Aalto University

MS-A0401 Foundations of discrete mathematics
Exercise 2
14–18.9.2015, week 38

Gripenberg, Potka, Pöllänen, Tuhkanen, Ääri

Return your solutions to the P-questions and answer the S-questions not later than 21.9.2015 at 16.00.
Remember to write your name, student number and group!

P1. The function $f : \mathbb{N}_0 \to \mathbb{Z}$ is defined recursively in the following way

$$f(n) = \begin{cases} 1, & n = 0, \\ 0, & n = 1, \\ 5 \cdot f(n-1) - 6 \cdot f(n-2), & n \geq 2. \end{cases}$$

(a) Calculate $f(5)$ by calculating $f(2)$, $f(3)$ etc.
(b) Check that the expression $3 \cdot 2^n - 2 \cdot 3^n$ satisfies the definition of f (so that you get another way of calculating $f(5)$).
(c) If $\lambda \neq 0$ is a number such that $y(n) = 5 \cdot y(n-1) - 6 \cdot y(n-2)$ when $y(n) = \lambda^n$, which equation (in which n does not appear) is satisfied by λ and what are the solutions to this equation?
(d) If y_1 and y_2 are solutions to the equation $y(n) = 5 \cdot y(n-1) - 6 \cdot y(n-2)$ then $y(n) = c_1 y_1(n) + c_2 y_2(n)$ is clearly (??) a solution as well. Derive the expression for the function f in part (b) using this result, the result of part (c) and the conditions $f(0) = 1$ and $f(1) = 0$.

P2. If we have a sequence of numbers $x = [x_1, x_2, \ldots x_n]$ then we can produce a new sequence where the elements are put in increasing order with the following algorithm:

```
function x=Jarj(x)
   n=max(size(x));
   for i=1:n
     for j=1:i
       if x(i)<x(j)
         x([i,j])=x([j,i]);
       end
     end
   end
endfunction
```

If x is a sequence of numbers with length n n then we let $v(n)$ be the number of comparisons (`x(i)<x(j)`) this algorithm performs when it calculates `Jarj(x)`. Determine the smallest possible nonnegative numbers a and b so that $v \in O(n^a \log(n)^b)$.

P3. Determine the smallest possible equivalence relation in the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ that contains the ordered pairs $[1, 4]$, $[2, 5]$, $[6, 8]$ and $[7, 4]$. (Smallest in the sense that it as a subset of $A \times A$ contains the least number of elements.) Determine the equivalence classes of the equivalence relation, i.e., a set of nonempty and disjoint subsets of A such that their union is A and two elements x and y belong to the same subset if and only if $[x, y]$ belongs to the equivalence relation.

P4. Let Sur(m, n) be the number of surjections from the set A to the set B when there are m elements in A and n elements in B. If you construct the surjections in the following way you can count how many they are as well:

(a) Fix some element $a_0 \in A$ and then choose $f(a_0)$. In how many ways can $f(a_0)$ be chosen? Observe that each choice gives a different function.

(b) Next you define the function f in the set $A \setminus \{a_0\}$. There are two cases (that exclude each other):

(b-1) For some $a_1 \neq a_0$ you define $f(a_1) = f(a_0)$ in which case f restricted to the set $A \setminus \{a_0\}$, i.e., the function $f_{|A\setminus\{a_0\}}$ is a surjection: $A \setminus \{a_0\} \to B$. How many such surjections are there? Express your answer with the aid of the Sur-function.

(b-2) For all $a \in A \setminus \{a_0\}$ you define $f(a)$ so that $f(a) \neq f(a_0)$ so that f restricted to the set $A \setminus \{a_0\}$, i.e., the function $f_{|A\setminus\{a_0\}}$ is a surjection: $A \setminus \{a_0\} \to B \setminus \{f(a_0)\}$. How many such surjections are there? Express your answer with the aid of the Sur-function.

(c) If you calculate Sur(m, n) with the aid of parts (a) and (b) as well as the sum and product rule, what formula do you get?

(d) Calculate Sur$(4, 3)$ with the aid of the result in part (c) taking into account that we, of course (?), have Sur$(m, 1) = 1$ when $m \geq 1$ and Sur$(m, n) = 0$ when $m < n$.

Answer: Sur$(4, 3) = 36$

P5. If X and Y are nonempty sets and $f : X \to Y$ is a function then we can define the functions $f^{\to} : \mathcal{P}(X) \to \mathcal{P}(Y)$ and $f^{\leftarrow} : \mathcal{P}(Y) \to \mathcal{P}(X)$ as follows:

$$f^{\to}(A) = \{\, f(x) \in Y : x \in A \,\}, \quad A \in \mathcal{P}(X),$$
$$f^{\leftarrow}(B) = \{\, x \in X : f(x) \in B \,\}, \quad B \in \mathcal{P}(Y),$$

where $\mathcal{P}(X)$ and $\mathcal{P}(Y)$ are the sets of subsets of X and Y.

(a) Show that if f is an injection then $f^{\leftarrow}(f^{\to}(A)) = A$ for all $A \subseteq X$.

(b) Show that if f is not an injection then there is a set $A \subseteq X$ so that $f^{\leftarrow}(f^{\to}(A)) \neq A$

Hint: In part (a) you have to show that $A \subseteq f^{\leftarrow}(f^{\to}(A))$ and $f^{\leftarrow}(f^{\to}(A)) \subseteq A$ and one of these claims follows directly from the definition and does not depend on whether f is an injection or not. In part (b) you can choose the set A to conatin exactly one element.

Note! *Usually one writes f instead of $f^{\to}$ and often f^{-1} instead of $f^{\leftarrow}$.*