S2:n III välikoe
Malliratkaisu

Tehtävä 1

a) $\bar{F}$ on konservatiivinen jos
1° $\nabla \times \bar{F} = 0$
2° $\bar{F}$:llä on olemassa potentiaali

Lasketaan

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & \bar{k} \\ \partial_x & \partial_y & \partial_z \\ 2xy - z^2 & 2yz + x^2 & -2zx + y^2 \end{vmatrix}$$

$$= i(2y - 2y) - j(-2z - (-2z)) + \bar{k}(2x - 2x)$$

$= 0$ OK.

+1p

Potentiaali $\bar{F} = \nabla\phi = F_1 i + F_2 j + F_3 \bar{k}$

$$F_1 = \frac{\partial\phi}{\partial x} = 2xy - z^2 \quad \| \int$$

$$\Rightarrow \phi(x,y,z) = x^2 y - z^2 x + C_1(y,z)$$

$$\Rightarrow F_2 = \frac{\partial\phi}{\partial y} = \cancel{x^2} + \frac{\partial C_1}{\partial y} = 2yz + \cancel{x^2}$$

$$\Rightarrow C_1(y,z) = y^2 z + C_2(z)$$

Eli $\phi = x^2 y - z^2 x + y^2 z + C_2(z)$

$$\Rightarrow F_3 = \frac{\partial\phi}{\partial z} = \cancel{-2zx + y^2} + C_2' = \cancel{-2zx + y^2}$$

$$\Rightarrow C_2' = 0 \Rightarrow C_2(z) = C = \text{vakio}$$

Nyt $\phi(x,y,z) = x^2 y - z^2 x + y^2 z + C$

$\Rightarrow \bar{F}$ on konservatiivinen

+2p

TEHTÄVÄ 1

b)

$\bar{F}$ ON KONSERVATIIVINEN

$$\Rightarrow \int_C \bar{F} \cdot d\bar{r} = \phi(2,3,1) - \phi(1,1,2) \qquad 2p$$

$$= (4 \cdot 3 - 1 \cdot 2 + 9 \cdot 1 + c) - (1 - 0 - 0 + c)$$

$$= \underline{\underline{18}}$$

3p

Tai

$$\begin{cases} x = t+1 \\ y = 2t+1 \\ z = t \end{cases} \quad 0 \le t \le 1$$

1p

$$d\bar{r} = (1,2,1)\,dt$$

$$\bar{F} = \dots = (3t^2+6t+2,\ 5t^2+4t+1,\ 2t^2+2t+1) \qquad 2p$$

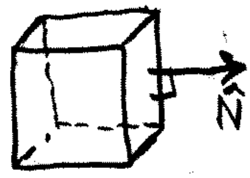
$$\Rightarrow \int_C \bar{F} \cdot d\bar{r} = \dots = 18$$

3p

2.

$$\bar{F} = (2x - z)\bar{i} + x^2 y \bar{j} - x z^2 \bar{k}$$

$\bar{F}$:n vuo ulos D:stä $= \iint_{\partial D} \bar{F} \cdot \hat{N}\, dS =$



$\hat{N}$ ulospäin kuutiosta

$$= \iiint_D \nabla \cdot \bar{F}\, dV = \iiint_D [2 + x^2 - 2xz]\, dV$$

$\uparrow$ divergenssilause

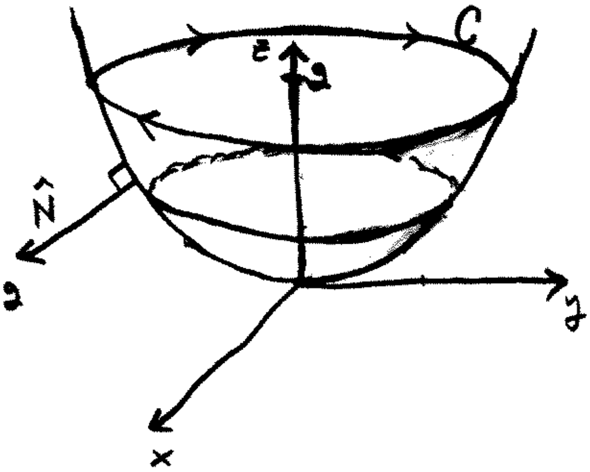
$$= 2 \underbrace{\iiint_D dV}_{=1} + \int_0^1\int_0^1\int_0^1 x^2\, dy\, dz\, dx - \int_0^1\int_0^1\int_0^1 2xz\, dy\, dz\, dx$$

$$= 2 + \int_0^1 x^2\, dx - \int_0^1\int_0^1 2xz\, dz\, dx = 2 + \frac{1}{3} - \int_0^1 x\, dx =$$

$$= 2 + \frac{1}{3} - \frac{1}{2} = \underline{\underline{\frac{11}{6}}}$$

Tehtävä on mahdollista laskea myös suoraan vuo-integraalista $\iint_{\partial D} \bar{F} \cdot \hat{N}\, dS$.

3. $\bar{F} = 3y\bar{i} - xz\bar{j} + yz^2\bar{k}$



S: paraboloidin $2z = x^2 + y^2$
se osa, joka jää tason $z=2$
alle.

C: S:n reunakäyrä, joka
on ympyrä (säde $\sqrt{x^2+y^2} = \sqrt{2\cdot 2} = 2$)

$$\begin{cases} x = 2\cos t \\ y = -2\sin t \\ z = 2 \end{cases}$$

($t: 0 \to 2\pi$, kiertosuunta huomioituna)

Stokes $\Rightarrow$ $$\iint_S \nabla \times \bar{F} \cdot d\bar{S} = \oint_C \bar{F} \cdot d\bar{r} =$$

$$= \oint_C 3y\,dx - xz\,dy + yz^2\,dz =$$

$$\begin{aligned} dx &= -2\sin t\,dt \\ dy &= -2\cos t\,dt \\ dz &= 0 \end{aligned}$$

$$= \int_0^{2\pi} [12\sin^2 t + 8\cos^2 t]\,dt =$$

$$= \int_0^{2\pi} [8 + 4\sin^2 t]\,dt =$$

$$= 16\pi + 4\int_0^{2\pi} \tfrac{1}{2}(1 - \cos 2t)\,dt = \underline{\underline{20\pi}}$$

4. $x = u^2\cos(2v)$, $y = u^2\sin(2v)$, $z = 3w$

a) $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$h_u = \left|\frac{d\bar{r}}{du}\right| = \sqrt{[2u\cos(2v)]^2 + [2u\sin(2v)]^2 + 0^2} =$$

$$= \sqrt{(2u)^2} = \underline{\underline{2u}}$$

$$h_v = \left|\frac{d\bar{r}}{dv}\right| = \sqrt{[-2u^2\sin(2v)]^2 + [2u^2\cos(2v)]^2 + 0^2} =$$

$$= \sqrt{(2u^2)^2} = \underline{\underline{2u^2}}$$

$$h_w = \left|\frac{d\bar{r}}{dw}\right| = \sqrt{0^2 + 0^2 + 3^2} = \underline{\underline{3}}$$

b)

$$dS_u = h_v h_w \, dv\, dw = 6u^2 \, dv\, dw$$

$$dS_v = h_u h_w \, du\, dw = 6u \, du\, dw$$

c) $f(u, v, w) = \frac{1}{2}u^2 + ve^w$

$$\text{grad}(f) = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{u} + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{v} + \frac{1}{h_w}\frac{\partial f}{\partial w}\hat{w} =$$

$$= \left(\frac{1}{2u}\cdot u\right)\hat{u} + \left(\frac{1}{2u^2}\cdot e^w\right)\hat{v} + \left(\frac{1}{3}\cdot ve^w\right)\hat{w} =$$

$$= \underline{\underline{\frac{1}{2}\hat{u} + \frac{e^w}{2u^2}\hat{v} + \frac{ve^w}{3}\hat{w}}}$$