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Mat-1.1620 Mathematics II

1st partial exam 20.2.2008, 16–19.

You may use a calculator but no "Formula books".

1. A curve called cycloid (see below) has a parametrization of the form

$$\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}, \quad t \geq 0$$

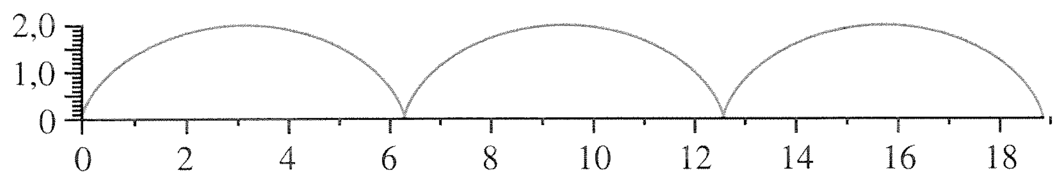
(the coefficient of $\mathbf{k}$ is zero: $z(t) = 0$).
a) Find the smallest parameter value $t_0 > 0$ such that the velocity $\mathbf{v}(t_0) = \overline{0}$.
b) Show that the acceleration $\mathbf{a}(t)$ has a constant magnitude $|\mathbf{a}(t)|$ independent of t.
c) Calculate the length of the cycloid corresponding to the parameter interval $t \in [0, 2\pi]$.
Hint: You may want to use $2 - 2\cos t = 4\sin^2(t/2)$.

2. Let $f(x, y) = xy$ and define $F(r, \theta) = f(r\cos\theta, r\sin\theta)$ using the polar coordinates. Calculate the partial derivatives F_r, F_θ and $F_{r\theta}$ by
a) substituting the expressions immediately;
b) using the chain rule and making the substitutions in the end.

3. The side length s and the height h of a pyramid (like the ones in Egypt) are each measured to within an accuracy of 2 % . Using the differential, find an (approximate) upper bound for the relative error $|\Delta V|/V$ for its volume.
Note: $V = s^2h/3$.

4. a) It can be shown that the equation $2\sin x + \sin y = x + 2y$ defines $y = y(x)$ uniquely for all $x \in \mathbf{R}$. Calculate $y'(0)$.
b) Find the equation of the tangent plane to the surface $z = x^2y - y^3x$ at the point $(x_0, y_0, z_0) = (2, 1, 2)$.

The path of a small stone attached to a bicycle tyre is a cycloid:



Math II, 1st Exam

1. a) $\bar{r}'(t) = (1-\cos t)\bar{i} + \sin t\,\bar{j} = \bar{0} \iff \begin{cases} \cos t = 1 \\ \sin t = 0 \end{cases}$ First pos. sol. is $t_0 = 2\pi$.

b) $\bar{a}(t) = \bar{r}''(t) = \sin t\,\bar{i} + \cos t\,\bar{j}$, $|\bar{a}(t)| = \sqrt{\sin^2 t + \cos^2 t} = \underline{1} \quad \forall t$

c) $$l = \int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt = \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} 2\sin(t/2)\, dt$$

$$= \Big/_{0}^{2\pi} -4\cos(t/2) = \underline{\underline{8}}.$$

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2. a) $F(r,\theta) = r^2 \sin\theta \cos\theta = \frac{1}{2} r^2 \sin(2\theta)$

$\Rightarrow F_r = r\sin(2\theta)$, $F_\theta = r^2\cos(2\theta)$, $F_{r\theta} = 2r\cos(2\theta)$

b) $f_x = y$, $f_y = x$, $f_{xy} = f_{yx} = 1$, $x_r = \cos\theta$, $x_\theta = -r\sin\theta$, $y_r = \sin\theta$, $y_\theta = r\cos\theta$

$\Rightarrow F_r = f_x x_r + f_y y_r = y\cos\theta + x\sin\theta$

$= r\sin\theta\cos\theta + r\cos\theta\sin\theta = 2r\sin\theta\cos\theta = r\sin(2\theta)$

$F_\theta = f_x x_\theta + f_y y_\theta = y(-r\sin\theta) + x\,r\cos\theta = -r^2\sin^2\theta + r^2\cos^2\theta = r^2\cos(2\theta)$

$F_{r\theta} = (f_{xx} x_\theta + f_{xy} y_\theta) x_r + f_x x_{r\theta} + (f_{yx} x_\theta + f_{yy} y_\theta) y_r + f_y y_{r\theta}$

$= \dots = 2r\cos(2\theta)$.

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3. $\Delta V \approx V_r \cdot \Delta r + V_h \cdot \Delta h = 2rh/3 \cdot \Delta r + r^2 h/3 \cdot \Delta h \quad (:V)$

$\Rightarrow \frac{\Delta V}{V} \approx 2 \cdot \frac{\Delta r}{r} + \frac{\Delta h}{h} \Rightarrow \frac{|\Delta V|}{V} \lesssim 2 \cdot \frac{|\Delta r|}{r} + \frac{|\Delta h|}{h} \le 2 \cdot 0.02 + 0.02 = \underline{\underline{6\%}}$

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4. a) $x = 0 \Rightarrow 2\sin 0 + \sin(y(0)) = 0 + 2y(0) \Rightarrow y(0) = 0$

$y = y(x) \Rightarrow 2\cos x + y'(x)\cos y(x) = 1 + 2y'(x)$. Subst. $x=0$: $2 + y'(0)\cdot 1 = 1 + 2y'(0)$

$\Rightarrow y'(0) = \underline{1}$

b) $\bar{n} = f_x \bar{i} + f_y \bar{j} - \bar{k} = (2xy - y^3)\bar{i} + (x^2 - 3xy^2)\bar{j} - \bar{k} = 3\bar{i} - 2\bar{j} - \bar{k}$

$\bar{n}\cdot(\bar{r} - \bar{r}_0) = 0 \iff 3x - 2y - z = \bar{n}\cdot\bar{r}_0 = 6 - 2 - 2 = 2 \iff \underline{z = 3x - 2y - 2}$