Math II, 2ND EXAM 1.4.2008, SOLUTIONS

1. $f(x,y) = \cos(x+y) + \sin(x-y)$

a) Critical points: $\begin{cases} f_x = -\sin(x+y) + \cos(x-y) = 0 \\ f_y = -\sin(x+y) - \cos(x-y) = 0 \end{cases}$

This is of the form $\begin{cases} u - v = 0 \\ u + v = 0 \end{cases} \iff u = v = 0$

b) $\sin(\pi/4 + 3\pi/4) = \sin \pi = 0$, $\cos(\pi/4 - 3\pi/4) = \cos(-\pi/2) = 0$

$\Rightarrow (\pi/4, 3\pi/4)$ is a critical point

Matrix of 2ND derivatives:

$$\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} -\cos(x+y) - \sin(x-y) & -\cos(x+y) + \sin(x-y) \\ -\cos(x+y) + \sin(x-y) & -\cos(x+y) - \sin(x-y) \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ at } x = \pi/4,\ y = 3\pi/4$$

$\Rightarrow$ eigenvalues $\lambda_1 = \lambda_2 = 2 > 0 \Rightarrow$ local min.

$\left(f(\pi/4 + h, 3\pi/4 + k) \approx f(\pi/4, 3\pi/4) + 2h^2 + 2k^2 \gtrsim f(\pi/4, 3\pi/4). \right)$

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2. Lagrange: $f(x,y,z) = x+y+z$, $g(x,y,z) = xyz - 1$, $x,y,z > 0$

$L = f + \lambda g = x + y + z + \lambda(xyz - 1)$

$\begin{cases} L_x = 1 + \lambda yz = 0 \\ L_y = 1 + \lambda xz = 0 \\ L_z = 1 + \lambda xy = 0 \\ L_\lambda = xyz - 1 = 0 \end{cases}$

$\lambda yz = \lambda xz \Rightarrow y = x$ since $z \neq 0, \lambda \neq 0$

$\lambda xy = \lambda xz \Rightarrow y = z$ since $x \neq 0, \lambda \neq 0$

$\Rightarrow x = y = z$ and since $xyz = 1$, we have $x = y = z = 1$.

Type: $x = 1 + h$, $y = 1 + k$, $z = \frac{1}{xy} = \frac{1}{1+h+k+hk} \approx 1 - (h+k+hk) + (h+k+hk)^2$ ← geom. series

$$\Rightarrow \quad x+y+z \approx 1+h+1+k+1-(h+k+hk)+(h+k+hk)^2$$

$$= 3 + h^2 + hk + k^2 + \text{terms of order } 3$$

$$= 3 + \left(h+\tfrac{1}{2}k\right)^2 + \tfrac{3}{4}k^2 + \dots$$

$$\geq 3 \quad \text{for small } |h| \text{ and } |k|$$

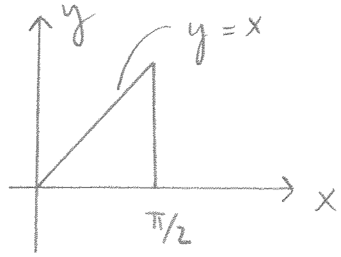
$\Rightarrow$ $(1,1,1)$ is a local min

But it is the only crit. point $\Rightarrow$ $(1,1,1)$ is a min

$\Rightarrow$ $x+y+z \geq 1+1+1 = 3.$

3.

$$\int_0^{\pi/2} dy \int_y^{\pi/2} \frac{\sin x}{x}\, dx = \int_0^{\pi/2} dx \int_0^{x} \frac{\sin x}{x}\, dy$$



$$= \int_0^{\pi/2} \left[y\,\frac{\sin x}{x} \right]_{y=0}^{y=x} dx = \int_0^{\pi/2} \sin x \, dx = \underline{1}.$$

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4. a) $A = \frac{1}{2}\pi R^2$

$$\bar{y} = \frac{1}{A}\iint_D y\, dA = \frac{1}{A}\int_0^{\pi} d\theta \int_0^R r\sin\theta \cdot r\, dr = \frac{1}{A}\int_0^{\pi}\sin\theta\, d\theta \cdot \int_0^R r^2\, dr$$

$$= \frac{1}{A}\cdot 2 \cdot \frac{1}{3}R^3 = \underline{\frac{4}{3\pi}R} \approx 0{,}42\, R$$

b)

$$\bar{T} = \frac{1}{V}\iiint_B T\, dV = \frac{1}{V}\int_0^{\pi}\int_0^{2\pi}\int_0^R 100\,(1-\rho/R)\cdot \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \frac{100}{V}\int_0^{\pi}\sin\varphi\, d\varphi \cdot \int_0^{2\pi} 1\, d\theta \cdot \int_0^R (1-\rho/R)\,\rho^2\, d\rho$$

$$= \frac{100}{V}\cdot 2\cdot 2\pi \cdot \left[\frac{1}{3}\rho^3 - \frac{1}{4R}\rho^4\right]_{\rho=0}^{R} = \frac{400\pi}{V}\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{1/12} R^3$$

$$= \frac{400\pi R^3}{12\cdot 4\pi R^3/3} = \underline{25}$$