<u>Note</u>: How to study MAX/MIN using the 2^{ND} order Taylor-polynomial?

Idea: critical point (x_0, y_0), where $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$

$$\Rightarrow f(x,y) = f(x_0,y_0) + \overset{=0}{f_x(x_0,y_0)} \cdot (x-x_0) + \overset{=0}{f_y(x_0,y_0)} \cdot (y-y_0)$$

$$+ \frac{1}{2} f_{xx}(x_0,y_0) \cdot (x-x_0)^2 + f_{xy}(x_0,y_0) \cdot (x-x_0)(y-y_0)$$

$$+ \frac{1}{2} f_{yy}(x_0,y_0) \cdot (y-y_0)^2 + \text{higher order terms}$$

Substitute $x = x_0 + h$, $y = y_0 + k$

$$\Rightarrow f(x_0+h, y_0+k) = f(x_0,y_0) + \frac{1}{2} f_{xx} h^2 + f_{xy} hk + \frac{1}{2} f_{yy} \cdot k^2 + \ldots$$

$$= f(x_0,y_0) + \frac{1}{2}\left(f_{xx} \cdot h^2 + 2 f_{xy} hk + f_{yy} \cdot k^2\right) + \ldots$$

The type of (x_0,y_0) is determined by the quadratic form

$$f_{xx} h^2 + 2 f_{xy} hk + f_{yy} k^2 = [h \ \ k] \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda_1 h'^2 + \lambda_2 k'^2,$$

where λ_1, λ_2 are the eigenvalues of Df, as follows:

(i) $\lambda_1, \lambda_2 > 0 \Rightarrow (x_0,y_0)$ is a local min.

(ii) $\lambda_1 < 0, \lambda_2 > 0 \Rightarrow (x_0,y_0)$ is a saddle point $\Rightarrow$ no local MAX/MIN

(iii) $\lambda_1, \lambda_2 < 0 \Rightarrow (x_0,y_0)$ is a local max.

Ex. 5

13.1.1 $f(x,y) = x^2 + 2y^2 - 4x + 4y$

$$\begin{cases} f_x = 2x - 4 = 0 \\ f_y = 4y + 4 = 0 \end{cases} \iff \begin{cases} x = 2 \\ y = -1 \end{cases}$$ one critical point $(2,-1)$

$$Df = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \Rightarrow \lambda_1 = 2,\ \lambda_2 = 4 \quad (\text{local min})$$

Also: $f(x,y) = (x-2)^2 + 2(y+1)^2 - 6 \Rightarrow$ local min

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13.15 $f(x,y) = \frac{x}{y} + \frac{8}{x} - y \quad (xy \neq 0)$

$$\begin{cases} f_x = 1/y - 8/x^2 = 0 \\ f_y = -x/y^2 - 1 = 0 \end{cases} \iff \begin{cases} x^2 - 8y = 0 \\ x + y^2 = 0 \Rightarrow x = -y^2 \end{cases}$$ subst.

$\Rightarrow y^4 - 8y = 0 \iff y(y^3 - 8) = 0 \iff y = 0$ or $y = 2$

not allowed!

critical point $(-4, 2)$

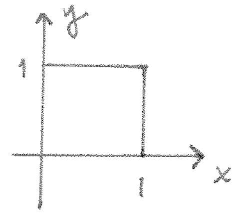
$$Df(-4,2) = \begin{bmatrix} +16/x^3 & -1/y^2 \\ -1/y^2 & +2x/y^3 \end{bmatrix}_{\substack{x=-4 \\ y=2}} = \begin{bmatrix} -1/4 & -1/4 \\ -1/4 & -1 \end{bmatrix}$$

Eigenvals: $$\begin{vmatrix} -\frac{1}{4} - \lambda & -1/4 \\ -1/4 & -1-\lambda \end{vmatrix} = \left(-\tfrac{1}{4} - \lambda\right)(-1-\lambda) - (-1/4)^2 = \lambda^2 + \frac{5}{4}\lambda + \frac{3}{16} = 0$$

$$\Rightarrow \lambda = \frac{-5/4 \pm \sqrt{25/16 - 4 \cdot 1 \cdot 3/16}}{2} < 0$$

$\Rightarrow (4, -2)$ is a local max.

13.2.5 $f(x,y) = xy - x^3y^2$, $0 \le x, y \le 1$, MAX/MIN?



Critical points?

$$\begin{cases} f_x = y - 3x^2y^2 = 0 \\ f_y = x - 2x^3y = 0 \end{cases} \iff \begin{cases} y(1-3x^2y) = 0 \\ x(1-2x^2y) = 0 \end{cases}$$

Only sol. $(0,0) \in$ boundary of the square

$\Rightarrow$ no loc. max or min inside $\Rightarrow$ global MAX and MIN are obtained on the boundary:

$\underline{x=0 \text{ or } y=0}$ $\Rightarrow f(x,y) \equiv 0$

$\underline{x=1,\ 0 \le y \le 1}$ $g(y) = f(1,y) = y - y^2$, $g'(y) = 1 - 2y = 0$?

$$\iff y = 1/2$$

$\Rightarrow (1, 1/2)$ is a possible MAX/MIN-point

$\underline{y=1,\ 0 \le x \le 1}$ $h(x) = f(x,1) = x - x^3$, $h'(x) = 1 - 3x^2 = 0$?

$$\iff x = \pm 1/\sqrt{3}$$

$\Rightarrow (1/\sqrt{3}, 1)$ is a possible point.

Now $f(0,0) = 0$ (or any other point with $x=0$ or $y=0$)

$f(1, 1/2) = 1/4$ all possible MAX/MIN values

$f(1/\sqrt{3}, 1) = 2/3\sqrt{3}$

$\Rightarrow f(0,0) = 0$ is MIN

$f(1/\sqrt{3}, 1) = 2/3\sqrt{3}$ is MAX.