**Mat-1.3604 Stationary Processes.**

*Exercise 20.9. 2007 Tikanmäki/Valkeila.*

1. Let $L^2(\mathbb{P})$ be a space of square integrable complex valued random variables. Let $Z_1, Z_2 \in L^2(\mathbb{P})$. Show that $Z_1 \perp Z_2$ if and only for every complex number $\alpha$ we have

$$||Z_1 + \alpha Z_2|| = ||Z_1 - \alpha Z_2||.$$

2. Prove that if $(Z_k)_{k\geq 1}$ is orthonormal, then
   (i) For any $Z \in L^2(\mathbb{P})$ $\lim_k (Z, Z_k) = 0$.
   (ii) For $j \neq k$ $||Z_k - Z_j|| = \sqrt{2}$.
3. Let $(Z_k)_{k=1}^n \in L^2(\mathbb{P})$ be orthonormal: $j \neq k \Rightarrow Z_j \perp Z_k$. Show that for every $Z \in L^2(\mathbb{P})$

$$\inf \left( ||Z - \sum_{k=1}^n \alpha_k Z_k|| \right)$$

is attained, if $\alpha_k = (Z, Z_k)$.

4. Let $\xi_k$ be square integrable complex valued random variables with

$$E\xi_k = 0,\ E\xi_k\bar{\xi}_j = 0, \quad \text{when } j \neq k, \ \text{and} \sigma_k^2 = E\xi_k\bar{\xi}_k.$$

Let $\alpha_k \in \mathbb{R}$. Show that the process

$$X_t = \sum_{k=1}^n e^{i\alpha_k t}\xi_k$$

is stationary [in the weak sense].

5. Let $h_1, \ldots, h_n$ be real functions and $a_k \in \mathbb{R}, a_k > 0$. Show that

$$C(s,t) = \sum_{k=1}^n a_k h_k(s) h_k(t)$$

is a covariance function.

6. Let $\psi_k$ be functions such that $\int_a^b \psi_k(s)\bar{\psi}_j(s)ds = 0$, $j,k = 1,\ldots,n$, $j \neq k$, and $\int_a^b \psi_k(s)\bar{\psi}_k(s)ds = \sigma_k^2$. Let $X_t$ be a $L^2$ process defined on $[a,b]$, $EX_t = 0$ and covariance

$$C(s,t) = \sum_{k=1}^n \psi_k(s)\bar{\psi}_k(t).$$

Put $\xi_k = \int_a^b X_u \bar{\psi}_k(u)du$. Compute $E(\xi_j\bar{\xi}_k)$.