posed boundary node Ξ, then S is energy preserving if and only if Ξ is energy preserving. However, since the dynamics is now described by a different equation (1.4), also the Green identity (1.3) takes a different form.

The standard definition of a *conservative* system node involves also the dual node[3] S^*. According to this definition, S is conservative if both S and S^* are energy preserving. This is the approach adopted in most systems theory papers, such as [2–4,6,10–12,26,34–40,42–44]. However, as we show in Proposition 4.3 below, this is equivalent to the requirement that S is time–flow invertible and both S and its time–flow inverse $S^{\leftarrow}$ are energy preserving. This leads to the following conclusion.

Theorem 1.7. *A boundary node Ξ is conservative if and only if the corresponding operator node S* (*see Theorems* 2.3 *and* 2.4) *is conservative in the usual operator node sense* (*see Definition* 4.1).

The results obtained in this article lead to the following two new theorems about time–flow invertible or conservative operator nodes. Note that the statements of these two theorems contain no reference to boundary nodes (in spite of the fact that their proofs depend heavily on such nodes).

Theorem 1.8. *If an operator node S of boundary control type is time–flow invertible, then the time–flow inverse $S^{\leftarrow}$ is also of boundary control type.*

This follows immediately from Theorem 1.6.

Theorem 1.9. *Let S be a conservative system node. Then S is of boundary control type if and only if the dual S^* is of boundary control type.*

This follows from Theorem 1.8 and Proposition 4.3 below.

The outline of this paper is the following. In Section 2 we introduce operator nodes and explain the relationship between a boundary node and an operator node of boundary control type, roughly following [23,31,38]. In Section 3 we discuss time–flow invertibility of boundary

[3] If S is a system node, then so is S^*; see [26, Proposition 2.4] or [38].

nodes, and connect this notion with the time–flow invertibility of operator nodes as presented in [38,40]. Conservative operator nodes are studied in Section 4 in the spirit of [26]; see also [34–38,40,43]. The proof of Theorems 1.5 and 1.7 are given in this section.

Finally, in Section 5 we apply Theorem 1.5 to a PDE describing a reflecting mirror, and we conclude that it induces a conservative system node. The same example has been treated earlier in [44] as an example of a "thin air" system. The strong and exponential stability of the semigroup generated by the same PDE (take $u \equiv 0$ in (5.2)) is studied by, e.g., Lagnese [19] and Triggiani [41], but they do not pay attention to system theoretic properties of this example, such as conservativity.

The boundary nodes that we present here have a long history. It started with the boundary control of parabolic and hyperbolic PDEs; for the early history we refer to [21,30]. The two volumes [20] contain a large collection of examples and references to more recent work, as does [8]. The origin of our abstract formulation dates back to Fattorini [13], and significant progress was made by Salamon [31].

Even earlier in the former Soviet Union, the study of Sturm–Liouville and related problems led Neumark [27] and Kreĭn [18] to the question of finding symmetric and self-adjoint extensions of a symmetric operator, as described in [14, Chapter 3] and [15]. The final results have natural interpretations in the context of *conservative* boundary nodes. We shall return to this in [25].

At the moment, not much has been written in the west on conservative boundary control systems. Typical parabolic boundary control systems (arising, e.g., from thermodynamics) are not time–flow invertible, hence not conservative. However, many hyperbolic systems (coming, e.g., from continuum mechanics) are conservative. More specifically, conservative hyperbolic boundary control systems (or parts of such systems where either the input or the output is either implicit or missing) are found in [7,9,16,29,42,44].

2. Operator nodes versus boundary nodes

The purpose of this section is to explain the one-to-one connection between all boundary nodes and all operator nodes with injective and strictly unbounded control operators. This connection is known in principle (see, e.g., [31] or [38, Section 5.2]), but it cannot be found in the literature in exactly the form that we need it.

2.1. Operator and system nodes

Let us first recall the notions of an operator node and a system node. This involves a densely defined unbounded (main) operator A on a Hilbert space $\mathcal{X}$ with a nonempty resolvent set. We define $\mathcal{X}_1$ to be the domain of A with the graph norm $\|z\|^2_{\mathcal{X}_1} = \|Az\|^2_{\mathcal{X}} + \|z\|^2_{\mathcal{X}}$, and define $\mathcal{X}_{-1}$ to be the dual of $\mathcal{D}om(A^*)$ with the graph norm when we identify the dual of $\mathcal{X}$ with itself. Then $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$ with continuous and dense embeddings. The operator A has a unique extension to an operator $A_{-1} \in \mathcal{L}(\mathcal{X}; \mathcal{X}_{-1})$.

Definition 2.1. Let $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{Y}$ be Hilbert spaces. An operator

$$S := \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} \supset \mathcal{D}om(S) \to \begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \end{bmatrix}$$

is called an *operator node* on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ if it has the following structure:

(i) A is a densely defined operator on $\mathcal{X}$ with a nonempty resolvent set (which we extend to an operator $A_{-1} \in \mathcal{L}(\mathcal{X}; \mathcal{X}_{-1})$ as explained above).

(ii) $B \in \mathcal{L}(\mathcal{U}; \mathcal{X}_{-1})$.

(iii) $\mathcal{D}om(S) = \left\{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]:\ A_{-1}x + Bu \in \mathcal{X}\right\}$, and $A\&B = [A_{-1}\ B] \mid \mathcal{D}om(S)$.

(iv) $C\&D \in \mathcal{L}(\mathcal{D}om(S); \mathcal{Y})$, where we use the graph norm

$$\left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{A\&B}^2 = \|A_{-1}x + Bu\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + \|u\|_{\mathcal{U}}^2 \tag{2.1}$$

of $A\&B$ on $\mathcal{D}om(S)$.

If, in addition to the above, A generates a strongly continuous semigroup on $\mathcal{X}$, then S is called a *system node*.

A system or operator node is *of boundary control type* if its control operator B is injective and *strictly unbounded*, i.e., it satisfies $\mathcal{R}an(B) \cap \mathcal{X} = \{0\}$.

Every operator node is closed (as an operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$). This follows from the facts that $A\&B$ is closed, that $C\&D$ has the same domain as $A\&B$, and that $C\&D$ is continuous with respect to the graph norm of $A\&B$. It is also true that the graph norm of $A\&B$ on $\mathcal{D}om(S)$ is equivalent to the full graph norm

$$\left\| \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{S}^2 = \left\| A\&B \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\mathcal{X}}^2 + \left\| C\&D \begin{bmatrix} x \\ u \end{bmatrix} \right\|_{\mathcal{X}}^2 + \|x\|_{\mathcal{X}}^2 + \|u\|_{\mathcal{U}}^2 \tag{2.2}$$

of S.

We call $A \in \mathcal{L}(\mathcal{X}_1; \mathcal{X})$ the *main operator* of S, $B \in \mathcal{L}(\mathcal{U}; \mathcal{X}_{-1})$ is its *control operator*, and $C\&D \in \mathcal{L}(\mathcal{D}om(S); \mathcal{Y})$ is its *combined observation/feedthrough operator*. From the last operator we can extract $C \in \mathcal{L}(\mathcal{X}_1; \mathcal{Y})$, the *observation operator* of S, defined by

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \quad x \in \mathcal{X}_1. \tag{2.3}$$

A short computation shows that for each $\alpha \in \rho(A)$, the operator

$$E_\alpha := \begin{bmatrix} 1 & (\alpha - A_{-1})^{-1}B \\ 0 & 1 \end{bmatrix} \tag{2.4}$$

is a bounded bijection from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ onto itself and also from $\left[\begin{smallmatrix} \mathcal{X}_1 \\ \mathcal{U} \end{smallmatrix}\right]$ onto $\mathcal{D}om(S)$. In particular, for each $u \in \mathcal{U}$ there is some $x \in \mathcal{X}$ such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}om(S)$. Since $\left[\begin{smallmatrix} \mathcal{X}_1 \\ \mathcal{U} \end{smallmatrix}\right]$ is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$, this implies that also $\mathcal{D}om(S)$ is dense in $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$. Since the second column of E_α maps $\mathcal{U}$ into $\mathcal{D}om(S)$, we can define the *transfer function* of S by

$$\widehat{\mathfrak{D}}(\alpha) := C\&D \begin{bmatrix} (\alpha - A_{-1})^{-1}B \\ 1 \end{bmatrix}, \quad \alpha \in \rho(A), \tag{2.5}$$

which is an $\mathcal{L}(\mathcal{U}; \mathcal{Y})$-valued analytic function. Clearly, for any two $\alpha, \beta \in \rho(A)$,

$$\widehat{\mathfrak{D}}(\alpha) - \widehat{\mathfrak{D}}(\beta) = C\big[(\alpha - A_{-1})^{-1} - (\beta - A_{-1})^{-1}\big]B. \tag{2.6}$$

Each system node $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ generates a family of smooth solutions of the differential/algebraic equation (1.4) of the following type:

Lemma 2.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. Then for all $z_0 \in \mathcal{X}$ and $u \in C^2(\mathbb{R}^+; \mathcal{U})$ with $\left[\begin{smallmatrix} z_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}om(S)$ the equation*

$$\dot{z}(t) = A\&B \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+, \qquad z(0) = z_0, \tag{2.7}$$

has a unique solution $z \in C^1(\mathbb{R}^+; \mathcal{X})$ such that $\left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right] \in C(\mathbb{R}^+; \mathcal{D}om(S))$. Hence we can define $y \in C(\mathbb{R}^+; \mathcal{Y})$ by

$$y(t) = C\&D \begin{bmatrix} z(t) \\ u(t) \end{bmatrix}, \quad t \in \mathbb{R}^+. \tag{2.8}$$

For a proof, see [26, Proposition 2.5] or [38, Lemma 4.7.8]. In the sequel, by "a smooth solution of (1.4) on $\mathbb{R}^+$" we mean a solution with the above properties. Additional information about system and operator nodes can be found in, e.g., [6,26,31–40,44].

2.2. The connection between operator and boundary nodes

We now show that there is an one-to-one correspondence between boundary nodes and operator nodes of boundary control type.

Let Ξ be a boundary node as in Definition 1.1. In that definition we denote the common domains of K, L, and G by $\mathcal{Z}$ and call it the solution space. In the sequel we shall throughout equip $\mathcal{Z}$ with the graph norm of $\left[\begin{smallmatrix} K \\ L \\ G \end{smallmatrix}\right]$, i.e.,

$$\begin{aligned} \mathcal{Z} &:= \mathcal{D}om(K) = \mathcal{D}om(L) = \mathcal{D}om(G), \\ \|z\|_{\mathcal{Z}}^2 &:= \|z\|_{\mathcal{X}}^2 + \|Kz\|_{\mathcal{Y}}^2 + \|Lz\|_{\mathcal{X}}^2 + \|Gz\|_{\mathcal{U}}^2. \end{aligned} \tag{2.9}$$

Clearly $K \in \mathcal{L}(\mathcal{Z}; \mathcal{Y})$, $L \in \mathcal{L}(\mathcal{Z}; \mathcal{X})$, and $G \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$. We call $A := L|\mathcal{N}(G)$ the (forward) *main operator* and $A^{\leftarrow} := -L|\mathcal{N}(K)$ the *backward main operator*.

As our following theorem shows, every boundary node induces as an operator node, and every internally well-posed boundary node induces as a system node.[4] A converse to this theorem is given in Theorem 2.4.

Theorem 2.3. *Let $\Xi := (G, L, K)$ be a boundary node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. Then*

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} := \begin{bmatrix} L \\ K \end{bmatrix} \begin{bmatrix} 1 \\ G \end{bmatrix}^{-1}, \qquad \mathcal{D}om(S) = \mathcal{R}an\left(\begin{bmatrix} 1 \\ G \end{bmatrix}\right), \tag{2.10}$$

is an operator node on $(\mathcal{U}; \mathcal{X}; \mathcal{Y})$ of boundary control type. This operator node is a system node if and only if Ξ is internally well-posed.

[4] This theorem resembles [38, Theorem 5.2.13]. That theorem was added to [38] in the proof reading process, and it was originally obtained as a part of the present work. A less precise version of this result is also found in [31, Section 2.2].

More precisely, the operator node S can be constructed as follows:

(i) *The main operator A of S is given by $A := L|\mathcal{D}om(A)$, where $\mathcal{D}om(A) = \mathcal{N}(G)$. The spaces $\mathcal{X}_1 \subset \mathcal{X} \subset \mathcal{X}_{-1}$ and the extended operator A_{-1} are constructed as described in the paragraph before Definition* 2.1. *The norm in $\mathcal{X}_1$ (i.e., the graph norm of A) is equivalent to the norm that $\mathcal{X}_1$ inherits from the space $\mathcal{Z}$ defined in* (2.9).

(ii) *The control operator $B \in \mathcal{L}(\mathcal{U}; \mathcal{X}_{-1})$ of S is uniquely determined by the identity $BG = L - A_{-1}|\mathcal{Z}$.*

(iii) $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]$ *is a boundedly invertible operator from $\mathcal{Z}$ onto*

$$V := \left\{ \begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix} :\ A_{-1}z + Bu \in \mathcal{X} \right\},$$

equipped with the norm (2.1). *In particular, V is continuously embedded in $\left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{U} \end{smallmatrix}\right]$.*

(iv) *The observation/feedthrough operator $C\&D$ of S is given by $C\&D = [K\ 0]|\mathcal{D}om(S)$.*

(v) *The space $\mathcal{Z}$ can be written as the direct sum of the closed subspaces*

$$\mathcal{Z} = \mathcal{X}_1 \dotplus \mathcal{R}an\big((\alpha - A_{-1})^{-1}B\big), \tag{2.11}$$

where α is an arbitrary number in $\rho(A_{-1}) = \rho(A)$, and

$$G(\alpha - A_{-1})^{-1}B = 1, \quad \alpha \in \rho(A_{-1}) = \rho(A). \tag{2.12}$$

Moreover, (u, x, y) is a smooth solution of (1.1) *if and only if (u, x, y) is a smooth solution of* (1.4).

Proof. We build the operator node S from its components as described in (i)–(v), and then, at the end of the proof, we show that S is given by (2.10).

We begin with condition (i). By the definition of a boundary node, $A = L|\mathcal{N}(G)$ has a non-empty resolvent set. Let $\mathcal{X}_1' := \mathcal{N}(G)$ with the norm inherited from $\mathcal{Z}$, and let $\mathcal{X}_1 := \mathcal{D}om(A) = \mathcal{N}(G)$ with the graph norm. Let $\alpha \in \rho(A)$. Then $(\alpha - A) \in \mathcal{L}(\mathcal{X}_1'; \mathcal{X})$ is a bounded bijection, and hence it has a bounded inverse in $\mathcal{L}(\mathcal{X}; \mathcal{X}_1')$. This implies that the norms in $\mathcal{X}_1'$ and $\mathcal{X}_1$ are equivalent.

We continue by defining $B = (L - A_{-1})G_{\text{right}}^{-1}$, where $G_{\text{right}}^{-1} \in \mathcal{L}(\mathcal{U}; \mathcal{Z})$ is an arbitrary right-inverse to G (such a right-inverse exists since G is bounded and surjective). Then $B \in \mathcal{L}(\mathcal{U}; \mathcal{X}_{-1})$, since $\mathcal{Z} \subset \mathcal{X} \subset \mathcal{X}_{-1}$ with continuous embeddings. The operator B defined this way satisfies $BG = L - A_{-1}|\mathcal{Z}$, and this equation determines B uniquely (since G is surjective).

Next we prove (2.11) and (2.12). We have

$$(\alpha - A_{-1})^{-1}B = (\alpha - A_{-1})^{-1}(L - A_{-1})G_{\text{right}}^{-1} = G_{\text{right}}^{-1} + (\alpha - A_{-1})^{-1}(L - \alpha)G_{\text{right}}^{-1}, \tag{2.13}$$

where $G_{\text{right}}^{-1} \in \mathcal{L}(\mathcal{U}; \mathcal{Z})$ and $L - \alpha \in \mathcal{L}(\mathcal{Z}; \mathcal{X})$. This implies that $(\alpha - A_{-1})^{-1}B$ maps $\mathcal{U}$ continuously into $\mathcal{Z}$. Moreover, since the last term in (2.13) belongs to $\mathcal{X}_1 = \mathcal{N}(G)$, we find that (2.12) holds. In particular, B is injective and $\mathcal{R}an((\alpha - A_{-1})^{-1}B)$ is closed in $\mathcal{Z}$.

To complete our proof of (2.11), we still need to show that

$$\mathcal{X}_1 \cap \mathcal{R}an\big((\alpha - A_{-1})^{-1}B\big) = \{0\}$$

and that $\mathcal{X}_1 + \mathcal{R}an((\alpha - A_{-1})^{-1}B) = \mathcal{Z}$. If $x \in \mathcal{X}_1 \cap \mathcal{R}an((\alpha - A_{-1})^{-1}B)$, then $Gx = 0$ (since $X_1 = \mathcal{N}(G)$), and $x = (\alpha - A_{-1})^{-1}Bu$ for some $u \in \mathcal{U}$. Therefore, by (2.12),

$$0 = Gx = G(\alpha - A_{-1})^{-1}Bu = u,$$

hence also $x = 0$. Thus $\mathcal{X}_1 \cap \mathcal{R}an((\alpha - A|\mathcal{X})^{-1}B) = \{0\}$, or equivalently, $\mathcal{X} \cap \mathcal{R}an(B) = \{0\}$. Given any $z \in \mathcal{Z}$, we can define $u = Gz$ and $x = z - (\alpha - A_{-1})^{-1}Bu$. Then $u \in \mathcal{U}$ and

$$Gx = Gz - G(\alpha - A_{-1})^{-1}Bu = u - u = 0,$$

and so $x \in \mathcal{X}_1$. This completes the proof of the direct sum decomposition (2.11) and property (2.12).

We proceed to prove (iii), and begin by showing that

$$\mathcal{R}an\left(\begin{bmatrix} 1 \\ G \end{bmatrix}\right) = \left\{\begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}:\ A_{-1}z + Bu \in \mathcal{X}\right\}. \tag{2.14}$$

One direction of this inclusion is immediate: if $z \in \mathcal{Z}$ and $u = Gz$, then, as we saw above, $w := A_{-1}z + Bu = Lz \in \mathcal{X}$. Thus,

$$\mathcal{R}an\left(\begin{bmatrix} 1 \\ G \end{bmatrix}\right) \subset \left\{\begin{bmatrix} z \\ u \end{bmatrix} \in \begin{bmatrix} \mathcal{X} \\ \mathcal{U} \end{bmatrix}:\ A_{-1}z + Bu \in \mathcal{X}\right\}.$$

For the converse inclusion we take some $z \in \mathcal{X}$ and $u \in \mathcal{U}$ and suppose that $w := A_{-1}z + Bu \in \mathcal{X}$. Then by (2.11),

$$z = (\alpha - A_{-1})^{-1}(\alpha z - w) + (\alpha - A_{-1})^{-1}Bu \in \mathcal{Z}.$$

This proves (2.14).

By the continuity of L, G, and the embedding $\mathcal{Z} \subset \mathcal{X}$, each of $\|w\|_{\mathcal{X}}$, $\|z\|_{\mathcal{X}}$ and $\|u\|_{\mathcal{U}}$ are dominated by $\|z\|_{\mathcal{Z}}$ up to multiplicative constants. Thus $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]$ is a bounded bijection from $\mathcal{Z}$ onto V equipped with the norm (2.1). Therefore it also has a bounded inverse.

Since $V = \mathcal{R}an(\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right])$, we find that $V \subset \left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{U} \end{smallmatrix}\right]$. The embedding of V into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ is continuous, and since the range of this embedding operator is contained in $\left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{U} \end{smallmatrix}\right]$ (where $\mathcal{Z}$ is continuously embedded in $\mathcal{X}$), also the embedding $V \subset \left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{U} \end{smallmatrix}\right]$ must be continuous.

We continue by defining $C\&D$ as described in (iv). Then $C\&D$ is bounded from V into $\mathcal{Y}$ (because of the continuous embedding $V \subset \left[\begin{smallmatrix} \mathcal{Z} \\ \mathcal{U} \end{smallmatrix}\right]$). Finally, we define $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$, where $A\&B = [A_{-1}\ B] \mid V$ and $\mathcal{D}om(S) = V$. It follows from what we have proved so far that S is a system node. It only remains to show that S is given by (2.10), or equivalently, that

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{bmatrix} 1 \\ G \end{bmatrix} = \begin{bmatrix} L \\ K \end{bmatrix}. \tag{2.15}$$

The top row of this identity holds because

$$A_{-1}z + BGz = A_{-1}z + (L - A_{-1})z = Lz, \quad z \in \mathcal{Z}.$$

The bottom row follows directly from our definition of $C\&D$.

The final claim about the equivalence of smooth solutions of (1.1) and (1.4) follows immediately from (2.10). □

In Theorem 2.4 we give a converse to Theorem 2.3. In this theorem we start with a system node S of boundary control type and construct the corresponding boundary node Ξ. This time we define the solution space $\mathcal{Z}$ to be the range of $(\alpha - A_{-1})^{-1}[1\ B] : \left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right] \to \mathcal{X}$, where A is the main operator and B the control operator of S. Thus, B is injective and $\mathcal{R}an(B) \cap \mathcal{X} = \{0\}$. This implies that, for each fixed $\alpha \in \rho(A)$, every $w \in \mathcal{Z}$ has a unique representation

$$w = x + (\alpha - A_{-1})^{-1} Bu, \quad x \in \mathcal{X}_1,\ u \in \mathcal{U}. \tag{2.16}$$

We can therefore define a Hilbert space norm on $\mathcal{Z}$ by

$$\|w\|_{\mathcal{Z}}^2 = \|x\|_{\mathcal{X}_1}^2 + \|u\|_{\mathcal{U}}^2, \quad \text{where } w = x + (\alpha - A_{-1})^{-1} Bu. \tag{2.17}$$

With this norm the space $\mathcal{Z}$ is densely and continuously embedded in $\mathcal{X}$, and (2.11) holds, so that the complementary projections in $\mathcal{Z}$ onto $\mathcal{X}_1$, respectively $(\alpha - A_{-1})^{-1} B\mathcal{U}$ are continuous. Furthermore, the operator $(\alpha - A_{-1})^{-1} B$ is a bounded linear operator mapping $\mathcal{U}$ one-to-one onto its closed range, and it has a bounded inverse defined on its range. Different values of α gives different but equivalent norms in (2.17). For more details, see, e.g., [31, p. 389] or [38, Lemma 5.2.2].

Theorem 2.4. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an operator node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ of boundary control type with main operator A, control operator B, observation operator C, and transfer function $\widehat{\mathfrak{D}}$. Define the spaces $\mathcal{X}_1$ and $\mathcal{X}_{-1}$ and the extended operator A_{-1} as described in the paragraph preceding Definition* 2.1. *Then S induces a* (*unique*) *boundary node $\Xi = (G, L, K)$ on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ in the following way:*[5]

(i) *The space $\mathcal{Z}$ is defined by* (2.11) (*as described above*), *with the norm defined in* (2.17).
(ii) *There exists a unique operator $G \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$ such that*

$$\mathcal{D}om(S) := \mathcal{R}an\left(\begin{bmatrix} 1 \\ G \end{bmatrix}\right).$$

The operator G surjective and $\mathcal{N}(G) = \mathcal{X}_1$ is dense in $\mathcal{X}$. The operator $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]$ is a bounded bijection of $\mathcal{Z}$ onto $\mathcal{D}om(S)$ (*with the graph norm* (2.1) *of $A\&B$*).
(iii) *The operator $L \in \mathcal{L}(\mathcal{Z}; \mathcal{X})$ is defined by*

$$L := A_{-1}|\mathcal{Z} + BG = [A_{-1}\ B]\begin{bmatrix} 1 \\ G \end{bmatrix}.$$

In particular, $L|\mathcal{N}(G) = A$ has a nonempty resolvent set.

[5] This is a slightly simplified version of [38, Theorem 5.2.6]. A slightly less precise version of this result is found in [31, Proposition 2.8].

(iv) *The operator* $K \in \mathcal{L}(\mathcal{Z}; \mathcal{Y})$ *is defined by*

$$K := C\&D \begin{bmatrix} 1 \\ G \end{bmatrix}.$$

The node Ξ *is an internally well-posed boundary node if and only if* S *is a system node. The operator node that we obtain by applying Theorem* 2.3 *to* Ξ *coincides with the given operator node* S.

Our proof of Theorem 2.4 uses the following alternative characterisation of a boundary node, which is also of independent interest (see Section 5).[6]

Proposition 2.5. *A triple* $\Xi := (G, L, K)$ *is a boundary node on the Hilbert spaces* $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ *if and only if the following conditions are satisfied*:

(i) *There exists a Hilbert space* $\mathcal{Z}$, *such that the embedding* $\mathcal{Z} \subset \mathcal{X}$ *is dense and continuous*;
(ii) $L \in \mathcal{L}(\mathcal{Z}; \mathcal{X})$, $G \in \mathcal{L}(\mathcal{Z}; \mathcal{X})$ *and* $K \in \mathcal{L}(\mathcal{Z}; \mathcal{Y})$;
(iii) G *is surjective and* $\mathcal{N}(G)$ *is dense in* $\mathcal{X}$; *and*
(iv) $(\alpha - L)|\mathcal{N}(G)$ *maps* $\mathcal{N}(G)$ *one-to-one onto* $\mathcal{X}$ *for some* $\alpha \in \mathbb{C}$.

Proof. It is clear that (i)–(iv) are necessary conditions when $\mathcal{Z}$ is the solution space of Ξ. For the sufficiency part we note that conditions (i), (ii) and (iv) imply (iv) of Definition 1.1 since bounded bijections have bounded inverses.

We complete the proof by showing that $\left[\begin{smallmatrix} G \\ L \\ K \end{smallmatrix}\right]$ is closed with domain $\mathcal{Z}$. Suppose that $z_n \in \mathcal{Z}$, $z_n \to z$ in $\mathcal{X}$, $x_n := Lz_n \to x$ in $\mathcal{X}$, $u_n := Gz_n \to u$ in $\mathcal{U}$ and $y_n := Kz_n \to y$ in $\mathcal{Y}$. Choose α as in condition (iv). Then $\left[\begin{smallmatrix} G \\ -\alpha+L \end{smallmatrix}\right] \in \mathcal{L}(\mathcal{Z}; \left[\begin{smallmatrix} \mathcal{U} \\ \mathcal{X} \end{smallmatrix}\right])$ is a bijection (see conditions (i) and (ii) for boundedness, and conditions (iii) and (iv) for bijectivity), and it has a bounded inverse. As $z_n = \left[\begin{smallmatrix} G \\ -\alpha+L \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} u_n \\ -\alpha z_n + x_n \end{smallmatrix}\right]$, we find that $z \in \mathcal{Z}$, $z_n \to z$ in $\mathcal{Z}$ and $z = \left[\begin{smallmatrix} G \\ -\alpha+L \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} u \\ -\alpha z + x \end{smallmatrix}\right]$. Thus $u = Gz$ and $x = Lz$, and by the continuity of K, $y = Kz$. Thus (ii) of Definition 1.1 holds. □

Proof of Theorem 2.4. We begin by proving (ii). Fix some $\alpha \in \rho(A)$. As we observed in the paragraph preceding Theorem 2.4, each $w \in \mathcal{Z}$ has a unique decomposition $w = x + (\alpha - A_{-1})^{-1}Bu$ where $x \in \mathcal{X}_1$ and $u \in \mathcal{U}$. Define $Gw := u$. Then $G \in \mathcal{L}(\mathcal{Z}; \mathcal{U})$ and $\mathcal{N}(G) = \mathcal{X}_1$ (G is the projection of $\mathcal{Z}$ onto $(\alpha - A_{-1})^{-1}B\mathcal{U}$ along $\mathcal{X}_1$ followed by the inverse of $(\alpha - A_{-1})^{-1}B$).

We next show that $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]\mathcal{Z} = \mathcal{D}om(S)$, and begin with the inclusion $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]\mathcal{Z} \subset \mathcal{D}om(S)$. Let $w \in \mathcal{Z}$, and split w as in (2.16). Then

$$A_{-1}w + BGw = A_{-1}\big(x + (\alpha - A_{-1})^{-1}Bu\big) + Bu = Ax + \alpha(\alpha - A_{-1})^{-1}Bu \in \mathcal{X}.$$

Thus, $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]\mathcal{Z} \subset \mathcal{D}om(S)$. Conversely, suppose that $\left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right] \in \mathcal{D}om(S)$, i.e., $w \in \mathcal{X}$, $u \in \mathcal{U}$, and $z := A_{-1}w + Bu \in \mathcal{X}$. Then

$$w = (\alpha - A_{-1})^{-1}(\alpha w - z) + (\alpha - A_{-1})^{-1}Bu \in \mathcal{X}_1 + (\alpha - A_{-1})^{-1}B\mathcal{U} = \mathcal{Z},$$

[6] This characterisation resembles the one used by Salamon [31].

and $Gw = u$. Thus, $\mathcal{D}om(S) = \left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right] \mathcal{Z}$. The same argument shows that G is surjective (as S is an operator node, for each $u \in \mathcal{U}$ there is some $x \in \mathcal{X}$ such $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}om(S)$, and $Gx = u$).

All of $\|A_{-1}w + BGw\|_{\mathcal{X}}$, $\|w\|_{\mathcal{X}}$, and $\|u\|$ are dominated by $\|w\|_{\mathcal{Z}}$ up to multiplicative constants, so the mapping $\left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]$ from $\mathcal{Z}$ into $\mathcal{D}om(S)$ is continuous with respect to the graph norm (2.1) of $A\&B$, hence a bounded bijection. This is a graph representation of $\mathcal{D}om(S)$ over $\mathcal{Z}$, and hence it determines G uniquely. This completes our proof of (ii).

The claim (iii) is obvious, and so is (iv). Since G, L and K, satisfy (i)–(iv) it follows from Proposition 2.5 that (G, L, K) is a boundary node. It is also clear that Ξ is internally well-posed if and only if S is a system node. The final claim of Theorem 2.4 is also easily verified. □

We end this section by using the one-to-one correspondence between internally well-posed boundary nodes and system nodes of boundary control type to get an existence result for solutions of (1.1).

Lemma 2.6. *Let $\Xi := (G, L, K)$ be an internally well-posed boundary node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. Then, for all $z_0 \in \mathcal{X}$ and $u \in C^2(\mathbb{R}^+; \mathcal{U})$ with $Gz_0 = u(0)$ the first, second and fourth equation in* (1.1) *have a unique solution $z \in C^1(\mathbb{R}^+; \mathcal{X}) \cap C(\mathbb{R}^+; \mathcal{Z})$. Hence we can define $y \in C(\mathbb{R}^+; \mathcal{Y})$ by the third equation in* (1.1).

Proof. This follows immediately from Lemma 2.2 and Theorem 2.3 (define $z = \left[\begin{smallmatrix} 1 \\ G \end{smallmatrix}\right]^{-1} \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, and use (2.10) to convert (1.4) into (1.1)). □

3. Time–flow invertibility

We now define what we mean by the *time–flow invertibility of an operator node* and prove Theorem 1.6.

Definition 3.1. Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be an operator node on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$. We call this operator node *time–flow invertible* if there exists an operator node

$$S^{\leftarrow} = \begin{bmatrix} [A\&B]^{\leftarrow} \\ [C\&D]^{\leftarrow} \end{bmatrix} \quad \text{on } (\mathcal{Y}, \mathcal{X}, \mathcal{U})$$

which together with S satisfies the following conditions: the operator $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ maps $\mathcal{D}om(S)$ continuously onto $\mathcal{D}om(S^{\leftarrow})$, its inverse is $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^{\leftarrow} \end{smallmatrix}\right]$, and

$$\begin{bmatrix} A\&B \\ C\&D \end{bmatrix} = \begin{bmatrix} -[A\&B]^{\leftarrow} \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ [C\&D]^{\leftarrow} \end{bmatrix}^{-1} \quad \big(\text{on } \mathcal{D}om(S)\big), \tag{3.1}$$

$$\begin{bmatrix} [A\&B]^{\leftarrow} \\ [C\&D]^{\leftarrow} \end{bmatrix} = \begin{bmatrix} -A\&B \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix}^{-1} \big(\text{on } \mathcal{D}om(S^{\leftarrow})\big). \tag{3.2}$$

In this case we call S and $S^{\leftarrow}$ *time–flow inverses* of each other.

For more details, see [38, Section 6.5].

Proof of Theorem 1.6. Suppose that Ξ is time–flow invertible. Define the operator node S by (2.10), and define $S^{\leftarrow}$ by

$$S^{\leftarrow}=\begin{bmatrix}[A\&B]^{\leftarrow}\\ [C\&D]^{\leftarrow}\end{bmatrix}:=\begin{bmatrix}-L\\ G\end{bmatrix}\begin{bmatrix}1\\ K\end{bmatrix}^{-1},\qquad \mathcal{D}om(S^{\leftarrow})=\mathcal{R}an\left(\begin{bmatrix}1\\ K\end{bmatrix}\right). \tag{3.3}$$

By Theorem 2.3, $S^{\leftarrow}$ is an operator node. Clearly $\left[\begin{smallmatrix}1 & 0\\ C\&D\end{smallmatrix}\right]=\left[\begin{smallmatrix}1\\ K\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ G\end{smallmatrix}\right]^{-1}$ maps $\mathcal{D}om(S)$ one-to-one onto $\mathcal{D}om(S^{\leftarrow})$ with the bounded inverse $\left[\begin{smallmatrix}1\\ G\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ K\end{smallmatrix}\right]^{-1}=\left[\begin{smallmatrix}1 & 0\\ [C\&D]^{\leftarrow}\end{smallmatrix}\right]$. Moreover,

$$\begin{aligned}\begin{bmatrix}-A\&B\\ 0\quad 1\end{bmatrix}\begin{bmatrix}1 & 0\\ C\&D\end{bmatrix}^{-1}&=\left(\begin{bmatrix}-L\\ G\end{bmatrix}\begin{bmatrix}1\\ G\end{bmatrix}\right)\left(\begin{bmatrix}1\\ G\end{bmatrix}^{-1}\begin{bmatrix}1\\ K\end{bmatrix}^{-1}\right)\\ &=\begin{bmatrix}-L\\ G\end{bmatrix}\begin{bmatrix}1\\ K\end{bmatrix}^{-1}=\begin{bmatrix}[A\&B]^{\leftarrow}\\ [C\&D]^{\leftarrow}\end{bmatrix},\end{aligned}$$

and a similar computation shows that also (3.1) holds. Thus, S is time–flow invertible with time–flow inverse $S^{\leftarrow}$.

Conversely, suppose that S is time–flow invertible with time–flow inverse $S^{\leftarrow}$. We claim that Ξ is then time–flow invertible. The time–flow invertibility of S implies that $\left[\begin{smallmatrix}1 & 0\\ C\&D\end{smallmatrix}\right]$ is a bijection between $\mathcal{D}om(S)$ and $\mathcal{D}om(S^{\leftarrow})$. It follows from (2.10) that $\left[\begin{smallmatrix}1 & 0\\ C\&D\end{smallmatrix}\right]=\left[\begin{smallmatrix}1\\ K\end{smallmatrix}\right]\left[\begin{smallmatrix}1\\ G\end{smallmatrix}\right]^{-1}$, and hence $\left[\begin{smallmatrix}1\\ K\end{smallmatrix}\right]\mathcal{Z}=\mathcal{D}om(S^{\leftarrow})$. Since $S^{\leftarrow}$ is an operator node, for every $y\in\mathcal{Y}$ there is some $x\in\mathcal{X}$ such that $\left[\begin{smallmatrix}x\\ y\end{smallmatrix}\right]\in\mathcal{D}om(S^{\leftarrow})$. Thus K is surjective. By (2.10) and (3.2), $S^{\leftarrow}$ is given by (3.3).

Denote the main operator of $S^{\leftarrow}$ by $A^{\leftarrow}$. It follows from part (iii) of Definition 2.1 that

$$\mathcal{D}om(A^{\leftarrow})=\left\{x\in\mathcal{X}\colon \begin{bmatrix}x\\ 0\end{bmatrix}\in\mathcal{D}om(S^{\leftarrow})\right\}.$$

Since $\left[\begin{smallmatrix}1\\ K\end{smallmatrix}\right]\mathcal{Z}=\mathcal{D}om(S^{\leftarrow})$, this means that $\mathcal{D}om(A^{\leftarrow})=\mathcal{N}(K)$. Finally, from (3.3) we also see that $A^{\leftarrow}=-L|\mathcal{N}(K)$. By the assumption that S is time–flow invertible, $\mathcal{D}om(A^{\leftarrow})=\mathcal{N}(K)$ is dense in $\mathcal{X}$, and $A^{\leftarrow}=-L|\mathcal{N}(K)$ has a nonempty resolvent set. Clearly, $\left[\begin{smallmatrix}K\\ -L\\ G\end{smallmatrix}\right]$ is closed since $\left[\begin{smallmatrix}G\\ L\\ K\end{smallmatrix}\right]$ is closed (with the same domain). By definition, Ξ is time–flow invertible. □

The preceding proof gives us a little more that what is explicitly stated in Theorem 1.6.

Corollary 3.2. *Suppose that the boundary node Ξ is time–flow invertible, and denote the corresponding time–flow invertible operator node by S. Then the time–flow inverse $S^{\leftarrow}$ of S is the operator node induced by the time–flow inverse $\Xi^{\leftarrow}$ in the way described in Theorem* 2.3.

Proof. See (3.1)–(3.3). □

4. Conservative systems

We now define what we mean by the *conservativity of an operator node* and prove Theorems 1.5 and 1.7.

Definition 4.1. An operator node S on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ is *energy preserving* if it is a system node and all smooth solutions of (1.4) on $\mathbb{R}^+$ satisfy (1.2). It is *conservative* if both S and S^* are energy preserving.

In the case of an energy preserving operator node, the Green–Lagrange identity (1.3) becomes

$$2\operatorname{Re}\left\langle z, A\&B \begin{bmatrix} z \\ u \end{bmatrix} \right\rangle_{\mathcal{X}} = \|u\|_{\mathcal{U}}^2 - \left\| C\&D \begin{bmatrix} z \\ u \end{bmatrix} \right\|_{\mathcal{Y}}^2, \quad \begin{bmatrix} z \\ u \end{bmatrix} \in \mathcal{D}om(S). \tag{4.1}$$

Proposition 4.2. *Let $\Xi = (G, L, K)$ be a boundary node, and let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be the corresponding operator node S (see Theorems* 2.3 *and* 2.4) *with main operator A. Then the following conditions are equivalent*:

(i) *Ξ is energy preserving* (*in the sense of Definition* 1.2).
(ii) *S is energy preserving* (*in the sense of Definition* 4.1).
(iii) *$\rho(L|\mathcal{N}(G)) \cap \overline{\mathbb{C}^+} \neq \emptyset$, and* (1.3) *holds* (*here $\overline{\mathbb{C}^+} = \{\alpha \in \mathbb{C} \mid \operatorname{Re}\alpha \geqslant 0\}$*).
(iv) *$\rho(A) \cap \overline{\mathbb{C}^+} \neq \emptyset$, and* (4.1) *holds.*

Proof. (i) $\Leftrightarrow$ (ii). The equivalence of (i) and (ii) is an immediate consequence of Definitions 1.2 and 4.1, and the one-to-one correspondence between solutions of (1.1) and solutions of (1.4) established in Theorem 2.3.

(ii) $\Rightarrow$ (iv). Assume (ii). Clearly the internal well-posedness of S implies that $\rho(A) \cap \overline{\mathbb{C}^+} \neq \emptyset$. Let $\left[\begin{smallmatrix} z_0 \\ u_0 \end{smallmatrix}\right] \in \mathcal{D}om(S)$, and let u be the constant function $u(t) = u_0$ for all $t \geqslant 0$. Let z be the solution of (2.7) with $z(0) = z_0$ given by Lemma 2.2, and define y by (2.8). Then (1.2) with $t = 0$ implies that (4.1) holds with $\left[\begin{smallmatrix} z \\ u \end{smallmatrix}\right]$ replaced by $\left[\begin{smallmatrix} z_0 \\ u_0 \end{smallmatrix}\right]$ (since $\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_{\mathcal{X}}^2 = 2\operatorname{Re}\langle z(t), \dot{z}(t)\rangle_{\mathcal{X}}$).

(iv) $\Rightarrow$ (ii). Taking $z \in \mathcal{D}om(A)$ and $u = 0$ in (4.1) we find that A is dissipative, i.e., $\operatorname{Re}\langle z, Az\rangle \leqslant 0$ for all $z \in \mathcal{D}om(A)$. This together with the condition $\rho(A) \cap \overline{\mathbb{C}^+} \neq \emptyset$ implies that A generates a contraction semigroup; see, e.g., [28, Theorem 4.3, p. 14] or [38, Theorem 3.4.8]. Thus, S is a system node. It follows from (4.1) that all smooth solutions of (1.4) satisfy (1.2), and hence S is energy-preserving.

(iii) $\Leftrightarrow$ (iv). This follows directly from Theorem 2.3 and 2.4 (take u in (4.1) to be $u = Gz$, and use (2.10)). □

Proof of Theorem 1.5. The necessity of (i)–(iv) for the conservativity of Ξ follows directly from Definitions 1.1 and 1.3, and Proposition 4.2. Conversely, if these conditions hold, then according to Definitions 1.1 and 1.3, Ξ is time–flow invertible. Proposition 4.2 can be applied both to Ξ and to the time–flow inverse $\Xi^{\leftarrow}$: conditions (ii) and (iv) imply that Ξ is energy preserving, and conditions (iii) and (iv) imply that $\Xi^{\leftarrow}$ is energy preserving. Thus, Ξ is conservative. □

Our proof of Theorem 1.7 is based on the following characterization of a conservative system node.

Proposition 4.3. *Let S be a system node. Then the following conditions are equivalent*:

(i) *S is conservative.*
(ii) *S is time–flow invertible, and the time–flow inverse $S^{\leftarrow}$ is given by $S^{\leftarrow} = S^*$.*

(iii) *S is energy preserving and time–flow invertible, and the time–flow inverse $S^{\leftarrow}$ is a system node.*
(iv) *S is time–flow invertible, and both S and the time–flow inverse $S^{\leftarrow}$ are energy preserving.*

This proposition is of some independent interest. It can be derived fairly easily from the results presented in [26], but unfortunately it was not included in [26]. Since a self-contained proof is would be rather long we assume below that the reader has access to [26].

Proof of Proposition 4.3. (i) $\Rightarrow$ (ii). Let (i) hold. We denote the dual system node by $S^* = \left[\begin{smallmatrix} [A\&B]^d \\ [C\&D]^d \end{smallmatrix}\right]$. By [26, Theorem 4.2] (and its proof), $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ is a bijection of $\mathcal{D}om(S)$ onto $\mathcal{D}om(S^*)$ with inverse $\left[\begin{smallmatrix} 1 & 0 \\ [C\&D]^d \end{smallmatrix}\right]$. The operator $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ is continuous from $\mathcal{D}om(S)$ into $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{Y} \end{smallmatrix}\right]$ with range equal to $\mathcal{D}om(S^*)$, so it is a bounded bijection between $\mathcal{D}om(S)$ and $\mathcal{D}om(S^*)$ (both domains being equipped with the respective graph norms). By [26, Theorem 4.2],

$$S^* = \begin{bmatrix} -A\&B \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix}^{-1}.$$

Applying the same argument to the dual system we get the same identity where S and S^* have changed places. This implies that S is time–flow invertible, with time–flow inverse $S^{\leftarrow} = S^*$.

(ii) $\Rightarrow$ (i). Let (ii) hold. Then, by Definition 3.1, $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ maps $\mathcal{D}om(S)$ onto $\mathcal{D}om(S^*) = \mathcal{D}om(S^{\leftarrow})$ and

$$S^* = S^{\leftarrow} = \begin{bmatrix} -A\&B \\ 0 \quad 1 \end{bmatrix} \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix}^{-1}.$$

By [26, Theorem 3.2], S is energy preserving. We can then apply [26, Theorem 4.2] to conclude that S is conservative.

(i)&(ii) $\Rightarrow$ (iv) $\Rightarrow$ (iii). These two implications follow directly from the definition of conservativity of an operator node.

(iii) $\Rightarrow$ (ii). Assume (iii). By Definition 1.3, $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ maps $\mathcal{D}om(S)$ onto $\mathcal{D}om(S^{\leftarrow})$ and

$$S^{\leftarrow} \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix} = \begin{bmatrix} -A\&B \\ 0 \quad 1 \end{bmatrix}.$$

By [26, Theorem 3.2], $\left[\begin{smallmatrix} 1 & 0 \\ C\&D \end{smallmatrix}\right]$ maps $\mathcal{D}om(S)$ into $\mathcal{D}om(S^*)$, and hence $\mathcal{D}om(S^{\leftarrow}) \subset \mathcal{D}om(S^*)$. Moreover, by the same theorem,

$$S^* \begin{bmatrix} 1 \quad 0 \\ C\&D \end{bmatrix} = \begin{bmatrix} -A\&B \\ 0 \quad 1 \end{bmatrix}.$$

Thus, $S^{\leftarrow} = S^*|\mathcal{D}om(S^{\leftarrow})$, and, in particular, $A^{\leftarrow} = A^*|\mathcal{D}om(A^{\leftarrow})$ where $A^{\leftarrow}$ and A^* are the main operators of $S^{\leftarrow}$ and S^*, respectively. But both $A^{\leftarrow}$ and A^* are the generators of C_0 semigroups on $\mathcal{X}$, and so their resolvent sets have a nonzero intersection. This implies that their domains coincide; hence $A^{\leftarrow} = A^*$, and the extended state spaces $X_{-1}^{\leftarrow}$ and X_{-1}^{d} also coincide. Recall that the control operator of S^* is C^*. Both $[\,A_{-1}^{\leftarrow} \; B^{\leftarrow}\,]$ and $[\,A_{-1}^{*} \; C^*\,]$ are bounded operator from $\left[\begin{smallmatrix} \mathcal{X} \\ \mathcal{U} \end{smallmatrix}\right]$ to $X_{-1}^{\leftarrow} = X_{-1}^{*}$, and they coincide on the dense subset $\mathcal{D}om(S^{\leftarrow})$. This implies that

$[\, A_{-1}^{\leftarrow} \; B^{\leftarrow} \,] = [\, A_{-1}^{*} \; C^{*} \,]$. Since these two operators determine the domains of $S^{\leftarrow}$ and S^{*}, we have $\mathcal{D}om(S^{\leftarrow}) = \mathcal{D}om(S^{*})$ and $S^{\leftarrow} = S^{*}$. □

Proof of Theorem 1.7. Theorem 1.7 follows immediately from Theorem 1.6 and Propositions 4.2 and 4.3. □

5. The reflecting mirror

In this section we apply Theorems 1.5 and 1.7 to a PDE describing a reflecting mirror, and we conclude that it induces a conservative system node. This example is classical. A more general version has been treated as an example of a "thin air" system in [44, Section 7] by means of a construction that bears some resemblance to feedback techniques appearing in [41]. Our approach resembles the techniques of [19].

Suppose that $n \geqslant 2$ and let $\Omega \subset \mathbb{R}^n$ be a bounded domain (open connected set) with C^2-boundary $\partial\Omega$. We assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$ where both Γ_0 and Γ_1 are nonempty.[7] Thus Ω is not simply connected. A simple example of this geometry in $\mathbb{R}^2$ is provided by the annulus

$$\Omega = \bigl\{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ 1/4 < \xi_1^2 + \xi_2^2 < 1\bigr\} \tag{5.1}$$

with $\Gamma_0 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ \xi_1^2 + \xi_2^2 = 1/4\}$ and $\Gamma_1 = \{(\xi_1, \xi_2) \in \mathbb{R}^2 \colon\ \xi_1^2 + \xi_2^2 = 1\}$, or the other way around.

We consider the linear system described by the system of equations

$$\begin{cases} z_{tt}(t,\xi) = \Delta z(t,\xi) \quad \text{for } \xi \in \Omega \text{ and } t \geqslant 0, \\ \sqrt{2}\, u(t,\xi) = z_t(t,\xi) + \frac{\partial z}{\partial \nu}(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geqslant 0, \\ \sqrt{2}\, y(t,\xi) = -z_t(t,\xi) + \frac{\partial z}{\partial \nu}(t,\xi) \quad \text{for } \xi \in \Gamma_1 \text{ and } t \geqslant 0, \\ z(t,\xi) = 0 \quad \text{for } \xi \in \Gamma_0 \text{ and } t \geqslant 0, \text{ and} \\ z(0,\xi) = z_0(\xi), \quad z_t(0,\xi) = w_0(\xi) \quad \text{for } \xi \in \Omega. \end{cases} \tag{5.2}$$

Here $z_t(t,\xi)$ stands for the time derivative and $\frac{\partial z}{\partial \nu}(t,\xi)$ for the normal derivative of z at time t at the boundary point ξ. Before going any further, let us recall the definitions of the Sobolev spaces and the boundary trace mappings that we need.

The spaces $H^m(\Omega) = W_2^m(\Omega)$ for $m = 1, 2$, are defined as usual, i.e.,

$$H^m(\Omega) := \bigl\{ f \in L^2(\Omega) \colon\ D^\alpha f \in L^2(\Omega) \text{ for all multi-indices } |\alpha| \leqslant m \bigr\} \tag{5.3}$$

where the differentiation D^α is understood in the sense of distributions; see, e.g., [17, Definition 1.3.2.1] or [22, p. 1]. There is yet another equivalent *closure definition* for $H^m(\Omega)$, see, e.g., [1, p. 60]. We use the Hilbert space norm $\|f\|_{H^m(\Omega)}^2 = \sum_{|\alpha| \leqslant m} \|D^\alpha f\|_{L^2(\Omega)}^2$ in $H^m(\Omega)$.

We shall also need the fractional Sobolev space $H^s(\Omega)$ with $s = 3/2$. This space can be defined in several different but equivalent ways. It can, for example, be characterized (for any $s > 0$)

[7] The sets Γ_1 and Γ_0 are allowed to have zero distance in [44], and there Ω can be simply connected. The analysis in [44] is based on stronger background results from [29].

as the restriction $H^s(\Omega) := \{f|\Omega\colon\ f \in H^s(\mathbb{R}^n)\}$ to Ω of the set of all functions in $H^s(\mathbb{R}^n)$, where

$$\widehat{H}^s(\mathbb{R}^n) := \{\hat{f} \in L^2(\mathbb{R}^n)\colon\ (1 + |\cdot|^2)^{s/2}\hat{f}(\cdot) \in L^2(\mathbb{R}^n)\}$$

is defined on the Fourier transform side; see [17, Definition 1.3.1.3] or [22, p. 30]. By Plancherels theorem (see [17, comment on p. 16]), $H^s(\mathbb{R}^n) = W_2^s(\mathbb{R}^n)$, where

$$W_2^s(\mathbb{R}^n) := \left\{ f \in W_2^m(\mathbb{R}^n)\colon\ \iint\limits_{\mathbb{R}^n \times R^n} \frac{|D^\alpha f(\xi) - D^\alpha f(\nu)|^2}{|\xi - \nu|^{n+2\sigma}}\, d\xi\, d\nu < \infty \right\},$$

$s = m + \sigma$, $m \in \mathbb{Z}_+$ and $\sigma \in (0, 1)$ for all $s \in \mathbb{R} \setminus \mathbb{Z}_+$; see [17, Definition 1.3.1.1]. We denote by $\overline{W}_2^s(\Omega) := \{f|\Omega\colon\ f \in W_2^s(\mathbb{R}^n)\}$ the restrictions to Ω of the set of functions in $W_2^s(\mathbb{R}^n)$, and define $W_2^s(\Omega)$ in the same way as $W_2^s(\mathbb{R}^n)$ with $\mathbb{R}^n$ replaced by Ω. Then, by [17, Theorem 1.4.3.1], $H^s(\Omega) = W_2^s(\Omega) = \overline{W}_2^s(\Omega)$ for all $s > 0$ and for all domains Ω that have a C^2-boundary. Still another way to characterize the same space $H^s(\Omega)$ is to interpolate between two spaces of type $H^m(\Omega)$ with integer m as is done in [22, Theorem 9.1, p. 40].[8] A Hilbert space norm for $H^s(\Omega)$ can be introduced in a number of equivalent ways[9] so that the embedding $H^s(\Omega) \subset L^2(\Omega)$ becomes continuous.

The boundary spaces $L^2(\partial\Omega)$, $L^2(\Gamma_0)$ and $L^2(\Gamma_1)$ are defined using the standard $(n-1)$-dimensional Hausdorff measure for $(n-1)$-dimensional hypersurfaces in $\mathbb{R}^n$. We shall write $L^2(\partial\Omega) = L^2(\Gamma_0) \oplus L^2(\Gamma_1)$ by extending functions in $L^2(\Gamma_0)$ or $L^2(\Gamma_1)$ by zero on the other component of Γ. The boundary Sobolev spaces $H^s(\partial\Omega)$, $H^s(\Gamma_0)$, and $H^s(\Gamma_1)$ are defined for $s > 0$ by covering the manifold $\partial\Omega$ with charts $(\mathcal{O}_j, \psi_j)$ of $\mathbb{R}^n$ such that $\phi_j(\mathcal{O}_j \cap \partial\Omega) \subset \mathbb{R}^{n-1} \times \{0\}$ for $j = 1, \dots, m$. Let $\alpha_j \in \mathcal{D}(\partial\Omega)$ be a partition of unity satisfying $\sum_{j=1}^m \alpha_j(\xi) = 1$ and $\operatorname{supp}\alpha_j \subset \mathcal{O}_j$ for $j = 1, \dots, m$. Given $f \in L^2(\partial\Omega)$ and $y' \in \mathbb{R}^{n-1}$, we define $\psi_{j,f}^*(y') := (\alpha_j f)(\psi_j^{-1}(y', 0))$ if $(y', 0) \in \psi_j(\mathcal{O}_j)$ and $\psi_{j,f}^*(y') := 0$ otherwise. Then

$$H^s(\partial\Omega) := \{f \in L^2(\partial\Omega)\colon\ \psi_{j,f}^* \in H^s(\mathbb{R}^{n-1}) \text{ for all } j = 1, \dots, m\}$$

with the Hilbert space norm $\|f\|^2_{H^s(\partial\Omega)} = \sum_{j=1}^m \|\psi_{j,f}\|^2_{H^s(\mathbb{R}^{n-1})}$. Recalling our standing assumption $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, we may choose the charts so that either $\mathcal{O}_j \cap \partial\Omega \subset \Gamma_0$ or $\mathcal{O}_j \cap \partial\Omega \subset \Gamma_1$ for all j, and thus $H^{1/2}(\partial\Omega) = H^{1/2}(\Gamma_0) \oplus H^{1/2}(\Gamma_1)$. For further details, see [22, pp. 34–35] for domains having a C^∞-boundary. Additional complications arise in the case of C^2-boundary, see [17, Definition 1.3.3.2] and the discussion following it.

If the domain Ω is the annulus in (5.1), then a more intuitive description can be given for $H^{1/2}(\Gamma_1)$ (and similarly for $H^{1/2}(\Gamma_0)$) using the fact that the associated Laplace–Beltrami operator is now given by $(\Delta_{\Gamma_1} f)(\cos\phi, \sin\phi) = -\sum_{j=-\infty}^{\infty} j^2 a_j e^{ij\phi}$ for all $f \in C^\infty(\Gamma_1)$ and $\phi \in (-\pi, \pi]$ where $\sum_{j=-\infty}^{\infty} a_j e^{ij\phi} := f(\cos\phi, \sin\phi)$. Indeed, then $f \in H^{1/2}(\Gamma_1)$ if and only if $\{|j|^{1/2} a_j\} \in \ell^2(\mathbb{Z})$ by [22, Remark 7.6, p. 37].

[8] Note that Ω has a C^∞-boundary in [22] but this assumption can be often relaxed.

[9] For example, by interpolation theory, or by restriction of the natural norm of the space $W_2^s(\mathbb{R}^n)$ above. The particular choice of the norm is irrelevant in this paper.

The *Dirichlet trace operator* γ is first defined for functions $f \in C^\infty(\overline{\Omega})$ by setting $\gamma f := f|\partial\Omega$. This operator has a unique extension to a bounded operator from $H^1(\Omega)$ to $L^2(\partial\Omega)$ that actually satisfies $\gamma \in \mathcal{L}(H^1(\Omega); H^{1/2}(\partial\Omega))$ by [17, Theorem 1.5.1.3]. Let π be the orthogonal projection of $L^2(\partial\Omega)$ onto its subspace $L^2(\Gamma_1)$. Since $\gamma \in \mathcal{L}(H^1(\Omega); L^2(\partial\Omega))$, we have $(I-\pi)\gamma \in \mathcal{L}(H^1(\Omega); L^2(\partial\Omega))$ and the space $H^1_{\Gamma_0}(\Omega) := \mathcal{N}((I-\pi)\gamma)$ is a closed subspace of $H^1(\Omega)$. With a slight misuse of notation, we write henceforth $\pi f = f|\Gamma_1$, $(I-\pi)f = f|\Gamma_0$, and

$$H^1_{\Gamma_0}(\Omega) = \{f \in H^1(\Omega)\colon\ f|\Gamma_0 = 0\}. \tag{5.4}$$

Similarly, the operator $\gamma_0 := \pi\gamma|H^1_{\Gamma_0}(\Omega)$ is in $\mathcal{L}(H^1_{\Gamma_0}(\Omega); L^2(\Gamma_1))$, and we abbreviate it by writing $\gamma_0 f = f|\Gamma_1$.

The *Neumann trace operator* $\gamma\frac{\partial}{\partial\nu}$ is first defined on $C^\infty(\overline{\Omega})$ (with values in $L^2(\partial\Omega)$) by setting $(\gamma\frac{\partial}{\partial\nu}f)(\xi) := \nu(\xi)\cdot\nabla f(\xi)$ for all $\xi \in \partial\Omega$ where $\nu(\xi)$ denotes the outward unit normal vector of $\partial\Omega$ at ξ.

For Ω having a C^2-boundary, the operator $\gamma\frac{\partial}{\partial\nu}$ has an extension from $C^\infty(\overline{\Omega})$ to a bounded operator (also denoted by $\gamma\frac{\partial}{\partial\nu}$) mapping $\mathcal{D}om(\Delta; L^2(\Omega)) := \{f \in L^2(\Omega)\colon\ \Delta f \in L^2(\Omega)\}$ into $H^{-3/2}(\partial\Omega)$; see [17, discussion on p. 54]. Here $\mathcal{D}om(\Delta; L^2(\Omega))$ is equipped with the norm $\|f\|^2_{\mathcal{D}om(\Delta;L^2(\Omega))} = \|f\|^2_{L^2(\Omega)} + \|\Delta f\|^2_{L^2(\Omega)}$, and the space $H^{-3/2}(\partial\Omega)$ is the dual of $H^{3/2}(\partial\Omega)$.

After these preparations, let us return to Eqs. (5.2). We obtain first order equations of the form (1.1) by noting that $z_{tt} = \Delta z$ is equivalent to the first order equation $\frac{d}{dt}\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 1 \\ \Delta & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} z \\ w \end{smallmatrix}\right]$. Let

$$\mathcal{Z}_0 := \left\{ f \in H^1_{\Gamma_0}(\Omega)\colon\ \Delta F \in L^2(\Omega) \text{ and } \frac{\partial f}{\partial\nu}\bigg|\Gamma_1 \in L^2(\Gamma_1)\right\} \tag{5.5}$$

with the norm $\|f\|^2_{\mathcal{Z}_0} = \|f\|^2_{H^1(\Omega)} + \|\Delta f\|^2_{L^2(\Omega)} + \|\frac{\partial f}{\partial\nu}|\Gamma_1\|^2_{L^2(\Gamma_1)}$. Here the Neumann trace $\frac{\partial f}{\partial\nu}$ is understood distributionally in the sense of [17, p. 54] as explained above.

The operator $\gamma_1 := \pi\gamma\frac{\partial}{\partial\nu}|\mathcal{Z}_0$ is in $\mathcal{L}(\mathcal{Z}_0; L^2(\Gamma_1))$, and we write $\gamma_1 f = \frac{\partial f}{\partial\nu}|\Gamma_1$. The spaces $\mathcal{Z}$, $\mathcal{X}$ and operator L are defined by

$$L := \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} : \mathcal{Z} \to \mathcal{X} \quad \text{with}$$
$$\mathcal{Z} := \mathcal{Z}_0 \times H^1_{\Gamma_0}(\Omega) \quad \text{and} \quad \mathcal{X} := H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$$

where $H^1_{\Gamma_0}(\Omega)$ and $\mathcal{Z}_0$ are given by (5.4) and (5.5), respectively. For the space $\mathcal{X}$, we use the energy norm

$$\left\|\begin{bmatrix} z_0 \\ w_0 \end{bmatrix}\right\|^2_{\mathcal{X}} := \|\nabla z_0\|^2_{L^2(\Omega)} + \|w_0\|^2_{L^2(\Omega)}. \tag{5.6}$$

By the Poincaré inequality, $\|z_0\|_{L^2(\Omega)} \leqslant M\|\nabla z_0\|_{L^2(\Omega)}$ for $z_0 \in H^1_{\Gamma_0}(\Omega)$. Therefore (5.6) defines a norm on $\mathcal{X}$, equivalent to the norm

$$\left\|\begin{bmatrix} z_0 \\ w_0 \end{bmatrix}\right\|^2_{H^1(\Omega)\times L^2(\Omega)} := \|z_0\|^2_{L^2(\Omega)} + \|\nabla z_0\|^2_{L^2(\Omega)} + \|w_0\|^2_{L^2(\Omega)},$$

see, e.g., [19, p. 168]. Thus $\mathcal{Z} \subset \mathcal{X}$ with a continuous embedding and $L \in \mathcal{L}(\mathcal{Z};\mathcal{X})$ with respect to the $\mathcal{Z}$-norm

$$\left\| \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \right\|_{\mathcal{Z}}^2 := \|z_0\|_{\mathcal{Z}_0}^2 + \|w_0\|_{L^2(\Omega)}^2 + \|\nabla w_0\|_{L^2(\Omega)}^2.$$

Defining $\mathcal{U} = \mathcal{Y} := L^2(\Gamma_1)$, the above properties of the trace mappings imply that $G \in \mathcal{L}(\mathcal{Z};\mathcal{U})$ and $K \in \mathcal{L}(\mathcal{Z};\mathcal{Y})$ when

$$G\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} := \frac{1}{\sqrt{2}}\left(\frac{\partial z_0}{\partial \nu}\Big|\Gamma_1 + w_0|\Gamma_1\right) \quad \text{and} \quad K\begin{bmatrix} z_0 \\ w_0 \end{bmatrix} := \frac{1}{\sqrt{2}}\left(\frac{\partial z_0}{\partial \nu}\Big|\Gamma_1 - w_0|\Gamma_1\right).$$

We have now constructed the triple $\Xi = (G, L, K)$. To show that Ξ is a boundary node on Hilbert spaces $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$, some facts from the elliptic regularity theory will be required. Following [41, p. 444], we denote the Neumann mapping $\widetilde{N}$ by

$$z_0 = \widetilde{N} g \quad \Leftrightarrow \quad \begin{cases} \Delta z_0 = 0 & \text{in } \Omega, \\ z_0|\Gamma_0 = 0 & \text{in } \Gamma_0, \\ \frac{\partial z_0}{\partial \nu}|\Gamma_1 = g & \text{in } \Gamma_1, \end{cases} \tag{5.7}$$

where $z_0 \in H^1_{\Gamma_0}(\Omega)$ is the unique variational solution. By the elliptic regularity theory, $\widetilde{N} \in \mathcal{L}(L^2(\Gamma_1); H^{3/2}(\Omega)) \cap \mathcal{L}(H^{1/2}(\Gamma_1); H^2(\Omega))$. Moreover, if $z_0 \in H^1_{\Gamma_0}(\Omega)$ is the unique variational solution of

$$\Delta z_0 = f \in L^2(\Omega), \qquad z_0|\Gamma_0 = 0, \qquad \frac{\partial z_0}{\partial \nu}\Big|\Gamma_1 = 0,$$

then $z_0 \in H^2(\Omega)$, see [19, Section 4]. Hence, the unique variational solution of

$$\Delta z_0 = f \in L^2(\Omega), \qquad z_0|\Gamma_0 = 0, \qquad \frac{\partial z_0}{\partial \nu}\Big|\Gamma_1 = g$$

belongs to $H^{3/2}(\Omega)$ (or to $H^2(\Omega)$) if $g \in L^2(\Gamma_1)$ (or $g \in H^{1/2}(\Gamma_1)$, respectively).

It is an interesting fact that both the spaces $\mathcal{Z}_0$ and $\mathcal{N}(G)$ have additional regularity:

Proposition 5.1. *Under the standing assumptions on Ω, we have*

$$\mathcal{N}(G) = \left\{ \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \in \left(H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)\right) \times H^1_{\Gamma_0}(\Omega) \colon \frac{\partial z_0}{\partial \nu}\Big|\Gamma_1 = w_0|\Gamma_1 \right\}. \tag{5.8}$$

Proof. That $\mathcal{Z}_0 \subset H^{3/2}(\Omega)$ follows from elliptic regularity.

To verify (5.8), we argue as follows: If $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{N}(G)$, then $w_0 \in H^1(\Omega)$ and hence $w_0|\Gamma_1 \in H^{1/2}(\Gamma_1)$. But then z_0 is the variational solution of

$$\Delta z_0 = f \in L^2(\Omega), \qquad z_0|\Gamma_0 = 0, \qquad \frac{\partial z_0}{\partial \nu}\Big|\Gamma_1 = w_0|\Gamma_1 \in H^{1/2}(\Gamma_1),$$

and thus $z_0 \in H^2(\Omega)$ by the elliptic regularity theory. $\square$

Proposition 5.2. *Let the operators L, G, K and spaces $\mathcal{Z}$, $\mathcal{X}$ be defined as above. Then $\Xi = (G, L, K)$ is a time–flow invertible boundary node that satisfies* $0 \in \rho(L|\mathcal{N}(G)) \cap \rho(-L|\mathcal{N}(K))$.

Proof. It has been already shown in the above discussion that conditions (i) and (ii) of Proposition 2.5 are satisfied.

Since $\widetilde{N} \in \mathcal{L}(L^2(\Gamma_1); H^{3/2}(\Omega))$, we have $\widetilde{N} L^2(\Gamma_1) \subset \mathcal{Z}_0$. Furthermore, for any $g \in L^2(\Gamma_1)$ we have $\gamma_1 \widetilde{N} g = g$. Thus $\gamma_1 \mathcal{Z}_0 = L^2(\Gamma_1)$ and G is surjective. It follows from (5.8) that $\mathcal{N}(G)$ is dense in $\mathcal{X} = H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, which can be seen as follows. Let $\epsilon > 0$, $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{X}$ and choose $\left[\begin{smallmatrix} \tilde{z} \\ \tilde{w} \end{smallmatrix}\right] \in (H^1_{\Gamma_0}(\Omega) \cap H^2(\Omega)) \times H^1_{\Gamma_0}(\Omega)$ with $\|\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} \tilde{z} \\ \tilde{w} \end{smallmatrix}\right]\|_{\mathcal{X}} < \epsilon$. It is possible to construct $\hat{w} \in H^1_{\Gamma_0}(\Omega)$ satisfying $\|\hat{w}\|_{L^2(\Omega)} < \epsilon$ and $\hat{w}|\Gamma_1 = \tilde{w}|\Gamma_1 - \frac{\partial \tilde{z}}{\partial \nu}|\Gamma_1$; indeed, such $\hat{w}$ could be made to vanish in almost all of Ω except for points very close to Γ_1 by using a suitable smooth "mollifier." Now

$$\begin{bmatrix} \tilde{z}_0 \\ \tilde{w}_0 \end{bmatrix} := \begin{bmatrix} \tilde{z} \\ \tilde{w} \end{bmatrix} - \begin{bmatrix} 0 \\ \hat{w} \end{bmatrix} \in \mathcal{N}(G) \quad \text{and} \quad \left\| \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} - \begin{bmatrix} \tilde{z}_0 \\ \tilde{w}_0 \end{bmatrix} \right\|_{\mathcal{X}} < 2\epsilon.$$

Thus condition (iii) of Proposition 2.5 is satisfied.

We proceed to show that $L\mathcal{N}(G) = \mathcal{X}$. Let $\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right] \in \mathcal{X}$ be arbitrary. By (5.8), $\left[\begin{smallmatrix} z_1 \\ w_1 \end{smallmatrix}\right] = L \left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} w_0 \\ \Delta z_0 \end{smallmatrix}\right]$ for $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{N}(G)$ if and only if $w_0 = z_1$ and the variational solution $z_0 \in H^1_{\Gamma_0}(\Omega)$ of the problem

$$\Delta z_0 = w_1, \qquad z_0|\Gamma_0 = 0, \qquad \frac{\partial z_0}{\partial \nu}\bigg|\Gamma_1 = -z_1|\Gamma_1$$

satisfies $z_0 \in H^2(\Omega)$. Since $w_1 \in L^2(\Omega)$ and $z_1|\Gamma_1 \in H^{1/2}(\Gamma_1)$, this follows from the same elliptic regularity result as Proposition 5.1.

Finally, $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{N}(L) \cap \mathcal{N}(G)$ if and only if $w_0 = 0$ together with

$$z_0 \in H^2(\Omega), \quad \Delta z_0 = 0, \qquad z_0|\Gamma_0 = 0 \quad \text{and} \quad \frac{\partial z_0}{\partial \nu}\bigg|\Gamma_1 = w_0|\Gamma_1 = 0$$

if and only if $w_0 = 0$ and $z_0 = \widetilde{N} 0 = 0$ in (5.7). Condition (ii) of Proposition 2.5 is now satisfied with $\alpha = 0$, and thus $\Xi = (G, L, K)$ is a boundary node. A similar argument shows that $\Xi^{\leftarrow} = (K, -L, G)$ is a boundary node, too. □

It is now almost trivial to check that $\Xi = (G, L, K)$ is conservative.

Proposition 5.3. *Let the operators L, G, and K together with spaces $\mathcal{U}$, $\mathcal{X}$, and $\mathcal{Y}$ be defined as above, and use the energy norm* (5.6) *for $\mathcal{X}$. Then the boundary node $\Xi = (G, L, K)$ associated to* (5.2) *is conservative. Consequently, it induces a conservative system node S.*

Proof. For an arbitrary $\left[\begin{smallmatrix} z_0 \\ w_0 \end{smallmatrix}\right] \in \mathcal{Z}$, Green's formula [17, p. 62] implies

$$\begin{aligned} 2\,\mathrm{Re}\left\langle \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, L \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \right\rangle_{\mathcal{X}} &= 2\,\mathrm{Re}\left\langle \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \begin{bmatrix} w_0 \\ \Delta z_0 \end{bmatrix} \right\rangle_{\mathcal{X}} \\ &= 2\,\mathrm{Re}\left(\langle \Delta \overline{z_0}, w_0 \rangle_{L^2(\Omega)} + \int_{\Omega} \nabla \overline{z_0} \cdot \nabla w_0 \, d\Omega \right) \end{aligned}$$

$$= 2\,\mathrm{Re}\left(\int\limits_{\Gamma_0 \cup \Gamma_1} \frac{\partial \overline{z_0}}{\partial \nu} w_0 \, d\omega \right) = 2\,\mathrm{Re}\left\langle \frac{\partial z_0}{\partial \nu} \Big| \Gamma_1, w_0 | \Gamma_1 \right\rangle_{L^2(\Gamma_1)} \tag{5.9}$$

because $w_0|\Gamma_0 = 0$. By the definition of operators G and K we obtain

$$\left\langle G \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, G \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \right\rangle_{L^2(\Gamma_1)} = \frac{1}{2} \left\| \frac{\partial z_0}{\partial \nu} \Big| \Gamma_1 \right\|^2_{L^2(\Gamma_1)} + \mathrm{Re}\left\langle \frac{\partial z_0}{\partial \nu} \Big| \Gamma_1, w_0 | \Gamma_1 \right\rangle_{L^2(\Gamma_1)} + \frac{1}{2} \| w_0 | \Gamma_1 \|^2_{L^2(\Gamma_1)} \tag{5.10}$$

and also

$$\left\langle K \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, K \begin{bmatrix} z_0 \\ w_0 \end{bmatrix} \right\rangle_{L^2(\Gamma_1)} \\ = \frac{1}{2} \left\| \frac{\partial z_0}{\partial \nu} \Big| \Gamma_1 \right\|^2_{L^2(\Gamma_1)} - \mathrm{Re}\left\langle \frac{\partial z_0}{\partial \nu} \Big| \Gamma_1, w_0 | \Gamma_1 \right\rangle_{L^2(\Gamma_1)} + \frac{1}{2} \| w_0 | \Gamma_1 \|^2_{L^2(\Gamma_1)}. \tag{5.11}$$

Putting (5.9)–(5.11) together yields the Green–Lagrange identity (1.3). Using then Proposition 5.2 and Theorem 1.5 completes the proof. □

We remark that the conservativity of the system node S in Proposition 5.3 opens up the possibility to apply operator theory techniques, developed especially for conservative systems, to this PDE, such as canonical realizations and unitary similarity of different conservative realizations.

The example discussed above has some additional important properties not mentioned above, such as the strong bi-stability of its semigroup. A more complete discussion is found in [23, Section 7.3].

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