Representation theory of compact groups

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Preface

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Chapter 1

Groups

1.1 Introduction

Perhaps the first non-trivial group that the mankind encountered was the set $\mathbb{Z}$ of integers; with the usual addition $(x, y) \mapsto x + y$ and "inversion" $x \mapsto -x$ this is a basic example of a (non-compact) group. Intuitively, a group is a set G that has two mappings $G \times G \to G$ and $G \to G$ generalizing the properties of the integers in a simple and natural way.

We start by defining the groups, and we study the mappings preserving such structures, i.e., group homomorphisms. Of special interest are representations, that is, those group homomorphisms that have values in groups of invertible linear operators on vector spaces. Representation theory is a key ingredient in the theory of groups.

In this framework we study analysis on compact groups, foremost measure theory and Fourier transform. Remarkably, on a compact group G there exists a unique translation-invariant linear functional functional on $C(G)$ corresponding to a probability measure. We shall construct this Haar measure, closely related to the Lebesgue measure of a Euclidean space. We shall also introduce Fourier series of functions on a group.

Groups having a smooth manifold structure (with smooth group operations) are called Lie groups, and their representation theory is especially interesting. Left-invariant first order partial differential operators on such a group can be identified with left-invariant vector fields on the group, and the corresponding set called the Lie algebra is studied.

Finally, we introduce Hopf algebras and study the Gelfand theory related to them.

Remark 1.1.1. If X, Y are spaces with the same kind of algebraic structure, the set $\mathrm{Hom}(X, Y)$ of *homomorphisms* consists of mappings $f : X \to Y$ respecting the structure. Bijective homomorphisms are called *isomorphisms*. Homomorphisms $f : X \to X$ are called *endomorphisms* of X, and their set is denoted by $\mathrm{End}(X) := \mathrm{Hom}(X, X)$. Isomorphism-endomorphisms are called *automorphisms*, and their set is $\mathrm{Aut}(X) \subset \mathrm{End}(X)$. If there exist the zero-elements $0_X, 0_Y$ in respective algebraic structures X, Y, the *null space* or the *kernel* of $f \in \mathrm{Hom}(X, Y)$ is

$$\mathrm{Ker}(f) := \{x \in X : \; f(x) = 0_Y\} .$$

Sometimes algebraic structures might have, say, topology, and then the homomorphisms are typically required to be continuous. Hence, for instance, a homomorphism $f : X \to Y$ between Banach spaces X, Y is usually assumed to be continuous and linear, denoted by $f \in \mathcal{L}(X, Y)$, unless otherwise mentioned; for short, let $\mathcal{L}(X) := \mathcal{L}(X, X)$. The assumptions in theorems etc. will still be explicitely stated.

Conventions. $\mathbb{N}$ is the set of non-negative integers, so that $0 \in \mathbb{N}$,
$\mathbb{Z}^+ := \mathbb{N} \setminus \{0\}$,
$\mathbb{Z}$ is the set of integers,
$\mathbb{Q}$ the set of rational numbers,
$\mathbb{R}$ the set of real numbers,
$\mathbb{C}$ the set of complex numbers, and
$\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

1.2 Groups without topology

Definition 1.2.1. A *group* consists of a set G having an element $e = e_G \in G$ and endowed with mappings

$$\begin{aligned} ((x,y) \mapsto xy) &: G \times G \to G, \\ (x \mapsto x^{-1}) &: G \to G \end{aligned}$$

satisfying

$$\begin{aligned} &x(yz) = (xy)z \\ &ex = x = xe \\ &x\,x^{-1} = e = x^{-1}x \end{aligned}$$

for every $x, y, z \in G$. We may freely write $xyz := x(yz) = (xy)z$; element $e \in G$ is called the *neutral element*, and x^{-1} is the *inverse* of $x \in G$. If the group operations are implicitely known, we may say that G is a *group*. If $xy = yx$ for every $x, y \in G$ then G is called *commutative* (or *Abelian*).

Example. Examples of groups:

1. The sets $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are commutative groups with operations $(x,y) \mapsto x+y$, $x \mapsto -x$. The neutral element is 0 in each case.

2. Any vector space is a commutative group with operations $(x,y) \mapsto x+y$, $x \mapsto -x$; the neutral element is 0.

3. Let V be a vector space. The set $\mathrm{Aut}(V)$ of invertible linear operators $V \to V$ forms a group with operations $(A,B) \mapsto AB$, $A \mapsto A^{-1}$; this group is non-commutative when $\dim(V) \geq 2$. The neutral element is $I = (v \mapsto v) : V \to V$.

4. Sets $\mathbb{Q}^\times := \mathbb{Q} \setminus \{0\}$, $\mathbb{R}^\times := \mathbb{R} \setminus \{0\}$, $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ (more generally, invertible elements of a unital ring) form multiplicative groups with operations $(x,y) \mapsto xy$ (ordinary multiplication) and $x \mapsto x^{-1}$ (as usual). The neutral element is 1 in each case.

5. The set

   $$\mathrm{Aff}(V) = \{A_a = (v \mapsto Av + a) : V \to V \mid A \in \mathrm{Aut}(V),\ a \in V\}$$

   of affine mappings forms a group with operations $(A_a, B_b) \mapsto (AB)_{Ab+a}$, $A_a \mapsto (A^{-1})_{A^{-1}a}$; this group is non-commutative when $\dim(V) \geq 1$. The neutral element is I_0.

6. Let $G = \{f : X \to X \mid f \text{ bijection}\}$, where $X \neq \emptyset$; this is a group with operations $(f, g) \mapsto f \circ g$, $f \mapsto f^{-1}$. This group G is called the *symmetric group of* X, and it is non-commutative if $|X| \geq 3$, where $|X|$ is the number of elements of X. The neutral element is $\mathrm{id}_X = (x \mapsto x) : X \to X$.

7. If G and H are groups then $G \times H$ has a natural group structure:

$$((g_1, h_1), (g_2, h_2)) \mapsto (g_1 h_1, g_2 h_2), \quad (g, h) \mapsto (g^{-1}, h^{-1}).$$

   The neutral element is $e_{G \times H} := (e_G, e_H)$.

Exercise 1.2.2. Let G be a group and $x, y \in G$. Prove:
(a) $(x^{-1})^{-1} = x$.
(b) If $xy = e$ then $y = x^{-1}$.
(c) $(xy)^{-1} = y^{-1} x^{-1}$.

Definition 1.2.3. Some notations: Let G be a group, $x \in A$, $A, B \subset G$ and $n \in \mathbb{Z}^+$; we define

$$\begin{aligned} AB &:= \{ab \mid a \in A,\ b \in B\}, \\ A^0 &:= \{e\}, \\ A^{-1} &:= \left\{a^{-1} \mid a \in A\right\}, \\ A^{n+1} &:= A^n A, \\ A^{-n} &:= (A^n)^{-1}. \end{aligned}$$

Definition 1.2.4. A set $H \subset G$ is a *subgroup* of a group G, denoted by $H < G$, if

$$e \in H, \quad xy \in H \quad \text{and} \quad x^{-1} \in H$$

for every $x, y \in H$ (hence H is a group with "inherited" operations). A subgroup $H < G$ is called *normal in* G if

$$xH = Hx$$

for all $x \in G$; then we write $H \lhd G$.

Exercise 1.2.5. Let $H < G$. Show that $H \lhd G$ if and only if $H = x^{-1} H x$ for every $x \in G$.

Example. Examples of subgroups:

1. We always have normal *trivial subgroups* $\{e\} \lhd G$ and $G \lhd G$. Subgroups of a commutative group are always normal.

2. The *center* $Z(G) \lhd G$, where $Z(G) := \{z \in G \mid \forall x \in G : \ xz = zx\}$.
3. If $F < H$ and $G < H$ then $F \cap G < H$.
4. If $F < H$ and $G \lhd H$ then $FG < H$.
5. $\{I_a \mid a \in V\} \lhd \mathrm{Aff}(V)$.
6. $\mathrm{SO}(n) < \mathrm{O}(n) < \mathrm{GL}(n, \mathbb{R}) \cong \mathrm{Aut}(\mathbb{R}^n)$, where the groups consist of real $n \times n$-matrices: $\mathrm{GL}(n, \mathbb{R})$ is the real *general linear* group consisting of invertible real matrices (i.e. determinant non-zero); $\mathrm{O}(n)$ is the *orthogonal* group, where the matrix columns (or rows) form an orthonormal basis for $\mathbb{R}^n$ (so that $A^T = A^{-1}$ for $A \in \mathrm{O}(n)$, $\det(A) \in \{-1, 1\}$); $\mathrm{SO}(n)$ is the *special orthogonal* group, the group of rotation matrices of $\mathbb{R}^n$ around the origin (so that $\mathrm{SO}(n) = \{A \in \mathrm{O}(n) : \ \det(A) = 1\}$).
7. $\mathrm{SU}(n) < \mathrm{U}(n) < \mathrm{GL}(n, \mathbb{C}) \cong \mathrm{Aut}(\mathbb{C}^n)$, where the groups consist of complex $n \times n$-matrices: $\mathrm{GL}(n, \mathbb{C})$ is the complex *general linear* group consisting of invertible complex matrices (i.e. determinant non-zero); $\mathrm{U}(n)$ is the *unitary* group, where the matrix columns (or rows) form an orthonormal basis for $\mathbb{C}^n$ (so that $A^* = A^{-1}$ for $A \in \mathrm{U}(n)$, $|\det(A)| = 1$); $\mathrm{SU}(n)$ is the *special unitary* group, $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) : \ \det(A) = 1\}$.

Remark 1.2.6. Mapping $(z \mapsto (z)) : \mathbb{C} \to \mathbb{C}^{1\times 1}$ identifies complex numbers with complex (1×1)-matrices. Thereby the complex unit circle group $\{z \in \mathbb{C} : \ |z| = 1\}$ is identified with the group $\mathrm{U}(1)$.

Definition 1.2.7. Let $H < G$. Then

$$x \sim y \qquad \Longleftrightarrow xH = yH$$

defines an equivalence relation on G, as can be easily verified. The *(right) quotient of G by H* is the set

$$G/H = \{xH \mid x \in G\} .$$

Notice that $xH = yH$ if and only if $x^{-1}y \in H$.

Proposition 1.2.8. *Let $H \lhd G$ be normal. The quotient G/H can be endowed with the group structure*

$$(xH, yH) \mapsto xyH, \quad xH \mapsto x^{-1}H.$$

Proof. The operations are well-defined mappings $(G/H) \times (G/H) \to G/H$ and $G/H \to G/H$, respectively, since

$$xHyH \overset{H \trianglelefteq G}{=} xyHH \overset{HH=H}{=} xyH,$$

and

$$(xH)^{-1} = H^{-1}x^{-1} \overset{H^{-1}=H}{=} Hx^{-1} \overset{H \trianglelefteq G}{=} x^{-1}H.$$

The group axioms follow, since by simple calculations

$$(xH)(yH)(zH) = xyzH,$$

$$(xH)(eH) = xH = (eH)(xH),$$

$$(x^{-1}H)(xH) = H = (xH)(x^{-1}H).$$

Notice that $e_{G/H} = e_G H = H$. □

Definition 1.2.9. Let G, H be groups. A mapping $\phi : G \to H$ is called a *homomorphism* (or a *group homomorphism*), denoted by $\phi \in \mathrm{Hom}(G, H)$, if

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in G$. A bijective homomorphism $\phi \in \mathrm{Hom}(G, H)$ is called an *isomorphism*, denoted by $\phi : G \cong H$.

Example. Examples of homomorphisms:

1. $(x \mapsto e_H) \in \mathrm{Hom}(G, H)$.
2. For $y \in G$, $(x \mapsto y^{-1}xy) \in \mathrm{Hom}(G, G)$.
3. If $H \triangleleft G$ then $x \mapsto xH$ is a surjective homomorphism $G \to G/H$.
4. For $x \in G$, $(n \mapsto x^n) \in \mathrm{Hom}(\mathbb{Z}, G)$.
5. If $\phi \in \mathrm{Hom}(F, G)$ and $\psi \in \mathrm{Hom}(G, H)$ then $\psi \circ \phi \in \mathrm{Hom}(F, H)$.

Theorem 1.2.10. *Let $\phi : G \to H$ be a group homomorphism. Then $\phi(G) < H$ and the kernel $K = \mathrm{Ker}(\phi) := \{x \in G : \phi(x) = e\} \subset G$ is a normal subgroup. Moreover, $(xK \mapsto \phi(x)) : G/K \to \phi(G)$ is a (well-defined) group isomorphism.*

Proof. Now $\phi(G)$ is a subgroup of H, because

$$\begin{aligned} e_H &= \phi(e_G) \in \phi(G), \\ \phi(x)\phi(y) &= \phi(xy) \in \phi(G), \\ \phi(x^{-1})\phi(x) &= \phi(x^{-1}x) = \phi(e_G) \\ &= e_H \\ &= \ldots = \phi(x)\phi(x^{-1}) \end{aligned}$$

for every $x, y \in G$; notice that $\phi(x)^{-1} = \phi(x^{-1})$. If $a, b \in \mathrm{Ker}(\phi)$ then

$$\begin{aligned} \phi(e_G) &= e_H, \\ \phi(ab) &= \phi(a)\phi(b) = e_H e_H = e_H, \\ \phi(a^{-1}) &= \phi(a)^{-1} = e_H^{-1} = e_H, \end{aligned}$$

so that $K = \mathrm{Ker}(\phi) < G$. If moreover $x \in G$ then

$$\phi(x^{-1}Kx) = \phi(x^{-1})\ \phi(K)\ \phi(x) = \phi(x)^{-1}\ \{e_H\}\ \phi(x) = \{e_H\},$$

meaning $x^{-1}Kx \subset K$. Thus $K \triangleleft G$ by Exercise 1.2.5. By Proposition 1.2.8, G/K is a group (with the natural operations). Since $\phi(xa) = \phi(x)$ for every $a \in K$, $\psi = (xK \mapsto \phi(x)) : G/K \to \phi(G)$ is a well-defined surjection. Furthermore,

$$\psi(xyK) = \phi(xy) = \phi(x)\phi(y) = \psi(xK)\psi(yK),$$

thus $\psi \in \mathrm{Hom}(G/K, \phi(G))$. Finally,

$$\psi(xK) = \psi(yK) \iff \phi(x) = \phi(y) \iff x^{-1}y \in K \iff xK = yK,$$

so that ψ is injective. □

1.3 Group actions and representations

Definition 1.3.1. A *(left) action of a group G on a set $M \neq \emptyset$* is a mapping

$$((x, p) \mapsto x \cdot p) : G \times M \to M,$$

for which

$$\begin{cases} x \cdot (y \cdot p) = (xy) \cdot p, \\ e \cdot p = p \end{cases}$$

for every $x, y \in G$ and $p \in M$; the action is *transitive* if

$$\forall p, q \in M \ \exists x \in G : \ x \cdot q = p.$$

If M is a vector space and the mapping $p \mapsto x \cdot p$ is linear for each $x \in G$, the action is called *linear*.

Example. Examples of actions:

1. $\mathrm{Aut}(V)$ acts on V by $(A, v) \mapsto Av$.
2. If $\phi \in \mathrm{Hom}(G, H)$ then G acts on H by $(x, y) \mapsto \phi(x)y$. Especially, G acts on G by $(x, y) \mapsto xy$.
3. $\mathrm{SO}(n)$ acts on the sphere $\mathbb{S}^{n-1} := \{x = (x_j)_{j=1}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$ by $(A, x) \mapsto Ax$ (interpretation: rotations of a sphere).
4. If $H < G$ and $((x, p) \mapsto x \cdot p) : G \times M \to M$ is an action then $((x, p) \mapsto x \cdot p) : H \times M \to M$ is an action.

Theorem 1.3.2. *Let $((x, p) \mapsto x \cdot p) : G \times M \to M$ be a transitive action. Let $q \in M$ and*

$$G_q := \{x \in G \mid x \cdot q = q\}.$$

Then $G_q < G$ (the so-called isotropy subgroup *of q), and*

$$f_q := (xG_q \mapsto x \cdot q) : G/G_q \to M$$

is a bijection.

Remark 1.3.3. If $G_q \lhd G$ then G/G_q is a group; otherwise the quotient is just a set. Notice also that the choice of $q \in M$ here is essentially irrelevant.

Example. Let $G = \mathrm{SO}(3)$, $M = \mathbb{S}^2$, and $q \in \mathbb{S}^2$ be the north pole (i.e. $q = (0, 0, 1) \in \mathbb{R}^3$). Then $G_q < \mathrm{SO}(3)$ consists of the rotations around the vertical axis (passing through the north and south poles). Since $\mathrm{SO}(3)$ acts transitively on $\mathbb{S}^2$, we get a bijection $\mathrm{SO}(3)/G_q \to \mathbb{S}^2$. The reader may think how $A \in \mathrm{SO}(3)$ moves the north pole $q \in \mathbb{S}^2$ to $Aq \in \mathbb{S}^2$...

Proof. Let $a, b \in G_q$. Then

$$\begin{aligned}
&e \cdot q = q, \\
&(ab) \cdot q = a \cdot (b \cdot q) = a \cdot q = q, \\
&a^{-1} \cdot q = a^{-1} \cdot (a \cdot q) = (a^{-1}a) \cdot q = e \cdot q = q,
\end{aligned}$$

so that $G_q < G$. Let $x, y \in G$. Since

$$(xa) \cdot q = x \cdot (a \cdot q) = x \cdot q,$$

$f = (xG_q \mapsto x \cdot q) : G/G_q \to M$ is a well-defined mapping. If $x \cdot q = y \cdot q$ then

$$(x^{-1}y) \cdot q = x^{-1} \cdot (y \cdot q) = x^{-1} \cdot (x \cdot q) = (x^{-1}x) \cdot q = e \cdot q = q,$$

i.e. $x^{-1}y \in G_q$, that is $xG_q = yG_q$; hence f is injective. Take $p \in M$. By transitivity, there exists $x \in G$ such that $x \cdot q = p$. Thereby $f(xG_q) = x \cdot q = p$, i.e. f is surjective. □

Remark 1.3.4. If an action $((x,p) \mapsto x \cdot p) : G \times M \to M$ is not transitive, it is often reasonable to study only the *orbit* of $q \in M$, defined by

$$G \cdot q := \{x \cdot q \mid x \in G\}.$$

Now

$$((x,p) \mapsto x \cdot p) : G \times (G \cdot q) \to (G \cdot q)$$

is transitive, and $(x \cdot q \mapsto xG_q) : G \cdot q \to G/G_q$ is a bijection. Notice that either $G \cdot p = G \cdot q$ or $(G \cdot p) \cap (G \cdot q) = \emptyset$; thus the action of G "cuts" M into a disjoint union of "slices" (orbits).

Definition 1.3.5. Let $(v,w) \mapsto \langle v,w\rangle_{\mathcal{H}}$ be the inner product of a complex vector space $\mathcal{H}$. Recall that the adjoint $A^* \in \mathrm{Aut}(\mathcal{H})$ of $A \in \mathrm{Aut}(\mathcal{H})$ is defined by

$$\langle A^*v, w\rangle_{\mathcal{H}} := \langle v, Aw\rangle_{\mathcal{H}}.$$

The *unitary group* of $\mathcal{H}$ is

$$\mathcal{U}(\mathcal{H}) := \{A \in \mathrm{Aut}(\mathcal{H}) \mid \forall v,w \in \mathcal{H}: \ \langle Av, Aw\rangle_{\mathcal{H}} = \langle v,w\rangle_{\mathcal{H}}\},$$

i.e. $\mathcal{U}(\mathcal{H})$ contains the unitary linear bijections $\mathcal{H} \to \mathcal{H}$. Clearly $A^* = A^{-1}$ for $A \in \mathcal{U}(\mathcal{H})$. The *unitary matrix group* for $\mathbb{C}^n$ is

$$\mathrm{U}(n) := \left\{A = (a_{ij})_{i,j=1}^n \in \mathrm{GL}(n,\mathbb{C}) \mid A^* = A^{-1}\right\};$$

here $A^* = (\overline{a_{ji}})_{i,j=1}^n = A^{-1}$, i.e.

$$\sum_{k=1}^n \overline{a_{ki}} a_{kj} = \delta_{ij}.$$

Definition 1.3.6. A *representation of a group G on a vector space V* is $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$; the *dimension* of ϕ is $\dim(\phi) := \dim(V)$. Representation $\psi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ is called a *unitary representation*, and $\psi \in \mathrm{Hom}(G, \mathrm{U}(n))$ is called a *unitary matrix representation*.

Remark 1.3.7. There is a bijective correspondence between the representations of G on V and linear actions of G on V: If $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ then

$$((x, v) \mapsto \phi(x)v) : G \times V \to V$$

is an action of G on V. Conversely, if $((x, v) \mapsto x \cdot v) : G \times V \to V$ is a linear action then

$$(x \mapsto (v \mapsto x \cdot v)) \in \mathrm{Hom}(G, \mathrm{Aut}(V)).$$

Example. Examples of representations:

1. If $G < \mathrm{Aut}(V)$ then $(A \mapsto A) \in \mathrm{Hom}(G, \mathrm{Aut}(V))$.
2. If $G < \mathcal{U}(\mathcal{H})$ then $(A \mapsto A) \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$.
3. There is always the trivial representation $(x \mapsto I) \in \mathrm{Hom}(G, \mathrm{Aut}(V))$.
4. Let $\mathcal{F}(G) = \mathbb{C}^G$, i.e. the vector space of functions $G \to \mathbb{C}$. Let us define $\pi_L, \pi_R \in \mathrm{Hom}(G, \mathrm{Aut}(\mathcal{F}(G)))$ by
$$\begin{aligned} (\pi_L(y)f)(x) &:= f(y^{-1}x), \\ (\pi_R(y)f)(x) &:= f(xy). \end{aligned}$$
5. Let us identify complex (1×1)-matrices with $\mathbb{C}$, $(z) \mapsto z \in \mathbb{C}$. Then $\mathrm{U}(1)$ is identified with the unit circle $\{z \in \mathbb{C} : \ |z| = 1\}$ and $(x \mapsto \mathrm{e}^{\mathrm{i}x\cdot\xi}) \in \mathrm{Hom}(\mathbb{R}^n, \mathrm{U}(1))$ for every $\xi \in \mathbb{R}^n$.
6. Analogously, $(x \mapsto \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}) \in \mathrm{Hom}(\mathbb{R}^n/\mathbb{Z}^n, \mathrm{U}(1))$ for every $\xi \in \mathbb{Z}^n$.
7. Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(W))$, where V, W are over the same field. Then
$$\phi \oplus \psi = (x \mapsto \phi(x) \oplus \psi(x)) \in \mathrm{Hom}(G, \mathrm{Aut}(V \oplus W)),$$
$$\phi \otimes \psi|_G = (x \mapsto \phi(x) \otimes \psi(x)) \in \mathrm{Hom}(G, \mathrm{Aut}(V \otimes W)),$$
where $V \oplus W$ is the direct sum and $V \otimes W$ is the tensor product space (to be introduced later).

8. If $\phi = (x \mapsto (\phi(x)_{ij})_{i,j=1}^n) \in \mathrm{Hom}(G, \mathrm{GL}(n,\mathbb{C}))$ then $\overline{\phi} = (x \mapsto (\overline{\phi(x)_{ij}})_{i,j=1}^n) \in \mathrm{Hom}(G, \mathrm{GL}(n,\mathbb{C}))$.

Definition 1.3.8. Let V be a vector space and $A \in \mathrm{End}(V)$. Subspace $W \subset V$ is called *A-invariant* if

$$AW \subset W,$$

where $AW = \{Aw : w \in W\}$. Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$. A subspace $W \subset V$ is called *ϕ-invariant* if W is $\phi(x)$-invariant for every $x \in G$ (abbreviated $\phi(G)W \subset W$); ϕ is *irreducible* if the only ϕ-invariant subspaces are the *trivial subspaces* $\{0\}$ and V.

Remark 1.3.9. If $W \subset V$ is ϕ-invariant for $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$, we may define the *restricted representation* $\phi|_W \in \mathrm{Hom}(G, \mathrm{Aut}(W))$ by $\phi|_W(x)w := \phi(x)w$. If ϕ is unitary then its restriction is also unitary.

Lemma 1.3.10. *Let $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$. Let $W \subset \mathcal{H}$ be a ϕ-invariant subspace. Then its orthocomplement*

$$W^\perp = \{v \in \mathcal{H} \mid \forall w \in W : \langle v, w\rangle_{\mathcal{H}} = 0\}$$

is ϕ-invariant.

Proof. If $x \in G$, $v \in W^\perp$ and $w \in W$ then

$$\langle \phi(x)v, w\rangle_{\mathcal{H}} = \langle v, \phi(x)^* w\rangle_{\mathcal{H}} = \langle v, \phi(x)^{-1} w\rangle_{\mathcal{H}} = \langle v, \phi(x^{-1}) w\rangle_{\mathcal{H}} = 0,$$

meaning that $\phi(x)v \in W^\perp$. □

Definition 1.3.11. Let V be an inner product space and let $\{V_j\}_{j\in J}$ be some family of its mutually orthogonal subspaces (i.e. $\langle v_i, v_j\rangle_V = 0$ if $v_i \in V_i$, $v_j \in V_j$ and $i \neq j$). The *(algebraic) direct sum* of $\{V_j\}_{j\in J}$ is the subspace

$$W = \bigoplus_{j\in J} V_j := \mathrm{span} \bigcup_{j\in J} V_j.$$

If $A_j \in \mathrm{End}(V_j)$ then define

$$A = \bigoplus_{j\in J} A_j \in \mathrm{End}(W)$$

by $A|_{V_j} v = A_j v$ for every $j \in J$ and $v \in V_j$. If $\phi_j \in \mathrm{Hom}(G, \mathrm{Aut}(V_j))$ then define

$$\phi = \bigoplus_{j \in J} \phi_j \in \mathrm{Hom}(G, \mathrm{Aut}(W))$$

by $\phi|_{V_j} = \phi_j$ for every $j \in J$, i.e. $\phi(x) := \bigoplus_{j\in J} \phi_j(x)$ for every $x \in G$.

Theorem 1.3.12. *Let $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ be finite-dimensional. Then ϕ is a direct sum of irreducible unitary representations.*

Proof (by induction). The claim is true for $\dim(\mathcal{H}) = 1$, since then the only subspaces of $\mathcal{H}$ are the trivial ones. Suppose the claim is true for representations of dimension n or less. Suppose $\dim(\mathcal{H}) = n + 1$. If ϕ is irreducible, there is nothing to prove. Hence assume that there exists a non-trivial ϕ-invariant subspace $W \subset \mathcal{H}$. Then also the orthocomplement $W^\perp$ is ϕ-invariant by Lemma 1.3.10. Due to the ϕ-invariance of the subspaces W and $W^\perp$, we may define restricted representations $\phi|_W \in \mathrm{Hom}(G,\mathcal{U}(W))$ and $\phi|_{W^\perp} \in \mathrm{Hom}(G,\mathcal{U}(W^\perp))$. Hence $\mathcal{H} = W \oplus W^\perp$ and $\phi = \phi|_W \oplus \phi|_{W^\perp}$. Moreover, $\dim(W) \le n$ and $\dim(W^\perp) \le n$; the proof is complete, since unitary representations up to dimension n are direct sums of irreducible unitary representations. □

Remark 1.3.13. Theorem 1.3.12 means that if $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is finite-dimensional then

$$\mathcal{H} = \bigoplus_{j=1}^{k} W_j, \quad \phi = \bigoplus_{j=1}^{k} \phi|_{W_j},$$

where each $\phi|_{W_j} \in \mathrm{Hom}(G,\mathcal{U}(W_j))$ is irreducible.

Definition 1.3.14. A linear mapping $A : V \to W$ is an *intertwining operator* between representations $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut(W)})$, denoted by $A \in \mathrm{Hom}(\phi, \psi)$, if

$$A\phi(x) = \psi(x)A$$

for every $x \in G$; if such A is invertible then ϕ and ψ are said to be *equivalent*, denoted by $\phi \sim \psi$.

Remark 1.3.15. Always $0 \in \mathrm{Hom}(\phi, \psi)$, and $\mathrm{Hom}(\phi, \psi)$ is a vector space. Moreover, if $A \in \mathrm{Hom}(\phi, \psi)$ and $B \in \mathrm{Hom}(\psi, \xi)$ then $BA \in \mathrm{Hom}(\phi, \xi)$.

Proposition 1.3.16. *Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V_\phi))$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(V_\psi))$ be irreducible. If $A \in \mathrm{Hom}(\phi, \psi)$ then either $A = 0$ or $A : V_\phi \to V_\psi$ is invertible.*

Proof. The image $AV_\phi \subset V_\psi$ of A is ψ-invariant, because

$$\psi(G)\ AV_\phi = A\ \phi(G)V_\phi = AV_\phi,$$

so that either $AV_\phi = \{0\}$ or $AV_\phi = V_\psi$, as ψ is irreducible. Hence either $A = 0$ or A is a surjection.

The kernel $\mathrm{Ker}(A) = \{v \in V_\phi \mid Av = 0\}$ is ϕ-invariant, since

$$A\ \phi(G)\ \mathrm{Ker}(A) = \psi(G)\ A\ \mathrm{Ker}(A) = \psi(G)\,\{0\} = \{0\}\,,$$

so that either $\mathrm{Ker}(A) = \{0\}$ or $\mathrm{Ker}(A) = V_\phi$, as ϕ is irreducible. Hence either A is injective or $A = 0$.

Thus either $A = 0$ or A is bijective. □

Corollary 1.3.17. (Schur's Lemma (finite-dimensional [1905]).) *Let* $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ *be irreducible and finite-dimensional. Then* $\mathrm{Hom}(\phi, \phi) = \mathbb{C}I = \{\lambda I \mid \lambda \in \mathbb{C}\}$.

Proof. Let $A \in \mathrm{Hom}(\phi, \phi)$. The finite-dimensional linear operator $A : V \to V$ has an eigenvalue $\lambda \in \mathbb{C}$: now $\lambda I - A : V \to V$ is not invertible. On the other hand, $\lambda I - A \in \mathrm{Hom}(\phi, \phi)$, so that $\lambda I - A = 0$ by Proposition 1.3.16. □

Corollary 1.3.18. *Let G be a commutative group. Irreducible finite-dimensional representations of G are one-dimensional.*

Proof. Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ be irreducible, $\dim(\phi) < \infty$. Due to the commutativity of G,

$$\phi(x)\phi(y) = \phi(xy) = \phi(yx) = \phi(y)\phi(x)$$

for every $x, y \in G$, so that $\phi(G) \subset \mathrm{Hom}(\phi, \phi)$. By Schur's Lemma 1.3.17, $\mathrm{Hom}(\phi, \phi) = \mathbb{C}I$. Hence if $v \in V$ then

$$\phi(G)\mathrm{span}\{v\} = \mathrm{span}\{v\},$$

i.e. $\mathrm{span}\{v\}$ is ϕ-invariant. Therefore either $v = 0$ or $\mathrm{span}\{v\} = V$. □

Corollary 1.3.19. *Let* $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}_\phi))$ *and* $\psi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}_\psi))$ *be finite-dimensional. Then* $\phi \sim \psi$ *if and only if there exists isometric isomorphism* $B \in \mathrm{Hom}(\phi, \psi)$.

Remark 1.3.20. An *isometry* $f : M \to N$ between metric spaces $(M, d_M), (N, d_N)$ satisfies $d_N(f(x), f(y)) = d_M(x, y)$ for every $x, y \in M$.

Proof. The "if"-part is trivial. Assume that $\phi \sim \psi$. Recall that there are direct sum decompositions

$$\phi = \bigoplus_{j=1}^{m} \phi_j, \quad \psi = \bigoplus_{k=1}^{n} \psi_k,$$

where ϕ_j, ψ_k are irreducible unitary representations on $\mathcal{H}_{\phi_j}, \mathcal{H}_{\psi_k}$, respectively. Now $n = m$, since $\phi \sim \psi$. Moreover, we may arrange the indeces so that $\phi_j \sim \psi_j$ for each j. Choose invertible $A_j \in \mathrm{Hom}(\phi_j, \psi_j)$. Then A_j^* is invertible, and $A_j^* \in \mathrm{Hom}(\psi_j, \phi_j)$: if $x \in G$, $v \in \mathcal{H}_{\phi_j}$ and $w \in \mathcal{H}_{\psi_j}$ then

$$\begin{aligned}
\langle A_j^* \psi_j(x) w, v \rangle_{\mathcal{H}_\phi} &= \langle w, \psi_j(x)^* A_j v \rangle_{\mathcal{H}_\psi} \\
&= \langle w, \psi_j(x^{-1}) A_j v \rangle_{\mathcal{H}_\psi} \\
&= \langle w, A_j \phi_j(x^{-1}) v \rangle_{\mathcal{H}_\psi} \\
&= \langle \phi_j(x^{-1})^* A_j^* w, v \rangle_{\mathcal{H}_\phi} \\
&= \langle \phi_j(x) A_j^* w, v \rangle_{\mathcal{H}_\phi}.
\end{aligned}$$

Thereby $A_j^* A_j \in \mathrm{Hom}(\phi_j, \phi_j)$ is invertible. By Schur's Lemma 1.3.17, $A_j^* A_j = \lambda_j I$, where $\lambda_j \neq 0$. Let $v \in \mathcal{H}_{\phi_j}$ such that $\|v\|_{\mathcal{H}_\phi} = 1$. Then

$$\lambda = \lambda \|v\|_{\mathcal{H}_\phi}^2 = \langle \lambda v, v \rangle_{\mathcal{H}_\phi} = \langle A_j^* A_j v, v \rangle_{\mathcal{H}_\phi} = \langle A_j v, A_j v \rangle_{\mathcal{H}_\psi} = \|A_j v\|_{\mathcal{H}_\psi}^2 > 0,$$

so that we may define $B_j := \lambda^{-1/2} A_j \in \mathrm{Hom}(\phi_j, \psi_j)$. Then $B_j : \mathcal{H}_{\phi_j} \to \mathcal{H}_{\psi_j}$ is an isometry, $B_j^* B_j = I$. Finally, define

$$B := \bigoplus_{j=1}^{m} B_j.$$

Clearly, $B : \mathcal{H}_\phi \to \mathcal{H}_\psi$ is an isometry, bijection, and $B \in \mathrm{Hom}(\phi, \psi)$. □

Exercise 1.3.21. Let G be a finite group and let $\mathcal{F}(G)$ be the vector space of functions $f : G \to \mathbb{C}$. Let

$$\int_G f \, \mathrm{d}\mu_G := \frac{1}{|G|} \sum_{x \in G} f(x),$$

when $f \in \mathcal{F}(G)$. Let us endow $\mathcal{F}(G)$ with the inner product

$$\langle f, g \rangle_{L^2(\mu_G)} := \int_G f \, \overline{g} \, \mathrm{d}\mu_G.$$

Define $\pi_L, \pi_R : G \to \mathrm{Aut}(\mathcal{F}(G))$ by

$$(\pi_L(y)\ f)(x) := f(y^{-1}x),$$

$$(\pi_R(y)\ f)(x) := f(xy).$$

Show that π_L and π_R are equivalent unitary representations.

Exercise 1.3.22. Let G be non-commutative and $|G| = 6$. Endow $\mathcal{F}(G)$ with the inner product given in Exercise 1.3.21. Find the π_L-invariant subspaces and give orthogonal bases for them.

Exercise 1.3.23. Let us endow the n-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ with the quotient group structure and with the Lebesgue measure. Let $\pi_L, \pi_R : \mathbb{T}^n \to \mathcal{L}(L^2(\mathbb{T}^n))$ be defined by

$$(\pi_L(y)\ f)(x) := f(x - y),$$

$$(\pi_R(y)\ f)(x) := f(x + y)$$

for almost every $x \in \mathbb{T}^n$. Show that π_L and π_R are equivalent reducible unitary representations. Describe the minimal π_L- and π_R-invariant subspaces containing the function $x \mapsto \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}$, where $\xi \in \mathbb{Z}^n$.

Chapter 2

Topological groups

2.1 Topological groups

Definition 2.1.1. A group and a topological space G is called a *topological group* if $\{e\} \subset G$ is closed and if the mappings

$$\begin{aligned} ((x,y) \mapsto xy) &\quad : \quad G \times G \to G, \\ (x \mapsto x^{-1}) &\quad : \quad G \to G \end{aligned}$$

are continuous.

Example. In the following, when not specified, the topologies and the group operations are the usual ones:

1. Any group G endowed with the discrete topology $\mathcal{P}(G) = \{U : U \subset G\}$ is a topological group.
2. $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are topological groups when the group operation is the addition and the topology is as usual.
3. $\mathbb{Q}^\times$, $\mathbb{R}^\times$, $\mathbb{C}^\times$ are topological groups when the group operation is the multiplication and the topology is as usual.
4. Topological vector spaces are topological groups with vector addition: such a space is both a vector space and a topological Hausdorff space such that the vector space operations continuous.
5. Let X be a Banach space. The set $\mathrm{AUT}(X) := \mathrm{Aut}(X) \cap \mathcal{L}(X)$ of invertible bounded linear operators $X \to X$ forms a topological group with respect to the norm topology.

6. Subgroups of topological groups are topological groups.
7. If G and H are topological groups then $G \times H$ is a topological group. Actually, Cartesian products always preserve the topological group structure.

Exercise 2.1.2. Show that a topological group is actually even a Hausdorff space.

Lemma 2.1.3. *Let G be a topological group and $y \in G$. Then*

$$x \mapsto xy, \quad x \mapsto yx, \quad x \mapsto x^{-1}$$

are homeomorphisms $G \to G$.

Proof. Mapping

$$(x \mapsto xy) : G \overset{x \mapsto (x,y)}{\to} G \times G \overset{(a,b) \mapsto ab}{\to} G$$

is continuous as a composition of continuous mappings. The inverse mapping is $(x \mapsto xy^{-1}) : G \to G$, being also continuous; hence this is a homeomorphism. Similarly, $(x \mapsto yx) : G \to G$ is a homeomorphism. The inversion $(x \mapsto x^{-1}) : G \to G$ is continuous by definition, and it is its own inverse. □

Corollary 2.1.4. *If $U \subset G$ is open and $S \subset G$ then $SU, US, U^{-1} \subset G$ are open.*

Proposition 2.1.5. *Let G be a topological group. If $H < G$ then $\overline{H} < G$. If $H \lhd G$ then $\overline{H} \lhd G$.*

Proof. Let $H < G$. Trivially $e \in H \subset \overline{H}$. Now

$$\overline{H}\,\overline{H} \subset \overline{HH} = \overline{H},$$

where the inclusion is due to the continuity of the mapping $((x, y) \mapsto xy) : G \times G \to G$. The continuity of the inversion $(x \mapsto x^{-1}) : G \to G$ gives

$$\overline{H}^{-1} \subset \overline{H^{-1}} = \overline{H}.$$

Thus $\overline{H} < G$.

Let $H \lhd G$, $y \in G$. Then

$$y\overline{H} = \overline{yH} = \overline{Hy} = \overline{H}y;$$

notice how homeomorphisms $(x \mapsto yx), (x \mapsto xy) : G \to G$ were exploited. □

Proposition 2.1.6. *Let G be a topological group and $C_e \subset G$ the component of e. Then $C_e \lhd G$ is closed.*

Proof. Components are always closed, and $e \in C_e$ by definition. Since $C_e \subset G$ is connected, also $C_e \times C_e \subset G \times G$ and is connected. By the continuity of the group operations, $C_e C_e \subset G$ and $C_e^{-1} \subset G$ are connected. Since $e = ee \in C_e C_e$, we have $C_e C_e \subset C_e$. And since $e = e^{-1} \in C_e^{-1}$, also $C_e^{-1} \subset C_e$. Take $y \in G$. Then $y^{-1} C_e y \subset G$ is connected, by the continuity of $(x \mapsto y^{-1} x y) : G \to G$. Now $e = y^{-1} e y \in y^{-1} C_e y$, so that $y^{-1} C_e y \subset C_e$; C_e is normal in G. □

Remark 2.1.7. Let $H < G$ and $S \subset G$. The mapping $(x \mapsto xH) : G \to G/H$ identifies the sets

$$\begin{array}{rcl} SH = \{sh : \ s \in S, \ h \in H\} & \subset & G, \\ \{sH : \ s \in S\} = \{\{sh : \ h \in H\} : \ s \in S\} & \subset & G/H. \end{array}$$

This provides a nice way to treat the quotient G/H.

Definition 2.1.8. Let G be a topological group, $H < G$. The *quotient topology* of G/H is

$$\tau_{G/H} := \{\{uH : \ u \in U\} : \ U \subset G \text{ open}\};$$

in other words, $\tau_{G/H}$ is the strongest (i.e. largest) topology for which the quotient map $(x \mapsto xH) : G \to G/H$ is continuous. If $U \subset G$ is open, we may identify sets $UH \subset G$ and $\{uH : u \in H\} \subset G/H$.

Theorem 2.1.9. *Let G be a topological group and $H \lhd G$. Then*

$$\begin{array}{rcl} ((xH, yH) \mapsto xyH) & : & (G/H) \times (G/H) \to G/H, \\ (xH \mapsto x^{-1}H) & : & G/H \to G/H \end{array}$$

are continuous. Moreover, G/H is a topological group if and only if H is closed.

Proof. We know already that the operations in Theorem are well-defined group operations, because H is normal in G. Recall Remark 2.1.7, how we may identify certain subsets of G with subsets of G/H. Then a neighbourhood of the point $xyH \in G/H$ is of the form UH for some open $U \subset G$, $U \ni xy$. Take open $U_1 \ni x$ and $U_2 \ni y$ such that $U_1 U_2 \subset U$. Then

$$(xH)(yH) \subset (U_1 H)(U_2 H) = U_1 U_2 H \subset UH,$$

so that $((xH, yH) \mapsto xyH) : (G/H) \times (G/H) \to G/H$ is continuous. A neighbourhood of the point $x^{-1}H \in G/H$ is of the form VH for some open $V \subset G$, $V \ni x^{-1}$. But $V^{-1} \ni x$ is open, and $(V^{-1})^{-1} = V$, so that $(xH \mapsto x^{-1}H) : G/H \to G/H$ is continuous.

Notice that $e_{G/H} = H$. If G/H is a topological group, then

$$H = (x \mapsto xH)^{-1} \left\{ e_{G/H} \right\} \subset G$$

is closed. On the other hand, if $H \triangleleft G$ is closed then

$$(G/H) \setminus \{e_{G/H}\} \cong (G \setminus H)H \subset G$$

is open, i.e. $\{e_{G/H}\} \subset G/H$ is closed. □

Definition 2.1.10. Let G_1, G_2 be topological groups. Let

$$\mathrm{HOM}(G_1, G_2) := \mathrm{Hom}(G_1, G_2) \cap C(G_1, G_2),$$

i.e. the set of continuous homomorphisms $G_1 \to G_2$.

Remark 2.1.11. By Theorem 2.1.9, closed subgroups of G correspond bijectively to continuous surjective homomorphisms from G to some other topological group (up to isomorphism).

Definition 2.1.12. Let G be a topological group and $\mathcal{H}$ be a Hilbert space. A representation $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ is *strongly continuous* if

$$(x \mapsto \phi(x)v) : G \to \mathcal{H}$$

is continuous for every $v \in \mathcal{H}$.

Remark 2.1.13. This means that $(x \mapsto \phi(x)) : G \to \mathcal{L}(\mathcal{H})$ is continuous, when $\mathcal{L}(\mathcal{H}) \supset \mathcal{U}(\mathcal{H})$ is endowed with the *strong operator topology*:

$$A_j \overset{\text{strongly}}{\to} A \overset{\text{definition}}{\iff} \forall v \in \mathcal{H} : \ \|A_j v - Av\|_{\mathcal{H}} \to 0.$$

Why we should not endow $\mathcal{U}(\mathcal{H})$ with the operator norm topology (which is even stronger, i.e. larger topology)? The reason is that there are interesting unitary representations, which are continuous in the strong operator topology, but not in the operator norm topology: this is exemplified by $\pi_L : \mathbb{R}^n \to \mathcal{U}(L^2(\mathbb{R}^n))$, defined by

$$(\pi_L(y)f)(x) := f(x-y)$$

for almost every $x \in \mathbb{R}^n$.

Definition 2.1.14. A strongly continuous $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is called *topologically irreducible* if the only closed ϕ-invariant subspaces are the trivial ones $\{0\}$ and $\mathcal{H}$.

Exercise 2.1.15. Let V be a topological vector space and let $W \subset V$ be an A-invariant subspace, where $A \in \mathrm{Aut}(V)$ is continuous. Show that the closure $\overline{W} \subset V$ is also A-invariant.

Definition 2.1.16. A strongly continuous $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is called a *cyclic representation* if

$$\mathrm{span}\ \phi(G)v \subset \mathcal{H}$$

is dense for some $v \in \mathcal{H}$; then such v is called a *cyclic vector*.

Example. If $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ is topologically irreducible then any non-zero $v \in \mathcal{H}$ is cyclic: Namely, if $V := \mathrm{span}\ \phi(G)v$ then $\phi(G)V \subset V$ and consequently $\phi(G)\overline{V} \subset \overline{V}$, so that $\overline{V}$ is ϕ-invariant. If $v \neq 0$ then $\overline{V} = \mathcal{H}$, because of the topological irreducibility.

Definition 2.1.17. A Hilbert space $\mathcal{H}$ is a *direct sum* of closed subspaces $(\mathcal{H}_j)_{j\in J}$, denoted by

$$\mathcal{H} = \bigoplus_{j\in J} \mathcal{H}_j$$

if the subspace family is pairwise orthogonal and $\mathrm{span} \cup_{j\in J} \mathcal{H}_j$ is dense in $\mathcal{H}$. Then

$$\forall x \in \mathcal{H}\ \forall j \in J\ \exists! x_j \in \mathcal{H}_j :\ x = \sum_{j\in J} x_j, \quad \|x\|_{\mathcal{H}}^2 = \sum_{j\in J} \|x_j\|_{\mathcal{H}}^2 .$$

If $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ and each $\mathcal{H}_j$ is ϕ-invariant then ϕ is said to be the *direct sum*

$$\phi = \bigoplus_{j\in J} \phi|_{\mathcal{H}_j}$$

where $\phi|_{\mathcal{H}_j} = (x \mapsto \phi(x)v) \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}_j))$.

Proposition 2.1.18. *Let $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ be strongly continuous. Then*

$$\phi = \bigoplus_{j\in J} \phi|_{\mathcal{H}_j},$$

where each $\phi|_{\mathcal{H}_j}$ is cyclic.

Proof. Let $\tilde{J}$ be the family of all closed ϕ-invariant subspaces $V \subset \mathcal{H}$ for which $\phi|_V$ is cyclic. Let

$$S = \left\{ s \subset \tilde{J} \;\middle|\; \forall V, W \in s : \; V = W \text{ or } V \perp W \right\}.$$

It is easy to see that $\{\{0\}\} \in S$, so that $S \neq \emptyset$. Let us introduce a partial order on S by inclusion:

$$s_1 \leq s_2 \overset{\text{definition}}{\iff} s_1 \subset s_2.$$

The chains in S have upper bounds: if $R \subset S$ is a chain then $r \leq \cup_{s\in R}\, s \in S$ for every $r \in R$. Therefore by **Zorn's Lemma**, there exists a maximal element $t \in S$. Let

$$V := \bigoplus_{W\in t} W.$$

To get a contradiction, suppose $V \neq \mathcal{H}$. Then there exists $v \in V^\perp \setminus \{0\}$. Since $\mathrm{span}(\phi(G)v)$ is ϕ-invariant, its closure W_0 is also ϕ-invariant (see Exercise 2.1.15). Clearly $W_0 \subset \overline{V^\perp} = V^\perp$, and $\phi|_{W_0}$ has cyclic vector v, yielding

$$s := t \cup \{W_0\} \in S,$$

where $t \leq s \not\leq t$. This contradicts the maximality of t; thus $V = \mathcal{H}$. □

Exercise 2.1.19. Fill in the details in the proof of Proposition 2.1.18.

Exercise 2.1.20. Assuming that $\mathcal{H}$ is separable, prove Proposition 2.1.18 by ordinary induction (without resorting to general Zorn's Lemma).

2.2 Some results for topological groups

Proposition 2.2.1. *Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be a continuous action of G on M, and let $q \in M$. If G_q and G/G_q are connected then G is connected.*

Proof. Suppose G is disconnected and G_q is connected. Then there are non-empty disjoint open sets $U, V \subset G$ such that $G = U \cup V$. The sets

$$U' := \{uG_q : \ u \in U\} \subset G/G_q, \quad V' := \{vG_q : \ v \in V\} \subset G/G_q$$

are non-empty and open, and $G/G_q = U' \cup V'$. Take $u \in U$ and $v \in V$. As a continuous image of a connected set, $uG_q = (x \mapsto ux)(G_q) \subset G$ is connected; moreover $u = ue \in uG_q$; thereby $uG_q \subset U$. In the same way we see that $vG_q \subset V$. Hence $U' \cap V' = \emptyset$, so that G/G_q is disconnected. □

Corollary 2.2.2. *If G is a topological group, $H < G$ is connected and G/H is connected then G is connected.*

Proof. Using the notation of Proposition 2.2.1, let $M = G/H$, $q = H$ and $x \cdot p = xp$, so that $G_q = H$ and $G/G_q = G/H$. □

Exercise 2.2.3. Show that $\mathrm{SO}(n)$, $\mathrm{SU}(n)$ and $\mathrm{U}(n)$ are connected for every $n \in \mathbb{Z}^+$. How about $\mathrm{O}(n)$?

Proposition 2.2.4. *Let G be a topological group and $H < G$. Then $f : G/H \to \mathbb{C}$ is continuous if and only if $(x \mapsto f(xH)) : G \to \mathbb{C}$ is continuous.*

Proof. If $f \in C(G/H)$ then $(x \mapsto f(xH)) \in C(G)$, since it is obtained by composing f and the continuous quotient map $(x \mapsto xH) : G \to G/H$.

Now suppose $(x \mapsto f(xH)) \in C(G)$. Take open $V \subset \mathbb{C}$. Then $U := (x \mapsto f(xH))^{-1}(V) \subset G$ is open, so that $U' := \{uH : \ u \in U\} \subset G/H$ is open. Trivially, $f(U') = V$. Hence $f \in C(G/H)$. □

Proposition 2.2.5. *Let G be a topological group and $H < G$. Then G/H is a Hausdorff space if and only if H is closed.*

Proof. If G/H is a Hausdorff space then $H = (x \mapsto xH)^{-1}(\{H\}) \subset G$ is closed, because the quotient map is continuous and $\{H\} \subset G/H$ is closed.

Next suppose H is closed. Take $xH, yH \in G/H$ such that $xH \neq yH$. Then $S := ((a,b) \mapsto a^{-1}b)^{-1}(H) \subset G \times G$ is closed, since $H \subset G$ is closed and $((a,b) \mapsto a^{-1}b) : G \times G \to G$ is continuous. Now $(x,y) \notin S$. Take open sets $U \ni x$ and $V \ni y$ such that $(U \times V) \cap S = \emptyset$. Then

$$U' := \{uH : \ u \in U\} \subset G/H, \quad V' := \{vH : \ v \in V\} \subset G/H$$

are disjoint open sets, and $xH \in U'$, $yH \in V'$; G/H is Hausdorff. □

2.3 Compact groups

Definition 2.3.1. A topological group is a *(locally) compact group* if it is (locally) compact as a topological space.

Example.

1. Any group G with the discrete topology is a locally compact group; then G is a compact group if and only if it is finite.
2. $\mathbb{Q}, \mathbb{Q}^\times$ are not locally compact groups; $\mathbb{R}, \mathbb{R}^\times, \mathbb{C}, \mathbb{C}^\times$ are locally compact groups, but non-compact.
3. A topological vector space is a locally compact group if and only if it is finite-dimensional.
4. $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n)$ are compact groups.
5. $\mathrm{GL}(n)$ is a locally compact group, but non-compact.
6. If G, H are locally compact groups then $G \times H$ is a locally compact group.
7. If $\{G_j\}_{j \in J}$ is a family of compact groups then $\prod_{j \in J} G_j$ is a compact group.
8. If G is a compact group and $H < G$ is closed then H is a compact group.

Proposition 2.3.2. *Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be a continuous action of a compact group G on a Hausdorff space M, and let $q \in M$. Then*

$$f := (xG_q \mapsto x \cdot q) : G/G_q \to G \cdot q$$

is a homeomorphism.

Proof. We already know that f is a well-defined bijection. We need to show that f is continuous. An open subset of $G \cdot q$ is of the form $V \cap (G \cdot q)$, where $V \subset M$ is open. Since the action is continuous, also $(x \mapsto x \cdot q) : G \to M$ is continuous, so that $U := (x \mapsto x \cdot q)^{-1}(V) \subset G$ is open. Thereby

$$f^{-1}(V \cap (G \cdot q)) = \{xG_q : \ x \in U\} \subset G/G_q$$

is open; f is continuous. Space G is compact and the quotient map $(x \mapsto xG_q) : G \to G/G_q$ is continuous, so that G/G_q is compact. From the general topology we know that a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. □

Corollary 2.3.3. *If G is compact, $\phi \in \mathrm{HOM}(G,H)$ and $K = \mathrm{Ker}(\phi)$ then*

$$\psi := (xK \mapsto \phi(x)) \in \mathrm{HOM}(G/K, \phi(G))$$

is a homeomorphism.

Proof. Using the notation of Proposition 2.3.2, we have $M = H$, $q = e_H$, $x \cdot p = \phi(x)p$, so that $G_q = K$, $G/G_q = G/K$, $G \cdot q = \phi(G)$, $\psi = f$. □

Remark 2.3.4. What could happen if we drop the compactness assumption in Corollary 2.3.3? If G and H are Banach spaces, $\phi \in \mathcal{L}(G,H)$ is compact and $\dim(\phi(G)) = \infty$ then $\psi = (x + \mathrm{Ker}(\phi) \mapsto \phi(x)) : G/\mathrm{Ker}(\phi) \to \phi(G)$ is a bounded linear bijection, but ψ^{-1} is not bounded! But if $\phi \in \mathcal{L}(G,H)$ is a bijection then ϕ^{-1} is bounded by the **Open Mapping Theorem**!

Definition 2.3.5. Let G be a topological group. A function $f : G \to \mathbb{C}$ is *uniformly continuous* if for every $\varepsilon > 0$ there exists open $U \ni e$ such that

$$\forall x, y \in G : \ x^{-1}y \in U \ \Rightarrow \ |f(x) - f(y)| < \varepsilon.$$

Exercise 2.3.6. Under which circumstances a polynomial $p : \mathbb{R} \to \mathbb{C}$ is uniformly continuous? Show that if a continuous function $f : \mathbb{R} \to \mathbb{C}$ is periodic or vanishes outside a bounded set then it is uniformly continuous.

Theorem 2.3.7. *If G is a compact group and $f \in C(G)$ then f is uniformly continuous.*

Proof. Take $\varepsilon > 0$. Define the open disk $\mathbb{D}(z,r) := \{w \in \mathbb{C} : \ |w - z| < r\}$, where $z \in \mathbb{C}$, $r > 0$. Since f is continuous,

$$V_x := f^{-1}(\mathbb{D}(f(x), \varepsilon)) \ni x$$

is open. Then $x^{-1}V_x \ni e = ee$ is open, so that there exist open $U_{1,x}, U_{2,x} \ni e$ such that $U_{1,x}U_{2,x} \subset x^{-1}V_x$, by the continuity of the group multiplication. Define $U_x := U_{1,x} \cap U_{2,x}$. Since $\{xU_x : \ x \in G\}$ is an open cover of compact G, there is a finite subcover $\{x_j U_{x_j}\}_{j=1}^n$. Now

$$U := \bigcap_{j=1}^{n} U_{x_j} \ni e$$

is open. Suppose $x, y \in G$ such that $x^{-1}y \in U$. There exists $k \in \{1, \ldots, n\}$ such that $x \in x_k U_{x_k}$, so that

$$x, y \in xU \subset x_k U_{x_k} U_{x_k} \subset x_k x_k^{-1} V_{x_k} = V_{x_k},$$

yielding

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_k)| + |f(x_k) - f(y)| \\ &< 2\varepsilon. \end{aligned}$$

□

Exercise 2.3.8. Let G be a compact group, $x \in G$ and $A = \{x^n\}_{n=1}^{\infty}$. Show that $\overline{A} < G$.

2.4 Haar measure

Definition 2.4.1. Let X be a compact Hausdorff space and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $C(X, \mathbb{K})$ is a Banach space over $\mathbb{K}$ with the norm

$$f \mapsto \|f\|_{C(X,\mathbb{K})} := \max_{x \in X} |f(x)|.$$

Its dual $C(X, \mathbb{K})' = \mathcal{L}(C(X, \mathbb{K}), \mathbb{K})$ consists of the bounded linear functionals $C(X, \mathbb{K}) \to \mathbb{K}$, and is endowed with the Banach space norm

$$L \mapsto \|L\|_{C(X,\mathbb{K})'} := \sup_{f \in C(X,\mathbb{K}):\ \|f\|_{C(X,\mathbb{K})} \leq 1} |Lf|.$$

A functional $L : C(X, \mathbb{K}) \to \mathbb{C}$ is called *positive* if $Lf \geq 0$ whenever $f \geq 0$.

By the Riesz Representation Theorem (see e.g. [17]), if $L \in C(X, \mathbb{K})'$ is positive then there exists a unique positive Borel regular measure μ on X such that

$$Lf = \int_X f \ \mathrm{d}\mu$$

for every $f \in C(X, \mathbb{K})$; moreover, $\mu(X) = \|L\|_{C(X,\mathbb{K})'}$. For short, $C(X) := C(X, \mathbb{C})$.

To be proven. *Let G be a compact group. There exists a unique positive linear functional* Haar $\in C(G)'$ *such that*

$$\begin{aligned} \mathrm{Haar}(f) &= \mathrm{Haar}(x \mapsto f(yx)), \\ \mathrm{Haar}(\mathbf{1}) &= 1, \end{aligned}$$

for every $y \in G$*, where* $\mathbf{1} = (x \mapsto 1) \in C(G)$*. Moreover, this* Haar integral *satisfies*

$$\begin{aligned} \mathrm{Haar}(f) &= \mathrm{Haar}(x \mapsto f(xy)) \\ &= \mathrm{Haar}(x \mapsto f(x^{-1})). \end{aligned}$$

Remark 2.4.2. By the Riesz Representation Theorem (see e.g. [17]), the Haar integral begets a unique Borel regular probability measure μ_G such that

$$\mathrm{Haar}(f) = \int_G f \ \mathrm{d}\mu_G.$$

This μ_G is called the *Haar measure* of G. Obviously,

$$\begin{aligned} \int_G \mathbf{1} \ \mathrm{d}\mu_G &= \mu_G(G) \ = \ 1, \\ \int_G f \ \mathrm{d}\mu_G &= \int_G f(yx) \ \mathrm{d}\mu_G(x) \\ &= \int_G f(xy) \ \mathrm{d}\mu_G(x) \\ &= \int_G f(x^{-1}) \ \mathrm{d}\mu_G(x). \end{aligned}$$

Thus the Haar integral $\mathrm{Haar}(f) = \int_G f \ \mathrm{d}\mu_G$ can be thought as the most natural average of $f \in C(G)$. In the real (but still idealized) world, we can know usually only finitely many values of f, i.e. we are able to take only samples $\{f(x) : \ x \in S\}$ for a finite set $S \subset G$. Then a natural idea for approximating $\mathrm{Haar}(f)$ would be computing

$$\sum_{x \in S} f(x) \ \alpha(x),$$

where sampling weights $\alpha(x) \geq 0$ satisfy $\sum_{x \in S} \alpha(x) = 1$. The problem is to find clever choices for sampling sets and weights, some sort of "almost uniformly distributed unit mass" on G is needed; for this end we shall introduce convolutions.

Example. If G is finite then

$$\int_G f \ \mathrm{d}\mu_G = \frac{1}{|G|} \sum_{x \in G} f(x).$$

For $\mathbb{T} = \mathbb{R}/\mathbb{Z}$,

$$\int_{\mathbb{T}} f \, \mathrm{d}\mu_{\mathbb{T}} = \int_0^1 f(x+\mathbb{Z}) \, \mathrm{d}x,$$

i.e. integration with respect to the Lebesgue measure on $[0,1[$.

Definition 2.4.3. Let G be a compact group. A function $\alpha : G \to [0,1]$ is a *sampling measure* on G, $\alpha \in \mathcal{SM}_G$, if

$$\mathrm{supp}(\alpha) = \{a \in G : \ \alpha(a) \neq 0\} \quad \text{is finite and} \quad \sum_{a\in G} \alpha(a) = 1.$$

The set $\mathrm{supp}(\alpha) \subset G$ is called the *support* of α. Naturally, $\alpha \in \mathcal{SM}_G$ can be regarded as a finitely supported probability measure on G, and

$$\int_G f \, \mathrm{d}\alpha = \check{\alpha} * f(e) = f * \check{\alpha}(e),$$

where $\check{\alpha}(a) := \alpha(a^{-1})$.

Definition 2.4.4. Let $\alpha, \beta \in \mathcal{SM}_G$ and $f \in C(G, \mathbb{K})$. The *convolutions*

$$\alpha * \beta, \alpha * f, f * \beta : G \to \mathbb{K}$$

are defined by

$$\begin{aligned} \alpha * \beta(b) &= \sum_{a\in G} \alpha(a)\beta(a^{-1}b), \\ \alpha * f(x) &= \sum_{a\in G} \alpha(a) f(a^{-1}x), \\ f * \beta(x) &= \sum_{b\in G} f(xb^{-1})\beta(b). \end{aligned}$$

Definition 2.4.5. A *semigroup* is a non-empty set S with an operation $((r,s) \mapsto rs) : S \times S \to S$ satisfying $r(st) = (rs)t$ for every $r, s, t \in S$. A semigroup is *commutative* if $rs = sr$ for every $r, s \in S$. Moreover, if there exists $e \in S$ such that $es = se = s$ for every $s \in S$ then S is called a *monoid*.

Example. $\mathbb{Z}^+ = \{n \in \mathbb{Z} : \ n > 0\}$ is a commutative monoid with respect to multiplication, and a commutative semigroup with respect addition. If V is a vector space then $(\mathrm{End}(V), (A, B) \mapsto AB)$ is a monoid with $e = I$.

Lemma 2.4.6. *$(\mathcal{SM}_G, (\alpha, \beta) \mapsto \alpha * \beta)$ is a monoid.*

Exercise 2.4.7. Prove Lemma 2.4.6. How $\mathrm{supp}(\alpha * \beta)$ is related to $\mathrm{supp}(\alpha)$ and $\mathrm{supp}(\beta)$? In which case $\mathcal{SM}_G$ is a group? Show that $\mathcal{SM}_G$ is commutative if and only if G is commutative.

Lemma 2.4.8. *If $\alpha \in \mathcal{SM}_G$ then $(f \mapsto \alpha * f), (f \mapsto f * \alpha) \in \mathcal{L}(C(G, \mathbb{K}))$ and*

$$\|\alpha * f\|_{C(G,\mathbb{K})} \leq \|f\|_{C(G,\mathbb{K})}, \quad \|f * \alpha\|_{C(G,\mathbb{K})} \leq \|f\|_{C(G,\mathbb{K})}.$$

Moreover, $\alpha * \mathbf{1} = \mathbf{1} = \mathbf{1} * \alpha$.

Proof. Trivially, $\alpha * \mathbf{1} = \mathbf{1}$. Because $(x \mapsto a^{-1}x) : G \to G$ is a homeomorphism and the summing is finite, $\alpha * f \in C(G, \mathbb{K})$. Linearity of $f \mapsto \alpha * f$ is clear. Next,

$$|\alpha * f(x)| \leq \sum_{a \in G} \alpha(a) |f(a^{-1}x)| \leq \sum_{a \in G} \alpha(a) \|f\|_{C(G,\mathbb{K})} = \|f\|_{C(G,\mathbb{K})}.$$

Similar conclusions hold for $f * \alpha$. □

Lemma 2.4.9. *If $f \in C(G, \mathbb{R})$ and $\alpha \in \mathcal{SM}_G$ then*

$$\min(f) \leq \min(\alpha * f) \leq \max(\alpha * f) \leq \max(f),$$

$$\min(f) \leq \min(f * \alpha) \leq \max(f * \alpha) \leq \max(f),$$

so that

$$p(\alpha * f) \leq p(f), \quad p(f * \alpha) \leq p(f),$$

where $p(g) := \max(g) - \min(g)$.

Proof. Now

$$\min(f) = \sum_{a \in G} \alpha(a) \ \min(f) \leq \min_{x \in G} \sum_{a \in G} \alpha(a) f(a^{-1}x) = \min(\alpha * f),$$

$$\max(\alpha * f) = \max_{x \in G} \sum_{a \in G} \alpha(a) f(a^{-1}x) \leq \sum_{a \in G} \alpha(a) \ \max(f) = \max(f),$$

and clearly $\min(\alpha * f) \leq \max(\alpha * f)$. The proof for $f * \alpha$ is symmetric. □

Exercise 2.4.10. Show that $p := (f \mapsto \max(f) - \min(f)) : C(G, \mathbb{R}) \to \mathbb{R}$ is a bounded seminorm on $C(G, \mathbb{R})$.

Proposition 2.4.11. *Let $f \in C(G,\mathbb{R})$. For every $\varepsilon > 0$ there exist $\alpha, \beta \in \mathcal{SM}_G$ such that*

$$p(\alpha * f) < \varepsilon, \quad p(f * \beta) < \varepsilon.$$

Remark 2.4.12. This is the decisive stage in the construction of the Haar measure. The idea is that for a non-constant $f \in C(G)$ we can find sampling measures α, β that "tame" the oscillations of f so that $\alpha * f$ and $f * \beta$ are almost constant functions. It will turn out that there exists a unique constant function approximated by convolutions, "the average" $\mathrm{Haar}(f)\mathbf{1}$ of f. In the sequel, notice how compactness is exploited!

Proof. Let $\varepsilon > 0$. By Theorem 2.3.7, a continuous function on a **compact group** is uniformly continuous. Thus there exists an open set $U \supset e$ such that $|f(x) - f(y)| < \varepsilon$, when $x^{-1}y \in U$. We notice easily that if $\gamma \in \mathcal{SM}_G$ then also $|\gamma * f(x) - \gamma * f(y)| < \varepsilon$, when $x^{-1}y \in U$:

$$\begin{aligned} |\gamma * f(x) - \gamma * f(y)| &= \left| \sum_{a \in G} \gamma(a) \left(f(a^{-1}x) - f(a^{-1}y) \right) \right| \\ &\leq \sum_{a \in G} \gamma(a) \left| f(a^{-1}x) - f(a^{-1}y) \right| \\ &< \sum_{a \in G} \gamma(a)\, \varepsilon \quad = \quad \varepsilon. \end{aligned}$$

Now $\{xU : x \in G\}$ is an open cover of the **compact** space G, hence having a finite subcover $\{x_jU\}_{j=1}^n$. The set $S := \{x_i x_j^{-1} : 1 \leq i,j \leq n\}$ has $|S| \leq n^2$ elements. Define $\gamma_1 \in \mathcal{SM}_G$ by

$$\gamma_1(a) = \begin{cases} |S|^{-1}, & \text{when } a \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\gamma_{k+1} := \gamma_k * \gamma_1 \in \mathcal{SM}_G$. Then

$$\begin{aligned} p(\gamma_{k+1} * f) &= \max(\gamma_{k+1} * f) - \min(\gamma_{k+1} * f) \\ &\leq \max(\gamma_{k+1} * f) - \min(\gamma_k * f) \\ &= \frac{1}{|S|} \max_{x \in G} \sum_{a \in S} \gamma_k * f(a^{-1}x) - \min(\gamma_k * f) \\ &\overset{(\star)}{<} \frac{1}{|S|} \left[(|S| - 1) \max(\gamma_k * f) + [\min(\gamma_k * f) + \varepsilon] \right] - \min(\gamma_k * f) \\ &= \frac{|S| - 1}{|S|}\, p(\gamma_k * f) + \frac{1}{|S|} \varepsilon, \end{aligned}$$

where the last inequality $(\star)$ was obtained by estimating $|S|-1$ terms in the sum trivially, and finally the remaining term was estimated by recalling the uniform continuity of $\gamma_k * f$. But $(p(\gamma_k * f))_{k=1}^{\infty} \subset \mathbb{R}$ is a non-increasing sequence bounded from below by 0. Thus there exists the limit $\delta := \lim_{k\to\infty} p(\gamma_k * f) \geq 0$, and

$$\delta \leq \frac{|S|-1}{|S|}\delta + \frac{1}{|S|}\varepsilon, \quad \text{i.e.} \quad \delta \leq \varepsilon.$$

Hence there exists k_0 such that, say, $p(\gamma_k * f) \leq 2\varepsilon$ for every $k \geq k_0$. This proves the claim. □

Exercise 2.4.13. In the proof above, check the validity of inequality $(\star)$ with details.

Corollary 2.4.14. *For $f \in C(G,\mathbb{R})$ there exists a unique constant function* $\mathrm{Haar}(f)\mathbf{1}$ *belonging to the closure of*

$$\{\alpha * f : \ \alpha \in \mathcal{SM}_G\} \subset C(G,\mathbb{R}).$$

Moreover, $\mathrm{Haar}(f)\mathbf{1}$ *is the unique constant function belonging to the closure of*

$$\{f * \beta : \ \beta \in \mathcal{SM}_G\} \subset C(G,\mathbb{R}).$$

Proof. Pick any $\alpha_1 \in \mathcal{SM}_G$. Suppose we have chosen $\alpha_k \in \mathcal{SM}_G$. Let $\alpha_{k+1} := \gamma_k * \alpha_k$, where $\gamma_k \in \mathcal{SM}_G$ satisfies

$$p(\alpha_{k+1} * f) = p(\gamma_k * (\alpha_k * f)) < 2^{-k}.$$

Now

$$\min(\alpha_k * f) \leq \min(\alpha_{k+1} * f) \leq \max(\alpha_{k+1} * f) \leq \max(\alpha_k * f),$$

so that there exists

$$\lim_{k\to\infty} \min(\alpha_k * f) = \lim_{k\to\infty} \max(\alpha_k * f) =: c_1 \in \mathbb{R}.$$

In the same way we may construct $(\beta_k)_{k=1}^{\infty} \subset \mathcal{SM}_G$ such that

$$\lim_{k\to\infty} \min(f * \beta_k) = \lim_{k\to\infty} \max(f * \beta_k) =: c_2 \in \mathbb{R}.$$

But

$$\begin{aligned}|c_1 - c_2| &= \|c_1\mathbf{1} - c_2\mathbf{1}\|_{C(G,\mathbb{R})}\\ &= \|(c_1\mathbf{1} - \alpha_k * f) * \beta_k + \alpha_k * (f * \beta_k - c_2\mathbf{1})\|_{C(G,\mathbb{R})}\\ &\leq \|(c_1\mathbf{1} - \alpha_k * f) * \beta_k\|_{C(G,\mathbb{R})} + \|\alpha_k * (f * \beta_k - c_2\mathbf{1})\|_{C(G,\mathbb{R})}\\ &\leq \|c_1\mathbf{1} - \alpha_k * f\|_{C(G,\mathbb{R})} + \|f * \beta_k - c_2\mathbf{1}\|_{C(G,\mathbb{R})}\\ &\xrightarrow[k\to\infty]{} 0.\end{aligned}$$

Thus $c_1 = c_2 \in \mathbb{R}$ is unique, depending only on $f \in C(G,\mathbb{R})$. □

Definition 2.4.15. The *Haar integral* of $f \in C(G)$ is

$$\mathrm{Haar}(f) := \mathrm{Haar}(\Re(f)) + \mathrm{i}\ \mathrm{Haar}(\Im(f)),$$

where $\Re(f), \Im(f)$ are the real and imaginary parts of f, respectively.

Theorem 2.4.16. *The Haar integral* $\mathrm{Haar} : C(G) \to \mathbb{C}$ *is the unique positive bounded linear functional satisfying*

$$\begin{aligned}\mathrm{Haar}(\mathbf{1}) &= 1,\\ \mathrm{Haar}(f) &= \mathrm{Haar}(x \mapsto f(yx)),\end{aligned}$$

for every $f \in C(G)$ *and* $y \in G$*. Moreover,*

$$\mathrm{Haar}(f) = \mathrm{Haar}(x \mapsto f(xy)) = \mathrm{Haar}(x \mapsto f(x^{-1})).$$

Proof. By the definition of Haar, it is enough to deal with real-valued functions here. From the construction, it is clear that

$$f \geq 0 \ \Rightarrow\ \mathrm{Haar}(f) \geq 0,$$

$$|\mathrm{Haar}(f)| \leq \|f\|_{C(G)},$$

$$\mathrm{Haar}(\lambda f) = \lambda\ \mathrm{Haar}(f),$$

$$\mathrm{Haar}(\mathbf{1}) = 1,$$

$$\mathrm{Haar}(f) = \mathrm{Haar}(x \mapsto f(yx)) = \mathrm{Haar}(x \mapsto f(xy)).$$

Choose $\alpha, \beta \in \mathcal{SM}_G$ such that

$$\|\alpha * f - \mathrm{Haar}(f)\mathbf{1}\|_{C(G)} < \varepsilon, \quad \|g * \beta - \mathrm{Haar}(g)\mathbf{1}\|_{C(G)} < \varepsilon.$$

Then

$$\begin{aligned}
& \|\alpha * (f+g) * \beta - (\mathrm{Haar}(f) + \mathrm{Haar}(g))\mathbf{1}\|_{C(G)} \\
= & \|(\alpha * f - \mathrm{Haar}(f)\mathbf{1}) * \beta + \alpha * (g * \beta - \mathrm{Haar}(g)\mathbf{1})\|_{C(G)} \\
\leq & \|(\alpha * f - \mathrm{Haar}(f)\mathbf{1}) * \beta\|_{C(G)} + \|\alpha * (g * \beta - \mathrm{Haar}(g)\mathbf{1})\|_{C(G)} \\
\leq & \|\alpha * f - \mathrm{Haar}(f)\mathbf{1}\|_{C(G)} + \|g * \beta - \mathrm{Haar}(g)\mathbf{1}\|_{C(G)} \\
< & 2\varepsilon,
\end{aligned}$$

so that $\mathrm{Haar}(f+g) = \mathrm{Haar}(f) + \mathrm{Haar}(g)$.

Suppose $L : C(G) \to \mathbb{C}$ is a positive bounded linear functional such that $L(\mathbf{1}) = 1$ and $L(f) = L(x \mapsto f(yx))$ for every $f \in C(G)$ and $y \in G$. Let $f \in C(G)$, $\varepsilon > 0$ and $\alpha \in \mathcal{SM}_G$ be as above. Then

$$\begin{aligned}
|L(f) - \mathrm{Haar}(f)| &= |L(\alpha * f - \mathrm{Haar}(f)\mathbf{1})| \\
&\leq \|L\|_{C(G)'}\ \|\alpha * f - \mathrm{Haar}(f)\mathbf{1}\|_{C(G)} \\
&< \|L\|_{C(G)'}\ \varepsilon
\end{aligned}$$

yields the uniqueness $L = \mathrm{Haar}$.

Finally, $(f \mapsto \mathrm{Haar}(x \mapsto f(x^{-1}))) : C(G) \to \mathbb{C}$ is a positive bounded linear translation-invariant normalized functional, hence equaling to Haar by the uniqueness. □

Exercise 2.4.17. In the previous proof, many properties were declared clear. Should any of the "clarities" appear uncertain, provide verification.

Definition 2.4.18. For $1 \leq p < \infty$, the *Lebesgue-p-space* $L^p(\mu_G)$ is the completion of $C(G)$ with respect to the norm

$$f \mapsto \|f\|_{L^p(\mu_G)} := \left(\int_G |f|^p \ \mathrm{d}\mu_G \right)^{1/p}.$$

The space $L^\infty(\mu_G)$ is the usual Banach space of μ_G-essentially bounded functions with the norm $f \mapsto \|f\|_{L^\infty(\mu_G)}$; on the closed subspace $C(G) \subset L^\infty(\mu_G)$ we have $\|f\|_{C(G)} = \|f\|_{L^\infty(\mu_G)}$. Notice that $L^p(\mu_G)$ is a Banach space, but it is a Hilbert space if and only if $p = 2$, having the inner product $(f,g) \mapsto \langle f, g\rangle_{L^2(\mu_G)}$ satisfying

$$\langle f, g\rangle_{L^2(\mu_G)} = \int_G f\overline{g}\ \mathrm{d}\mu_G$$

for $f, g \in C(G)$.

Remark 2.4.19. We have now seen that for a compact group G there exists a unique translation-invariant probability functional on $C(G)$, the Haar integral! We also know that it is enough to demand only either left- or right-invariance, since one follows from the other. Moreover, the Haar integral is also inversion-invariant. It must be noted that an inversion-invariant probability functional on $C(G)$ is not necessarily translation-invariant: e.g.

$$f \mapsto f(e) = \int_G f(x) \, \mathrm{d}\delta_e(x)$$

is inversion-invariant but clearly the point mass δ_e at $e \in G$ cannot be translation-invariant (unless we have the triviality $G = \{e\}$). Next we observe that the Haar integral distinguishes continuous functions $f, g \in C(G)$ in the sense that if $\int_G |f - g| \, \mathrm{d}\mu_G = 0$ then $f = g$:

Theorem 2.4.20. *Let G be a compact group and $f \in C(G)$. If $\int_G |f| \, \mathrm{d}\mu_G = 0$ then $f = 0$.*

Proof. The set $U := f^{-1}(\mathbb{C} \setminus \{0\}) \subset G$ is open, since f is continuous and $\{0\} \subset \mathbb{C}$ is closed. Suppose $f \neq 0$: Then $U \neq \emptyset$, and $\{xU : \ x \in G\}$ is an open cover for G. By the compactness, there exists a subcover $\{x_j U\}_{j=1}^n$. Define $g \in C(G)$ by

$$g(x) := \sum_{j=1}^n \left| f(x_j^{-1} x) \right| .$$

Now $g(x) > 0$ for every $x \in G$, so that $c := \min_{x \in G} g(x) > 0$ by the compactness. We use the normalization, positivity and translation-invariance of μ_G to obtain

$$0 < c = \int_G c\mathbf{1} \, \mathrm{d}\mu_G \leq \int_G g \, \mathrm{d}\mu_G = n \int_G |f| \, \mathrm{d}\mu_G,$$

so that $0 < \int_G |f| \, \mathrm{d}\mu_G$. □

Exercise 2.4.21. Let G, H be compact groups. Show that $\mu_{G \times H} = \mu_G \times \mu_H$ (i.e. the Haar measure of the product group is the product of the original Haar measures).

Exercise 2.4.22. Let $\mathcal{M}_G$ denote the σ-algebra of the Haar-measurable sets on the compact group G. Consider mappings $m, p_1, p_2 : G \times G \to G$, where

$$m(x, y) = xy, \quad p_1(x, y) = x, \quad p_2(x, y) = y.$$

Show that they are Haar measurable (that is, $(\mathcal{M}_{G\times G}, \mathcal{M}_G)$-measurable). Moreover, show that

$$\mu_G(E) = \mu_{G\times G}(m^{-1}(E)) = \mu_{G\times G}(p_1^{-1}(E)) = \mu_{G\times G}(p_2^{-1}(E)).$$

for every $E \in \mathcal{M}_G$.

2.4.1 Integration on quotient spaces

We have already noticed that the "good" subgroups of a topological group are the closed ones. Moreover, by now we know that a transitive action of a compact topological group G on a Hausdorff space X begets a homeomorphism $G/H \cong X$ of compact Hausdorff spaces, where H is a closed subgroup of G; effectively, spaces G/H and X are the same. What we are about to do is to show that for X there exists a unique G-action-invariant probability functional on $C(X)$, which might be called the Haar functional of the action; the corresponding measure on G/H will accordingly be denoted by $\mu_{G/H}$. We have seen that continuous functions on G/H (and hence on X) can be interpreted as continuous right-H-translation-invariant functions on G, i.e. $f(xh) = f(x)$ for every $x \in G$ and $h \in H$. Next we show how $f \in C(G)$ "casts a shadow" $f_{G/H} \in C(G/H)$ in a canonical way...

Lemma 2.4.23. *Let G be a compact group and $H < G$ closed. If $f \in C(G)$ then $P_{G/H}f \in C(G)$ and $f_{G/H} \in C(G/H)$, where*

$$f_{G/H}(xH) = P_{G/H}f(x) := \int_H f(xh)\ \mathrm{d}\mu_H(h).$$

Furthermore, the projection $P_{G/H} : C(G) \to C(G)$ is bounded, more precisely $\|f_{G/H}\|_{C(G/H)} = \left\|P_{G/H}f\right\|_{C(G)} \leq \|f\|_{C(G)}$.

Proof. First, H is a compact group having the Haar measure μ_H. The integration in the definition is legitimate since $f_x := (h \mapsto f(xh)) \in C(H)$ for each $x \in G$. If $x \in G$ and $h_0 \in H$ then

$$P_{G/H}f(xh_0) = \int_H f_x(h_0 h)\ \mathrm{d}\mu_H(h) = \int_H f_x(h)\ \mathrm{d}\mu_H(h) = P_{G/H}f(x),$$

so that $f_{G/H} : G/H \to \mathbb{C}$. Next we prove the continuities. Let $\varepsilon > 0$. A continuous on a compact group is uniformly continuous, so that for

$f \in C(G)$ there exists an open $U \ni e$ such that

$$\forall x, y \in G: \ xy^{-1} \in U \ \Rightarrow \ |f(x) - f(y)| < \varepsilon$$

(apparently, this slightly deviates from our definition of the uniform continuity; however, this is almost trivially equivalent). Suppose $xy^{-1} \in U$. Then

$$\begin{aligned} |P_{G/H}f(x) - P_{G/H}f(y)| &= \left| \int_H f(xh) - f(yh) \ \mathrm{d}\mu_H(h) \right| \\ &\le \int_H |f(xh) - f(yh)| \ \mathrm{d}\mu_H(h) \ < \ \varepsilon, \end{aligned}$$

so that $P_{G/H}f \in C(G)$ and $f_{G/H} \in C(G/H)$. Finally,

$$|P_{G/H}f(x)| \le \int_H |f(xh)| \ \mathrm{d}\mu_H(h) \le \int_H \|f\|_{C(G)} \ \mathrm{d}\mu_H(h) = \|f\|_{C(G)}.$$

□

Remark 2.4.24. Projection $P_{G/H} \in \mathcal{L}(C(G))$ uniquely extends to an orthogonal projection $P_{G/H} \in \mathcal{L}(L^2(\mu_G))$.

Theorem 2.4.25. *Let $((x,p) \mapsto x \cdot p) : G \times M \to M$ be a continuous transitive action of a compact group G on a Hausdorff space M. Then there exists a unique Borel-regular probability measure μ_M on M which is the action-invariant in the sense that*

$$\int_M f_M \ \mathrm{d}\mu_M = \int_M f_M(x \cdot p) \ \mathrm{d}\mu_M(p)$$

for every $f_M \in C(M)$ and $x \in G$.

Proof. Given $q \in M$, we know that $M \cong G/G_q$. Hence it is enough to deal with $M = G/H$, where $H < G$ is closed and the action is $((x, yH) \mapsto xyH) : G \times G/H \to G/H$.

We first prove the existence of a G-action-invariant Borel regular probability measure $\mu_{G/H}$ on the compact Hausdorff space G/H. Define $\mathrm{Haar}_{G/H} : C(G/H) \to \mathbb{C}$ by

$$\mathrm{Haar}_{G/H}(f_{G/H}) := \int_G f_{G/H}(xH) \ \mathrm{d}\mu_G(x).$$

Notice that

$$\begin{aligned}
\mathrm{Haar}_{G/H}(f_{G/H}) &= \int_G \int_H f(xh)\ \mathrm{d}\mu_H(h)\ \mathrm{d}\mu_G(x) \\
&\overset{\text{Fubini}}{=} \int_H \int_G f(xh)\ \mathrm{d}\mu_G(x)\ \mathrm{d}\mu_H(h) \\
&= \int_H \mathrm{Haar}_G(f)\ \mathrm{d}\mu_H(h) \\
&= \mathrm{Haar}_G(f).
\end{aligned}$$

It is clear that $\mathrm{Haar}_{G/H}$ is a bounded linear functional, and $\mathrm{Haar}_{G/H}(\mathbf{1}_{G/H}) = \mathrm{Haar}_G(\mathbf{1}_G) = 1$. By the Riesz Representation Theorem, there exists the unique Borel-regular probability measure $\mu_{G/H}$ on G/H such that

$$\mathrm{Haar}_{G/H}(f_{G/H}) = \int_{G/H} f_{G/H}\ \mathrm{d}\mu_{G/H}$$

for every $f_{G/H} \in C(G/H)$. Action-invariance follows by the left-invariance of Haar_G: if $g(y) = f(xy)$ for every $x, y \in G$ then

$$\begin{aligned}
\mathrm{Haar}_{G/H}(y \mapsto f_{G/H}(xyH)) &= \mathrm{Haar}_{G/H}(g_{G/H}) \\
&= \mathrm{Haar}_G(g) \\
&= \mathrm{Haar}_G(f) \\
&= \mathrm{Haar}_{G/H}(f_{G/H}).
\end{aligned}$$

Now we prove the uniqueness part. So suppose $L : C(G/H) \to \mathbb{C}$ is an action-invariant bounded linear functional for which $L(\mathbf{1}_{G/H}) = 1$. Recall the mapping $(f \mapsto f_{G/H}) : C(G) \to C(G/H)$ from Lemma 2.4.23. Then

$$\tilde{L}(f) := L(f_{G/H})$$

defines a bounded linear functional $\tilde{L} : C(G) \to \mathbb{C}$ such that $\tilde{L}(\mathbf{1}_G) = 1$ and

$$\tilde{L}(y \mapsto f(xy)) = L(y \mapsto f_{G/H}(xyH)) = L(f_{G/H}) = \tilde{L}(f).$$

Hence $\tilde{L} = \mathrm{Haar}_G$ by Theorem 2.4.16. Consequently,

$$L(f_{G/H}) = \tilde{L}(f) = \mathrm{Haar}_G(f) = \mathrm{Haar}_{G/H}(f_{G/H}),$$

yielding $L = \mathrm{Haar}_{G/H}$. □

Remark 2.4.26. Let G be a compact group and $H < G$ closed. From the proof of Theorem 2.4.25 we see that

$$\int_G f \, \mathrm{d}\mu_G = \int_{G/H} \int_H f(xh) \, \mathrm{d}\mu_H(h) \, \mathrm{d}\mu_{G/H}(xH)$$

for every $f \in C(G)$.

Exercise 2.4.27. Let $\omega_j(t) \in \mathrm{SO}(3)$ denote the rotation of $\mathbb{R}^3$ by angle $t \in \mathbb{R}$ around the jth coordinate axis, $j \in \{1, 2, 3\}$. Show that $x \in \mathrm{SO}(3)$ can be represented in the form

$$x = x(\phi, \theta, \psi) = \omega_3(\phi) \, \omega_2(\theta) \, \omega_3(\psi)$$

where $0 \leq \phi, \psi < 2\pi$ and $0 \leq \theta \leq \pi$.

Exercise 2.4.28. Let $G = \mathrm{SO}(3)$ and $M = \mathbb{S}^2$. Let $((x, p) \mapsto xp) : G \times M \to M$ be the usual action. Let $q = (0, 0, 1) \in M$, i.e. q is the north pole. Show that $G_q = \{\omega_3(\psi) : \ 0 \leq \psi < 2\pi\}$. We know that the Lebesgue measure is rotation-invariant. Using the normalized angular part of the Lebesgue measure of $\mathbb{R}^3$, deduce that here

$$\int_G f \, \mathrm{d}\mu_G = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{2\pi} f(x(\phi, \theta, \psi)) \, \sin(\theta) \, \mathrm{d}\psi \, \mathrm{d}\theta \, \mathrm{d}\phi,$$

i.e. $\mathrm{d}\mu_{\mathrm{SO}(3)} = \frac{1}{8\pi^2} \, \sin(\theta) \, \mathrm{d}\psi \, \mathrm{d}\theta \, \mathrm{d}\phi$.

2.5 Fourier transforms on compact groups

In this section, we exploit the Haar integral in studying unitary representations of compact groups. The main result is the Peter–Weyl Theorem 2.5.13, begetting canonical Fourier series representations for functions on a compact group.

Exercise 2.5.1. Let $\phi \in \mathrm{Hom}(G, \mathrm{Aut}(\mathcal{H}))$ be a representation of a compact group G on a finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$. Construct a G-invariant inner product $((u, v) \mapsto \langle u, v \rangle_{\mathcal{H}}) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, that is

$$\langle \phi(x)u, \phi(x)v \rangle_{\mathcal{H}} = \langle u, v \rangle_{\mathcal{H}}$$

for every $x \in G$ and $u, v \in \mathcal{H}$. Notice that now ϕ is unitary with respect to this inner product!

Lemma 2.5.2. *Let G be a compact group and $\mathcal{H}$ be a Hilbert space with the inner product $(u,v) \mapsto \langle u,v\rangle_{\mathcal{H}}$. Let $\phi \in \mathrm{Hom}(G,\mathcal{U}(\mathcal{H}))$ be cyclic and $w \in \mathcal{H}$ a ϕ-cyclic vector with $\|w\|_{\mathcal{H}} = 1$. Then*

$$\langle u,v\rangle_\phi := \int_G \langle \phi(x)u,w\rangle_{\mathcal{H}} \ \langle w,\phi(x)v\rangle_{\mathcal{H}} \ \mathrm{d}\mu_G(x)$$

defines an inner product $(u,v) \mapsto \langle u,v\rangle_\phi$ for $\mathcal{H}$. Moreover, ϕ is unitary also with respect to this new inner product, and $\|u\|_\phi \leq \|u\|_{\mathcal{H}}$ for every $u \in \mathcal{H}$, where $\|u\|_\phi^2 := \langle u,u\rangle_\phi$.

Proof. Defining $f_u(x) := \langle \phi(x)u,w\rangle_{\mathcal{H}}$, we notice that $f \in C(G)$, because

$$\begin{aligned} |f_u(x) - f_u(y)| &= |\langle(\phi(x)-\phi(y))u,w\rangle_{\mathcal{H}}| \\ &\leq \|(\phi(x)-\phi(y))u\|_{\mathcal{H}} \ \|w\|_{\mathcal{H}} \\ &\xrightarrow[x\to y]{} 0 \end{aligned}$$

due to the strong continuity of ϕ. Thereby $f_u\overline{f_v}$ is Haar integrable, justifying the definition of $\langle u,v\rangle_\phi$.

Let $\lambda \in \mathbb{C}$ and $t,u,v \in \mathcal{H}$. Then it is easy to verify that

$$\begin{aligned} \langle \lambda u,v\rangle_\phi &= \lambda\langle u,v\rangle_\phi, \\ \langle t+u,v\rangle_\phi &= \langle t,v\rangle_\phi + \langle u,v\rangle_\phi, \\ \langle u,v\rangle_\phi &= \overline{\langle v,u\rangle_\phi}, \\ \|u\|_\phi^2 &= \int_G |f_u|^2 \ \mathrm{d}\mu_G \ \geq \ 0. \end{aligned}$$

What if $0 = \|u\|_\phi^2 = \int_G |f_u|^2 \mathrm{d}\mu_G$? Then $f_u \equiv 0$ by Theorem 2.4.20, i.e.

$$0 = \langle \phi(x)u,w\rangle_{\mathcal{H}} = \langle u,\phi(x^{-1})w\rangle_{\mathcal{H}}$$

for every $x \in G$. Since w is a cyclic vector, $u = 0$ follows. Thus $(u,v) \mapsto \langle u,v\rangle_\phi$ is an inner product on $\mathcal{H}$.

The original norm dominates the ϕ-norm, since

$$\begin{aligned} \|u\|_\phi^2 &= \int_G |\langle\phi(x)u,w\rangle_{\mathcal{H}}|^2 \ \mathrm{d}\mu_G(x) \\ &\leq \int_G \|\phi(x)u\|_{\mathcal{H}}^2 \ \|w\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_G(x) \\ &= \int_G \|u\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_G(x) \quad = \quad \|u\|_{\mathcal{H}}^2. \end{aligned}$$

The ϕ-unitarity of ϕ follows by

$$\begin{aligned}
\langle u, \phi(y)^* v\rangle_\phi &= \langle \phi(y)u, v\rangle_\phi \\
&= \int_G \langle \phi(xy)u, w\rangle_{\mathcal{H}} \ \langle w, \phi(x)v\rangle_{\mathcal{H}} \ \mathrm{d}\mu_G(x) \\
&\overset{z=xy}{=} \int_G \langle \phi(z)u, w\rangle_{\mathcal{H}} \ \langle w, \phi(zy^{-1})v\rangle_{\mathcal{H}} \ \mathrm{d}\mu_G(z) \\
&= \langle u, \phi(y)^{-1} v\rangle_\phi,
\end{aligned}$$

where we applied the translation invariance of the Haar integral. □

Exercise 2.5.3. Check the missing details in the proof of Lemma 2.5.2.

Lemma 2.5.4. *Let $\langle u, v\rangle_\phi$ be as above. Then*

$$\langle u, Av\rangle_{\mathcal{H}} := \langle u, v\rangle_\phi \tag{2.1}$$

defines a compact operator $A = A^* \in \mathcal{L}(\mathcal{H})$. Moreover, A is positive definite and $A \in \mathrm{Hom}(\phi, \phi)$.

Proof. By Lemma 2.5.2, if $v \in \mathcal{H}$ then $F_v(u) := \langle u, v\rangle_\phi$ defines a linear functional $F_v : \mathcal{H} \to \mathbb{C}$, which is bounded in both norms, since

$$|F_v(u)| = |\langle u, v\rangle_\phi| \le \|u\|_\phi \ \|v\|_\phi \le \|u\|_{\mathcal{H}} \ \|v\|_\phi.$$

By the Riesz Representation Theorem 5.7.3, F_v is represented by a unique vector $A(v) \in \mathcal{H}$, i.e. $F_v(u) = \langle u, A(v)\rangle_{\mathcal{H}}$ for every $u \in \mathcal{H}$. Thus we have an operator $A : \mathcal{H} \to \mathcal{H}$, which is clearly linear. We obtain a bound $\|A\|_{\mathcal{L}(\mathcal{H})} \le 1$ from

$$\|Av\|_{\mathcal{H}}^2 = \langle Av, Av\rangle_{\mathcal{H}} = \langle Av, v\rangle_\phi \le \|Av\|_\phi \ \|v\|_\phi \le \|Av\|_{\mathcal{H}} \ \|v\|_{\mathcal{H}}.$$

Self-adjointness follows from

$$\langle u, A^* v\rangle_{\mathcal{H}} = \langle Au, v\rangle_{\mathcal{H}} = \overline{\langle v, Au\rangle_{\mathcal{H}}} = \overline{\langle v, u\rangle_\phi} = \langle u, v\rangle_\phi = \langle u, Av\rangle_{\mathcal{H}}.$$

Moreover, A is positive definite, because $\langle u, Au\rangle_{\mathcal{H}} = \langle u, u\rangle_\phi = \|u\|_\phi^2 \ge 0$, where $\|u\|_\phi = 0$ if and only if $u = 0$.

The intertwining property $A \in \mathrm{Hom}(\phi, \phi)$ is seen from

$$\begin{aligned}
\langle u, A\phi(y)v\rangle_{\mathcal{H}} &= \langle u, \phi(y)v\rangle_\phi \\
&= \langle \phi(y)^{-1}u, v\rangle_\phi \\
&= \langle \phi(y)^{-1}u, Av\rangle_{\mathcal{H}} \\
&= \langle u, \phi(y)Av\rangle_{\mathcal{H}}.
\end{aligned}$$

Let $\mathbb{B} = \{u \in \mathcal{H} : \ \|u\|_{\mathcal{H}} \leq 1\}$, the closed unit ball of $\mathcal{H}$. To show that $A \in \mathcal{L}(\mathcal{H})$ is compact, we must show that $\overline{A(\mathbb{B})} \subset \mathcal{H}$ is a compact set. So take a sequence $(v_j)_{j=1}^{\infty} \subset A(\mathbb{B})$; we have to find a converging subsequence. Take a sequence $(u_j)_{j=1}^{\infty} \subset \mathbb{B}$ such that $Au_j = v_j$. By the Banach–Alaoglu Theorem 5.7.5, the closed ball $\mathbb{B}$ is weakly compact: there exists a subsequence $(u_{j_k})_{k=1}^{\infty}$ such that $u_{j_k} \xrightarrow[k\to\infty]{} u \in \mathbb{B}$ weakly, i.e.

$$\langle u_{j_k}, v\rangle_{\mathcal{H}} \xrightarrow[k\to\infty]{} \langle u, v\rangle_{\mathcal{H}}$$

for every $v \in \mathcal{H}$. Then

$$\begin{aligned} \|v_{j_k} - Au\|_{\mathcal{H}}^2 &= \|A(u_{j_k} - u)\|_{\mathcal{H}}^2 \\ &= \langle A(u_{j_k} - u), u_{j_k} - u\rangle_{\phi} \\ &= \int_G g_k \ \mathrm{d}\mu_G \end{aligned}$$

where

$$g_k(x) := \langle \phi(x) A(u_{j_k} - u), w\rangle_{\mathcal{H}} \ \langle w, \phi(x)(u_{j_k} - u)\rangle_{\mathcal{H}}.$$

Let us show that $\int_G g_k \ \mathrm{d}\mu_G \to_{k\to\infty} 0$. First, $g_k \in C(G)$ (hence g_k is integrable) and for each $x \in G$

$$\begin{aligned} |g_k(x)| &= \left|\langle u_{j_k} - u, A^*\phi(x^{-1})w\rangle_{\mathcal{H}}\right| \ \left|\langle \phi(x^{-1})w, u_{j_k} - u\rangle_{\mathcal{H}}\right| \\ &\xrightarrow[k\to\infty]{} 0 \end{aligned}$$

by the weak convergence. Second,

$$\begin{aligned} |g_k(x)| &\leq \|\phi(x)\|_{\mathcal{L}(\mathcal{H})}^2 \ \|A^*\|_{\mathcal{L}(\mathcal{H})} \ \|w\|_{\mathcal{H}}^2 \ \|u_{j_k} - u\|_{\mathcal{H}}^2 \\ &\leq 4, \end{aligned}$$

because $\|\phi(x)\|_{\mathcal{L}(\mathcal{H})} = 1$, $\|A\|_{\mathcal{L}(\mathcal{H})} = \|A\|_{\mathcal{L}(\mathcal{H})} \leq 1$, $\|w\|_{\mathcal{H}} = 1$ and $u_{j_k}, u \in \mathbb{B}$. Thus $\int_G g_k \ \mathrm{d}\mu_G \to_{k\to\infty} 0$ by the **Lebesgue Dominated Convergence Theorem**. Equivalently, $v_{j_k} \to_{k\to\infty} Au \in A(\mathbb{B})$. We have shown that $\overline{A(\mathbb{B})} = A(\mathbb{B}) \subset \mathcal{H}$ is compact. □

Theorem 2.5.5. *Let G be a compact group and $\mathcal{H}$ a Hilbert space. Let $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ be strongly continuous. Then ϕ is a direct sum of finite-dimensional irreducible unitary representations.*

Proof. We know that ϕ is a direct sum of cyclic representations. Therefore it is enough to assume that ϕ itself is cyclic. With the notation of Lemma 2.5.4, $A \in \mathcal{L}(\mathcal{H})$ is a compact self-adjoint operator, hence by the Hilbert–Schmidt Spectral Theorem 5.7.6, we have

$$\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} \mathrm{Ker}(\lambda I - A),$$

where $\dim(\mathrm{Ker}(\lambda I - A)) < \infty$ for each $\lambda \in \sigma(A)$. Since $A \in \mathrm{Hom}(\phi, \phi)$, the subspace $\mathrm{Ker}(\lambda I - A) \subset \mathcal{H}$ is ϕ-invariant. Thereby

$$\phi = \bigoplus_{\lambda \in \sigma(A)} \phi|_{\mathrm{Ker}(\lambda I - A)},$$

where $\phi|_{\mathrm{Ker}(\lambda I - A)}$ is finite-dimensional and unitary for every $\lambda \in \sigma(A)$. The proof is concluded, since we know that a finite-dimensional unitary representation is a direct sum of irreducible unitary representations. □

Corollary 2.5.6. *A strongly continuous irreducible unitary representation of a compact group is always finite-dimensional.* □

Definition 2.5.7. The *(unitary) dual* $\widehat{G}$ of a locally compact group G is the set consisting of the equivalence classes of the strongly continuous irreducible unitary representations of G.

Remark 2.5.8. For compact G, $\widehat{G}$ consists of the equivalence classes of continuous irreducible unitary representations (due to the finite-dimensionality), i.e.

$$\widehat{G} = \{[\phi] \mid \phi \text{ continuous irreducible unitary representation of } G\},$$

where $[\phi] = \{\psi \mid \psi \sim \phi\}$ is the equivalence class of ϕ.

Example. It can be proven that

$$\widehat{\mathbb{R}^n} = \left\{[\mathrm{e}_\xi] \mid \xi \in \mathbb{R}^n,\ \mathrm{e}_\xi : \mathbb{R}^n \to \mathrm{U}(1),\ \mathrm{e}_\xi(x) := \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\right\},$$

so that $\widehat{\mathbb{R}^n} \cong \mathbb{R}^n$. Similarly,

$$\widehat{\mathbb{T}^n} = \left\{[\mathrm{e}_\xi] \mid \xi \in \mathbb{Z}^n,\ \mathrm{e}_\xi : \mathbb{R}^n \to \mathrm{U}(1),\ \mathrm{e}_\xi(x) := \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\right\},$$

so that $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$.

Remark 2.5.9. For a commutative locally compact group G the unitary dual $\widehat{G}$ has a natural structure of a commutative locally compact group, and $\widehat{\widehat{G}} \cong G$; this is so called *Pontryagin duality*. For a compact non-commutative group G, the unitary dual $\widehat{G}$ is never a group, but still has a sort of "weak algebraic structure"; we do not consider this in these lecture notes.

Definition 2.5.10. Let G be a compact group. For the equivalence class $\xi \in \widehat{G}$ we may find a representative $\phi \in \xi = [\phi]$ such that $\phi = (\phi_{ij})_{i,j=1}^m \in \mathrm{Hom}(G, \mathrm{U}(m))$, with $m = \dim(\phi)$: namely, if $\psi \in \xi$, $\psi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ and $\{e_j\}_{j=1}^m \subset \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$ then we can define

$$\phi_{ij}(x) := \langle e_i, \phi(x) e_j \rangle_{\mathcal{H}}.$$

Notice that the matrix element $\phi_{ij} \in C(G)$, because ϕ is continuous. Next we present an L^2-orthogonality result for these continuous functions $\phi_{ij} : G \to \mathbb{C}$.

Lemma 2.5.11. *Let G be a compact group. Let $\xi, \eta \in \widehat{G}$, where $\xi \ni \phi = (\phi_{ij})_{i,j=1}^m \in \mathrm{Hom}(G, \mathrm{U}(m))$ and $\eta \ni \psi = (\psi_{kl})_{k,l=1}^n \in \mathrm{Hom}(G, \mathrm{U}(n))$. Then*

$$\langle \phi_{ij}, \psi_{kl} \rangle_{L^2(\mu_G)} = \begin{cases} 0, & \text{if } \xi \not\sim \eta, \\ \frac{1}{m}\ \delta_{ik}\delta_{jl}, & \text{if } \phi = \psi. \end{cases}$$

Proof. Let $1 \leq j \leq m$ and $1 \leq l \leq n$. Define matrix $E \in \mathbb{C}^{m \times n}$ by $E_{pq} := \delta_{pj}\delta_{lq}$ (i.e. the matrix elements of E are zero except for the (j,l)-element, which is 1.) Define matrix $A \in \mathbb{C}^{m \times n}$ by

$$A := \int_G \phi(y)\ E\ \psi(y^{-1})\ \mathrm{d}\mu_G(y).$$

Now $A \in \mathrm{Hom}(\psi, \phi)$, since

$$\begin{aligned} \phi(x)A &= \int_G \phi(xy)\ E\ \psi(y^{-1})\ \mathrm{d}\mu_G(y) \\ &= \int_G \phi(z)\ E\ \psi(z^{-1}x)\ \mathrm{d}\mu_G(z) \\ &= A\psi(x). \end{aligned}$$

Since ϕ, ψ are finite-dimensional irreducible unitary representations, Schur's Lemma 1.3.17 implies

$$A = \begin{cases} 0, & \text{if } \phi \not\sim \psi, \\ \lambda I, & \text{if } \phi = \psi \end{cases}$$

for some $\lambda \in \mathbb{C}$. We notice that

$$\begin{aligned} A_{ik} &= \int_G \sum_{p=1}^{m} \sum_{q=1}^{n} \phi_{ip}(y)\ E_{pq}\ \psi_{qk}(y^{-1})\ \mathrm{d}\mu_G(y) \\ &= \int_G \phi_{ij}(y)\ \overline{\psi_{kl}(y)}\ \mathrm{d}\mu_G(y) \\ &= \langle \phi_{ij}, \psi_{kl} \rangle_{L^2(\mu_G)}. \end{aligned}$$

Now suppose $\phi = \psi$. Then $m = n$ and

$$\begin{aligned} \langle \phi_{kj}, \psi_{kl} \rangle_{L^2(\mu_G)} &= A_{kk} \ = \ \lambda \ = \ \frac{1}{m}\ \mathrm{Tr}(A) \\ &= \frac{1}{m} \int_G \mathrm{Tr}(\phi(y)\ E\ \phi(y^{-1}))\ \mathrm{d}\mu_G(y) \\ &= \frac{1}{m} \int_G \mathrm{Tr}(E)\ \mathrm{d}\mu_G(y) \\ &= \frac{1}{m}\delta_{jl}, \end{aligned}$$

where we used the property $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$ of the trace functional. The results can be collected from above. □

Definition 2.5.12. Let G be a compact group. Its *left* and *right regular representations* $\pi_L, \pi_R : G \to \mathcal{U}(L^2(\mu_G))$ are defined, respectively, by

$$(\pi_L(y)\ f)(x) := f(y^{-1}x),$$

$$(\pi_R(y)\ f)(x) := f(xy)$$

for μ_G-almost every $x \in G$.

The idea here is that G is represented as a natural group of operators on a Hilbert space, enabling the use of functional analytic techniques in studying G. And now for the main result of this section, the Peter–Weyl Theorem (1927):

Theorem 2.5.13. *Let G be a compact group. Then*

$$\mathcal{B} := \left\{ \sqrt{\dim(\phi)}\phi_{ij} \mid \phi = (\phi_{ij})_{i,j=1}^{\dim(\phi)},\ [\phi] \in \widehat{G} \right\}$$

is an orthonormal basis for $L^2(\mu_G)$. Let $\phi = (\phi_{ij})_{i,j=1}^{m}$, $\phi \in [\phi] \in \widehat{G}$. Then

$$\mathcal{H}_{i,\cdot}^{\phi} := \mathrm{span}\{\phi_{ij} \mid 1 \leq j \leq m\} \subset L^2(\mu_G)$$

is π_R-invariant and

$$\phi \sim \pi_R|_{\mathcal{H}^{\phi}_{i,\cdot}},$$

$$L^2(\mu_G) = \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{m} \mathcal{H}^{\phi}_{i,\cdot},$$

$$\pi_R \sim \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{i=1}^{m} \phi.$$

Remark 2.5.14. Here $\bigoplus_{i=1}^{m} \phi := \phi \oplus \cdots \oplus \phi$, the m-fold direct sum of ϕ; in literature, this is sometimes denoted even by $m\phi$.

Left Peter–Weyl. We can formulate the Peter–Weyl Theorem 2.5.13 analogously for the left regular representation, as follows: Let $\phi = (\phi_{ij})_{i,j=1}^{m}$, $\phi \in [\phi] \in \widehat{G}$. Then

$$\mathcal{H}^{\phi}_{\cdot,j} := \mathrm{span}\{\phi_{ij} \mid 1 \leq i \leq m\} \subset L^2(\mu_G)$$

is π_L-invariant and

$$\phi \sim \pi_L|_{\mathcal{H}^{\phi}_{\cdot,j}},$$

$$L^2(\mu_G) = \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{j=1}^{m} \mathcal{H}^{\phi}_{\cdot,j},$$

$$\pi_L \sim \bigoplus_{[\phi]\in\widehat{G}} \bigoplus_{j=1}^{m} \phi.$$

Example. Let $G = \mathbb{T}^n$. Recall that

$$\widehat{\mathbb{T}^n} = \left\{[\mathrm{e}_\xi] \mid \xi \in \mathbb{Z}^n,\ \mathrm{e}_\xi(x) = \mathrm{e}^{\mathrm{i}2\pi x\cdot\xi}\right\}.$$

Now $\mathcal{B} = \{\mathrm{e}_\xi \mid \xi \in \mathbb{Z}^n\}$ is an orthonormal basis for $L^2(\mu_{\mathbb{T}^n})$,

$$L^2(\mu_{\mathbb{T}^n}) = \bigoplus_{\xi\in\mathbb{Z}^n} \mathrm{span}\{\mathrm{e}_\xi\},$$

$$\pi_L \sim \bigoplus_{\xi\in\mathbb{Z}^n} \mathrm{e}_\xi \sim \pi_R.$$

Moreover, for $f \in L^2(\mu_{\mathbb{T}^n})$, we have

$$f = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) \ \mathrm{e}_\xi,$$

where the Fourier coefficients $\widehat{f}(\xi)$ are calculated by

$$\widehat{f}(\xi) = \int_{\mathbb{T}^n} f \ \overline{\mathrm{e}_\xi} \ \mathrm{d}\mu_{\mathbb{T}^n} = \langle f, \mathrm{e}_\xi \rangle_{L^2(\mu_{\mathbb{T}^n})}.$$

We shall return to the *Fourier series* theme after the proof of the Peter–Weyl Theorem...

Proof for the Peter–Weyl Theorem 2.5.13. The π_R-invariance of $\mathcal{H}^\phi_{i,\cdot}$ follows due to

$$\pi_R(y)\phi_{ij}(x) = \phi_{ij}(xy) = \sum_{k=1}^{\dim(\phi)} \phi_{ik}(x)\phi_{kj}(y),$$

i.e.

$$\pi_R(y)\phi_{ij} = \sum_{k=1}^{\dim(\phi)} \lambda_k(y) \ \phi_{ik} \in \mathrm{span}\{\phi_{ik}\}_{k=1}^{\dim(\phi)} = \mathcal{H}^\phi_{i,\cdot}.$$

If $\{e_j\}_{j=1}^{\dim(\phi)} \subset \mathbb{C}^{\dim(\phi)}$ is the standard orthonormal basis then

$$\phi(y)e_j = \sum_{k=1}^{\dim(\phi)} \phi_{kj}(y)e_k,$$

so that

$$A \sum_{j=1}^{\dim(\phi)} \lambda_j e_j := \sum_{j=1}^{\dim(\phi)} \lambda_j \phi_{ij}$$

defines an intertwining isomorphism $A \in \mathrm{Hom}(\phi, \pi_R|_{\mathcal{H}^\phi_{i,\cdot}})$, i.e. $\phi \sim \pi_R|_{\mathcal{H}^\phi_{i,\cdot}}$.

By Lemma 2.5.11, $\mathcal{B} \subset L^2(\mu_G)$ is orthonormal. Let

$$\mathcal{H} := \bigoplus_{[\phi] \in \widehat{G}} \bigoplus_{i=1}^{\dim(\phi)} \mathcal{H}^\phi_{i,\cdot}.$$

We assume that $\mathcal{H} \neq L^2(\mu_G)$, and show that this leads to a contradiction (so that $\mathcal{H} = L^2(\mu_G)$ and $\mathcal{B}$ must be a basis): Clearly $\mathcal{H}$ is π_R-invariant. By our assumption, $\mathcal{H}^\perp$ is a non-trivial π_R-invariant closed subspace. Since $\pi_R|_{\mathcal{H}^\perp}$ is a direct sum of irreducible unitary representations, there exists a non-trivial subspace $E \subset \mathcal{H}^\perp$ and a unitary matrix representation $\phi = (\phi_{ij})_{i,j=1}^m \in \mathrm{HOM}(G, \mathrm{U}(m))$ such that $\phi \sim \pi_R|_E$. The subspace E has an orthonormal basis $\{f_j\}_{j=1}^m$ such that

$$\pi_R(y) f_j = \sum_{i=1}^m \phi_{ij}(y) f_i$$

for every $y \in G$ and $j \in \{1, \ldots, m\}$. Notice that $f_j \in L^2(\mu_G)$ has pointwise values perhaps only μ_G-almost everywhere, so that

$$f_j(xy) = \sum_{i=1}^m \phi_{ij}(y) f_i(x)$$

may hold for only μ_G-almost every $x \in G$. Let us define measurable sets

$$N(y) := \left\{ x \in G : \ f_j(xy) \neq \sum_{i=1}^m \phi_{ij}(y) f_i(x) \right\},$$

$$M(x) := \left\{ y \in G : \ f_j(xy) \neq \sum_{i=1}^m \phi_{ij}(y) f_i(x) \right\},$$

$$K := \left\{ (x,y) \in G \times G : \ f_j(xy) \neq \sum_{i=1}^m \phi_{ij}(y) f_i(x) \right\}.$$

By Exercise 2.4.22, we may exploit the Fubini Theorem to change the order of integration, to get

$$\begin{aligned}
\int_G \mu_G(M(x)) \, \mathrm{d}\mu_G(x) &= \mu_{G\times G}(K) \\
&= \int_G \mu_G(N(y)) \, \mathrm{d}\mu_G(y) \\
&= \int_G 0 \, \mathrm{d}\mu_G(y) \\
&= 0,
\end{aligned}$$

meaning that $\mu_G(M(x)) = 0$ for almost every $x \in G$. But it is enough for us to pick just one $x_0 \in G$ such that $\mu_G(M(x_0)) = 0$. Then

$$f_j(x_0 y) = \sum_{i=1}^{m} \phi_{ij}(y) f_i(x_0)$$

for μ_G-almost every $y \in G$. If we denote $z := x_0 y$ then

$$\begin{aligned} f_j(z) &= \sum_{i=1}^{m} \phi_{ij}(x_0^{-1} z) f_i(x_0) \\ &= \sum_{i=1}^{m} \sum_{k=1}^{m} \phi_{ik}(x_0^{-1}) \phi_{kj}(z) f_i(x_0) \\ &= \sum_{k=1}^{m} \phi_{kj}(z) \sum_{i=1}^{m} \phi_{ik}(x_0^{-1}) f_i(x_0) \end{aligned}$$

for μ_G-almost every $z \in G$. Hence

$$f_j \in \operatorname{span}\{\phi_{kj}\}_{k=1}^{m} \subset \bigoplus_{k=1}^{m} \mathcal{H}^{\phi}_{k,\cdot} \subset \mathcal{H}$$

for every $j \in \{1, \ldots, m\}$. Thereby

$$E = \operatorname{span}\{f_j\}_{j=1}^{m} \subset \mathcal{H};$$

at the same time $E \subset \mathcal{H}^{\perp}$, yielding $E = \{0\}$. This is a contradiction, since E should be non-trivial. Hence $\mathcal{H} = L^2(\mu_G)$ and $\mathcal{B}$ is a basis. □

Exercise 2.5.15. Check the details of the proof of the Peter–Weyl Theorem. In particular, pay attention to verify the conditions for applying the Fubini Theorem.

2.6 Trigonometric polynomials and Fourier series

Let G be a compact group and

$$\mathcal{B} := \left\{ \sqrt{\dim(\phi)}\,\phi_{ij} \mid \phi = (\phi_{ij})_{i,j=1}^{\dim(\phi)},\ [\phi] \in \widehat{G} \right\}$$

as in the Peter–Weyl Theorem 2.5.13. The space of *trigonometric polynomials* on G is

$$\operatorname{TrigPol}(G) = \operatorname{span}(\mathcal{B}).$$

For instance, $f \in \mathrm{TrigPol}(\mathbb{T}^n)$ is of the form

$$f(x) = \sum_{\xi \in \mathbb{Z}^n} \widehat{f}(\xi) \mathrm{e}^{\mathrm{i}2\pi x \cdot \xi},$$

where $\widehat{f}(\xi) \neq 0$ for only finitely many $\xi \in \mathbb{Z}^n$.

Theorem 2.6.1. $\mathrm{TrigPol}(G)$ *is a dense subalgebra of* $C(G)$.

Proof. It is enough to verify that $\mathrm{TrigPol}(G)$ is a involutive subalgebra of $C(G)$; the Stone–Weierstrass Theorem 5.9.3 provides then the density. We already know that $\mathrm{TrigPol}(G)$ is a subspace of $C(G)$.

First, $\phi = (x \mapsto (1)) \in \mathrm{Hom}(G, \mathrm{U}(1))$ is a continuous irreducible unitary representation, so that $\mathbf{1} = (x \mapsto 1) \in C(G)$ belongs to $\mathcal{B} \subset \mathrm{TrigPol}(G)$.

Let $[\phi] \in \widehat{G}$, $\phi = (\phi_{ij})_{i,j=1}^m$. Then $[\overline{\phi}] \in \widehat{G}$, where $\overline{\phi} = (\overline{\phi_{ij}})_{i,j=1}^m$, as it is easy to verify. Thereby we get the involutivity: $\overline{f} \in \mathrm{TrigPol}(G)$ whenever $f \in \mathrm{TrigPol}(G)$.

Let $[\psi] \in \widehat{G}$, $\psi = (\psi_{kl})_{k,l=1}^n$. Then $\phi \otimes \psi|_G = (x \mapsto \phi(x) \otimes \psi(x)) \in \mathrm{Hom}(G, \mathcal{U}(\mathbb{C}^m \otimes \mathbb{C}^n))$. Let $\{e_i\}_{i=1}^m \subset \mathbb{C}^m$ and $\{f_k\}_{k=1}^n \subset \mathbb{C}^n$ be orthonormal bases. Then $\{e_i \otimes f_k \mid 1 \leq i \leq m,\ 1 \leq k \leq n\}$ is an orthonormal basis for $\mathbb{C}^m \otimes \mathbb{C}^n$, and the $(ik)(jl)$-matrix element of $\phi \otimes \psi|_G$ is calculated by

$$\begin{aligned}
(\phi \otimes \psi|_G)_{(ik)(jl)}(x) &= \langle (\phi \otimes \psi|_G)(x)(e_j \otimes f_l), e_i \otimes f_k \rangle_{\mathbb{C}^m \otimes \mathbb{C}^n} \\
&= \langle \phi(x) e_j, e_i \rangle_{\mathbb{C}^m} \ \langle \psi(x) f_l, f_k \rangle_{\mathbb{C}^n} \\
&= \phi_{ij}(x) \psi_{kl}(x).
\end{aligned}$$

Hence $\phi_{ij}\psi_{kl}$ is a matrix element of $\phi \otimes \psi|_G$. Representation $\phi \otimes \psi|_G$ can be decomposed as a finite direct sum of irreducible unitary representations. Hence the matrix elements of $\phi \otimes \psi|_G$ can be written as linear combinations of elements of $\mathcal{B}$. Thus $\phi_{ij}\psi_{kl} \in \mathrm{TrigPol}(G)$, so that $fg \in \mathrm{TrigPol}(G)$ for every $f, g \in \mathrm{TrigPol}(G)$. □

Corollary 2.6.2. $\mathrm{TrigPol}(G)$ *is dense in* $L^2(\mu_G)$. □

Another proof for the Peter–Weyl Theorem. Notice that we did not need the Peter–Weyl Theorem 2.5.13 to show that TrigPol(G) $\subset L^2(\mu_G)$ is dense. Therefore this density provides another proof for that $\mathcal{B}$ in the Peter–Weyl Theorem is a basis!

Remark 2.6.3. By now, we have encountered plenty of translation- and inversion-invariant function spaces on G. For instance, TrigPol(G), $C(G)$ and $L^p(G)$, and more: namely, if $[\phi] \in \widehat{G}$, $\phi = (\phi_{ij})_{i,j=1}^m$, then

$$\pi_L(y)\phi_{i_0 j_0}, \pi_R(y)\phi_{i_0 j_0} \in \mathrm{span}\{\phi_{ij}\}_{i,j=1}^m$$

for every $y \in G$ (and inversion-invariance is clear!).

Exercise 2.6.4. Prove that $f \in C(G)$ is a trigonometric polynomial if and only if

$$\dim\left(\mathrm{span}\{\pi_R(y)f : \ y \in G\}\right) < \infty.$$

Corollary 2.6.5. (Corollary to the Peter–Weyl Theorem (Fourier series).) *$f \in L^2(\mu_G)$ can be represented as the* Fourier series

$$f = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} \langle f, \phi_{ij}\rangle_{L^2(\mu_G)} \ \phi_{ij};$$

in the series here, we pick just one unitary matrix representation $\phi = (\phi_{ij})_{i,j=1}^{\dim(\phi)}$ from each equivalence class $[\phi] \in \widehat{G}$. Moreover, there is the Plancherel equality *(sometimes called the* Parseval equality*)*

$$\|f\|_{L^2(\mu_G)} = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} |\langle f, \phi_{ij}\rangle_{L^2(\mu_G)}|^2.$$

□

Remark 2.6.6. In $L^2(\mu_G)$, also clearly

$$f = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} \langle f, \overline{\phi_{ij}}\rangle_{L^2(\mu_G)} \ \overline{\phi_{ij}}.$$

A nice thing about the Fourier series is that the basis functions $\overline{\phi_{ij}}$ are well-behaved under translations and inversions.

Definition 2.6.7. Let G be a compact group, $f \in L^1(\mu_G)$ and $\phi = (\phi_{ij})_{i,j=1}^m$, $[\phi] \in \widehat{G}$. The *ϕ-Fourier coefficient of f* is

$$\widehat{f}(\phi) := \int_G f(x)\ \phi(x)\ \mathrm{d}\mu_G(x) \in \mathbb{C}^{m\times m},$$

where the integration of the matrix-valued function is element-wise. The matrix-valued function $\widehat{f}$ is called the the *Fourier transform* of $f \in L^1(\mu_G)$.

Corollary 2.6.8. (Corollary again to the Peter–Weyl Theorem.) *$f \in L^2(\mu_G)$ can be presented by the series*

$$f(x) = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\left(\widehat{f}(\phi)\ \phi(x)^*\right)$$

converging for μ_G-almost every $x \in G$. The Plancherel equality takes the form

$$\|f\|^2_{L^2(\mu_G)} = \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\left(\widehat{f}(\phi)\ \widehat{f}(\phi)^*\right).$$

Proof. Now

$$\widehat{f}(\phi)_{ij} = \int_G f(x)\ \phi_{ij}(x)\ \mathrm{d}\mu_G(x) = \langle f, \overline{\phi_{ij}}\rangle_{L^2(\mu_G)},$$

so that

$$\begin{aligned}
\mathrm{Tr}\left(\widehat{f}(\phi)\ \phi(x)^*\right) &= \sum_{i=1}^{\dim(\phi)} \left(\widehat{f}(\phi)\ \phi(x)^*\right)_{ii} \\
&= \sum_{i,j=1}^{\dim(\phi)} \widehat{f}(\phi)_{ij}\ \overline{\phi_{ij}(x)} \\
&= \sum_{i,j=1}^{\dim(\phi)} \langle f, \overline{\phi_{ij}}\rangle_{L^2(\mu_G)}\ \overline{\phi_{ij}(x)}.
\end{aligned}$$

Finally, if $A = (A_{kl})_{k,l=1}^m \in \mathbb{C}^{m\times m}$ then

$$\|A\|^2_{\mathbb{C}^{m\times m}} := \mathrm{Tr}(A^*A) = \sum_{k,l=1}^m |A_{kl}|^2.$$

□

Definition 2.6.9. The natural inner product for $\mathbb{C}^{m\times m}$ is

$$(A,B) \mapsto \langle A, B\rangle_{\mathbb{C}^{m\times m}} := \mathrm{Tr}\left(A\ B^*\right) = \sum_{k,l=1}^m A_{kl}\ \overline{B_{kl}}.$$

Definition 2.6.10. Let G be a compact group. Let $L^2(\widehat{G})$ be the space containing mappings

$$F : \widehat{G} \to \bigcup_{m=1}^{\infty} \mathbb{C}^{m \times m}$$

satisfying $F([\phi]) \in \mathbb{C}^{\dim(\phi)\times\dim(\phi)}$ such that

$$\sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \|F([\phi])\|^2_{\mathbb{C}^{\dim(\phi)\times\dim(\phi)}} < \infty.$$

Then $L^2(\widehat{G})$ is a Hilbert space with the inner product

$$\langle E, F\rangle_{L^2(\widehat{G})} := \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \langle E([\phi]), F([\phi])\rangle_{\mathbb{C}^{\dim(\phi)\times\dim(\phi)}}.$$

Theorem 2.6.11. *Let G be a compact group. The Fourier transform $f \mapsto \widehat{f}$ defines a surjective isometry $L^2(\mu_G) \to L^2(\widehat{G})$.*

Proof. Let us choose one unitary matrix representation ϕ from each $[\phi] \in \widehat{G}$. If we define $F([\phi]) := \widehat{f}(\phi)$ then $F \in L^2(\widehat{G})$, and $f \mapsto F$ is isometric by the Plancherel equality.

Now take any $F \in L^2(\widehat{G})$. We have to show that $F([\phi]) = \widehat{f}(\phi)$ for some $f \in L^2(\mu_G)$, where $\phi \in [\phi] \in \widehat{G}$. Define

$$f(x) := \sum_{[\phi]\in\widehat{G}} \dim(\phi)\ \mathrm{Tr}\left(F([\phi])\ \phi(x)^*\right)$$

for μ_G-almost every $x \in G$. This can be done, since

$$f = \sum_{[\phi]\in\widehat{G}} \dim(\phi) \sum_{i,j=1}^{\dim(\phi)} F([\phi])_{ij}\ \overline{\phi_{ij}}$$

belongs to $L^2(\mu_G)$ by

$$\|f\|^2_{L^2(\mu_G)} = \|F\|^2_{L^2(\widehat{G})} < \infty.$$

Clearly $\widehat{f}(\phi) = F([\phi])$, the Fourier transform is surjective. □

2.7 Convolutions

Let G be a compact group and $f \in L^1(\mu_G)$, $g \in C(G)$ (or $f \in C(G)$ and $g \in L^1(\mu_G)$. The *convolution* $f * g : G \to \mathbb{C}$ is defined by

$$f * g(x) := \int_G f(y)\ g(y^{-1}x)\ \mathrm{d}\mu_G(y).$$

Now $f * g \in C(G)$: Due to the uniform continuity, for each $\varepsilon > 0$ there exists open $U \ni e$ such that $|g(x) - g(z)| < \varepsilon$ when $z^{-1}x \in U$. Thereby

$$\begin{aligned} |f * g(x) - f * g(z)| &\leq \int_G |f(y)|\ |g(y^{-1}x) - g(y^{-1}z)|\ \mathrm{d}\mu_G(y) \\ &\leq \|f\|_{L^1(\mu_G)}\ \varepsilon, \end{aligned}$$

when $z^{-1}x \in U$. Moreover, linear mapping $g \mapsto f * g$ satisfies

$$\|f * g\|_{C(G)} \leq \|f\|_{L^1(\mu_G)}\ \|g\|_{C(G)}, \quad \|f * g\|_{L^1(\mu_G)} \leq \|f\|_{L^1(\mu_G)}\ \|g\|_{L^1(\mu_G)}.$$

Hence we can consider $g \mapsto f * g$ as a bounded operator on $C(G)$ and $L^1(\mu_G)$; of course, we have symmetrical results for $g \mapsto g * f$.

It is also easy to show other L^p-boundedness results, like

$$\|f * g\|_{L^2(\mu_G)} \leq \|f\|_{L^2(\mu_G)}\ \|g\|_{L^2(\mu_G)}$$

and so on. Notice that the convolution product is commutative if and only if G is commutative.

Proposition 2.7.1. *Let $f, g, h \in L^1(\mu_G)$. Then $f * g \in L^1(\mu_G)$,*

$$\|f * g\|_{L^1(\mu_G)} \leq \|f\|_{L^1(\mu_G)}\ \|g\|_{L^1(\mu_G)},$$

and $f * g(x) = \int_G f(y^{-1})\ g(yx)\ \mathrm{d}\mu_G(y)$ for almost every $x \in G$. Moreover, for μ_G-almost every $x \in G$,

$$\begin{aligned} f * g(x) &= \int_G f(xy^{-1})\ g(y)\ \mathrm{d}\mu_G(y) \\ &= \int_G f(y^{-1})\ g(yx)\ \mathrm{d}\mu_G(y) \\ &= \int_G f(xy)\ g(y^{-1})\ \mathrm{d}\mu_G(y). \end{aligned}$$

The convolution product is also associative: $f * (g * h) = (f * g) * h$.

Exercise 2.7.2. Prove Proposition 2.7.1.

Proposition 2.7.3. *For $f, g \in L^1(\mu_G)$, $\widehat{f * g}(\phi) = \widehat{f}(\phi)\ \widehat{g}(\phi)$.*

Proof. It is enough to assume that $f, g \in C(G)$. Then

$$\begin{aligned}
\widehat{f * g}(\phi) &= \int_G f * g(x)\ \phi(x)\ \mathrm{d}\mu_G(x) \\
&= \int_G \int_G f(y)\ g(y^{-1}x)\ \mathrm{d}\mu_G(y)\ \phi(x)\ \mathrm{d}\mu_G(x) \\
&= \int_G f(y)\phi(y)\ \int_G g(y^{-1}x)\ \phi(y^{-1}x)\ \mathrm{d}\mu_G(x)\ \mathrm{d}\mu_G(y) \\
&= \int_G f(y)\ \phi(y)\ \mathrm{d}\mu_G(y) \int_G g(z)\ \phi(z)\ \mathrm{d}\mu_G(z) \\
&= \widehat{f}(\phi)\ \widehat{g}(\phi).
\end{aligned}$$

□

Remark 2.7.4. There are plenty of other interesting results concerning the Fourier transform and convolutions on compact groups. For instance, one can study approximate identities for $L^1(\mu_G)$ and prove that the Fourier transform $f \mapsto \widehat{f}$ is injective on $L^1(\mu_G)$.

Definition 2.7.5. Let $f \in L^2(\mu_G)$. For μ_G-almost every $x \in G$,

$$f(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\widehat{f}(\xi)\ \xi^*(x)\right),$$

where $\xi^*(x) := \xi(x)^*$. If $A \in \mathcal{L}(L^2(\mu_G))$ then

$$Af(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\widehat{f}(\xi)\ (A(\xi^*))(x)\right),$$

where $(A(\xi^*))(x) = \left((A\overline{\xi_{ji}})(x)\right)_{i,j=1}^{\dim(\xi)}$. The *(Fourier) symbol* of $A \in \mathcal{L}(L^2(\mu_G))$ is

$$(x,\xi) \mapsto \sigma_A(x,\xi), \quad \sigma_A(x,\xi) := \xi(x)(A(\xi^*))(x).$$

Then it is easy to see that

$$Af(x) = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ \mathrm{Tr}\left(\sigma_A(x,\xi)\ \widehat{f}(\xi)\ \xi^*(x)\right);$$

one may think that this a σ_A-weighted inverse Fourier transform formula for f.

Example. Show that

$$\sigma_{f\mapsto f}(x,\xi) = I \in \mathbb{C}^{\dim(\xi)\times\dim(\xi)},$$

$$\sigma_{f\mapsto g*f}(x,\xi) = \widehat{g}(\xi),$$

$$\sigma_{f\mapsto gf}(x,\xi) = g(x)I.$$

Symbols are often easier to study than the corresponding linear operators, and in many cases the symbols behave almost like ordinary functions. Symbols are used in e.g. the theory of partial differential equations; the classical "freezing-coefficients-technique" is related to symbol calculus. But this is another story, and we move on.

2.8 Characters

Let $\phi : G \to \mathrm{Aut}(\mathcal{H})$ be a finite-dimensional representation of a group G on a Hilbert space $\mathcal{H}$; the *character* of ϕ is the function $\chi_\phi : G \to \mathbb{C}$ defined by

$$\chi_\phi(x) := \mathrm{Tr}\,(\phi(x))\,.$$

Notice that here G is just any group. It turns out that on a compact group, characters provide a way of recognizing equivalence of representations: namely, for finite-dimensional unitary representations, $\phi \sim \psi$ if and only if $\chi_\phi = \chi_\psi$, as we shall see.

Proposition 2.8.1. *Let ϕ, ψ be finite-dimensional representations of a group G.*
(1) $\chi_\phi = \chi_\psi$ *if* $\phi \sim \psi$.
(2) $\chi_\phi(xyx^{-1}) = \chi_\phi(y)$ *for every* $x, y \in G$.
(3) $\chi_{\phi\oplus\psi} = \chi_\phi + \chi_\psi$.
(4) $\chi_{\phi\otimes\psi|_G} = \chi_\phi\ \chi_\psi$.
(5) $\chi_\phi(e) = \dim(\phi)$.

Proof. The results follow from the properties of the trace functionals. □

Remark 2.8.2. Since the character depends only on the equivalence class of a representation, we may define $\chi_{[\phi]} := \chi_\phi$, where $[\phi]$ is the equivalence class of ϕ.

Proposition 2.8.3. (Orthonormality of characters.) *Let G be a compact group and $\xi, \eta \in \widehat{G}$. Then*

$$\langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} = \begin{cases} 1 & \text{if } \xi = \eta, \\ 0 & \text{if } \xi \not\sim \eta. \end{cases}$$

Proof. Let $\phi = (\phi_{ij})_{i,j=1}^m \in \xi$ and $\psi = (\psi_{kl})_{k,l=1}^n \in \eta$. Then

$$\begin{aligned} \langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} &= \sum_{j=1}^m \sum_{k=1}^n \langle \phi_{jj}, \psi_{kk} \rangle_{L^2(\mu_G)} \\ &= \begin{cases} 0 & \text{if } \phi \not\sim \psi, \\ 1 & \text{if } \phi = \psi. \end{cases} \end{aligned}$$

□

Theorem 2.8.4. (Irreducibility and equivalence characterizations.) *Let ϕ, ψ be finite-dimensional continuous unitary representations of a compact group G. Then ϕ is irreducible if and only if $\|\chi_\phi\|_{L^2(\mu_G)} = 1$. Moreover, $\phi \sim \psi$ if and only if $\chi_\phi = \chi_\psi$.*

Proof. We already know the "only if"-parts of the proof. So suppose ϕ is a finite-dimensional unitary representation. Then

$$\phi \sim \bigoplus_{[\xi] \in \widehat{G}} m_{[\xi]} \xi,$$

where $m_{[\xi]} \in \mathbb{N}$ is non-zero for only finitely many $[\xi] \in \widehat{G}$. Then

$$\chi_\phi = \sum_{[\xi] \in \widehat{G}} m_{[\xi]} \chi_\xi,$$

and if $[\eta] \in \widehat{G}$ then

$$\langle \chi_\phi, \chi_\eta \rangle_{L^2(\mu_G)} = \sum_{[\xi] \in \widehat{G}} m_{[\xi]} \langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} = m_{[\eta]}.$$

This implies that the multiplicities $m_{[\xi]} \in \mathbb{N}$ can be uniquely obtained by knowing only χ_ϕ; hence if $\chi_\phi = \chi_\psi$ then $\phi \sim \psi$. Moreover,

$$\begin{aligned}
\|\chi_\phi\|^2_{L^2(\mu_G)} &= \langle \chi_\phi, \chi_\phi \rangle_{L^2(\mu_G)} \\
&= \sum_{[\xi],[\eta]\in\widehat{G}} m_{[\xi]}\ m_{[\eta]}\ \langle \chi_\xi, \chi_\eta \rangle_{L^2(\mu_G)} \\
&= \sum_{[\xi]\in\widehat{G}} m^2_{[\xi]},
\end{aligned}$$

so that ϕ is irreducible if and only if $\|\chi_\phi\|_{L^2(\mu_G)} = 1$. □

Remark 2.8.5. If $f \in L^2(\mu_G)$ then

$$f = \sum_{[\xi]\in\widehat{G}} \dim(\xi)\ f * \chi_\xi.$$

2.9 Induced representations

A group representation trivially begets a representation of its subgroup: if $H < G$ and $\psi \in \mathrm{Hom}(G, \mathrm{Aut}(V))$ then the restriction $\mathrm{Res}^G_H \psi := (h \mapsto \psi(h)) \in \mathrm{Hom}(H, \mathrm{Aut}(V))$. In this section, we show how a representation of a subgroup sometimes *induces* a representation for the whole group. This induction process has also plenty of nice properties. Induced representations were defined and studied by Ferdinand Georg Frobenius in 1898 for finite groups, and by George Mackey in 1949 for (most of the) locally compact groups.

The technical assumptions here are that G is a compact group, $H < G$ is closed and $\phi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}))$ is a strongly continuous; then ϕ induces a strongly continuous unitary representation

$$\mathrm{Ind}^G_H \phi \in \mathrm{Hom}\left(G, \mathcal{U}(\mathrm{Ind}^G_\phi \mathcal{H})\right),$$

where the notation will be explained in the sequel. We start by a lengthy definition of the induced representation space $\mathrm{Ind}^G_\phi \mathcal{H}$.

Definition 2.9.1. Since G is a compact group, continuous functions $G \to \mathcal{H}$ are *uniformly continuous* in the following sense: Let $f \in C(G, \mathcal{H})$ and $\varepsilon > 0$. Then there exists open $U \ni e$ such that $\|f(x) - f(y)\|_{\mathcal{H}} < \varepsilon$ when $xy^{-1} \in U$ (or $x^{-1}y \in U$); the proof of this fact is as in the scalar-valued case.

Proposition 2.9.2. *If $f \in C(G,\mathcal{H})$ then $f_\phi \in C(G,\mathcal{H})$, where*

$$f_\phi(x) := \int_H \phi(h) f(xh) \, \mathrm{d}\mu_H(h).$$

Moreover, $f_\phi(xh) = \phi(h)^* f_\phi(x)$ for every $x \in G$ and $h \in H$.

Proof. The integral here is to be understood in a weak sense (*Pettis integral*; see e.g. [16] (3.26–3.29): $f_\phi(x) \in \mathcal{H}$ is the unique vector defined by inner products, and

$$\begin{aligned} f_\phi(x) &= \sum_{j\in J} \langle f_\phi(x), e_j\rangle_{\mathcal{H}} \; e_j \\ &= \sum_{j\in J} \int_H \langle \phi(h) f(xh), e_j\rangle_{\mathcal{H}} \; \mathrm{d}\mu_H(h) \; e_j, \end{aligned}$$

where $\{e_j\}_{j\in J} \subset \mathcal{H}$ is an orthonormal basis. The integrals here are sound, since $(h \mapsto \langle \phi(h) f(xh), e_j\rangle_{\mathcal{H}}) \in C(H)$ because $f \in C(G,\mathcal{H})$ and ϕ is strongly continuous (this is easy to prove). If $h_0 \in H$ then

$$\begin{aligned} f_\phi(xh_0) &= \int_H \phi(h) \; f(xh_0h) \; \mathrm{d}\mu_H(h) \\ &= \int_H \phi(h_0^{-1}h) f(xh) \; \mathrm{d}\mu_H(h) \\ &= \phi(h_0)^* f_\phi(x). \end{aligned}$$

Take $\varepsilon > 0$. By the uniform continuity of $f \in C(G,\mathcal{H})$, there exists an open set $U \ni e$ such that $\|f(a) - f(b)\|_{\mathcal{H}} < \varepsilon$ whenever $ab^{-1} \in U$. If $x \in yU$ then

$$\begin{aligned} \|f_\phi(x) - f_\phi(y)\|_{\mathcal{H}}^2 &= \left\| \int_H \phi(h)(f(xh) - f(yh)) \; \mathrm{d}\mu_H(h) \right\|_{\mathcal{H}}^2 \\ &\le \left(\int_H \|f(xh) - f(yh)\|_{\mathcal{H}} \; \mathrm{d}\mu_H(h) \right)^2 \\ &\le \varepsilon^2, \end{aligned}$$

sealing the continuity of f_ϕ. □

Lemma 2.9.3. *If $f, g \in C(G,\mathcal{H})$ then $(xH \mapsto \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}) \in C(G/H)$.*

Proof. Let $x \in G$ and $h \in H$. Then

$$\begin{aligned}\langle f_\phi(xh), g_\phi(xh)\rangle_{\mathcal{H}} &= \langle \phi(h)^* f_\phi(x), \phi(h)^* g_\phi(x)\rangle_{\mathcal{H}} \\ &= \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}},\end{aligned}$$

so that $(xH \mapsto \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}) : G/H \to \mathbb{C}$ is well-defined. There exists a constant $C < \infty$ such that $\|f_\phi(y)\|_{\mathcal{H}}, \|g_\phi(x)\|_{\mathcal{H}} \leq C$ because G is compact and $f_\phi, g_\phi \in C(G, \mathcal{H})$. Thereby

$$\begin{aligned} & |\langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}} - \langle f_\phi(y), g_\phi(y)\rangle_{\mathcal{H}}| \\ \leq\ & |\langle f_\phi(x) - f_\phi(y), g_\phi(x)\rangle_{\mathcal{H}}| + |\langle f_\phi(y), g_\phi(x) - g_\phi(y)\rangle_{\mathcal{H}}| \\ \leq\ & C\left(\|f_\phi(x) - f_\phi(y)\|_{\mathcal{H}} + \|g_\phi(x) - g_\phi(y)\|_{\mathcal{H}}\right) \\ \xrightarrow[x \to y]{}\ & 0 \end{aligned}$$

by the continuities of f_ϕ and g_ϕ. □

Definition 2.9.4. Let us endow the vector space

$$\begin{aligned} C_\phi(G, \mathcal{H}) &:= \{f_\phi \mid f \in C(G, \mathcal{H})\} \\ &= \{e \in C(G, \mathcal{H}) \mid \forall x \in G\ \forall h \in H :\ e(xh) = \phi(h)^* e(x)\} \end{aligned}$$

with the inner product defined by

$$\langle f_\phi, g_\phi\rangle_{\mathrm{Ind}_\phi^G \mathcal{H}} := \int_{G/H} \langle f_\phi(x), g_\phi(x)\rangle_{\mathcal{H}}\ \mathrm{d}\mu_{G/H}(xH).$$

Let $\mathrm{Ind}_\phi^G \mathcal{H}$ be the completion of $C_\phi(G, \mathcal{H})$ with respect to the corresponding norm

$$f_\phi \mapsto \|f_\phi\|_{\mathrm{Ind}_\phi^G \mathcal{H}} := \sqrt{\langle f_\phi, f_\phi\rangle_{\mathrm{Ind}_\phi^G \mathcal{H}}};$$

this Hilbert space is called the *induced representation space.*

Remark 2.9.5. If $\mathcal{H} \neq \{0\}$ then $\{0\} \neq C_\phi(G, \mathcal{H}) \subset \mathrm{Ind}_\phi^G \mathcal{H}$: Let $0 \neq u \in \mathcal{H}$. Due to the strong continuity of ϕ, choose open $U \subset G$ such that $e \in U$ and $\|(\phi(h) - \phi(e))u\|_{\mathcal{H}} < \|u\|_{\mathcal{H}}$ for every $h \in H \cap U$. Choose $w \in C(G)$ such that $w \geq 0$, $w|_{G \setminus U} = 0$ and $\int_H w(h)\ \mathrm{d}\mu_H(h) = 1$. Let $f(x) := w(x)u$ for every $x \in G$. Then

$$\begin{aligned} \|f_\phi(e) - u\|_{\mathcal{H}} &= \left\| \int_H w(h)\ (\phi(h) - \phi(e))u\ \mathrm{d}\mu_H(h) \right\|_{\mathcal{H}} \\ &= \int_H w(h)\ \|(\phi(h) - \phi(e))u\|_{\mathcal{H}}\ \mathrm{d}\mu_H(h) \\ &< \|u\|_{\mathcal{H}}, \end{aligned}$$

so that $f_\phi(e) \neq 0$, yielding $f_\phi \neq 0$.

Theorem 2.9.6. *If $x, y \in G$ and $f_\phi \in C_\phi(G, \mathcal{H})$, let*

$$\left(\mathrm{Ind}_H^G \phi(y) f_\phi\right)(x) := f_\phi(y^{-1}x).$$

This begets a unique strongly continuous $\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\left(G, \mathcal{U}(\mathrm{Ind}_\phi^G \mathcal{H})\right)$, *called the* representation of G induced by ϕ.

Proof. If $y \in G$ and $f_\phi \in C_\phi(G, \mathcal{H})$ then $\mathrm{Ind}_H^G \phi(y) f_\phi = g_\phi \in C_\phi(G, \mathcal{H})$, where $g \in C(G, \mathcal{H})$ is defined by $g(x) := f(y^{-1}x)$. Thus we have a linear mapping $\mathrm{Ind}_H^G \phi(y) : C_\phi(G, \mathcal{H}) \to C_\phi(G, \mathcal{H})$. Clearly

$$\mathrm{Ind}_H^G \phi(yz) f_\phi = \mathrm{Ind}_H^G \phi(y)\ \mathrm{Ind}_H^G \phi(z) f_\phi.$$

Hence $\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\,(G, \mathrm{Aut}(C_\phi(G, \mathcal{H})))$.

If $f, g \in C(G, \mathcal{H})$ then

$$\begin{aligned}
\left\langle \mathrm{Ind}_H^G \phi(y) f_\phi, g_\phi \right\rangle_{\mathrm{Ind}_\phi^G \mathcal{H}} &= \int_{G/H} \langle f_\phi(y^{-1}x), g_\phi(x) \rangle_{\mathcal{H}}\ \mathrm{d}\mu_{G/H}(xH) \\
&= \int_{G/H} \langle f_\phi(z), g_\phi(yz) \rangle_{\mathcal{H}}\ \mathrm{d}\mu_{G/H}(zH) \\
&= \left\langle f_\phi, \mathrm{Ind}_H^G \phi(y)^{-1} g_\phi \right\rangle_{\mathrm{Ind}_\phi^G \mathcal{H}};
\end{aligned}$$

hence we have an extension $\mathrm{Ind}_H^G \phi \in \mathrm{Hom}\left(G, \mathcal{U}(\mathrm{Ind}_\phi^G \mathcal{H})\right)$. Next we exploit the uniform continuity of $f \in C(G, \mathcal{H})$: Let $\varepsilon > 0$. Take an open set $U \ni e$ such that $\|f(a) - f(b)\|_{\mathcal{H}} < \varepsilon$ when $ab^{-1} \in U$. Thereby, if $y^{-1}z \in U$ then

$$\begin{aligned}
&\left\| \left(\mathrm{Ind}_H^G \phi(y) - \mathrm{Ind}_H^G \phi(z)\right) f_\phi \right\|_{\mathrm{Ind}_\phi^G \mathcal{H}}^2 \\
= & \int_{G/H} \left\| f_\phi(y^{-1}x) - f_\phi(z^{-1}x) \right\|_{\mathcal{H}}^2\ \mathrm{d}\mu_{G/H}(xH) \\
\leq & \ \varepsilon^2.
\end{aligned}$$

This shows the strong continuity of the induced representation. □

Remark 2.9.7. In the sequel, some of the elementary properties of induced representations are deduced. Briefly: induced representations of equivalent representations are equivalent, and induction process can be taken in stages leading to the same result modulo equivalence.

Proposition 2.9.8. *Let G be a compact group and $H < G$ a closed subgroup. Let $\phi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}_\phi))$ and $\psi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}_\psi))$ be strongly continuous. If $\phi \sim \psi$ then $\mathrm{Ind}_H^G \phi \sim \mathrm{Ind}_H^G \psi$.*

Proof. Since $\phi \sim \psi$, there is an isometric isomorphism $A \in \mathrm{Hom}(\phi, \psi)$. Then

$$(Bf_\phi)(x) := A(f_\phi(x))$$

defines a linear mapping $B : C_\phi(G, \mathcal{H}_\phi) \to C_\psi(G, \mathcal{H}_\psi)$, because if $x \in G$ and $h \in H$ then

$$\begin{aligned}
(Bf_\phi)(xh) &= A(f_\phi(xh)) \\
&= A(\phi(h)^* f_\phi(x)) \\
&= A(\phi(h)^* A^* A(f_\phi(x))) \\
&= A(A^* \psi(h)^* A(f_\phi(x))) \\
&= \psi(h)^* A(f_\phi(x)) \\
&= \psi(h)^* (Bf_\phi)(x).
\end{aligned}$$

Furthermore, B begets a unique linear isometry $C : \mathrm{Ind}_\phi^G \mathcal{H}_\phi \to \mathrm{Ind}_\psi^G \mathcal{H}_\psi$, since

$$\begin{aligned}
\|Bf_\phi\|^2_{\mathrm{Ind}_\psi^G \mathcal{H}_\psi} &= \int_{G/H} \|(Bf_\phi)(x)\|^2_{\mathcal{H}_\psi} \ \mathrm{d}\mu_{G/H}(xH) \\
&= \int_{G/H} \|A(f_\phi(x))\|^2_{\mathcal{H}_\psi} \ \mathrm{d}\mu_{G/H}(xH) \\
&= \int_{G/H} \|f_\phi(x)\|^2_{\mathcal{H}_\phi} \ \mathrm{d}\mu_{G/H}(xH) \\
&= \|f_\phi\|^2_{\mathrm{Ind}_\phi^G \mathcal{H}_\phi}.
\end{aligned}$$

Next, C is a surjection: if $F \in C_\psi(G, \mathcal{H}_\psi)$ then $(y \mapsto A^{-1}(F(y)) \in C_\phi(G, \mathcal{H}_\phi)$ and $\left(C\left(y \mapsto A^{-1}(F(y))\right)\right)(x) = AA^{-1}(F(x)) = F(x)$, and this is enough due to the density. Finally,

$$\begin{aligned}(C\ \mathrm{Ind}_H^G\phi(y)f_\phi)(x) &= A(\mathrm{Ind}_H^G\phi(y)f_\phi(x)) \\ &= A(f_\phi(y^{-1}x)) \\ &= (Cf_\phi)(y^{-1}x) \\ &= (\mathrm{Ind}_H^G\phi(y)Cf_\phi)(x),\end{aligned}$$

so that $C \in \mathrm{Hom}\left(\mathrm{Ind}_H^G\phi, \mathrm{Ind}_H^G\psi\right)$ is an isometric isomorphism. □

Corollary 2.9.9. *Let G be a compact group and $H < G$ closed. Let ϕ_1 and ϕ_2 be strongly continuous unitary representations of H. Then $\mathrm{Ind}_H^G(\phi_1 \oplus \phi_2) \sim \left(\mathrm{Ind}_H^G\phi_1\right) \oplus \left(\mathrm{Ind}_H^G\phi_2\right)$.*

Exercise 2.9.10. Prove Corollary 2.9.9.

Corollary 2.9.11. *$\mathrm{Ind}_H^G\phi$ is irreducible only if ϕ is irreducible.*

Remark 2.9.12. Let G_1, G_2 be compact groups and $H_1 < G_1, H_2 < G_2$ be closed. If ϕ_1, ϕ_2 are strongly continuous unitary representations of H_1, H_2, respectively, then

$$\mathrm{Ind}_{H_1\times H_2}^{G_1\times G_2}(\phi_1 \otimes \phi_2) \sim \left(\mathrm{Ind}_{H_1}^{G_1}\phi_1\right) \otimes \left(\mathrm{Ind}_{H_2}^{G_2}\phi_2\right);$$

this is not proved in these lecture notes.

Theorem 2.9.13. *Let G be a compact group and $H < K < G$, where H, K are closed. If $\phi \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}))$ is strongly continuous then $\mathrm{Ind}_H^G\phi \sim \mathrm{Ind}_K^G\mathrm{Ind}_H^K\phi$.*

Proof. In this proof, $x \in G$, $k, k_0 \in K$ and $h \in H$. Let $\psi := \mathrm{Ind}_H^K\phi$ and $\mathcal{H}_\psi := \mathrm{Ind}_\phi^K\mathcal{H}$. Let $f_\phi \in C_\phi(G, \mathcal{H})$. Since $(k \mapsto f_\phi(xk)) : K \to \mathcal{H}$ is continuous and $f_\phi(xkh) = \phi(h)^* f_\phi(xk)$, we obtain $(k \mapsto f_\phi(xk)) \in C_\phi(K, \mathcal{H}) \subset \mathcal{H}_\psi$. Let us define $f_\phi^K : G \to \mathcal{H}_\psi$ by

$$f_\phi^K(x) := (k \mapsto f_\phi(xk)).$$

If $x \in G$ and $k_0 \in K$ then

$$\begin{aligned}f_\phi^K(xk_0)(k) &= f_\phi(xk_0k) \\ &= f_\phi^K(x)(k_0k) \\ &= \left(\psi(k_0)^* f_\phi^K(x)\right)(k),\end{aligned}$$

i.e. $f_\phi^K(xk_0) = \psi(k_0)^* f_\phi^K(x)$. Let $\varepsilon > 0$. By the uniform continuity of f_ϕ, take open $U \ni e$ such that $\|f_\phi(a) - f_\phi(b)\|_{\mathcal{H}} < \varepsilon$ if $ab^{-1} \in U$. Thereby if $xy^{-1} \in U$ then

$$\begin{aligned}
\left\|f_\phi^K(x) - f_\phi^K(y)\right\|_{\mathcal{H}_\psi}^2 &= \int_{K/H} \left\|f_\phi^K(x)(k) - f_\phi^K(y)(k)\right\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_{K/H}(kH) \\
&= \int_{K/H} \|f_\phi(xk) - f_\phi(yk)\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_{K/H}(kH) \\
&\leq \varepsilon^2.
\end{aligned}$$

Hence $f_\phi^K \in C_\psi(G, \mathcal{H}_\psi) \subset \mathrm{Ind}_\psi^G \mathcal{H}_\psi$, so that we have a mapping $(f_\phi \mapsto f_\phi^K) : C_\phi(G,\mathcal{H}) \to C_\psi(G, \mathcal{H}_\psi)$.

We claim that $f_\phi \mapsto f_\phi^K$ defines a surjective linear isometry $\mathrm{Ind}_\phi^G \mathcal{H} \to \mathrm{Ind}_\psi^G \mathcal{H}_\psi$. Isometricity follows by

$$\begin{aligned}
\left\|f_\phi^K\right\|_{\mathrm{Ind}_\psi^G \mathcal{H}_\psi}^2 &= \int_{G/K} \left\|f_\phi^K(x)\right\|_{\mathcal{H}_\psi}^2 \ \mathrm{d}\mu_{G/K}(xK) \\
&= \int_{G/K} \int_{K/H} \left\|f_\phi^K(x)(k)\right\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_{K/H}(kH) \ \mathrm{d}\mu_{G/K}(xK) \\
&= \int_{G/K} \int_{K/H} \|f_\phi(xk)\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_{K/H}(kH) \ \mathrm{d}\mu_{G/K}(xK) \\
&= \int_{G/H} \|f_\phi(x)\|_{\mathcal{H}}^2 \ \mathrm{d}\mu_{G/H}(xH) \\
&= \|f_\phi\|_{\mathrm{Ind}_\phi^G \mathcal{H}}^2 .
\end{aligned}$$

How about the surjectivity? The representation space $\mathrm{Ind}_\psi^G \mathcal{H}_\psi$ is the closure of $C_\psi(G, \mathcal{H}_\psi)$, and $\mathcal{H}_\psi$ is the closure of $C_\phi(K, \mathcal{H})$. Consequently, $\mathrm{Ind}_\psi^G \mathcal{H}_\psi$ is the closure of the vector space

$$\begin{aligned}
C_\psi(G, C_\phi(K,\mathcal{H})) \quad &:= \\
\{g \in C(G, C(K,\mathcal{H})) \quad &| \quad \forall x \in G \ \forall k \in K \ \forall h \in H : \\
&\qquad g(xk) = \psi(k)^* g(x), \ g(x)(kh) = \phi(h)^* g(x)(k)\}.
\end{aligned}$$

Given $g \in C_\psi(G, C_\phi(K,\mathcal{H}))$, define $f_\phi \in C_\phi(G,\mathcal{H})$ by $f_\phi(x) := g(x)(e)$. Then $f_\phi^K = g$, because

$$f_\phi^K(x)(k) = f_\phi(xk) = g(xk)(e) = \psi(k)^* g(x)(e) = g(x)(k).$$

Thus $(f_\phi \mapsto f_\phi^K) : C_\phi(G,\mathcal{H}) \to C_\psi(G, C_\phi(K,\mathcal{H}))$ is a linear isometric bijection. Hence this mapping can be extended uniquely to a linear isometric bijection $A : \mathrm{Ind}_\phi^G \mathcal{H} \to \mathrm{Ind}_\psi^G \mathcal{H}_\psi$.

Finally, $A \in \mathrm{Hom}\left(\mathrm{Ind}_H^G \phi, \mathrm{Ind}_K^G \mathrm{Ind}_H^K \phi\right)$, since

$$\begin{aligned} A\left(\mathrm{Ind}_H^G \phi(y) f_\phi(x)\right) &= A f_\phi(y^{-1}x) \\ &= f_\phi^K(y^{-1}x) \\ &= \mathrm{Ind}_K^G \psi(y) f_\phi^K(x) \\ &= \mathrm{Ind}_K^G \psi(y) A f_\phi(x). \end{aligned}$$

□

Exercise 2.9.14. Let G be a compact group, $H < G$ closed. Let $\phi = (h \mapsto I) \in \mathrm{Hom}(H, \mathcal{U}(\mathcal{H}))$, where $I = (u \mapsto u) : \mathcal{H} \to \mathcal{H}$.
(a) Show that $\mathrm{Ind}_\phi^G \mathcal{H} \cong L^2(G/H, \mathcal{H})$, where the inner product for $L^2(G/H,\mathcal{H})$ is given by

$$\langle f_{G/H}, g_{G/H} \rangle_{L^2(G/H,\mathcal{H})} := \int_{G/H} \langle f_{G/H}(xH), g_{G/H}(xH)\rangle_{\mathcal{H}} \, \mathrm{d}\mu_{G/H}(xH),$$

when $f_{G/H}, g_{G/H} \in C(G/H,\mathcal{H})$.
(b) Let $K < G$ be closed. Let π_K and π_G be the left regular representations of K and G, respectively. Prove that $\pi_G \sim \mathrm{Ind}_K^G \pi_K$.

Remark 2.9.15. A fundamental result for induced representations is the *Frobenius Reciprocity Theorem* 2.9.16, stated below without a proof. Let G be a compact group and $\phi \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{H}))$ be strongly continuous. Let $n([\xi], \phi) \in \mathbb{N}$ denote the *multiplicity of* $[\xi] \in \widehat{G}$ *in* ϕ, defined as follows: if $\phi = \bigoplus_{j=1}^k \phi_j$, where each ϕ_j is a continuous irreducible unitary representation, then

$$n([\xi], \phi) := \left|\{j \in \{1, \ldots, k\} : \ [\phi_j] = [\xi]\}\right|.$$

That is, $n([\xi], \phi)$ is the number of how many times ξ may occur in a direct sum decomposition of ϕ as an irreducible component.

Theorem 2.9.16. (Frobenius Reciprocity Theorem.) *Let G be a compact group and $H < G$ be closed. Let ξ, η be continuous such that $[\xi] \in \widehat{G}$ and $[\eta] \in \widehat{H}$. Then*

$$n\left([\xi], \mathrm{Ind}_H^G \eta\right) = n\left([\eta], \mathrm{Res}_H^G \xi\right).$$

Example. Let $[\xi] \in \widehat{G}$, $H = \{e\}$ and $\eta = (e \mapsto I) \in \mathrm{Hom}(H, \mathcal{U}(\mathbb{C}))$. Then $\pi_L \sim \mathrm{Ind}_H^G \eta$ by Exercise 2.9.14, and $\widehat{H} = \{[\eta]\}$, so that

$$\begin{aligned} n\left([\xi], \mathrm{Ind}_H^G \eta\right) &= n([\xi], \pi_L) \\ &\overset{\text{Peter-Weyl}}{=} \dim(\xi) \\ &= \dim(\xi)\ n([\eta], \eta) \\ &= n\left([\eta], \bigoplus_{j=1}^{\dim(\xi)} \eta\right) \\ &= n\left([\eta], \mathrm{Res}_H^G \xi\right) \end{aligned}$$

which is in accordance with the Frobenius Reciprocity Theorem 2.9.16!

Example. Let $[\xi], [\eta] \in \widehat{G}$. Then by the Frobenius Reciprocity Theorem 2.9.16,

$$\begin{aligned} n\left([\xi], \mathrm{Ind}_G^G \eta\right) &= n\left([\eta], \mathrm{Res}_G^G \xi\right) \\ &= n([\eta], \xi) \\ &= \begin{cases} 1, & \text{when } [\xi] = [\eta], \\ 0, & \text{when } [\xi \neq [\eta]. \end{cases} \end{aligned}$$

Let ϕ be a finite-dimensional continuous unitary representation of G. Then $\phi = \bigoplus_{j=1}^{k} \xi_k$, where each ξ_k is irreducible. Thereby

$$\mathrm{Ind}_G^G \phi \sim \bigoplus_{j=1}^{k} \mathrm{Ind}_G^G \xi_j \sim \bigoplus_{j=1}^{k} \xi_j \sim \phi;$$

induction "does nothing" in the finite-dimensional case.

Chapter 3

Linear Lie groups

3.1 Exponential map

A *Lie group*, by definition, is a group and a C^∞-manifold such that the group operations are C^∞-smooth. A *linear Lie group* means a closed subgroup of $\mathrm{GL}(n,\mathbb{C})$. There is a result stating that a Lie group is diffeomorphic to a linear Lie group, and thereby the matrix groups are especially interesting. The fundamental tool for studying these groups is the matrix exponential map, treated below.

Let us endow $\mathbb{C}^n$ with the usual inner product

$$(x,y) \mapsto \langle x,y\rangle_{\mathbb{C}^n} := \sum_{j=1}^{n} x_j \overline{y_j}.$$

The corresponding norm is $x \mapsto \|x\|_{\mathbb{C}^n} := \langle x,x\rangle_{\mathbb{C}^n}$. We identify the matrix algebra $\mathbb{C}^{n\times n}$ with $\mathcal{L}(\mathbb{C}^n)$, the algebra of linear operators $\mathbb{C}^n \to \mathbb{C}^n$. Let us endow $\mathbb{C}^{n\times n} \cong \mathcal{L}(\mathbb{C}^n)$ with the usual operator norm

$$Y \mapsto \|Y\|_{\mathcal{L}(\mathbb{C}^n)} := \sup_{x\in\mathbb{C}^n:\ \|x\|_{\mathbb{C}^n}\leq 1} \|Yx\|_{\mathbb{C}^n}.$$

Notice that $\|XY\|_{\mathcal{L}(\mathbb{C}^n)} \leq \|X\|_{\mathcal{L}(\mathbb{C}^n)}\|Y\|_{\mathcal{L}(\mathbb{C}^n)}$. For a matrix $X \in \mathbb{C}^{n\times n}$, the *exponential* $\exp(X) \in \mathbb{C}^{n\times n}$ is defined by the usual power series

$$\exp(X) := \sum_{k=0}^{\infty} \frac{1}{k!} X^k,$$

where $X^0 := I$; this series converges in the Banach space $\mathbb{C}^{n\times n} \cong \mathcal{L}(\mathbb{C}^n)$, because

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left\|X^k\right\|_{\mathcal{L}(\mathbb{C}^n)} \leq \sum_{k=0}^{\infty} \frac{1}{k!} \|X\|_{\mathcal{L}(\mathbb{C}^n)}^k = \mathrm{e}^{\|X\|_{\mathcal{L}(\mathbb{C}^n)}} < \infty.$$

Proposition 3.1.1. *Let* $X, Y \in \mathbb{C}^{n\times n}$. *If* $XY = YX$ *then* $\exp(X+Y) = \exp(X)\exp(Y)$. *Consequently,* $\exp : \mathbb{C}^{n\times n} \to \mathrm{GL}(n,\mathbb{C})$ *such that* $\exp(-X) = \exp(X)^{-1}$.

Proof. Now

$$\begin{aligned}
\exp(X+Y) &= \lim_{l\to\infty} \sum_{k=0}^{2l} \frac{1}{k!}(X+Y)^k \\
&\overset{XY=YX}{=} \lim_{l\to\infty} \sum_{k=0}^{2l} \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} X^i Y^{k-i} \\
&= \lim_{l\to\infty} \left(\sum_{i=0}^{l} \frac{1}{i!} X^i \sum_{j=0}^{l} \frac{1}{j!} Y^j + \sum_{\substack{i,j:\ i+j\leq 2l,\\ \max(i,j)>l}} \frac{1}{i!\,j!} X^i Y^j \right) \\
&= \lim_{l\to\infty} \left(\sum_{i=0}^{l} \frac{1}{i!} X^i \sum_{j=0}^{l} \frac{1}{j!} Y^j \right) \\
&= \exp(X)\exp(Y),
\end{aligned}$$

since the remainder term satisfies

$$\begin{aligned}
\left\| \sum_{\substack{i,j:\ i+j\leq 2l,\\ \max(i,j)>l}} \frac{1}{i!\,j!} X^i Y^j \right\|_{\mathcal{L}(\mathbb{C}^n)} &\leq \sum_{\substack{i,j:\ i+j\leq 2l,\\ \max(i,j)>l}} \frac{1}{i!\,j!} \|X\|_{\mathcal{L}(\mathbb{C}^n)}^i \|Y\|_{\mathcal{L}(\mathbb{C}^n)}^j \\
&\leq l(l+1) \frac{1}{(l+1)!} c^{2l} \\
&\xrightarrow[l\to\infty]{} 0,
\end{aligned}$$

where $c := \max\left(1, \|X\|_{\mathcal{L}(\mathbb{C}^n)}, \|Y\|_{\mathcal{L}(\mathbb{C}^n)}\right)$.

Consequently, $I = \exp(0) = \exp(X)\exp(-X) = \exp(-X)\exp(X)$, so that we get $\exp(-X) = \exp(X)^{-1}$. □

Lemma 3.1.2. *If* $X \in \mathbb{C}^{n\times n}$ *then* $\exp\left(X^{\mathrm{T}}\right) = \exp(X)^{\mathrm{T}}$ *and* $\exp(X^*) = \exp(X)^*$*; if* $P \in \mathrm{GL}(n, \mathbb{C})$ *then* $\exp(PXP^{-1}) = P\exp(X)P^{-1}$.

Proof. For the adjoint X^*,

$$\exp(X^*) = \sum_{k=0}^{\infty} \frac{1}{k!}(X^*)^k = \sum_{k=0}^{\infty} \frac{1}{k!}(X^k)^* = \left(\sum_{k=0}^{\infty} \frac{1}{k!}X^k\right)^* = \exp(X)^*,$$

and similarly for the transpose X^{T}. Finally,

$$\exp(PXP^{-1}) = \sum_{k=0}^{\infty} \frac{1}{k!}(PXP^{-1})^k = \sum_{k=0}^{\infty} \frac{1}{k!}PX^kP^{-1} = P\exp(X)P^{-1}.$$

□

Proposition 3.1.3. *If* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $X \in \mathbb{C}^{n\times n}$ *then* e^{λ} *is an eigenvalue of* $\exp(X)$*. Consequently*

$$\mathrm{Det}(\exp(X)) = \mathrm{e}^{\mathrm{Tr}(X)}.$$

Proof. Choose $P \in \mathrm{GL}(n, \mathbb{C})$ such that $Y := PXP^{-1} \in \mathbb{C}^{n\times n}$ is upper triangular; the eigenvalues of X and Y are the same, and for triangular matrices the eigenvalues are the diagonal elements. Since Y^k is upper triangular for every $k \in \mathbb{N}$, $\exp(Y)$ is upper triangular. Moreover, $(Y^k)_{jj} = (Y_{jj})^k$, so that $(\exp(Y))_{jj} = \mathrm{e}^{Y_{jj}}$. The eigenvalues of $\exp(X)$ and $\exp(Y) = P\exp(X)P^{-1}$ are the same.

The determinant of a matrix is the product of its eigenvalues; the trace of a matrix is the sum of its eigenvalues; this implies the last claim. □

Theorem 3.1.4. $\mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C})) = \{t \mapsto \exp(tX) \mid X \in \mathbb{C}^{n\times n}\}$.

Proof. It is clear that $(t \mapsto \exp(tX)) \in \mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C}))$, since it is continuous and $\exp(sX)\exp(tX) = \exp((s+t)X)$.

Let $\phi \in \mathrm{HOM}(\mathbb{R}, \mathrm{GL}(n, \mathbb{C}))$. Then $\phi(s+t) = \phi(s)\phi(t)$ implies

$$\left(\int_0^h \phi(s)\ \mathrm{d}s\right)\ \phi(t) = \int_0^h \phi(s+t)\ \mathrm{d}s = \int_t^{t+h} \phi(u)\ \mathrm{d}u.$$

Recall that $A \in \mathbb{C}^{n\times n}$ is invertible if $\|I - A\|_{\mathcal{L}(\mathbb{C}^n)} < 1$; now

$$\begin{aligned} \left\| I - \frac{1}{h}\int_0^h \phi(s)\ \mathrm{d}s \right\|_{\mathcal{L}(\mathbb{C}^n)} &= \left\| \frac{1}{h}\int_0^h (I - \phi(s))\ \mathrm{d}s \right\|_{\mathcal{L}(\mathbb{C}^n)} \\ &\le \sup_{s:\ |s|\le|h|} \|I - \phi(s)\|_{\mathcal{L}(\mathbb{C}^n)} \\ &< 1 \end{aligned}$$

when $|h|$ is small enough, because $\phi(0) = 1$ and ϕ is continuous. Hence $\int_0^h \phi(s)\ \mathrm{d}s$ is invertible for small $|h|$, and we get

$$\phi(t) = \left(\int_0^h \phi(s)\ \mathrm{d}s\right)^{-1} \int_t^{t+h} \phi(u)\ \mathrm{d}u.$$

Since ϕ is continuous, this formula states that ϕ is differentiable. Now

$$\phi'(t) = \lim_{s\to 0} \frac{\phi(s+t) - \phi(t)}{s} = \lim_{s\to 0} \frac{\phi(s) - \phi(0)}{s}\ \phi(t) = X\ \phi(t),$$

where $X := \phi'(0)$. Hence the initial value problem

$$\begin{cases} \psi'(t) = X\ \psi(t), & \psi : \mathbb{R} \to \mathrm{GL}(n,\mathbb{C}), \\ \psi(0) = I \end{cases}$$

has solutions $\psi = \phi$ and $\psi = \phi_X := (t \mapsto \exp(tX))$. Define $\alpha : \mathbb{R} \to \mathrm{GL}(n,\mathbb{C})$ by $\alpha(t) := \phi(t)\ \phi_X(-t)$. Then $\alpha(0) = \phi(0)\ \phi_X(0) = I$ and

$$\begin{aligned} \alpha'(t) &= \phi'(t)\ \phi_X(-t) - \phi(t)\ \phi_X'(-t) \\ &= X\ \phi(t)\ \phi_X(-t) - \phi(t)\ X\ \phi_X(-t) \\ &= 0, \end{aligned}$$

since $X\ \phi(t) = \phi(t)\ X$. Therefore $\alpha(t) = I$ for every $t \in \mathbb{R}$, so that $\phi = \phi_X$. □

Proposition 3.1.5. Logarithm

$$\log(A) := -\sum_{k=1}^{\infty} \frac{1}{k}(I - A)^k$$

is well-defined for matrices $A \in \mathbb{C}^{n\times n}$ *satisfying* $\|I - A\|_{\mathcal{L}(\mathbb{C}^n)} < 1$. *Moreover,* $\exp(\log(A)) = A$, *and* $\log(\exp(X)) = X$ *if* $\|X\|_{\mathcal{L}(\mathbb{C}^n)} < \ln(2)$.

Proof. When $c := \|I - A\|_{\mathcal{L}(\mathbb{C}^n)} < 1$,

$$\sum_{k=1}^{\infty} \frac{1}{k} \left\|(I-A)^k\right\|_{\mathcal{L}(\mathbb{C}^n)} \leq \sum_{k=1}^{\infty} \frac{1}{k} \|I-A\|_{\mathcal{L}(\mathbb{C}^n)}^k \leq \sum_{k=1}^{\infty} c^k = \frac{c}{1-c} < \infty,$$

so that $\log(A)$ is well-defined. Because *numbers* $a \in \mathbb{C}$ satisfy

$$\mathrm{e}^{\ln a} = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\sum_{l=1}^{\infty} \frac{1}{l}(1-a)^l\right)^k = a,$$

when $|1-a| < 1$, respectively for a matrix $A \in \mathbb{C}^{n\times n}$

$$\exp(\log(A)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\sum_{l=1}^{\infty} \frac{1}{l}(I-A)^l\right)^k = A,$$

when $\|I-A\| < 1$ (notice that I and A commute). Now

$$\ln(\mathrm{e}^x) = -\sum_{l=1}^{\infty} \frac{1}{l}(1-\mathrm{e}^x)^l = -\sum_{l=1}^{\infty} \frac{1}{l}\left(1-\sum_{k=0}^{\infty} \frac{1}{k!}x^k\right)^l = x,$$

when $|1-\mathrm{e}^x| < 1$, i.e. $x < \ln(2)$, so that if $X \in \mathbb{C}^{n\times n}$ and $\|X\|_{\mathcal{L}(\mathbb{C})} < \ln(2)$ then

$$\log(\exp(X)) = -\sum_{l=1}^{\infty} \frac{1}{l}(I-\exp(X))^l = -\sum_{l=1}^{\infty} \frac{1}{l}\left(I-\sum_{k=0}^{\infty} \frac{1}{k!}X^k\right)^l = X.$$

□

Corollary 3.1.6. *Let* $\mathbb{B} := \left\{X \in \mathbb{C}^{n\times n} : \ \|X\|_{\mathcal{L}(\mathbb{C}^n)} < \ln(2)\right\}$. *Then* $(X \mapsto \exp(X)) : \mathbb{B} \to \exp(\mathbb{B})$ *is a diffeomorphism (i.e. a bijective* C^∞*-smooth mapping).*

Proof. As exp and log are defined by power series, they are not just C^∞-smooth but also analytic. □

Lemma 3.1.7. *Let* $X, Y \in \mathbb{C}^{n\times n}$. *Then*

$$\exp(X+Y) = \lim_{m\to\infty} \left(\exp(X/m)\ \exp(Y/m)\right)^m$$

and

$$\exp([X,Y]) = \lim_{m\to\infty} \{\exp(X/m), \exp(Y/m)\}^{m^2},$$

where $[X,Y] := XY - YX$ *and* $\{a,b\} := aba^{-1}b^{-1}$.

Proof. As $t \to 0$,

$$\begin{aligned}\exp(tX)\ \exp(tY) &= \left(I + tX + \frac{t^2}{2}X^2 + \mathcal{O}(t^3)\right) \\ &\quad \left(I + tY + \frac{t^2}{2}Y^2 + \mathcal{O}(t^3)\right) \\ &= I + t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \mathcal{O}(t^3),\end{aligned}$$

so that

$$\begin{aligned}&\{\exp(tX), \exp(tY)\} \\ = &\left(I + t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \mathcal{O}(t^3)\right) \\ &\left(I - t(X+Y) + \frac{t^2}{2}(X^2 + 2XY + Y^2) + \mathcal{O}(t^3)\right) \\ = &\ I + t^2(XY - YX) + \mathcal{O}(t^3) \\ = &\ I + t^2[X,Y] + \mathcal{O}(t^3).\end{aligned}$$

Since exp is an injection in a neighbourhood of the origin $0 \in \mathbb{C}^{n\times n}$, we have

$$\exp(tX)\ \exp(tY) = \exp\left(t(X+Y) + \mathcal{O}(t^2)\right),$$

$$\{\exp(tX), \exp(tY)\} = \exp\left(t^2[X,Y] + \mathcal{O}(t^3)\right)$$

as $t \to 0$. Notice that $\exp(X)^m = \exp(mX)$ for every $m \in \mathbb{N}$. Therefore we get

$$\begin{aligned}\lim_{m\to\infty} \left(\exp(X/m)\ \exp(Y/m)\right)^m &= \lim_{m\to\infty} \exp\left(X + Y + \mathcal{O}(m^{-1})\right) \\ &= \exp(X+Y),\end{aligned}$$

$$\begin{aligned}\lim_{m\to\infty} \{\exp(X/m), \exp(Y/m)\}^{m^2} &= \lim_{m\to\infty} \exp\left([X,Y] + \mathcal{O}(m^{-1})\right) \\ &= \exp([X,Y]).\end{aligned}$$

□

3.2 No small subgroups for Lie, please

A topological group is said to have the "*no small subgroups*" property if there exists a neighbourhood of the neutral element containing no non-trivial subgroups. Next we show that this property characterizes Lie groups among compact groups.

Example. Let $\{G_j\}_{j\in J}$ be an infinite family of compact groups each having more than one element. Let us consider the compact product group $G := \prod_{j\in J} G_j$. If

$$H_j := \{x \in G \mid \forall i \in J \setminus \{j\} : x_i = e_{G_i}\}$$

then $G_j \cong H_j < G$; H_j is a non-trivial subgroup of G. If $V \subset G$ is a neighbourhood of $e \in G$ then it contains all but perhaps finitely many H_j, due to the definition of the product topology. Hence in this case G "has small subgroups" (i.e. has not "no small subgroups" property).

Theorem 3.2.1. *Let G be a compact group and $V \subset G$ open such that $e \in V$. Then there exists $n \in \mathbb{Z}^+$ and $\phi \in \mathrm{HOM}(G, \mathrm{U}(n))$ such that $\mathrm{Ker}(\phi) \subset V$.*

Proof. First, $\{e\} \subset G$ and $G \setminus V \subset G$ are disjoint closed subsets of a compact Hausdorff space G (case $V = G$ is also ok). By Urysohn's Lemma, there exists $f \in C(G)$ such that $f(e) = 1$ and $f(G \setminus V) = \{0\}$. Since trigonometric polynomials are dense in $C(G)$, we may take $p \in \mathrm{TrigPol}(G)$ such that $\|p - f\|_{C(G)} < 1/2$. Then

$$\mathcal{H} := \mathrm{span}\,\{\pi_R(x)p \mid x \in G\} \subset L^2(\mu_G)$$

is a finite-dimensional vector space, and $\mathcal{H}$ inherits the inner product from $L^2(\mu_G)$. Let $A : \mathcal{H} \to \mathbb{C}^n$ be a linear isometry, where $n = \dim(\mathcal{H})$. Let us identify $\mathcal{U}(\mathbb{C}^n)$ with $\mathrm{U}(n)$. Define $\phi \in \mathrm{Hom}(G, \mathrm{U}(n))$ by

$$\phi(x) := A\ \pi_R(x)|_{\mathcal{H}}\ A^{-1}.$$

Then ϕ is clearly a continuous unitary representation. For every $x \in G \setminus V$,

$$|p(x) - 0| = |p(x) - f(x)| \le \|p - f\|_{C(G)} < 1/2,$$

so that $p(x) \ne p(e)$, because

$$|p(e) - 1| = |p(e) - f(e)| \le \|p - f\|_{C(G)} < 1/2;$$

consequently $\pi_R(x)p \ne p$. Thus $\mathrm{Ker}(\phi) \subset V$. □

Corollary 3.2.2. *Let G be a compact group. Then G has no small subgroups if and only if it is isomorphic to a linear Lie group.*

Proof. Let G be a compact group without small subgroups. By Theorem 3.2.1, there exists injective $\phi \in \mathrm{HOM}(G, \mathrm{U}(n))$ for some $n \in \mathbb{Z}^+$. Then $(x \mapsto \phi(x)) : G \to \phi(G)$ is an isomorphism and a homeomorphism, because ϕ is continuous, G is compact and $\mathrm{U}(n)$ is Hausdorff. Thus $\phi(G) < \mathrm{U}(n) < \mathrm{GL}(n, \mathbb{C})$ is a compact linear Lie group.

Conversely, suppose $G < \mathrm{GL}(n, \mathbb{C})$ is closed. Recall that $(X \mapsto \exp(X)) : \mathbb{B} \to \exp(\mathbb{B})$ is a homeomorphism, where

$$\mathbb{B} = \left\{ X \in \mathbb{C}^{n \times n} : \ \|X\|_{\mathcal{L}(\mathbb{C}^n)} < \ln(2) \right\}.$$

Thereby $V := \exp(\mathbb{B}/2) \cap G$ is a neighbourhood of $I \in G$. In the search of a contradiction, suppose there exists $H < G$ such that $I \neq A \in H \subset V$. Then $0 \neq \log(A) \in \mathbb{B}/2$, so that $m \log(A) \in \mathbb{B} \setminus (\mathbb{B}/2)$ for some $m \in \mathbb{Z}^+$. Then

$$\exp(m \log(A)) = \exp(\log(A))^m = A^m \in H \subset V \subset \exp(\mathbb{B}/2),$$

but also

$$\exp(m \log(A)) \in \exp(\mathbb{B} \setminus (\mathbb{B}/2)) = \exp(\mathbb{B}) \setminus \exp(\mathbb{B}/2);$$

this is a contradiction. □

Remark 3.2.3. Actually, it is shown above that Lie groups have no small subgroups; compactness played no role in this part of the proof.

Exercise 3.2.4. Use the Peter–Weyl Theorem 2.5.13 to provide an alternative proof for Theorem 3.2.1. Hint: For each $x \in G \setminus V$ there exists $\phi_x \in \mathrm{HOM}(G, \mathrm{U}(n_x))$ such that $x \notin \mathrm{Ker}(\phi_x)$, because...

3.3 Lie groups and Lie algebras

This section deals with representation theory of Lie groups. We introduce Lie algebras, which sometimes still bear the archaic label "*infinitesimal groups*", quite adequately describing their essence: a Lie algebra is a sort of "locally linearized" version of a Lie group.

Definition 3.3.1. A $\mathbb{K}$-*Lie algebra* is a $\mathbb{K}$-vector space V endowed with a bilinear mapping $((a, b) \mapsto [a, b]_V = [a, b]) : V \times V \to V$ satisfying

$$[a, a] = 0 \quad \text{and} \quad [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$

for every $a, b, c \in V$; the second identity is called the *Jacobi identity.* Notice that here $[a, b] = -[b, a]$ for every $a, b \in V$. A vector subspace $W \subset V$ of a Lie algebra V is called a *Lie subalgebra* if $[a, b] \in W$ for every $a, b \in W$ (and thus W is a Lie algebra in its own right). A linear mapping $A : V_1 \to V_2$ between Lie algebras V_1, V_2 is called a *Lie algebra homomorphism* if $[Aa, Ab]_{V_2} = A[a, b]_{V_1}$ for every $a, b \in V_1$.

Example. 1. For a $\mathbb{K}$-vector space V, the trivial Lie product $[a, b] := 0$ begets a trivial Lie algebra.

2. A $\mathbb{K}$-algebra $\mathcal{A}$ can be endowed with the canonical Lie product

$$(a, b) \mapsto [a, b] := ab - ba;$$

this Lie algebra is denoted by $\mathrm{Lie}_{\mathbb{K}}(\mathcal{A})$. Important special cases of such Lie algebras are

$$\mathrm{Lie}_{\mathbb{K}}(\mathbb{C}^{n\times n}) \cong \mathrm{Lie}_{\mathbb{K}}(\mathrm{End}(\mathbb{C}^n)), \quad \mathrm{Lie}_{\mathbb{K}}(\mathrm{End}(V)), \quad \mathrm{Lie}_{\mathbb{K}}(\mathcal{L}(X)),$$

where X is a normed space and $\mathrm{End}(V)$ is the algebra of linear operators $V \to V$ on a vector space V. For short, let

$$\mathfrak{gl}(V) := \mathrm{Lie}_{\mathbb{R}}(\mathrm{End}(V)).$$

3. Let $\mathcal{D}(\mathcal{A})$ be the $\mathbb{K}$-vector space of *derivations* of a $\mathbb{K}$-algebra $\mathcal{A}$; that is, $D \in \mathcal{D}(\mathcal{A})$ is a linear mapping $\mathcal{A} \to \mathcal{A}$ satisfying the *Leibniz property*

$$D(ab) = D(a)\ b + a\ D(b)$$

for every $a, b \in \mathcal{A}$. Then $\mathcal{D}(\mathcal{A})$ has a Lie algebra structure given by $[D, E] := DE - ED$. An important special case is $\mathcal{A} = C^\infty(M)$, where M is a C^∞-manifold; if $C^\infty(M)$ is endowed with the usual test function topology then $D \in \mathcal{D}(C^\infty(M))$ is continuous if and only if it is a linear first-order partial differential operator with smooth coefficients (alternatively, a smooth vector field on M).

4. The following theorem introduces the *Lie algebra of a Lie group.*

Theorem 3.3.2. *Let $G < \mathrm{GL}(n, \mathbb{C})$ be closed. The $\mathbb{R}$-vector space*

$$\mathfrak{Lie}(G) = \mathfrak{g} := \left\{ X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R} :\ \exp(tX) \in G \right\}$$

is a Lie subalgebra of the $\mathbb{R}$-Lie algebra $\mathrm{Lie}_{\mathbb{R}}(\mathbb{C}^{n\times n}) \cong \mathfrak{gl}(\mathbb{C}^n)$; $\mathfrak{g}$ is called the Lie algebra of *G.*

Proof. Let $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Trivially, $\exp(t\lambda X) \in G$ for every $t \in \mathbb{R}$, yielding $\lambda X \in \mathfrak{g}$. Since G is closed and exp is continuous,

$$G \ni (\exp(tX/m)\ \exp(tY/m))^m \xrightarrow[m\to\infty]{} \exp\left(t(X+Y)\right) \in G$$
$$G \ni \{\exp(tX/m), \exp(tY/m)\}^{m^2} \xrightarrow[m\to\infty]{} \exp\left(t[X,Y]\right) \in G$$

by the known properties of the exponential map. Thereby $X+Y, [X,Y] \in \mathfrak{g}$. □

Definition 3.3.3. Let G be a linear Lie group and $\mathfrak{g} = \mathfrak{Lie}(G)$. Notice that

$$\mathrm{HOM}(\mathbb{R}, G) = \{t \mapsto \exp(tX) \mid X \in \mathfrak{g}\}.$$

Mapping $(X \mapsto \exp(X)) : \mathfrak{g} \to G$ is a diffeomorphism in a small neighbourhood of $0 \in \mathfrak{g}$. Hence, given a vector space basis for $\mathfrak{g} \cong \mathbb{R}^k$, a small neighbourhood of $\exp(0) = I \in G$ is endowed with so called *exponential coordinates*. The *dimension of* G is $\dim(G) := \dim(\mathfrak{g}) = k$. If G is compact and connected then $\exp(\mathfrak{g}) = G$, so that the exponential map may "wrap $\mathfrak{g}$ around G"; we do not prove this.

Remark 3.3.4. Informally speaking, if $X, Y \in \mathfrak{g}$ are "near $0 \in \mathfrak{g}$", $x := \exp(X)$ and $y := \exp(Y)$ then $x, y \in G$ are "near $I \in G$" and

$$\exp(X+Y) \approx xy, \quad \exp([X,Y]) \approx \{x,y\} = xyx^{-1}y^{-1}.$$

In a sense, the Lie algebra $\mathfrak{g}$ is the "infinitesimal linearized G nearby $I \in G$".

Remark 3.3.5. Interpreting the Lie algebra: The Lie algebra $\mathfrak{g}$ can be identified with the tangent space of G at $I \in G$. Using left-translations (resp. right-translations), $\mathfrak{g}$ can be identified with the set of left-invariant (resp. right-invariant) vector fields on G, and vector fields have a natural interpretation as first-order partial differential operators on G: For $x \in G$, $X \in \mathfrak{g}$ and $f \in C^\infty(G)$, define

$$L_X f(x) := \frac{\mathrm{d}}{\mathrm{d}t} f\left(x\ \exp(tX)\right)|_{t=0},$$
$$R_X f(x) := \frac{\mathrm{d}}{\mathrm{d}t} f\left(\exp(tX)\ x\right)|_{t=0}.$$

Then $\pi_L(y) L_X f = L_X \pi_L(y) f$ and $\pi_R(y) R_X f = R_X \pi_R(y) f$ for every $y \in G$, where π_L, π_R are the left and right regular representations of G, respectively.

Definition 3.3.6. Notations: $\mathfrak{gl}(n,\mathbb{K}) = \mathfrak{Lie}(\mathrm{GL}(n,\mathbb{K}))$, $\mathfrak{sl}(n,\mathbb{K}) = \mathfrak{Lie}(\mathrm{SL}(n,\mathbb{K}))$, $\mathfrak{o}(n) = \mathfrak{Lie}(\mathrm{O}(n))$, $\mathfrak{so}(n) = \mathfrak{Lie}(\mathrm{SO}(n))$, $\mathfrak{u}(n) = \mathfrak{Lie}(\mathrm{U}(n))$, $\mathfrak{su}(n) = \mathfrak{Lie}(\mathrm{SU}(n))$ etc.

Exercise 3.3.7. Calculate the dimensions of the linear Lie groups mentioned in Definition 3.3.6.

Proposition 3.3.8. *Let G, H be linear Lie groups having the respective Lie algebras $\mathfrak{g}, \mathfrak{h}$. Let $\psi \in \mathrm{HOM}(G,H)$. Then for every $X \in \mathfrak{g}$ there exists a unique $Y \in \mathfrak{h}$ such that $\psi(\exp(tX)) = \exp(tY)$ for every $t \in \mathbb{R}$.*

Proof. Let $X \in \mathfrak{g}$. Then $\phi := (t \mapsto \psi(\exp(tX))) : \mathbb{R} \to H$ is a continuous homomorphism, so that $\phi = (t \mapsto \exp(tY))$, where $Y = \phi'(0) \in \mathfrak{h}$. □

Proposition 3.3.9. *Let F, G, H be closed subgroups of $\mathrm{GL}(n,\mathbb{C})$, with the respective Lie algebras $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$. Then*
(a) $H < G \Rightarrow \mathfrak{h} \subset \mathfrak{g}$,
(b) *the Lie algebra of $F \cap G$ is $\mathfrak{f} \cap \mathfrak{g}$,*
(c) *the Lie algebra $\mathfrak{c}_I$ of the component $C_I < G$ of the neutral element I is $\mathfrak{g}$.*

Proof.
(a): If $H < G$ and $X \in \mathfrak{h}$ then $\exp(tX) \in H \subset G$ for every $t \in \mathbb{R}$, so that $X \in \mathfrak{g}$.
(b): Let $\mathfrak{e}$ be the Lie algebra of $F \cap G$. By (a), $\mathfrak{e} \subset \mathfrak{f} \cap \mathfrak{g}$. If $X \in \mathfrak{f} \cap \mathfrak{g}$ then $\exp(tX) \in F \cap G$ for every $t \in \mathbb{R}$, so that $X \in \mathfrak{a}$. Hence $\mathfrak{e} = \mathfrak{f} \cap \mathfrak{g}$.
(c): By (a), $\mathfrak{c}_I \subset \mathfrak{g}$. Let $X \in \mathfrak{g}$. Now the connectedness of $\mathbb{R}$ and the continuity of $t \mapsto \exp(tX)$ imply the connectedness of $\{\exp(tX) : t \in \mathbb{R}\} \ni \exp(0) = I$. Thereby $\{\exp(tX) : t \in \mathbb{R}\} \subset C_I$, so that $X \in \mathfrak{c}_I$. □

Example. Let us compute the Lie algebra $\mathfrak{sl}(n,\mathbb{K})$ of

$$\mathrm{SL}(n,\mathbb{K}) = \{A \in \mathrm{GL}(n,\mathbb{K}) \mid \mathrm{Det}(A) = 1\}.$$

Now

$$\begin{aligned}\mathfrak{sl}(n,\mathbb{K}) &:= \left\{X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R} :\ \exp(tX) \in \mathrm{SL}(n,\mathbb{K})\right\} \\ &= \left\{X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R} :\ \exp(tX) \in \mathbb{K}^{n\times n},\ \mathrm{Det}(\exp(tX)) = 1\right\}.\end{aligned}$$

Let $\{\lambda_j\}_{j=1}^n \subset \mathbb{C}$ be the set of the eigenvalues of $X \in \mathbb{K}^{n\times n}$. The characteristic polynomial $(z \mapsto \mathrm{Det}(zI - X)) : \mathbb{C} \to \mathbb{C}$ of X satisfies

$$\begin{aligned} \mathrm{Det}(zI - X) &= \prod_{j=1}^n (z - \lambda_j) \\ &= z^n - z^{n-1} \sum_{j=1}^n \lambda_j + \ldots + (-1)^n \prod_{j=1}^n \lambda_j \\ &= z^n - z^{n-1} \mathrm{Tr}(X) + \ldots + (-1)^n \mathrm{Det}(X), \end{aligned}$$

We know that X is similar to an upper triangular matrix $Y = PXP^{-1}$ for some $P \in \mathrm{GL}(n, \mathbb{K})$. Since

$$\begin{aligned} \mathrm{Det}(zI - PXP^{-1}) &= \mathrm{Det}(P(zI - X)P^{-1}) \\ &= \mathrm{Det}(P)\ \mathrm{Det}(zI - X)\ \mathrm{Det}(P^{-1}) \\ &= \mathrm{Det}(zI - X), \end{aligned}$$

the eigenvalues of X and Y are the same, and they are on the diagonal of Y. Evidently, $\{\mathrm{e}^{\lambda_j}\}_{j=1}^n \subset \mathbb{C}$ is the set of the eigenvalues of both $\exp(Y)$ and $\exp(X) = P^{-1} \exp(Y) P$. Since the determinant is the product of the eigenvalues and the trace is the sum of the eigenvalues, we have

$$\mathrm{Det}(\exp(X)) = \prod_{j=1}^n \mathrm{e}^{\lambda_j} = \mathrm{e}^{\sum_{j=1}^n \lambda_j} = \mathrm{e}^{\mathrm{Tr}(X)}.$$

Therefore $X \in \mathfrak{sl}(n, \mathbb{K})$ if and only if $\mathrm{Tr}(X) = 0$ and $\exp(tX) \in \mathbb{K}^{n\times n}$ for every $t \in \mathbb{R}$. Thus

$$\mathfrak{sl}(n, \mathbb{K}) = \left\{ X \in \mathbb{K}^{n\times n} \mid \mathrm{Tr}(X) = 0 \right\}$$

as the reader may check.

Remark 3.3.10. Differentiating homomorphisms: Next we ponder the relationship between Lie group and Lie algebra homomorphisms. Let G, H be linear Lie groups with respective Lie algebras $\mathfrak{g}, \mathfrak{h}$. The *differential homomorphism of* $\psi \in \mathrm{HOM}(G, H)$ is $\psi' = \mathfrak{Lie}(\psi) : \mathfrak{g} \to \mathfrak{h}$ defined by

$$\psi'(X) := \frac{\mathrm{d}}{\mathrm{d}t} \psi(\exp(tX))|_{t=0};$$

this is well-defined since $f := (t \mapsto \psi(\exp(tX))) \in \mathrm{HOM}(\mathbb{R}, H)$ is of the form $t \mapsto \exp(tY)$ for some $Y \in \mathfrak{h}$. Moreover, $Y = f'(0) = \psi'(X)$ holds, so that

$$\psi(\exp(tX)) = \exp(t\psi'(X)).$$

Theorem 3.3.11. *Let F, G, H be linear Lie groups with respective Lie algebras $\mathfrak{f}, \mathfrak{g}, \mathfrak{h}$. Let $\phi \in \mathrm{HOM}(F, G)$ and $\psi \in \mathrm{HOM}(G, H)$. Mapping $\psi' : \mathfrak{g} \to \mathfrak{h}$ defined above in Remark 3.3.10 is a Lie algebra homomorphism. Moreover,*

$$(\psi \circ \phi)' = \psi' \phi' \quad \text{and} \quad \mathrm{Id}_G' = \mathrm{Id}_{\mathfrak{g}},$$

where $\mathrm{Id}_G = (x \mapsto x) : G \to G$ *and* $\mathrm{Id}_{\mathfrak{g}} = (X \mapsto X) : \mathfrak{g} \to \mathfrak{g}$.

Proof. Let $X, Y \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} \psi'(\lambda X) &= \frac{\mathrm{d}}{\mathrm{d}t} \psi(\exp(t\lambda X))|_{t=0} \\ &= \lambda \frac{\mathrm{d}}{\mathrm{d}t} \psi(\exp(tX))|_{t=0} \\ &= \lambda \psi'(X). \end{aligned}$$

If $t \in \mathbb{R}$ then

$$\begin{aligned} \exp(t\psi'(X+Y)) &= \psi(\exp(tX + tY)) \\ &= \psi\left(\lim_{m\to\infty} (\exp(tX/m)\ \exp(tY/m))^m\right) \\ &= \lim_{m\to\infty} (\psi(\exp(tX/m))\ \psi(\exp(tY/m)))^m \\ &= \lim_{m\to\infty} (\exp(t\psi'(X)/m)\ \exp(t\psi'(Y)/m))^m \\ &= \exp(t(\psi'(X) + \psi'(Y))), \end{aligned}$$

so that $t\psi'(X+Y) = t(\psi'(X) + \psi'(Y))$ for "small enough" $|t|$, as we recall that exp is injective in a small neighbourhood of $0 \in \mathfrak{g}$. Consequently, $\psi' : \mathfrak{g} \to \mathfrak{h}$ is linear. Next,

$$\begin{aligned} \exp(t\psi'([X,Y])) &= \psi(\exp(t[X,Y])) \\ &= \psi\left(\lim_{m\to\infty} \{\exp(tX/m), \exp(tY/m)\}^{m^2}\right) \\ &= \lim_{m\to\infty} \{\exp(t\psi'(X)/m), \exp(t\psi'(Y)/m)\}^{m^2} \\ &= \exp(t[\psi'(X), \psi'(Y)]), \end{aligned}$$

so that we get $\psi'([X,Y]) = [\psi'(X), \psi'(Y)]$. Thus $\psi' : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.

If $Z \in \mathfrak{f}$ then

$$\begin{aligned}(\psi \circ \phi)'(Z) &= \frac{\mathrm{d}}{\mathrm{d}t} \psi\left(\phi(\exp(tZ))\right)|_{t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \psi\left(\exp(t\phi'(Z))\right)|_{t=0} \\ &= \psi'(\phi'(Z)).\end{aligned}$$

Finally, $\frac{\mathrm{d}}{\mathrm{d}t} \exp(tX)|_{t=0} = X$, yielding $\mathrm{Id}'_G = \mathrm{Id}_{\mathfrak{g}}$. □

Remark 3.3.12. Now we know that a continuous Lie group homomorphism ψ can naturally be "linearized" to get a Lie algebra homomorphism ψ', so that we have a commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\psi} & H, \\ {\scriptstyle\exp}\uparrow & & \uparrow{\scriptstyle\exp} \\ \mathfrak{g} & \xrightarrow{\psi'} & \mathfrak{h}. \end{array}$$

What if we are given a Lie algebra homomorphism $f : \mathfrak{g} \to \mathfrak{h}$, does there exists $\phi \in \mathrm{HOM}(G, H)$ such that $\phi' = f$? The answer is affirmative for simply connected G; this is a demanding but feasible exercise. But in general there can be other sorts of Lie algebra homomorphisms, too.

Notice also that isomorphic linear Lie groups must have isomorphic Lie algebras.

Lemma 3.3.13. *Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group G, and*

$$S := \left\{ \exp(X_1) \cdots \exp(X_m) \mid m \in \mathbb{Z}^+,\ \{X_j\}_{j=1}^m \subset \mathfrak{g} \right\}.$$

Then $S = C_I$, the component of $I \in G$.

Proof. Now $S < G$ is path-connected, since

$$(t \mapsto \exp(tX_1) \cdots \exp(tX_m)) : [0, 1] \to S$$

is continuous, connecting $I \in S$ to the point $\exp(X_1) \cdots \exp(X_m) \in S$. For a "small enough" neighbourhood $U \subset \mathfrak{g}$ of $0 \in \mathfrak{g}$, we have a homeomorphism $(X \mapsto \exp(X)) : U \to \exp(U)$. Because of

$$\exp(X_1) \cdots \exp(X_m) \ \in\ \exp(X_1) \cdots \exp(X_m) \exp(U) \ \subset\ S,$$

it follows that $S < G$ is open. But open subgroups are always closed, as the reader easily verifies. Thus $S \ni I$ is connected, closed and open, so that $S = C_I$, the component of $I \in G$. $\square$

Corollary 3.3.14. *Let G, H be linear Lie groups and $\phi, \psi \in \mathrm{HOM}(G, H)$. Then:*
(a) $\mathfrak{Lie}(\mathrm{Ker}(\psi)) = \mathrm{Ker}(\psi')$.
(b) *If G is connected and $\phi' = \psi'$ then $\phi = \psi$.*
(c) *Let H be connected; then ψ' is surjective if and only if ψ is surjective.*

Proof.
(a) $\mathrm{Ker}(\psi) < G < \mathrm{GL}(n, \mathbb{C})$ is a closed subgroup, since ψ is a continuous homomorphism. Thereby

$$\begin{aligned}
\mathfrak{Lie}(\mathrm{Ker}(\psi)) &= \left\{X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R}:\ \exp(tX) \in \mathrm{Ker}(\psi)\right\} \\
&= \left\{X \in \mathbb{C}^{n\times n} \mid \forall t \in \mathbb{R}:\ \exp(t\psi'(X)) = \psi(\exp(tX)) = I\right\} \\
&= \left\{X \in \mathbb{C}^{n\times n} \mid \psi'(X) = 0\right\} \\
&= \mathrm{Ker}(\psi').
\end{aligned}$$

(b) Take $A \in G$. Then $A = \exp(X_1)\cdots\exp(X_m)$ for some $\{X_j\}_{j=1}^m \subset \mathfrak{g}$ by Lemma 3.3.13, so that

$$\begin{aligned}
\phi(A) &= \exp\left(\phi'(X_1)\right)\cdots\exp\left(\phi'(X_m)\right) \\
&= \exp\left(\psi'(X_1)\right)\cdots\exp\left(\psi'(X_m)\right) \\
&= \psi(A).
\end{aligned}$$

(c) Suppose $\psi' : \mathfrak{g} \to \mathfrak{h}$ is surjective. Let $B \in H$. Now H is connected, so that Lemma 3.3.13 says that $B = \exp(Y_1)\cdots\exp(Y_m)$ for some $\{Y_j\}_{j=1}^m \subset \mathfrak{h}$. Exploit the surjectivity of ψ' to obtain $X_j \in \mathfrak{g}$ such that $\psi'(X_j) = Y_j$. Then

$$\begin{aligned}
\psi\left(\exp(X_1)\cdots\exp(X_m)\right) &= \psi\left(\exp(X_1)\right)\cdots\psi\left(\exp(X_m)\right) \\
&= \exp(Y_1)\cdots\exp(Y_m) \\
&= B.
\end{aligned}$$

Conversely, suppose $\psi : G \to H$ is surjective. Trivially, $\psi'(0) = 0 \in \mathfrak{h}$; let $0 \neq Y \in \mathfrak{h}$. Let $r := \ln(2)/\|Y\|$; recall that if $|t| < r$ then $\log(\exp(tY)) = tY$. The surjectivity of ψ guarantees that for every $t \in \mathbb{R}$ there exists $A_t \in G$ such that $\psi(A_t) = \exp(tY)$. The set $R := \{A_t :\ 0 < t < r\}$ is uncountable, so that it has an accumulation point $x \in \mathbb{C}^{n\times n}$; and $x \in G$, because $R \subset G$ and $G \subset \mathbb{C}^{n\times n}$ is closed. Let $\varepsilon > 0$. Then there exist $s, t \in]0, r[$ such that $s \neq t$ and

$$\|A_s - x\| < \varepsilon, \quad \|A_t - x\| < \varepsilon, \quad \left\|A_s^{-1} - x^{-1}\right\| < \varepsilon.$$

Thereby

$$\begin{aligned} \left\|A_s^{-1}A_t - I\right\| &= \left\|A_s^{-1}(A_t - A_s)\right\| \\ &\leq \left\|A_s^{-1}\right\| (\|A_t - x\| + \|x - A_s\|) \\ &\leq \left(\|x^{-1}\| + \varepsilon\right)\ 2\varepsilon. \end{aligned}$$

Hence we demand $\|A_s^{-1}A_t - I\| < 1$ and $\|\psi(A_s^{-1}A_t) - I\| < 1$, yielding

$$\psi(A_s^{-1}A_t) = \psi(A_s)^{-1}\psi(A_t) = \exp((t-s)Y).$$

Consequently

$$\psi'\left(\log(A_s^{-1}A_t)\right) = (t-s)Y.$$

Therefore $\psi'\left(\frac{1}{t-s}\log(A_s^{-1}A_t)\right) = Y$. □

Definition 3.3.15. The *adjoint representation of a linear Lie group G* is mapping $\mathrm{Ad} \in \mathrm{HOM}(G, \mathrm{Aut}(\mathfrak{g}))$ defined by

$$\mathrm{Ad}(A)X := AXA^{-1} \quad (A \in G,\ X \in \mathfrak{g}).$$

Indeed, $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$, because

$$\exp\left(t\mathrm{Ad}(A)X\right) = \exp\left(tAXA^{-1}\right) = A\exp\left(tX\right)A^{-1}$$

belongs to G if $A \in G$, $X \in \mathfrak{g}$ and $t \in \mathbb{R}$. It is a homomorphism, since

$$\mathrm{Ad}(AB)X = ABXB^{-1}A^{-1} = \mathrm{Ad}(A)(BXB^{-1}) = \mathrm{Ad}(A)\ \mathrm{Ad}(B)\ X,$$

and it is trivially continuous.

Exercise 3.3.16. Let $\mathfrak{g}$ be a Lie algebra. Consider $\mathrm{Aut}(\mathfrak{g})$ as a linear Lie group. Show that $\mathfrak{Lie}(\mathrm{Aut}(\mathfrak{g}))$ and $\mathfrak{gl}(\mathfrak{g})$ are isomorphic as Lie algebras.

Definition 3.3.17. The *adjoint representation of the Lie algebra $\mathfrak{g}$ of a linear Lie group G* is the differential representation

$$\mathrm{ad} = \mathrm{Ad}' : \mathfrak{g} \to \mathfrak{Lie}(\mathrm{Aut}(\mathfrak{g})) \cong \mathfrak{gl}(\mathfrak{g}),$$

that is $\mathrm{ad}(X) := \mathrm{Ad}'(X)$, so that

$$\begin{aligned}
\mathrm{ad}(X)Y &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(tX)Y\exp(-tX)\right)|_{t=0} \\
&= \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)\right)Y\exp(-tX) + \exp(tX)Y\frac{\mathrm{d}}{\mathrm{d}t}\exp(-tX)\right)|_{t=0} \\
&= XY - YX \\
&= [X,Y].
\end{aligned}$$

Remark 3.3.18. Higher order partial differential operators: Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group G. Next we construct a natural associative algebra $\mathcal{U}(\mathfrak{g})$ generated by $\mathfrak{g}$ modulo an ideal, enabling embedding $\mathfrak{g}$ into $\mathcal{U}(\mathfrak{g})$. Recall that $\mathfrak{g}$ can be interpreted as the vector space of first-order left (or right) -translation invariant partial differential operators on G. Consequently, $\mathcal{U}(\mathfrak{g})$ can be interpreted as the vector space of finite-order left (or right) -translation invariant partial differential operators on G.

Definition 3.3.19. Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Let

$$\mathcal{T} := \bigoplus_{m=0}^{\infty} \otimes^m \mathfrak{g}$$

be the tensor product algebra of $\mathfrak{g}$, where $\otimes^m\mathfrak{g}$ denotes the m-fold tensor product $\mathfrak{g}\otimes\cdots\otimes\mathfrak{g}$; that is, $\mathcal{T}$ is the linear span of the elements of the form

$$\lambda_{00}\mathbf{1} + \sum_{m=1}^{M}\sum_{k=1}^{K_m}\lambda_{mk}\ X_{mk1}\otimes\cdots\otimes X_{mkm},$$

where $\mathbf{1}$ is the formal unit element of $\mathcal{T}$, $\lambda_{mj}\in\mathbb{K}$, $X_{mkj}\in\mathfrak{g}$ and $M, K_m \in \mathbb{Z}^+$; the product of $\mathcal{T}$ is begotten by the tensor product, i.e.

$$(X_1\otimes\cdots\otimes X_p)(Y_1\otimes\cdots\otimes Y_q) := X_1\otimes\cdots\otimes X_p\otimes Y_1\otimes\cdots\otimes Y_q$$

is extended to a unique bilinear mapping $\mathcal{T}\times\mathcal{T}\to\mathcal{T}$. Let $\mathcal{J}$ be the (two-sided) ideal in $\mathcal{T}$ spanned by the set

$$\mathcal{O} := \{X\otimes Y - Y\otimes X - [X,Y] :\ X,Y\in\mathfrak{g}\};$$

i.e. $\mathcal{J}\subset\mathcal{T}$ is the smallest vector subspace such that $\mathcal{O}\subset\mathcal{J}$ and $DE, ED\in\mathcal{J}$ for every $D\in\mathcal{J}$ and $E\in\mathcal{T}$ (in a sense, $\mathcal{J}$ is a "huge zero" in $\mathcal{T}$). The quotient algebra

$$\mathcal{U}(\mathfrak{g}) := \mathcal{T}/\mathcal{J}$$

is called the *universal enveloping algebra of* $\mathfrak{g}$.

Let $\iota : \mathcal{T} \to \mathcal{U}(\mathfrak{g}) = \mathcal{T}/\mathcal{J}$ be the canonical projection $t \mapsto t + \mathcal{J}$. A natural interpretation is that $\mathfrak{g} \subset \mathcal{T}$. The restricted mapping $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is called the *canonical mapping of* $\mathfrak{g}$. It is easy to verify that $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{Lie}_{\mathbb{K}}(\mathcal{U}(\mathfrak{g}))$ is a Lie algebra homomorphism: it is linear and

$$\begin{aligned}
\iota|_{\mathfrak{g}}([X,Y]) &= \iota([X,Y]) \\
&= \iota(X \otimes Y - Y \otimes X) \\
&= \iota(X)\iota(Y) - \iota(Y)\iota(X) \\
&= \iota|_{\mathfrak{g}}(X)\iota|_{\mathfrak{g}}(Y) - \iota|_{\mathfrak{g}}(Y)\iota|_{\mathfrak{g}}(X) \\
&= [\iota|_{\mathfrak{g}}(X), \iota|_{\mathfrak{g}}(Y)].
\end{aligned}$$

Theorem 3.3.20. (Universality of $\mathcal{U}(\mathfrak{g})$.) *Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra, $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ its canonical mapping, $\mathcal{A}$ an associative $\mathbb{K}$-algebra, and*

$$\sigma : \mathfrak{g} \to \mathrm{Lie}_{\mathbb{K}}(\mathcal{A})$$

a Lie algebra homomorphism. Then there exists an algebra homomorphism

$$\tilde{\sigma} : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$$

satisfying $\tilde{\sigma}\,(\iota|_{\mathfrak{g}}(X)) = \sigma(X)$ for every $X \in \mathfrak{g}$, i.e.

$$\begin{array}{ccc}
\mathcal{U}(\mathfrak{g}) & \xrightarrow{\tilde{\sigma}} & \mathcal{A} \\
{\scriptstyle \iota|_{\mathfrak{g}}}\uparrow & & \Vert \\
\mathfrak{g} & \xrightarrow{\sigma} & \mathrm{Lie}_{\mathbb{K}}(\mathcal{A}).
\end{array}$$

Proof. Let us define a linear mapping $\sigma_0 : \mathcal{T} \to \mathcal{A}$ by

$$\sigma_0(X_1 \otimes \cdots \otimes X_m) := \sigma(X_1) \cdots \sigma(X_m).$$

Then $\sigma_0(\mathcal{J}) = \{0\}$, since

$$\begin{aligned}
\sigma_0(X \otimes Y - Y \otimes X - [X,Y]) &= \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) - \sigma([X,Y]) \\
&= \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X) - [\sigma(X), \sigma(Y)] \\
&= 0.
\end{aligned}$$

Hence if $t, u \in \mathcal{T}$ and $t - u \in \mathcal{J}$ then $\sigma_0(t) = \sigma_0(u)$. Thereby we may define $\tilde{\sigma} := (t + \mathcal{J} \mapsto \sigma_0(t)) : \mathcal{U}(\mathfrak{g}) \to \mathcal{A}$. Finally, it is clear that $\tilde{\sigma}$ is an algebra homomorphism making the diagram above commute. □

Corollary 3.3.21. (The Ado–Iwasawa Theorem.) *Let $\mathfrak{g}$ be the Lie algebra of a linear Lie group G. Then the canonical mapping $\iota|_{\mathfrak{g}} : \mathfrak{g} \to \mathcal{U}(\mathfrak{g})$ is injective.*

Proof. Let $\sigma = (X \mapsto X) : \mathfrak{g} \to \mathfrak{gl}(n,\mathbb{C})$. Due to the universality of $\mathcal{U}(\mathfrak{g})$ there exists an $\mathbb{R}$-algebra homomorphism $\tilde{\sigma} : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}^{n\times n}$ such that $\sigma(X) = \tilde{\sigma}\left(\iota|_{\mathfrak{g}}(X)\right)$ for every $X \in G$. Then $\iota|_{\mathfrak{g}}$ is injective because σ is injective. □

Remark 3.3.22. By the Ado–Iwasawa Theorem 3.3.21, the Lie algebra $\mathfrak{g}$ of a linear Lie group can be considered as a Lie subalgebra of $\mathrm{Lie}_{\mathbb{R}}\left(\mathcal{U}(\mathfrak{g})\right)$.

Remark 3.3.23. Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra. Let us define the linear mapping $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ by $\mathrm{ad}(X)Z := [X,Z]$. Since

$$\begin{aligned} 0 &= [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \\ &= [[X,Y],Z] - ([X,[Y,Z]] - [Y,[X,Z]]) \\ &= \mathrm{ad}([X,Y])Z - [\mathrm{ad}(X),\mathrm{ad}(Y)]Z; \end{aligned}$$

we notice that

$$\mathrm{ad}([X,Y]) = [\mathrm{ad}(X),\mathrm{ad}(Y)],$$

i.e. ad is a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. The *Killing form* of the Lie algebra $\mathfrak{g}$ is the bilinear mapping $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$, defined by

$$B(X,Y) := \mathrm{Tr}\left(\mathrm{ad}(X)\ \mathrm{ad}(Y)\right)$$

(recall that by Exercise 5.8.4, on a finite-dimensional vector space the trace can be defined independent of any inner product). A ($\mathbb{R}$- or $\mathbb{C}$-)Lie algebra $\mathfrak{g}$ is *semisimple* if its Killing form is *non-degenerate*, i.e. if

$$\forall X \in \mathfrak{g} \setminus \{0\}\ \exists Y \in \mathfrak{g} :\ B(X,Y) \neq 0;$$

equivalently, B is non-degenerate if $\left(B(X_i,X_j)\right)_{i,j=1}^n \in \mathrm{GL}(n,\mathbb{K})$, where $\{X_j\}_{j=1}^n \subset \mathfrak{g}$ is a vector space basis. A connected linear Lie group is called *semisimple* if its Lie algebra is semisimple.

Remark 3.3.24. Since $\mathrm{Tr}(ab) = \mathrm{Tr}(ba)$, we have

$$B(X,Y) = B(Y,X).$$

We have also

$$B(X,[Y,Z]) = B([X,Y],Z),$$

because

$$\mathrm{Tr}(a(bc-cb)) = \mathrm{Tr}(abc) - \mathrm{Tr}(acb) = \mathrm{Tr}(abc) - \mathrm{Tr}(bac) = \mathrm{Tr}((ab-ba)c)$$

yields

$$\begin{aligned} B(X,[Y,Z]) &= \mathrm{Tr}\left(\mathrm{ad}(X)\ \mathrm{ad}([Y,Z])\right) \\ &= \mathrm{Tr}\left(\mathrm{ad}(X)\ [\mathrm{ad}(Y),\mathrm{ad}(Z)]\right) \\ &= \mathrm{Tr}\left([\mathrm{ad}(X),\mathrm{ad}(Y)]\ \mathrm{ad}(Z)\right) \\ &= \mathrm{Tr}\left(\mathrm{ad}([X,Y])\ \mathrm{ad}(Z)\right) \\ &= B([X,Y],Z). \end{aligned}$$

It can be proven that the Killing form of a compact Lie group is negative semi-definite, i.e. $B(X,X) \leq 0$. On the other hand, if the Killing form of a Lie group is negative definite, i.e. $X \neq 0 \ \Rightarrow \ B(X,X) < 0$, then the group is compact.

Definition 3.3.25. Let $\mathfrak{g}$ be a semisimple $\mathbb{K}$-Lie algebra with a vector space basis $\{X_j\}_{j=1}^n \subset \mathfrak{g}$. Let $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ be the Killing form of $\mathfrak{g}$, and define $R = (R_{ij})_{i,j=1}^n := (B(X_i,X_j))_{i,j=1}^n$. Let us write $R^{-1} = \left((R^{-1})_{ij}\right)_{i,j=1}^n$. Then the *Casimir element* $\Omega \in \mathcal{U}(\mathfrak{g})$ *of* $\mathfrak{g}$ is defined by

$$\Omega := \sum_{i,j=1}^n (R^{-1})_{ij} X_i X_j.$$

Theorem 3.3.26. *The Casimir element of a finite-dimensional semisimple $\mathbb{K}$-Lie algebra $\mathfrak{g}$ is independent of the choice of the vector space basis $\{X_j\}_{j=1}^n \subset \mathfrak{g}$. Moreover,*

$$\forall D \in \mathcal{U}(\mathfrak{g}) : \ D\Omega = \Omega D.$$

Proof. To simplify notation, we consider only the case $\mathbb{K} = \mathbb{R}$. Let $\{Y_i\}_{i=1}^n \subset \mathfrak{g}$ be a vector space basis of $\mathfrak{g}$. Then there exists $A = (A_{ij})_{i,j=1}^n \in \mathrm{GL}(n,\mathbb{R})$ such that

$$\left\{ Y_i := \sum_{j=1}^n A_{ij} X_j \right\}_{i=1}^n .$$

Then

$$\begin{aligned} S &:= (B(Y_i, Y_j))_{i,j=1}^n \\ &= \left(B\Big(\sum_{k=1}^n A_{ik} X_k, \sum_{l=1}^n A_{jl} X_l\Big) \right)_{i,j=1}^n \\ &= \left(\sum_{k,l=1}^n A_{ik}\ B(X_k, X_l)\ A_{jl} \right)_{i,j=1}^n \\ &= ARA^{\mathrm{T}}; \end{aligned}$$

hence

$$S^{-1} = ((S^{-1})_{ij})_{i,j=1}^n = (A^{\mathrm{T}})^{-1} R^{-1} A^{-1}.$$

Let us now compute the Casimir element of $\mathfrak{g}$ with respect to the basis $\{Y_j\}_{j=1}^n$:

$$\begin{aligned} \sum_{i,j=1}^n (S^{-1})_{ij} Y_i Y_j &= \sum_{i,j=1}^n (S^{-1})_{ij} \sum_{k=1}^n A_{ik} X_k \sum_{l=1}^n A_{jl} X_l \\ &= \sum_{k,l=1}^n X_k X_l \sum_{i,j=1}^n A_{ik} (S^{-1})_{ij} A_{jl} \\ &= \sum_{k,l=1}^n X_k X_l \sum_{i,j=1}^n (A^{\mathrm{T}})_{ki} ((A^{\mathrm{T}})^{-1} R^{-1} A^{-1})_{ij} A_{jl} \\ &= \sum_{k,l=1}^n X_k X_l (R^{-1})_{kl}; \end{aligned}$$

thus the definition of the Casimir element does not depend of the choice of a vector space basis!

We still have to prove that Ω commutes with every $D \in \mathcal{U}(\mathfrak{g})$. First, using the Killing form, we construct a nice inner product for $\mathfrak{g}$: Let $X^i := \sum_{j=1}^n (R^{-1})_{ij} X_j$, so that $\{X^i\}_{i=1}^n$ is also a vector space basis for $\mathfrak{g}$. Then

$$\Omega = \sum_{i=1}^n X_i X^i,$$

and

$$B(X^i, X_j) = \sum_{k=1}^n (R^{-1})_{ik} B(X_k, X_j) = \sum_{k=1}^n (R^{-1})_{ik} R_{kj} = \delta_{ij}.$$

Hence $(X_i, X_j) \mapsto \langle X_i, X_j \rangle_{\mathfrak{g}} := B(X^i, X_j)$ can uniquely be extended to an inner product

$$((X, Y) \mapsto \langle X, Y \rangle_{\mathfrak{g}}) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},$$

and $\{X_i\}_{i=1}^n$ is an orthonormal basis for $\mathfrak{g}$ with respect to this inner product. For the Lie product $(x, y) \mapsto [x, y] := xy - yx$ of $\mathrm{Lie}_{\mathbb{R}}(\mathcal{U}(\mathfrak{g}))$ we have

$$[x, yz] = [x, y]z + y[x, z],$$

so that for $D \in \mathfrak{g}$ we get

$$[D, \Omega] = [D, \sum_{i=1}^n X_i X^i] = \sum_{i=1}^n \left([D, X_i]X^i + X_i[D, X^i]\right).$$

Let $c_{ij}, d_{ij} \in \mathbb{R}$ be defined by

$$[D, X_i] = \sum_{j=1}^n c_{ij} X_j, \quad [D, X^i] = \sum_{j=1}^n d_{ij} X^j.$$

Then

$$\begin{aligned}
c_{ij} &= \langle X_j, [D, X_i] \rangle_{\mathfrak{g}} \\
&= B(X^j, [D, X_i]) \\
&= B([X^j, D], X_i) \\
&= B(-[D, X^j], X_i) \\
&= B(-\sum_{k=1}^n d_{jk} X^k, X_i) \\
&= -\sum_{k=1}^n d_{jk} B(X^k, X_i) \\
&= -\sum_{k=1}^n d_{jk} \langle X_k, X_i \rangle_{\mathfrak{g}} \\
&= -d_{ji},
\end{aligned}$$

so that

$$\begin{aligned}
[D, \Omega] &= \sum_{i,j=1}^n (c_{ij} X_j X^i + d_{ij} X_i X^j) \\
&= \sum_{i,j=1}^n (c_{ij} + d_{ji}) X_j X^i \\
&= 0,
\end{aligned}$$

i.e. $D\Omega = \Omega D$ for every $D \in \mathfrak{g}$. By induction, we may prove that

$$[D_1 D_2 \cdots D_m, \Omega] \;=\; D_1[D_2 \cdots D_m, \Omega] + [D_1, \Omega] D_2 \cdots D_m \;=\; 0$$

for every $\{D_j\}_{j=1}^m \subset \mathfrak{g}$, so that $D\Omega = \Omega D$ for every $D \in \mathcal{U}(\mathfrak{g})$. □

Remark 3.3.27. The Casimir element $\Omega \in \mathcal{U}(\mathfrak{g})$ for the Lie algebra $\mathfrak{g}$ of a compact semisimple linear Lie group G can be considered as an elliptic linear second-order (left and right) translation invariant partial differential operator. In a sense, the Casimir operator is an analogy of the Euclidean Laplace operator

$$\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n).$$

Such a "Laplace operator" can be constructed for any compact Lie group G, and with it we may define Sobolev spaces on G nicely, etc.

We have seen only some basic features of the theory of Lie groups and Lie algebras. Unfortunately, we have to abandon the Lie theory, and move on to finish the course with elements of Hopf algebra theory, as presented in the next section.

Chapter 4

Hopf algebras

Instead of studying a compact group G, we may consider the algebra $C(G)$ of continuous functions $G \to \mathbb{C}$. The structure of the group is encoded in the function algebra, but we shall see that this approach paves way for a more general functional analytic theory of Hopf algebras, which possess nice duality properties.

4.1 Commutative C^*-algebras

Let X be a compact Hausdorff space and $\mathcal{A} := C(X)$. Without proofs, we present some fundamental results:

- All the algebra homomorphisms $\mathcal{A} \to \mathbb{C}$ are of the form

  $$f \mapsto f(x),$$

  where $x \in X$.

- All the closed ideals of $\mathcal{A}$ are of the form

  $$I(K) := \{f \in \mathcal{A} \mid f(K) = \{0\}\},$$

  where $K \subset X$ (with convention $I(\emptyset) := C(X)$). Moreover, $\overline{K} = V(I(K))$, where

  $$V(J) = \bigcap_{f \in J} f^{-1}(\{0\});$$

  these results follow by Urysohn's Lemma.

- Probability functionals $\mathcal{A} \to \mathbb{C}$ are of the form

$$f \mapsto \int_X f \, \mathrm{d}\mu,$$

where μ is a Borel-regular probability measure on X; this is called the Riesz Representation Theorem.

All in all, we might say that the topology and measure theory of a compact Hausdorff space X is encoded in the algebra $\mathcal{A} = C(X)$, with a "dictionary":

Space X	Algebra $\mathcal{A} = C(X)$
point	algebra functional
closed set	closed ideal
Borel-regular probability measure	probability functional
⋮	⋮

Remark 4.1.1. noncommgeometry In the light of the "dictionary" above, one is bound to ask:

1. If X is a group, how this is reflected in $C(X)$?
2. Could we study non-commutative algebras just like the commutative ones?

We might call the traditional topology and measure theory by the name "commutative geometry", referring to the commutative function algebras; "non-commutative geometry" would refer to the study of non-commutative algebras.

Answering problem 1. Let G be a compact group. By Urysohn's Lemma, $C(G)$ separates the points of X, so that the associativity of the group operation $((x, y) \mapsto xy) : G \times G \to G$ is encoded by

$$\forall x, y, z \in G \quad \forall f \in C(G) : \quad f((xy)z) = f(x(yz)).$$

Similarly,

$$\exists e \in G \quad \forall x \in G \quad \forall f \in C(G) : \quad f(xe) = f(x) = f(ex)$$

encodes the neutral element $e \in G$. Finally,

$$\forall x \in G \quad \exists x^{-1} \in G \quad \forall f \in C(G) : \quad f(x^{-1}x) = f(e) = f(xx^{-1})$$

encodes the inversion $(x \mapsto x^{-1}) : G \to G$. Thereby let us define linear operators

$$\begin{aligned} \tilde{\Delta} : C(G) \to C(G \times G), &\quad \tilde{\Delta} f(x,y) := f(xy), \\ \tilde{\varepsilon} : C(G) \to \mathbb{C}, &\quad \tilde{\varepsilon} f := f(e), \\ \tilde{S} : C(G) \to C(G), &\quad \tilde{S} f(x) := f(x^{-1}); \end{aligned}$$

the interactions of these algebra homomorphisms contain all the information about the structure of the underlying group! This is a key ingredient in the Hopf algebra theory.

Answering problem 2. Our algebras always have a unit element $\mathbf{1}$. An involutive $\mathbb{C}$-algebra $\mathcal{A}$ is a *C^*-algebra* if it has a Banach space norm satisfying

$$\|ab\| \leq \|a\| \, \|b\| \quad \text{and} \quad \|a^* a\| = \|a\|^2$$

for every $a, b \in \mathcal{A}$. By Gelfand and Naimark (1943), up to an isometric $*$-isomorphism a C^*-algebra is a closed involutive subalgebra of $\mathcal{L}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space; moreover, if $\mathcal{A}$ is a commutative unital C^*-algebra then $\mathcal{A} \cong C(X)$ for a compact Hausdorff space X, as explained below:

The *spectrum of* $\mathcal{A}$ is the set $\mathrm{Spec}(\mathcal{A})$ of the algebra homomorphisms $\mathcal{A} \to \mathbb{C}$ (automatically bounded functionals!), endowed with the *Gelfand topology*, which is the relative weak*-topology of $\mathcal{L}(\mathcal{A}, \mathbb{C})$. It turns out that $\mathrm{Spec}(\mathcal{A})$ is a compact Hausdorff space. For $a \in \mathcal{A}$ we define the *Gelfand transform*

$$\widehat{a} : \mathrm{Spec}(\mathcal{A}) \to \mathbb{C}, \quad \widehat{a}(x) := x(a).$$

It turns out that $\widehat{a}$ is continuous, and that

$$(a \mapsto \widehat{a}) : \mathcal{A} \to C(\mathrm{Spec}(\mathcal{A}))$$

is isometric $*$-algebra isomorphism!

If $\mathcal{B}$ is a non-commutative C^*-algebra, it still has plenty of interesting commutative C^*-subalgebras so that the Gelfand transform enables the nice tools of classic analysis on compact Hausdorff spaces in the study of the algebra. Namely, if $a \in \mathcal{B}$ is *normal*, i.e. $a^* a = a a^*$, then closure of the algebraic span (polynomials) of $\{a, a^*\}$ is a commutative C^*-subalgebra. E.g. $b^* b \in \mathcal{B}$ is normal for every $b \in \mathcal{B}$.

Synthesis of problems 1 and 2. By Gelfand and Naimark, the archetypal commutative C^*-algebra is $C(X)$ for a compact Hausdorff space X. In the sequel, we introduce Hopf algebras. In a sense, they are a not-necessarily-commutative analogy of $C(G)$, where G is a compact group. We begin by formally dualizing the category of *algebras*, to obtain the category of *co-algebras*. By marrying these concepts in a subtle way, we obtain the category of *Hopf algebras*.

4.2 Hopf algebras

The definition of a Hopf algebra is a lengthy one, yet quite natural. In the sequel, notice the evident dualities in the commutative diagrams.

For $\mathbb{C}$-vector spaces V, W, we define $\tau_{V,W} : V \otimes W \to W \otimes V$ by the linear extension of

$$\tau_{V,W}(v \otimes w) := w \otimes v.$$

Moreover, in the sequel the identity operation $(v \mapsto v) : V \to V$ for any vector space V is denoted by I. We constantly identify $\mathbb{C}$-vector spaces V and $\mathbb{C} \otimes V$ (and respectively $V \otimes \mathbb{C}$), since $(\lambda \otimes v) \mapsto \lambda v$ defines a linear isomorphism $\mathbb{C} \otimes V \to V$.

In the usual definition of an algebra, the multiplication is regarded as a bilinear map. In order to use dualization techniques for algebras, we want to linearize the multiplication. Let us therefore give a new, equivalent definition for an algebra:

Definition 4.2.1. The triple

$$(\mathcal{A}, m, \eta)$$

is an *algebra* (more precisely, an *associative unital* $\mathbb{C}$*-algebra*) if $\mathcal{A}$ is a $\mathbb{C}$-vector space, and

$$\begin{aligned} m &: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}, \\ \eta &: \mathbb{C} \to \mathcal{A} \end{aligned}$$

are linear mappings such that the following diagrams commute: the *associativity diagram*

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{I \otimes m} & \mathcal{A} \otimes \mathcal{A} \\ {\scriptstyle m \otimes I}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A} \otimes \mathcal{A} & \xrightarrow{m} & \mathcal{A} \end{array}$$

and the *unit diagrams*

$$\begin{array}{ccc} \mathcal{A}\otimes\mathbb{C} & \xrightarrow{I\otimes\eta} & \mathcal{A}\otimes\mathcal{A} \\ {\scriptstyle a\otimes\lambda\mapsto\lambda a}\downarrow & & \downarrow{\scriptstyle m} \\ \mathcal{A} & = & \mathcal{A}, \end{array} \qquad \begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xleftarrow{\eta\otimes I} & \mathbb{C}\otimes\mathcal{A} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \lambda\otimes a\mapsto\lambda a} \\ \mathcal{A} & = & \mathcal{A}. \end{array}$$

The mapping m is called the *multiplication* and η the *unit mapping*; the algebra $\mathcal{A}$ is *commutative* if $m\tau_{\mathcal{A},\mathcal{A}} = m$. The *unit* of an algebra $(\mathcal{A}, m, \eta)$ is

$$\mathbf{1}_{\mathcal{A}} := \eta(1),$$

and the usual abbreviation for the multiplication is $ab := m(a \otimes b)$. For algebras $(\mathcal{A}_1, m_1, \eta_1)$ and $(\mathcal{A}_2, m_2, \eta_2)$ the *tensor product algebra* $(\mathcal{A}_1 \otimes \mathcal{A}_2, m, \eta)$ is defined by

$$m := (m_1 \otimes m_2)(I \otimes \tau_{\mathcal{A}_1,\mathcal{A}_2} \otimes I)$$

i.e. $(a_1 \otimes a_2)(b_1 \otimes b_2) = (a_1b_1) \otimes (a_2b_2)$, and

$$\eta(1) := \mathbf{1}_{\mathcal{A}_1} \otimes \mathbf{1}_{\mathcal{A}_2}.$$

Remark 4.2.2. If an algebra $\mathcal{A} = (\mathcal{A}, m, \eta)$ is finite-dimensional, we can formally dualize its structural mappings m and η; this inspires the concept co-algebra:

Definition 4.2.3. The triple

$$(\mathcal{C}, \Delta, \varepsilon)$$

is a *co-algebra* (more precisely, a *co-associative co-unital* $\mathbb{C}$*-co-algebra*) if $\mathcal{C}$ is a $\mathbb{C}$-vector space and

$$\begin{array}{rcl} \Delta & : & \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}, \\ \varepsilon & : & \mathcal{C} \to \mathbb{C} \end{array}$$

are linear mappings such that the following diagrams commute: the *co-associativity diagram* (notice the duality to the associativity diagram)

$$\begin{array}{ccc} \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} & \xleftarrow{I\otimes\Delta} & \mathcal{C}\otimes\mathcal{C} \\ {\scriptstyle \Delta\otimes I}\uparrow & & \uparrow{\scriptstyle \Delta} \\ \mathcal{C}\otimes\mathcal{C} & \xleftarrow{\Delta} & \mathcal{C} \end{array}$$

and the *co-unit diagrams* (notice the duality to the unit diagrams)

$$
\begin{array}{ccc}
\mathcal{C}\otimes\mathbb{C} & \xleftarrow{I\otimes\varepsilon} & \mathcal{C}\otimes\mathcal{C} \\
{\scriptstyle \lambda c\mapsto c\otimes\lambda}\uparrow & & \uparrow{\scriptstyle \Delta} \\
\mathcal{C} & = & \mathcal{C},
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{C}\otimes\mathcal{C} & \xrightarrow{\varepsilon\otimes I} & \mathbb{C}\otimes\mathcal{C} \\
{\scriptstyle \Delta}\uparrow & & \uparrow{\scriptstyle \lambda c\mapsto\lambda\otimes c} \\
\mathcal{C} & = & \mathcal{C}.
\end{array}
$$

The mapping Δ is called the *co-multiplication* and ε the *co-unit mapping*; the co-algebra $\mathcal{C}$ is *co-commutative* if $\tau_{\mathcal{C},\mathcal{C}}\Delta = \Delta$. For co-algebras $(\mathcal{C}_1, \Delta_1, \varepsilon_1)$ and $(\mathcal{C}_2, \Delta_2, \varepsilon_2)$ the *tensor product co-algebra* $(\mathcal{C}_1 \otimes \mathcal{C}_2, \Delta, \varepsilon)$ is defined by

$$\Delta := (I \otimes \tau_{\mathcal{C}_1,\mathcal{C}_2} \otimes I)(\Delta_1 \otimes \Delta_2)$$

and

$$\varepsilon(c_1 \otimes c_2) := \varepsilon_1(c_1)\varepsilon_2(c_2).$$

Example. A trivial co-algebra example: If $(\mathcal{A}, m, \eta)$ is a finite-dimensional algebra then the vector space dual $\mathcal{A}' = \mathcal{L}(\mathcal{A}, \mathbb{C})$ has a natural co-algebra structure: Let us identify $(\mathcal{A} \otimes \mathcal{A})'$ and $\mathcal{A}' \otimes \mathcal{A}'$ naturally, so that $m' : \mathcal{A}' \to \mathcal{A}' \otimes \mathcal{A}'$ is the dual mapping to $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Let us identify $\mathbb{C}'$ and $\mathbb{C}$ naturally, so that $\eta' : \mathcal{A}' \to \mathbb{C}$ is the dual mapping to $\eta : \mathbb{C} \to \mathcal{A}$. Then

$$(\mathcal{A}', m', \eta')$$

is a co-algebra (draw the commutative diagrams!). We shall give more interesting examples of co-algebras after the definition of Hopf algebras.

Definition 4.2.4. Let $(\mathcal{B}, m, \eta)$ be an algebra and $(\mathcal{B}, \Delta, \varepsilon)$ be a co-algebra. Let $\mathcal{L}(\mathcal{B})$ denote the vector space of linear operators $\mathcal{B} \to \mathcal{B}$. Let us define the *convolution* $A * B \in \mathcal{L}(\mathcal{B})$ *of linear operators* $A, B \in \mathcal{L}(\mathcal{B})$ by

$$A * B := m(A \otimes B)\Delta.$$

Then we see that $\mathcal{L}(\mathcal{B})$ can be endowed with a structure of an algebra, with unit element $\eta\varepsilon$, i.e. $A * \eta\varepsilon = A = \varepsilon\eta * A$!

Exercise 4.2.5. Show that $\mathcal{L}(\mathcal{B})$ above in Definition 4.2.4 is an algebra, when endowed with the convolution product of operators.

Definition 4.2.6. A structure

$$(\mathcal{H}, m, \eta, \Delta, \varepsilon, S)$$

is a *Hopf algebra* if

- $(\mathcal{H}, m, \eta)$ is an algebra,
- $(\mathcal{H}, \Delta, \varepsilon)$ is a co-algebra,
- $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $\varepsilon : \mathcal{H} \to \mathbb{C}$ are algebra homomorphisms, i.e.

$$\Delta(fg) = \Delta(f)\Delta(g), \quad \Delta(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{H}\otimes\mathcal{H}},$$
$$\varepsilon(fg) = \varepsilon(f)\varepsilon(g), \quad \varepsilon(\mathbf{1}_{\mathcal{H}}) = 1,$$

- and $S : \mathcal{H} \to \mathcal{H}$ is a linear mapping, called the *antipode*, satisfying

$$S * I = \eta\varepsilon = I * S;$$

  i.e. $I \in \mathcal{L}(\mathcal{H})$ and $S \in \mathcal{L}(\mathcal{H})$ are inverses to each other in the convolution algebra $\mathcal{L}(\mathcal{H})$.

For Hopf algebras $(\mathcal{H}_1, m_1, \eta_1, \Delta_1, \varepsilon_1, S_1)$ and $(\mathcal{H}_2, m_2, \eta_2, \Delta_2, \varepsilon_2, S_2)$ we define the *tensor product Hopf algebra* $(\mathcal{H}_1 \otimes \mathcal{H}_2, m, \eta, \Delta, \varepsilon, S)$ such that

$$(\mathcal{H}_1 \otimes \mathcal{H}_2, m, \eta)$$

is the usual tensor product algebra,

$$(\mathcal{H}_1 \otimes \mathcal{H}_2, \Delta, \varepsilon)$$

is the usual tensor product co-algebra, and

$$S := S_{\mathcal{H}_1} \otimes S_{\mathcal{H}_2}.$$

Exercise 4.2.7. (Uniqueness of the antipode.) Let $(\mathcal{H}, m, \eta, \Delta, \varepsilon, S_j)$ be Hopf algebras, where $j \in \{1, 2\}$. Show that $S_1 = S_2$.

Remark 4.2.8. Commutative diagrams for Hopf algebras: Notice that we now have the *multiplication and co-multiplication diagram*

$$\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} & \xrightarrow{\Delta m} & \mathcal{H} \otimes \mathcal{H} \\
{\scriptstyle \Delta\otimes\Delta}\big\downarrow & & \big\uparrow{\scriptstyle m\otimes m} \\
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} & \xrightarrow{I\otimes\tau_{\mathcal{H},\mathcal{H}}\otimes I} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H},
\end{array}$$

the *co-multiplication and unit diagram*

$$\begin{array}{ccc}
\mathcal{H} & \xleftarrow{\eta} & \mathbb{C} \\
{\scriptstyle \Delta}\big\downarrow & & \big\| \\
\mathcal{H} \otimes \mathcal{H} & \xleftarrow{\eta\otimes\eta} & \mathbb{C} \otimes \mathbb{C},
\end{array}$$

the *multiplication and co-unit diagram*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\varepsilon} & \mathbb{C} \\ {\scriptstyle m}\uparrow & & \| \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow{\varepsilon\otimes\varepsilon} & \mathbb{C}\otimes\mathbb{C} \end{array}$$

and the *"everyone with the antipode" diagrams*

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\eta\varepsilon} & \mathcal{H} \\ {\scriptstyle \Delta}\downarrow & & \uparrow{\scriptstyle m} \\ \mathcal{H}\otimes\mathcal{H} & \xrightarrow[S\otimes I]{I\otimes S} & \mathcal{H}\otimes\mathcal{H}. \end{array}$$

Example. A monoid co-algebra example: Let G be a finite group and $\mathcal{F}(G)$ be the $\mathbb{C}$-vector space of functions $G \to \mathbb{C}$. Notice that $\mathcal{F}(G)\otimes\mathcal{F}(G)$ and $\mathcal{F}(G\times G)$ are naturally isomorphic by

$$\sum_{j=1}^{m}(f_j\otimes g_j)(x,y) := \sum_{j=1}^{m} f_j(x)g_j(y).$$

Then we can define mappings $\Delta : \mathcal{F}(G) \to \mathcal{F}(G)\otimes\mathcal{F}(G)$ and $\varepsilon : \mathcal{F}(G) \to \mathbb{C}$ by

$$\Delta f(x,y) := f(xy), \quad \varepsilon f := f(e).$$

In the next example we show that $(\mathcal{F}(G),\Delta,\varepsilon)$ is a co-algebra. But there is still more structure in the group to exploit: let us define an operator $S : \mathcal{F}(G) \to \mathcal{F}(G)$ by $(Sf)(x) := f(x^{-1})$...

Example. Hopf algebra for finite group: Let G be a finite group. Now $\mathcal{F}(G)$ from the previous example has a structure of a commutative Hopf algebra; it is co-commutative if and only if G is a commutative group. The algebra mappings are given by

$$\eta(\lambda)(x) := \lambda, \quad m(f\otimes g)(x) := f(x)g(x)$$

for every $\lambda \in \mathbb{C}$, $x \in G$ and $f,g \in \mathcal{F}(G)$. Notice that $\mathcal{F}(G\times G) \cong \mathcal{F}(G)\otimes\mathcal{F}(G)$ gives interpretation $(ma)(x) = a(x,x)$ for $a \in \mathcal{F}(G\times G)$. Clearly $(\mathcal{F}(G),m,\eta)$ is a commutative algebra. Let $x,y,z \in G$ and $f,g \in \mathcal{F}(G)$.

Then

$$\begin{aligned}((\Delta\otimes I)\Delta f)(x,y,z) &= (\Delta f)(xy,z)\\ &= f((xy)z)\\ &= f(x(yz))\\ &= (\Delta f)(x,yz)\\ &= ((I\otimes\Delta)\Delta f)(x,y,z),\end{aligned}$$

so that $(\Delta\otimes I)\Delta = (I\otimes\Delta)\Delta$. Next, $(\varepsilon\otimes I)\Delta \cong I \cong (I\otimes\varepsilon)\Delta$, because

$$\begin{aligned}(m(\eta\varepsilon\otimes I)\Delta f)(x) &= ((\eta\varepsilon\otimes I)\Delta f)(x,x)\\ &= \Delta f(e,x)\\ &= f(ex) \ = \ f(x) \ = \ f(xe)\\ &= \ \ldots \ = \ (m(I\otimes\eta\varepsilon)\Delta f)(x).\end{aligned}$$

Thereby $(\mathcal{F}(G),\Delta,\varepsilon)$ is a co-algebra. Moreover,

$$\varepsilon(fg) = (fg)(e) = f(e)g(e) = \varepsilon(f)\varepsilon(g),$$

$$\varepsilon(\mathbf{1}_{\mathcal{F}(G)}) = \mathbf{1}_{\mathcal{F}(G)}(e) = 1,$$

so that $\varepsilon : \mathcal{F}(G) \to \mathbb{C}$ is an algebra homomorphism. The co-multiplication $\Delta : \mathcal{F}(G) \to \mathcal{F}(G)\otimes\mathcal{F}(G) \cong \mathcal{F}(G\times G)$ is an algebra homomorphism, because

$$\Delta(fg)(x,y) = (fg)(xy) = f(xy)\ g(xy) = (\Delta f)(x,y)\ (\Delta g)(x,y),$$

$$\Delta(\mathbf{1}_{\mathcal{F}(G)})(x,y) = \mathbf{1}_{\mathcal{F}(G)}(xy) = 1 = \mathbf{1}_{\mathcal{F}(G\times G)}(x,y) \cong (\mathbf{1}_{\mathcal{F}(G)}\otimes\mathbf{1}_{\mathcal{F}(G)})(x,y).$$

Finally,

$$\begin{aligned}((I*S)f)(x) &= (m(I\otimes S)\Delta f)(x)\\ &= ((I\otimes S)\Delta f)(x,x)\\ &= (\Delta f)(x,x^{-1})\\ &= f(xx^{-1}) \ = \ f(e) \ = \ \varepsilon f\\ &= \ \ldots \ = \ ((S*I)f)(x),\end{aligned}$$

so that $I*S = \eta\varepsilon = S*I$. Thereby $\mathcal{F}(G)$ can be endowed with a Hopf algebra structure.

Example. Hopf algebra for a compact group: Let G be a compact group. We shall endow the dense subalgebra $\mathcal{H} := \mathrm{TrigPol}(G) \subset C(G)$ of trigonometric polynomials with a natural structure of a commutative Hopf algebra; $\mathcal{H}$ will be co-commutative if and only if G is commutative. Notice that $\mathcal{F}(G) = \mathrm{TrigPol}(G) = C(G)$ for a finite group G; actually, for a finite group, this trigonometric polynomial Hopf algebra coincides with the Hopf algebra of the previous example. It can be shown that here $\mathcal{H} \otimes \mathcal{H} \cong \mathrm{TrigPol}(G \times G)$, where the isomorphism is given by

$$\sum_{j=1}^{m} (f_j \otimes g_j)(x,y) := \sum_{j=1}^{m} f_j(x) g_j(y).$$

The algebra structure

$$(\mathcal{H}, m, \eta)$$

is the usual one for the trigonometric polynomials, i.e. $m(f \otimes g) := fg$ and $\eta(\lambda) = \lambda \mathbf{1}$, where $\mathbf{1}(x) = 1$ for every $x \in G$. By the Peter–Weyl Theorem 2.5.13, the $\mathbb{C}$-vector space $\mathcal{H}$ is spanned by

$$\left\{ \phi_{ij} : \ \phi = (\phi_{ij})_{i,j}^{\dim(\phi)}, \ [\phi] \in \widehat{G} \right\}.$$

Let us define the co-multiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ by

$$\Delta \phi_{ij} := \sum_{k=1}^{\dim(\phi)} \phi_{ik} \otimes \phi_{kj};$$

we see that then

$$\begin{aligned} (\Delta \phi_{ij})(x,y) &= \sum_{k=1}^{\dim(\phi)} (\phi_{ik} \otimes \phi_{kj})(x,y) \\ &= \sum_{k=1}^{\dim(\phi)} \phi_{ik}(x) \phi_{kj}(y) \\ &= \phi_{ij}(xy). \end{aligned}$$

The co-unit $\varepsilon : \mathcal{H} \to \mathbb{C}$ is defined by

$$\varepsilon f := f(e),$$

and the antipode $S : \mathcal{H} \to \mathcal{H}$ by

$$(Sf)(x) := f(x^{-1}).$$

Exercise 4.2.9. In the Example above, check the validity of the Hopf algebra axioms.

Theorem 4.2.10. *Let $\mathcal{H}$ be a commutative C^*-algebra. If $(\mathcal{H}, m, \eta, \Delta, \varepsilon, S)$ is a finite-dimensional Hopf algebra then there exists a Hopf algebra isomorphism $\mathcal{H} \cong C(G)$, where G is a finite group and $C(G)$ is endowed with the Hopf algebra structure given above.*

Proof. Let $G := \mathrm{Spec}(\mathcal{H})$. As $\mathcal{H}$ is a commutative C^*-algebra, it is isometrically $*$-isomorphic to the C^*-algebra $C(G)$ via the Gelfand transform

$$(f \mapsto \widehat{f}) : \mathcal{H} \to C(G), \quad \widehat{f}(x) := x(f).$$

The space G must be finite, because $\dim(C(G)) = \dim(\mathcal{H}) < \infty$.

Now

$$e := \varepsilon \in G,$$

because $\varepsilon : \mathcal{H} \to \mathbb{C}$ is an algebra homomorphism. This $e \in G$ will turn out to be the neutral element of our group.

Let $x, y \in G$. We identify the spaces $\mathbb{C} \otimes \mathbb{C}$ and $\mathbb{C}$, and get an algebra homomorphism $x \otimes y : \mathcal{H} \otimes \mathcal{H} \to \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$. Now $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is an algebra homomorphism, so that $(x \otimes y)\Delta : \mathcal{H} \to \mathbb{C}$ is an algebra homomorphism! Let us denote

$$xy := (x \otimes y)\Delta,$$

so that $xy \in G$. This defines the group operation $((x, y) \mapsto xy) : G \times G \to G$!

Inversion $x \mapsto x^{-1}$ will be defined via the antipode $S : \mathcal{H} \to \mathcal{H}$. We shall show that for a *commutative* Hopf algebra, the antipode is an algebra isomorphism. First we prove that $S(\mathbf{1}_{\mathcal{H}}) = \mathbf{1}_{\mathcal{H}}$:

$$\begin{aligned} S\mathbf{1}_{\mathcal{H}} &= m(\mathbf{1}_{\mathcal{H}} \otimes S\mathbf{1}_{\mathcal{H}}) \\ &= m(I \otimes S)(\mathbf{1}_{\mathcal{H}} \otimes \mathbf{1}_{\mathcal{H}}) \\ &= m(I \otimes S)\Delta\mathbf{1}_{\mathcal{H}} \\ &= (I * S)\mathbf{1}_{\mathcal{H}} = \eta\varepsilon\mathbf{1}_{\mathcal{H}} \\ &= \mathbf{1}_{\mathcal{H}}. \end{aligned}$$

Then we show that $S(gh) = S(h)S(g)$, where $g, h \in \mathcal{H}$, $gh := m(g \otimes h)$. Let us use the so-called *Sweedler notation*

$$\Delta f =: \sum f_{(1)} \otimes f_{(2)} =: f_{(1)} \otimes f_{(2)};$$

consequently

$$\begin{aligned}(\Delta\otimes I)\Delta f &= (\Delta\otimes I)(f_{(1)}\otimes f_{(2)}) = f_{(1)(1)}\otimes f_{(1)(2)}\otimes f_{(2)},\\ (I\otimes\Delta)\Delta f &= (I\otimes\Delta)(f_{(1)}\otimes f_{(2)}) = f_{(1)}\otimes f_{(2)(1)}\otimes f_{(2)(2)},\end{aligned}$$

and due to the *co-associativity* we may re-index as follows:

$$(\Delta\otimes I)\Delta f =: f_{(1)}\otimes f_{(2)}\otimes f_{(3)} := (I\otimes\Delta)\Delta f$$

(notice that e.g. $f_{(2)}$ appears in different meanings above, this is just notation!). Then

$$\begin{aligned}S(gh) &= S(\varepsilon((gh)_{(1)})(gh)_{(2)})\\ &= \varepsilon((gh)_{(1)})\ S((gh)_{(2)})\\ &= \varepsilon(g_{(1)}h_{(1)})\ S(g_{(2)}h_{(2)})\\ &= \varepsilon(g_{(1)})\ \varepsilon(h_{(1)})\ S(g_{(2)}h_{(2)})\\ &= \varepsilon(g_{(1)})\ S(h_{(1)(1)})\ h_{(1)(2)}\ S(g_{(2)}h_{(2)})\\ &= \varepsilon(g_{(1)})\ S(h_{(1)})\ h_{(2)}\ S(g_{(2)}h_{(3)})\\ &= S(h_{(1)})\ \varepsilon(g_{(1)})\ h_{(2)}\ S(g_{(2)}h_{(3)})\\ &= S(h_{(1)})\ S(g_{(1)(1)})\ g_{(1)(2)}\ h_{(2)}\ S(g_{(2)}h_{(3)})\\ &= S(h_{(1)})\ S(g_{(1)})\ g_{(2)}\ h_{(2)}\ S(g_{(3)}h_{(3)})\\ &= S(h_{(1)})\ S(g_{(1)})\ (gh)_{(2)}\ S((gh)_{(3)})\\ &= S(h_{(1)})\ S(g_{(1)})\ \varepsilon((gh)_{(2)})\\ &= S(h_{(1)})\ S(g_{(1)})\ \varepsilon(g_{(2)}h_{(2)})\\ &= S(h_{(1)})\ S(g_{(1)})\ \varepsilon(g_{(2)})\ \varepsilon(h_{(2)})\\ &= S(h_{(1)}\varepsilon(h_{(2)}))\ S(g_{(1)}\varepsilon(g_{(2)}))\\ &= S(h)\ S(g);\end{aligned}$$

this computation can be compared to

$$\begin{aligned}(xy)^{-1} &= e(xy)^{-1}\\ &= y^{-1}y(xy)^{-1}\\ &= y^{-1}ey(xy)^{-1}\\ &= y^{-1}x^{-1}xy(xy)^{-1}\\ &= y^{-1}x^{-1}e\\ &= y^{-1}x^{-1}\end{aligned}$$

for $x, y \in G$! Since $\mathcal{H}$ is commutative, we have proven that $S : \mathcal{H} \to \mathcal{H}$ is an algebra homomorphism. Thereby $xS : \mathcal{H} \to \mathbb{C}$ is an algebra homomorphism. Let us denote

$$x^{-1} := xS \in G;$$

this is the inverse of $x \in G$!

We leave it for the reader to show that $(G, (x, y) \mapsto xy, x \mapsto x^{-1})$ is indeed a group. □

Exercise 4.2.11. Finish the proof of Theorem 4.2.10.

Exercise 4.2.12. Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Let $X \in \mathfrak{g}$; extend definitions

$$\Delta X := X \otimes \mathbf{1}_{\mathcal{U}(\mathfrak{g})} + \mathbf{1}_{\mathcal{U}(\mathfrak{g})} \otimes X, \quad \varepsilon X := 0, \quad SX := -X$$

so that you obtain a Hopf algebra structure $(\mathcal{U}(\mathfrak{g}), m, \eta, \Delta, \varepsilon, S)$.

Exercise 4.2.13. Let $(\mathcal{H}, m, \eta, \Delta, \varepsilon, S)$ be a finite-dimensional Hopf algebra.
(a) Endow the dual $\mathcal{H}' = \mathcal{L}(\mathcal{H}, \mathbb{C})$ with a natural Hopf algebra structure via the duality

$$(f, \phi) \mapsto \langle f, \phi \rangle_{\mathcal{H}} := \phi(f)$$

where $f \in \mathcal{H}$, $\phi \in \mathcal{H}'$.
(b) If G is a finite group and $\mathcal{H} = \mathcal{F}(G)$, what are the Hopf algebra operations for $\mathcal{H}'$?
(c) With a suitable choice for $\mathcal{H}$, give an example of a non-commutative non-co-commutative Hopf algebra $\mathcal{H} \otimes \mathcal{H}'$.

Exercise 4.2.14. Let $(\mathcal{H}, m, \eta)$ be the algebra spanned by the set $\{\mathbf{1}, g, x, gx\}$, where $\mathbf{1}$ is the unit element and $g^2 = \mathbf{1}$, $x^2 = 0$ and $xg = -gx$. Let us define algebra homomorphisms $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ and $\varepsilon : \mathcal{H} \to \mathbb{C}$ by

$$\Delta(g) := g \otimes g, \quad \Delta(x) := x \otimes \mathbf{1} + g \otimes x,$$

$$\varepsilon(g) := 1, \quad \varepsilon(x) := 0.$$

Let us define a linear mapping $S : \mathcal{H} \to \mathcal{H}$ by

$$S(\mathbf{1}) := \mathbf{1}, \quad S(g) := g, \quad S(x) := -gx, \quad S(gx) := -x.$$

Show that $(\mathcal{H}, m, \eta, \Delta, \varepsilon, S)$ is a non-commutative non-co-commutative Hopf algebra (this example is by M. E. Sweedler).

Remark 4.2.15. In Exercise 4.2.14, a nice concrete matrix example can be given. Let us define $A \in \mathbb{C}^{2\times 2}$ by

$$A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $g, x \in \mathbb{C}^{4\times 4}$ be given by

$$g := \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}, \quad x := \begin{pmatrix} 0 & I_{\mathbb{C}^2} \\ 0 & 0 \end{pmatrix}.$$

Then it is easy to see that $\mathcal{H} = \operatorname{span}\{I_{\mathbb{C}^4}, g, x, gx\}$ is a four-dimensional subalgebra of $\mathbb{C}^{4\times 4}$ such that $g^2 = I_{\mathbb{C}^4}$, $x^2 = 0$ and $xg = -gx$.

Chapter 5

Appendices

5.1 Appendix on set theoretical notation

When X is a set, $\mathcal{P}(X)$ denotes the family of all subsets of X (the *power set*, sometimes denoted by 2^X). The cardinality of X is denoted by $|X|$. If J is a set and $S_j \subset X$ for every $j \in J$, we write

$$\bigcup \{S_j \mid j \in J\} = \bigcup_{j \in J} S_j, \quad \bigcap \{S_j \mid j \in J\} = \bigcap_{j \in J} S_j.$$

If $f : X \to Y$, $U \subset X$, and $V \subset Y$, we define

$$f(U) := \{f(x) \mid x \in U\} \quad \text{(image)},$$

$$f^{-1}(V) := \{x \in X \mid f(x) \in V\} \quad \text{(preimage)}.$$

5.2 Appendix on Axiom of Choice

It may be surprising, but the Zermelo-Fraenkel axiom system does not imply the following statement (nor its negation):

Axiom of Choice for Cartesian Products: *The Cartesian product of non-empty sets is non-empty.*

Nowadays there are hundreds of equivalent formulations for the Axiom of Choice. Next we present other famous variants: the classical Axiom of Choice, the Law of Trichotomy, the Well-Ordering Axiom, the Hausdorff Maximal Principle and Zorn's Lemma. Their equivalence is shown in [20].

Axiom of Choice: *For every non-empty set J there is a function $f : \mathcal{P}(J) \to J$ such that $f(I) \in I$ when $I \neq \emptyset$.*

Let A, B be sets. We write $A \sim B$ if there exists a bijection $f : A \to B$, and $A \leq B$ if there is a set $C \subset B$ such that $A \sim C$. Notion $A < B$ means $A \leq B$ such that not $A \sim B$.

Law of Trichotomy: *Let A, B be sets. Then $A < B$, $A \sim B$ or $B < A$.*

A set X is *partially ordered* with an *order relation* $R \subset X \times X$ if R is reflexive ($(x,x) \in R$), antisymmetric ($(x,y),(y,x) \in R \Rightarrow x = y$) and transitive ($(x,y),(y,z) \in R \Rightarrow (x,z) \in R$). A subset $C \subset X$ is a *chain* if $(x,y) \in R$ or $(y,x) \in R$ for every $x, y \in C$. An element $x \in X$ is *maximal* if $(x,y) \in R$ implies $y = x$.

Well-Ordering Axiom: *Every set is a chain for some order relation.*

Hausdorff Maximal Principle: *Any chain is contained in a maximal chain.*

Zorn's Lemma: *A non-empty partially ordered set where every chain has an upper bound has a maximal element.*

5.3 Appendix on algebras

A $\mathbb{K}$-vector space $\mathcal{A}$ with an element $\mathbf{1}_{\mathcal{A}} \in \mathcal{A} \setminus \{0\}$ and endowed with a bilinear mapping $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $(x,y) \mapsto xy$ is called an *algebra* (more precisely, an *associative unital* $\mathbb{K}$-*algebra*) if $x(yz) = (xy)z$ and if $\mathbf{1}_{\mathcal{A}}x = x = x\mathbf{1}_{\mathcal{A}}$ for every $x, y, z \in \mathcal{A}$. Then $\mathbf{1}_{\mathcal{A}}$ is called the *unit* of $\mathcal{A}$, and we write $xyz := (xy)z$. An algebra $\mathcal{A}$ is *commutative* if $xy = yx$ for every $x, y \in \mathcal{A}$. An element $x \in \mathcal{A}$ is *invertible* with inverse $x^{-1} \in \mathcal{A}$ if $x^{-1}x = \mathbf{1}_{\mathcal{A}} = xx^{-1}$.

An *algebra homomorphism* $\phi : \mathcal{A} \to \mathcal{B}$ is a linear mapping between algebras $\mathcal{A}, \mathcal{B}$ satisfying $\phi(xy) = \phi(x)\phi(y)$ and $\phi(\mathbf{1}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{B}}$ for every $x, y \in \mathcal{A}$. If $x \in \mathcal{A}$ is invertible then $\phi(x) \in \mathcal{B}$ is also invertible, since $\phi(x^{-1})\phi(x) = \mathbf{1}_{\mathcal{B}} = \phi(x)\phi(x^{-1})$!

An *ideal* (more precisely, a *two-sided ideal*) in an algebra $\mathcal{A}$ is a vector subspace $\mathcal{J} \subset \mathcal{A}$ such that $xj \in \mathcal{J}$ and $jx \in \mathcal{J}$ for every $x \in \mathcal{A}$ and $j \in \mathcal{J}$. An ideal $\mathcal{J}$ of an algebra $\mathcal{A}$ is *proper* if $\mathcal{J} \neq \mathcal{A}$; in such a case, the vector space $\mathcal{A}/\mathcal{J} := \{x + \mathcal{J} \mid x \in \mathcal{A}\}$ becomes an algebra with the operation $(x+\mathcal{J}, y+\mathcal{J}) \mapsto xy+\mathcal{J}$ and the unit element $\mathbf{1}_{\mathcal{A}/\mathcal{J}} := \mathbf{1}_{\mathcal{A}}+\mathcal{J}$. It is evident that no proper ideal contains any invertible elements. It is also evident that the *kernel* $\mathrm{Ker}(\phi) := \{x \in \mathcal{A} \mid \phi(x) = 0\}$ of an algebra homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is an ideal of $\mathcal{A}$.

A proper ideal is *maximal* if it is not contained in any larger proper ideal. The *radical* $\mathrm{Rad}(\mathcal{A})$ of an algebra $\mathcal{A}$ is the intersection of all the maximal ideals of $\mathcal{A}$; $\mathcal{A}$ is called *semisimple* if $\mathrm{Rad}(\mathcal{A}) = \{0\}$. In general, any intersection of ideals is an ideal. Hence for any set $\mathcal{S} \subset \mathcal{A}$ in an algebra $\mathcal{A}$ there exists a smallest possible ideal $\mathcal{J} \subset \mathcal{A}$ such that $\mathcal{S} \subset \mathcal{J}$; this $\mathcal{J}$ is called the *ideal spanned by the set* $\mathcal{S}$.

The *tensor product algebra* of a $\mathbb{K}$-vector space V is the $\mathbb{K}$-vector space

$$T := \bigoplus_{m=0}^{\infty} \otimes^m V,$$

where $\otimes^0 V := \mathbb{K}$, $\otimes^{m+1} V := (\otimes^m V) \otimes V$; the multiplication of this algebra is given by

$$(x, y) \mapsto xy := x \otimes y$$

with the identifications $W \otimes \mathbb{K} \cong W \cong \mathbb{K} \otimes W$ for a $\mathbb{K}$-vector space W, so that the unit element $\mathbf{1}_T \in T$ is the unit element $1 \in \mathbb{K}$.

5.4 Appendix on multilinear algebra

The basic idea in multilinear algebra is to “linearize” multilinear operators.

Definition 5.4.1. Let X_j $(1 \leq j \leq r)$ and V be $\mathbb{K}$-vector spaces (that is, vector spaces over the field $\mathbb{K}$). A mapping $A : X_1 \times X_2 \to V$ is 2-*linear* (or *bilinear*) if $x \mapsto A(x, x_2)$ and $x \mapsto A(x_1, x)$ are linear mappings for each $x_j \in X_j$. The reader may guess what an r-*linear* mapping

$$X_1 \times \cdots \times X_r \to V$$

satisfies...

Definition 5.4.2. The *(algebraic) tensor product* of $\mathbb{K}$-vector spaces $X_1, \ldots, X_r$ is a $\mathbb{K}$-vector space V endowed with an r-linear mapping i such that for every $\mathbb{K}$-vector space W and for every r-linear mapping

$$A : X_1 \times \cdots \times X_r \to W$$

there exists a (unique) linear mapping $\tilde{A} : V \to W$ satisfying $\tilde{A}i = A$. (Draw a commutative diagram involving the vector spaces and mappings $i, A, \tilde{A}$!) Any two tensor products for $X_1, \ldots, X_r$ can easily be seen isomorphic, so that we may denote *the* tensor product of these vector spaces by

$$X_1 \otimes \cdots \otimes X_r.$$

In fact, such a tensor product always exists: Let X, Y be $\mathbb{K}$-vector spaces. We may formally define the set $B := \{x \otimes y \mid x \in X,\ y \in Y\}$, where $x \otimes y = a \otimes b$ if and only if $x = a$ and $y = b$. Let Z be the $\mathbb{K}$-vector space with basis B, i.e.

$$\begin{aligned} Z &= \left\{ \sum_{j=0}^{n} \lambda_j (x_j \otimes y_j) :\ n \in \mathbb{N},\ \lambda_j \in \mathbb{K},\ x_j \in X,\ y_j \in Y \right\} \\ &= \operatorname{span}\{x \otimes y \mid x \in X,\ y \in Y\}. \end{aligned}$$

Let

$$\begin{aligned} [0 \otimes 0] \quad &:= \\ \operatorname{span} \quad &\big\{\alpha_1\beta_1(x_1 \otimes y_1) + \alpha_1\beta_2(x_1 \otimes y_2) + \alpha_2\beta_1(x_2 \otimes y_1) + \alpha_2\beta_2(x_2 \otimes y_2) \\ &\quad -(\alpha_1 x_1 + \alpha_2 x_2) \otimes (\beta_1 y_1 + \beta_2 y_2) : \\ &\qquad \alpha_j, \beta_j \in \mathbb{K},\ x_j \in X,\ y_j \in Y\big\}. \end{aligned}$$

For $z \in Z$, let $[z] := z + [0 \otimes 0]$. The *tensor product* of X, Y is the $\mathbb{K}$-vector space

$$X \otimes Y := Z/[0 \otimes 0] = \{[z] \mid z \in Z\},$$

where $([z_1], [z_2]) \mapsto [z_1 + z_2]$ and $(\lambda, [z]) \mapsto [\lambda z]$ are well-defined mappings $(X \otimes Y) \times (X \otimes Y) \to X \otimes Y$ and $\mathbb{K} \times (X \otimes Y) \to X \otimes Y$, respectively.

Definition 5.4.3. Let X, Y, V, W be $\mathbb{K}$-vector spaces, and let $A : X \to V$ and $B : Y \to W$ be linear operators. The *tensor product* of A, B is the linear operator $A \otimes B : X \otimes Y \to V \otimes W$, which is the unique linear extension of the mapping $x \otimes y \mapsto Ax \otimes By$, where $x \in X$ and $y \in Y$.

Example. Let X and Y be finite-dimensional $\mathbb{K}$-vector spaces with bases $\{x_i\}_{i=1}^{\dim(X)}$ and $\{y_j\}_{j=1}^{\dim(Y)}$, respectively. Then $X \otimes Y$ has a basis

$$\{x_i \otimes y_j \mid 1 \le i \le \dim(X),\ 1 \le j \le \dim(Y)\}.$$

Let S be a finite set. Let $\mathcal{F}(S)$ be the $\mathbb{K}$-vector space of functions $S \to \mathbb{K}$; it has a basis $\{\delta_x \mid x \in S\}$, where $\delta_x(y) = 1$ if $x = y$, and $\delta_x(y) = 0$ otherwise. Now it is easy to see that for finite sets S_1, S_2 the vector spaces $\mathcal{F}(S_1) \otimes \mathcal{F}(S_2)$ and $\mathcal{F}(S_1 \times S_2)$ are isomorphic; for $f_j \in \mathcal{F}(S_j)$, we may regard $f_1 \otimes f_2 \in \mathcal{F}(S_1) \otimes \mathcal{F}(S_2)$ as a function $f_1 \otimes f_2 \in \mathcal{F}(S_1 \times S_2)$ by

$$(f_1 \otimes f_2)(x_1, x_2) := f_1(x_1)\ f_2(x_2).$$

Definition 5.4.4. Suppose V, W are finite-dimensional inner product spaces over $\mathbb{K}$. The natural inner product for $V \otimes W$ is obtained by extending

$$\langle v_1 \otimes w_1, v_2 \otimes w_2 \rangle_{V \otimes W} := \langle v_1, v_2 \rangle_V\ \langle w_1, w_2 \rangle_W.$$

Definition 5.4.5. The dual $(V \otimes W)'$ of a tensor product space $V \otimes W$ is canonically identified with $V' \otimes W'$...

5.5 Topology (and metric), basics

The reader should know metric spaces; topological spaces are their generalization, which we soon introduce. Feel free to draw some clarifying schematic pictures on the margins!

Definition 5.5.1. A function $d : X \times X \to [0, \infty[$ is called a *metric* on the set X if for every $x, y, z \in X$ we have

- $d(x, y) = 0 \Leftrightarrow x = y$;
- $d(x, y) = d(y, x)$;
- $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Then (X, d) (or simply X when d is evident) is called a *metric space*. Sometimes a metric is called a *distance function*.

Definition 5.5.2. A family of sets $\tau \subset \mathcal{P}(X)$ is called a *topology* on the set X if

1. $\emptyset, X \in \tau$;
2. $\mathcal{U} \subset \tau \Rightarrow \bigcup \mathcal{U} \in \tau$;
3. $U, V \in \tau \Rightarrow U \cap V \in \tau$.

Then (X, τ) (or simply X when τ is evident) is called a *topological space*. The sets $U \in \tau$ are called *open sets*, and their complements $X \setminus U$ are *closed sets*.

Thus in a topological space, the empty set and the whole space are always open, **any** union of open sets is open, and an intersection of **finitely** many open sets is open. Equivalently, the whole space and the empty set are always closed, **any** intersection of closed sets is closed, and a union of **finitely** many closed sets is closed.

Definition 5.5.3. Let (X, d) be a metric space. We say that the *open ball of radius* $r > 0$ *centered at* $x \in X$ is

$$B_d(x, r) := \{y \in X \mid d(x, y) < r\}\,.$$

The *metric topology* τ_d of (X, d) is given by

$$U \in \tau_d \overset{\text{definition}}{\Leftrightarrow} \forall x \in U\ \exists r > 0 :\ B_d(x, r) \subset U.$$

A topological space (X, τ) is called *metrizable* if there is a metric d on X such that $\tau = \tau_d$.

Example. There are plenty of non-metrizable topological spaces, the easiest example being X with more than one point and with $\tau = \{\emptyset, X\}$. If X is an infinite-dimensional Banach space then the weak*-topology of $X' := \mathcal{L}(X, \mathbb{C})$ is not metrizable. The distribution spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are non-metrizable topological spaces. We shall later prove that for the compact Hausdorff spaces metrizability is equivalent to the existence of a countable base.

Definition 5.5.4. Let (X, τ) be a topological space. A family $\mathcal{B} \subset \tau$ of open sets is called a *base* (or *basis*) for the topology τ if any open set is a union of some members of $\mathcal{B}$, i.e.

$$\forall U \in \tau\ \exists \mathcal{B}' \subset \mathcal{B} :\ U = \bigcup \mathcal{B}'.$$

Example. Trivially a topology τ is a base for itself ($\forall U \in \tau : U = \bigcup \{U\}$). If (X, d) is a metric space then

$$\mathcal{B} := \{B_d(x, r) \mid x \in X,\ r > 0\}$$

constitutes a base for τ_d.

Definition 5.5.5. Let (X, τ) be a topological space. A *neighbourhood of* $x \in X$ is any open set$U \subset X$ containing x. The family of neighbourhoods of $x \in X$ is denoted by

$$\mathcal{V}_\tau(x) := \{U \in \tau \mid x \in U\}$$

(or simply $\mathcal{V}(x)$, when τ is evident).

The natural mappings (or the morphisms) between topological spaces are **continuous mappings**.

Definition 5.5.6. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is *continuous at* $x \in X$ if

$$\forall V \in \mathcal{V}_{\tau_Y}(f(x))\ \exists U \in \mathcal{V}_{\tau_X}(x) :\ f(U) \subset V.$$

Exercise 5.5.7. Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : X \to Y$ is continuous at $x \in X$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in X :\ d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$$

if and only if

$$d_X(x_n, x) \xrightarrow[n\to\infty]{} 0 \quad \Rightarrow \quad d_Y(f(x_n), f(x)) \xrightarrow[n\to\infty]{} 0$$

for every sequence $(x_n)_{n=1}^{\infty} \subset X$ (that is, $x_n \to x \Rightarrow f(x_n) \to f(x)$).

Definition 5.5.8. Let (X, τ_X) and (Y, τ_Y)be topological spaces. A mapping $f : X \to Y$ is *continuous*, denoted by $f \in C(X, Y)$, if

$$\forall V \in \tau_Y : f^{-1}(V) \in \tau_X,$$

where $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$; i.e. f is continuous if preimages of open sets are open (equivalently, preimages of closed sets are closed). In the sequel, we briefly write

$$C(X) := C(X, \mathbb{C}),$$

where $\mathbb{C}$ has the metric topology with the usual metric $(\lambda, \mu) \mapsto |\lambda - \mu|$.

Proposition 5.5.9. *Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is continuous at every $x \in X$ if and only if it is continuous.*

Proof. Suppose $f : X \to Y$ is continuous, $x \in X$, and $V \in \mathcal{V}_{\tau_Y}(f(x))$. Then $U := f^{-1}(V)$ is open, $x \in U$, and $f(U) = V$, implying the continuity at $x \in X$.

Conversely, suppose $f : X \to Y$ is continuous at every $x \in X$, and let $V \subset Y$ be open. Choose $U_x \in \mathcal{V}_{\tau_X}(x)$ such that $f(U_x) \subset V$ for every $x \in f^{-1}(V)$. Then

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x$$

is open in X. □

Exercise 5.5.10. Let X be a topological space. Show that $C(X)$ is an algebra.

Exercise 5.5.11. Prove that if $f : X \to Y$ and $g : Y \to Z$ are continuous then $g \circ f : X \to Z$ is continuous.

Definition 5.5.12. Let (X, τ_X) and (Y, τ_Y) be topological spaces. A mapping $f : X \to Y$ is called a *homeomorphism* if it is a bijection, $f \in C(X, Y)$ and $f^{-1} \in C(Y, X)$. Then X and Y are called *homeomorphic* or *topologically equivalent*, denoted by $X \cong Y$ or $f : X \cong Y$; more specifically, $f : (X, \tau_X) \cong (Y, \tau_Y)$.

Note that from the topology point of view, homeomorphic spaces can be considered equal.

Example. Of course $(x \mapsto x) : (X, \tau) \cong (X, \tau)$. The reader may check that $(x \mapsto x/(1 + |x|)) : \mathbb{R} \cong]-1, 1[$. Using algebraic topology, one can prove that $\mathbb{R}^m \cong \mathbb{R}^n$ if and only if $m = n$ (this is not trivial!).

Definition 5.5.13. Metrics d_1, d_2 on a set X are called *equivalent* if there exists $M < \infty$ such that

$$M^{-1}\ d_1(x, y) \leq d_2(x, y) \leq M\ d_1(x, y)$$

for every $x, y \in X$. An *isometry* between metric spaces (X, d_X) and (Y, d_Y) is a mapping $f : X \to Y$ satisfying $d_Y(f(x), f(y)) = d_X(x, y)$ for every $x, y \in X$; f is called an *isometric isomorphism* if it is a surjective isometry (hence a bijection with an isometric isomorphism as the inverse mapping).

Example. Any isometric isomorphism is a homeomorphism. Clearly the unbounded $\mathbb{R}$ and the bounded $]-1, 1[$ are not isometrically isomorphic. An orthogonal linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ is an isometric isomorphism, when $\mathbb{R}^n$ is endowed with the Euclidean norm. The forward shift operator on $\ell^p(\mathbb{Z})$ is an isometric isomorphism, but the forward shift operator on $\ell^p(\mathbb{N})$ is only a non-surjective isometry.

Definition 5.5.14. A topological space (X, τ) is a *Hausdorff space* if any two distinct points have some disjoint neighbourhoods, i.e.

$$\forall x, y \in X\ \exists U \in \mathcal{V}(x)\ \exists V \in \mathcal{V}(y) :\ x \neq y \Rightarrow U \cap V = \emptyset.$$

Example.

1. If τ_1 and τ_2 are topologies of X, $\tau_1 \subset \tau_2$, and (X, τ_1) is a Hausdorff space then (X, τ_2) is a Hausdorff space.
2. $(X, \mathcal{P}(X))$ is a Hausdorff space.

3. If X has more than one point and $\tau = \{\emptyset, X\}$ then (X, τ) is not Hausdorff.

4. Clearly any metric space (X, d) is a Hausdorff space; if $x, y \in X$, $x \neq y$, then $B_d(x, r) \cap B_d(y, r) = \emptyset$, when $r \leq d(x, y)/2$.

5. The distribution spaces $\mathcal{D}'(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{E}'(\mathbb{R}^n)$ are non-metrizable Hausdorff spaces.

Exercise 5.5.15. Let X be a Hausdorff space and $x \in X$. Then $\{x\} \subset X$ is a closed set.

Definition 5.5.16. Let X, Y be topological spaces with bases $\mathcal{B}_X, \mathcal{B}_Y$, respectively. Then a base for the *product topology* of $X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$ is

$$\{U \times V \mid U \in \mathcal{B}_X,\ V \in \mathcal{B}_Y\}.$$

Exercise 5.5.17. Let X, Y be metrizable. Prove that $X \times Y$ is metrizable, and that

$$(x_n, y_n) \overset{X \times Y}{\to} (x, y) \quad \Leftrightarrow \quad x_n \overset{X}{\to} x \text{ and } y_n \overset{Y}{\to} y.$$

Definition 5.5.18. Let (X, τ) be a topological space. Let $S \subset X$; its *closure* $\mathrm{cl}_\tau(S) = \overline{S}$ is the smallest closed set containing S. The set S is *dense* in X if $\overline{S} = X$; X is *separable* if it has a countable dense subset. The *boundary* of S is $\partial_\tau S = \partial S := \overline{S} \cap \overline{X \setminus S}$.

Exercise 5.5.19. Let (X, τ) be a topological space. Let $S, S_1, S_2 \subset X$. Show that
(a) $\overline{\emptyset} = \emptyset$,
(b) $S \subset \overline{S}$,
(c) $\overline{\overline{S}} = \overline{S}$,
(d) $\overline{S_1 \cup S_2} = \overline{S_1} \cup \overline{S_2}$.

Exercise 5.5.20. Let X be a set, $S, S_1, S_2 \subset X$. Let $c : \mathcal{P}(X) \to \mathcal{P}(X)$ satisfy Kuratowski's closure axioms (a-d):
(a) $c(\emptyset) = \emptyset$,
(b) $S \subset c(S)$,
(c) $c(c(S)) = c(S)$,
(d) $c(S_1 \cup S_2) = c(S_1) \cup c(S_2)$.
Show that $\tau := \{U \subset X \mid c(X \setminus U) = X \setminus U\}$ is a topology of X, and that $\mathrm{cl}_\tau(S) = c(S)$ for every $S \subset X$.

Exercise 5.5.21. Let (X, τ) be a topological space. Prove that
(a) $x \in \overline{S} \;\Leftrightarrow\; \forall U \in \mathcal{V}(x):\; U \cap S \neq \emptyset$.
(b) $\overline{S} = S \cup \partial S$.

Exercise 5.5.22. Let X, Y be topological spaces. Show that $f : X \to Y$ is continuous if and only if $f(\overline{S}) \subset \overline{f(S)}$ for every $S \subset X$.

Definition 5.5.23. A topological space (X, τ) is *disconnected* if $X = U \cup V$ for some disjoint non-empty $U, V \in \tau$; otherwise X is called *connected.* The *component* C_x of $x \in X$ is the largest connected subset containing x, i.e.

$$C_x = \bigcup \{S \subset X \mid x \in S,\; S \text{ connected}\}\,.$$

Exercise 5.5.24. Show that X is disconnected if and only if there exists $f \in C(X)$ such that $f^2 = f$, $f \not\equiv 0$, $f \not\equiv 1$.

Exercise 5.5.25. Prove that images of connected sets under continuous mappings are connected.

Exercise 5.5.26. Show that if X, Y are connected then $X \times Y$ is connected.

Exercise 5.5.27. Show that components are always closed, but sometimes they may fail to be open.

5.6 Compact Hausdorff spaces

In this section we mainly concentrate on compact Hausdorff spaces, though some results deal with more general classes of topological spaces. Roughly, Hausdorff spaces have enough open sets to distinguish between any two points, while compact spaces "do not have too many open sets". Combining these two properties, compact Hausdorff spaces form an extremely beautiful class to study.

Definition 5.6.1. Let X be a set and $K \subset X$. A family $\mathcal{S} \subset \mathcal{P}(X)$ is called a *cover of* K if

$$K \subset \bigcup \mathcal{S};$$

if the cover $\mathcal{S}$ is a finite set, it is called a *finite cover.* A cover $\mathcal{S}$ of $K \subset X$ has a *subcover* $\mathcal{S}' \subset \mathcal{S}$ if $\mathcal{S}'$ itself is a cover of K. Let (X, τ) be a topological space. An *open cover of* X is a cover $\mathcal{U} \subset \tau$ of X. A subset $K \subset X$ is *compact* (more precisely τ*-compact*) if every open cover of K has a finite subcover, i.e.

$$\forall \mathcal{U} \subset \tau\; \exists \mathcal{U}' \subset \mathcal{U}:\; K \subset \bigcup \mathcal{U} \Rightarrow K \subset \bigcup \mathcal{U}' \quad \text{and} \quad |\mathcal{U}'| < \infty.$$

We say that (X, τ) is a *compact space* if X itself is τ-compact. A topological space (X, τ) is *locally compact* if for each $x \in X$ has an neighbourhood $U \in \mathcal{V}_\tau(x)$ and a compact set $K \subset X$ such that $U \subset K$.

Example.

1. If τ_1 and τ_2 are topologies of X, $\tau_1 \subset \tau_2$, and (X, τ_2) is a compact space then (X, τ_1) is a compact space.
2. $(X, \{\emptyset, X\})$ is a compact space.
3. If $|X| = \infty$ then $(X, \mathcal{P}(X))$ is not a compact space. Clearly any space with a finite topology is compact. Even though a compact topology can be of *any* cardinality, it is in a sense "not far away from being finite".
4. A metric space is compact if and only if it is sequentially compact (i.e. every sequence contains a converging subsequence).
5. A subset $X \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded (Heine–Borel Theorem).
6. A theorem due to Frigyes Riesz asserts that a closed ball in a normed vector space over $\mathbb{C}$ (or $\mathbb{R}$) is compact if and only if the vector space is finite-dimensional.

Exercise 5.6.2. A union of two compact sets is compact.

Proposition 5.6.3. *An intersection of a compact set and a closed set is compact.*

Proof. Let $K \subset X$ be a compact set, and $C \subset X$ be a closed set. Let $\mathcal{U}$ be an open cover of $K \cap C$. Then $\{X \setminus C\} \cup \mathcal{U}$ is an open cover of K, thus having a finite subcover $\mathcal{U}'$. Then $\mathcal{U}' \setminus \{X \setminus C\} \subset \mathcal{U}$ is a finite subcover of $K \cap C$; hence $K \cap C$ is compact. □

Proposition 5.6.4. *Let X be a compact space and $f : X \to Y$ continuous. Then $f(X) \subset Y$ is compact.*

Proof. Let $\mathcal{V}$ be an open cover of $f(X)$. Then $\mathcal{U} := \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is an open cover of X, thus having a finite subcover $\mathcal{U}'$. Hence $f(X)$ is covered by $\{f(U) \mid U \in \mathcal{U}'\} \subset \mathcal{V}$. □

Corollary 5.6.5. *If X is compact and $f \in C(X)$ then $|f|$ attains its greatest value on X (here $|f|(x) := |f(x)|$).* □

5.6.1 Compact Hausdorff spaces

Theorem 5.6.6. *Let X be a Hausdorff space, $A, B \subset X$ compact subsets, and $A \cap B = \emptyset$. Then there exist open sets $U, V \subset X$ such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$. (In particular, compact sets in a Hausdorff space are closed.)*

Proof. The proof is trivial if $A = \emptyset$ or $B = \emptyset$. So assume $x \in A$ and $y \in B$. Since X is a Hausdorff space and $x \neq y$, we can choose neighbourhoods $U_{xy} \in \mathcal{V}(x)$ and $V_{xy} \in \mathcal{V}(y)$ such that $U_{xy} \cap V_{xy} = \emptyset$. The collection $\mathcal{P} = \{V_{xy} \mid y \in B\}$ is an open cover of the compact set B, so that it has a finite subcover

$$\mathcal{P}_x = \left\{V_{xy_j} \mid 1 \leq j \leq n_x\right\} \subset \mathcal{P}$$

for some $n_x \in \mathbb{N}$. Let

$$U_x := \bigcap_{j=1}^{n_x} U_{xy_j}.$$

Now $\mathcal{O} = \{U_x \mid x \in A\}$ is an open cover of the compact set A, so that it has a finite subcover

$$\mathcal{O}' = \{U_{x_i} \mid 1 \leq i \leq m\} \subset \mathcal{O}.$$

Then define

$$U := \bigcup \mathcal{O}', \quad V := \bigcap_{i=1}^{m} \bigcup \mathcal{P}_{x_i}.$$

It is an easy task to check that U and V have desired properties. □

Corollary 5.6.7. *Let X be a compact Hausdorff space, $x \in X$, and $W \in \mathcal{V}(x)$. Then there exists $U \in \mathcal{V}(x)$ such that $\overline{U} \subset W$.*

Proof. Now $\{x\}$ and $X \setminus W$ are closed sets in a compact space, thus they are compact. Since these sets are disjoint, there exist open disjoint sets $U, V \subset X$ such that $x \in U$ and $X \setminus W \subset V$; i.e.

$$x \in U \subset X \setminus V \subset W.$$

Hence $x \in U \subset \overline{U} \subset X \setminus V \subset W$. □

Proposition 5.6.8. *Let (X, τ_X) be a compact space and (Y, τ_Y) a Hausdorff space. A bijective continuous mapping $f : X \to Y$ is a homeomorphism.*

Proof. Let $U \in \tau_X$. Then $X \setminus U$ is closed, hence compact. Consequently, $f(X \setminus U)$ is compact, and due to the Hausdorff property $f(X \setminus U)$ is closed. Therefore $(f^{-1})^{-1}(U) = f(U)$ is open. $\square$

Corollary 5.6.9. *Let X be a set with a compact topology τ_2 and a Hausdorff topology τ_1. If $\tau_1 \subset \tau_2$ then $\tau_1 = \tau_2$.*

Proof. The identity mapping $(x \mapsto x) : X \to X$ is a continuous bijection from (X, τ_2) to (X, τ_1). $\square$

A more direct proof of the Corollary. Let $U \in \tau_2$. Since (X, τ_2) is compact and $X \setminus U$ is τ_2-closed, $X \setminus U$ must be τ_2-compact. Now $\tau_1 \subset \tau_2$, so that $X \setminus U$ is τ_1-compact. (X, τ_1) is Hausdorff, implying that $X \setminus U$ is τ_1-closed, thus $U \in \tau_1$; this yields $\tau_2 \subset \tau_1$. $\square$

5.6.2 Functional separation

A family $\mathcal{F}$ of mappings $X \to \mathbb{C}$ is said to *separate the points of the set* X if there exists $f \in \mathcal{F}$ such that $f(x) \neq f(y)$ whenever $x \neq y$. Later in these notes we shall discover that a compact space X is metrizable if and only if $C(X)$ is separable and separates the points of X.

Urysohn's Lemma is the key result of this section:

Theorem 5.6.10. (Urysohn's Lemma (1923?).) *Let X be a compact Hausdorff space, $A, B \subset X$ closed non-empty sets, $A \cap B = \emptyset$. Then there exists $f \in C(X)$ such that*

$$0 \leq f \leq 1, \quad f(A) = \{0\}, \quad f(B) = \{1\}.$$

Proof. The set $\mathbb{Q} \cap [0,1]$ is countably infinite; let $\phi : \mathbb{N} \to \mathbb{Q} \cap [0,1]$ be a bijection satisfying $\phi(0) = 0$ and $\phi(1) = 1$. Choose open sets $U_0, U_1 \subset X$ such that

$$A \subset U_0 \subset \overline{U_0} \subset U_1 \subset \overline{U_1} \subset X \setminus B.$$

Then we proceed inductively as follows: Suppose we have chosen open sets $U_{\phi(0)}, U_{\phi(1)}, \ldots, U_{\phi(n)}$ such that

$$\phi(i) < \phi(j) \Rightarrow \overline{U_{\phi(i)}} \subset U_{\phi(j)}.$$

Let us choose an open set $U_{\phi(n+1)} \subset X$ such that

$$\phi(i) < \phi(n+1) < \phi(j) \Rightarrow \overline{U_{\phi(i)}} \subset U_{\phi(n+1)} \subset \overline{U_{\phi(n+1)}} \subset U_{\phi(j)}$$

whenever $0 \leq i, j \leq n$. Let us define

$$r < 0 \Rightarrow U_r := \emptyset, \quad s > 1 \Rightarrow U_s := X.$$

Hence for each $q \in \mathbb{Q}$ we get an open set $U_q \subset X$ such that

$$\forall r, s \in \mathbb{Q}: \ r < s \Rightarrow \overline{U_r} \subset U_s.$$

Let us define a function $f : X \to [0,1]$ by

$$f(x) := \inf \{r : \ x \in U_r\}.$$

Clearly $0 \leq f \leq 1$, $f(A) = \{0\}$ and $f(B) = \{1\}$.

Let us prove that f is continuous. Take $x \in X$ and $\varepsilon > 0$. Take $r, s \in \mathbb{Q}$ such that

$$f(x) - \varepsilon < r < f(x) < s < f(x) + \varepsilon;$$

then f is continuous at x, since $x \in U_s \setminus \overline{U_r}$ and for every $y \in U_s \setminus \overline{U_r}$ we have $|f(y) - f(x)| < \varepsilon$. Thus $f \in C(X)$. □

Corollary 5.6.11. *Let X be a compact space. Then $C(X)$ separates the points of X if and only if X is Hausdorff.*

Exercise 5.6.12. Prove Corollary 5.6.11.

5.7 Some results from analysis

The reader probably already knows the results in this section, but if not, proving them provides nice challenges. Proofs can also be found in many books on measure theory or functional analysis.

Theorem 5.7.1. (Lebesgue Dominated Convergence Theorem.) *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $f_k, f : X \to [-\infty, \infty]$ be $\mathcal{M}$-measurable functions such that $f_k \to_{k\to\infty} f$ μ -almost everywhere, and let $|f_k| \leq g$ μ -almost everywhere, with $g : X \to [-\infty, \infty]$ being μ-integrable. Then*

$$\int |f_k - f| \, \mathrm{d}\mu \xrightarrow[k\to\infty]{} 0.$$

Proof. See e.g. [5] or [17]. □

Theorem 5.7.2. (Fubini Theorem.) *Let $(X, \mathcal{M}_X, \mu), (Y, \mathcal{M}_Y, \nu)$ be complete measure spaces. Let $\mu \times \nu$ be the complete product measure obtained from the product outer measure of μ and ν, and let $\mathcal{M}_{X\times Y}$ be the corresponding σ-algebra of measurable sets. If $A \in \mathcal{M}_X$ and $B \in \mathcal{M}_Y$ then $A \times B \in \mathcal{M}_{X\times Y}$ and*

$$(\mu \times \nu)(A \times B) = \mu(A)\ \nu(B).$$

If $S \in \mathcal{M}_{X\times Y}$ is σ-finite with respect to $\mu \times \nu$ then

$$\begin{aligned}
S_y := \{x \in X :\ (x,y) \in S\} \in \mathcal{M}_X & \qquad \text{for } \nu\text{−almost every } y \in Y,\\
S^x := \{y \in Y :\ (x,y) \in S\} \in \mathcal{M}_Y & \qquad \text{for } \mu\text{−almost every } x \in X,\\
y \mapsto \mu(S_y) & \qquad \text{is } \mathcal{M}_Y\text{−measurable},\\
x \mapsto \nu(S^x) & \qquad \text{is } \mathcal{M}_X\text{−measurable and}\\
(\mu \times \nu)(S) &= \int_X \nu(S^x)\ \mathrm{d}\mu(x)\\
&= \int_Y \mu(S_y)\ \mathrm{d}\nu(y).
\end{aligned}$$

If $f : X \times Y \to [-\infty, \infty]$ is $(\mu \times \nu)$-integrable then

$$\begin{aligned}
y \mapsto f(x,y) & \qquad \text{is } \nu\text{−integrable for } \mu\text{−almost every } x \in X,\\
x \mapsto f(x,y) & \qquad \text{is } \mu\text{−integrable for } \nu\text{−almost every } y \in Y,\\
x \mapsto \int_Y f(x,y)\ \mathrm{d}\nu(y) & \qquad \text{is } \mu\text{−integrable},\\
y \mapsto \int_X f(x,y)\ \mathrm{d}\mu(x) & \qquad \text{is } \nu\text{−integrable and}\\
\int_{X\times Y} f\ \mathrm{d}(\mu \times \nu) &= \int_X \int_Y f(x,y)\ \mathrm{d}\nu(y)\ \mathrm{d}\mu(x)\\
&= \int_Y \int_X f(x,y)\ \mathrm{d}\mu(x)\ \mathrm{d}\nu(y).
\end{aligned}$$

Proof. See e.g. [5]. □

Theorem 5.7.3. (Riesz Representation Theorem [F. Riesz].) *Let $\mathcal{H}$ be a Hilbert space and $F : \mathcal{H} \to \mathbb{C}$ bounded and linear. Then there exists a unique $w \in \mathcal{H}$ such that $F(u) = \langle u, w\rangle_{\mathcal{H}}$ for every $u \in \mathcal{H}$.*

Proof. See e.g. [12] or [16]. □

Definition 5.7.4. The *weak topology* of a Hilbert space $\mathcal{H}$ is the smallest topology for which $(u \mapsto \langle u, v\rangle_{\mathcal{H}}) : \mathcal{H} \to \mathbb{C}$ is continuous whenever $v \in \mathcal{H}$.

Theorem 5.7.5. (Banach–Alaoglu Theorem.) *Let $\mathcal{H}$ be a Hilbert space. Its closed unit ball*

$$\mathbb{B} = \{v \in \mathcal{H} : \ \|v\|_{\mathcal{H}} \leq 1\}$$

is compact in the weak topology.

Proof. See e.g. [16]. □

Theorem 5.7.6. (Hilbert–Schmidt Spectral Theorem.) *Let $\mathcal{H}$ be a Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be a compact self-adjoint operator. Then the spectrum*

$$\sigma(A) = \{\lambda \in \mathbb{C} : \ \lambda I - A \text{ not invertible}\}$$

is at most countable and the only possible accumulation point of $\sigma(A)$ is $0 \in \mathbb{C}$. Moreover, if $0 \neq \lambda \in \sigma(A)$ then $\dim(\mathrm{Ker}(\lambda I - A)) < \infty$ and

$$\mathcal{H} = \bigoplus_{\lambda \in \sigma(A)} \mathrm{Ker}(\lambda I - A).$$

Proof. See [4]. □

5.8 Appendix on trace

Definition 5.8.1. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_j \mid j \in J\}$. Let $A \in \mathcal{L}(\mathcal{H})$. Let us denote

$$\|A\|_{\mathcal{L}^1} := \sum_{j \in J} |\langle A e_j, e_j\rangle_{\mathcal{H}}|\,;$$

this is the *trace norm* of A, and the *trace class* is the (Banach) space

$$\mathcal{L}^1 = \mathcal{L}^1(\mathcal{H}) := \{A \in \mathcal{L}(\mathcal{H}) : \ \|A\|_{\mathcal{L}^1} < \infty\}\,.$$

The *trace* is the linear functional $\mathrm{Tr} : \mathcal{L}^1(\mathcal{H}) \to \mathbb{C}$,

$$A \mapsto \sum_{j \in J} \langle A e_j, e_j\rangle_{\mathcal{H}}.$$

Exercise 5.8.2. Verify that the definition of the trace is independent of the choice of the orthonormal basis for $\mathcal{H}$. Consequently, if $(a_{ij})_{i,j\in J}$ is the matrix representation of $A \in \mathcal{L}^1$ with respect to the chosen basis, then $\mathrm{Tr}(A) = \sum_{j\in J} a_{jj}$.

Exercise 5.8.3. Prove the following properties of the trace functional:

$$\begin{aligned}
\mathrm{Tr}(AB) &= \mathrm{Tr}(BA),\\
\mathrm{Tr}(A^*) &= \overline{\mathrm{Tr}(A)},\\
\mathrm{Tr}(A^*A) &\geq 0,\\
\mathrm{Tr}(A\oplus B) &= \mathrm{Tr}(A)+\mathrm{Tr}(B),\\
\dim(\mathcal{H})<\infty &\Rightarrow \begin{cases}\mathrm{Tr}(I_{\mathcal{H}}) = \dim(\mathcal{H}),\\ \mathrm{Tr}(A\otimes B) = \mathrm{Tr}(A)\ \mathrm{Tr}(B).\end{cases}
\end{aligned}$$

Exercise 5.8.4. Show that the trace on a finite-dimensional vector space is independent of the choice of inner product. Thus, the trace of a square matrix is defined to be the sum of its diagonal elements; moreover, the trace is the sum of the eigenvalues (with multiplicities counted).

Exercise 5.8.5. Let $\mathcal{H}$ be finite-dimensional. Let $f : \mathcal{L}(\mathcal{H}) \to \mathbb{C}$ be a linear functional satisfying

$$\begin{cases} f(AB) = f(BA),\\ f(A^*A) \geq 0,\\ f(I_{\mathcal{H}}) = \dim(\mathcal{H}) \end{cases}$$

for every $A, B \in \mathcal{L}(\mathcal{H})$. Show that $f = \mathrm{Tr}$.

Definition 5.8.6. The space of *Hilbert–Schmidt operators* is

$$\mathcal{L}^2 = \mathcal{L}^2(\mathcal{H}) := \left\{A \in \mathcal{L}(\mathcal{H}) :\ A^*A \in \mathcal{L}^1(\mathcal{H})\right\},$$

and it can be endowed with a Hilbert space structure via the inner product

$$\langle A, B\rangle_{\mathcal{L}^2} := \mathrm{Tr}(AB^*).$$

The Hilbert–Schmidt norm is then

$$\|A\|_{\mathcal{L}^2} := \langle A, A\rangle_{\mathcal{L}^2}^{1/2}.$$

Remark 5.8.7. In general, there are inclusions $\mathcal{L}^1 \subset \mathcal{L}^2 \subset \mathcal{K} \subset \mathcal{L}^\infty$, where $\mathcal{L}^\infty := \mathcal{L}(\mathcal{H})$ and $\mathcal{K} \subset \mathcal{L}^\infty$ is the subspace of compact linear operators. Moreover,

$$\|A\|_{\mathcal{L}^\infty} \leq \|A\|_{\mathcal{L}^2} \leq \|A\|_{\mathcal{L}^1}$$

for every $A \in \mathcal{L}^\infty$. One can show that the dual $\mathcal{K}' = \mathcal{L}(\mathcal{K}, \mathbb{C})$ is isometrically isomorphic to $\mathcal{L}^1$, and that $(\mathcal{L}^1)'$ is isometrically isomorphic to $\mathcal{L}^\infty$. In the latter case, it turns out that a bounded linear functional on $\mathcal{L}^1$ is of the form $A \mapsto \mathrm{Tr}(AB)$ for some $B \in \mathcal{L}^\infty$. These phenomena are related to properties of the sequence spaces $\ell^p = \ell^p(\mathbb{Z}^+)$. In analogy to the operator spaces, $\ell^1 \subset \ell^2 \subset c_0 \subset \ell^\infty$, where c_0 is the space of sequences converging to 0, playing the counterpart of space $\mathcal{K}$.

5.9 Appendix on polynomial approximation.

In this section we study densities of subalgebras in $C(X)$ for a compact Hausdorff space X. These results will be applied in characterizing function algebras among Banach algebras. First we study continuous functions on $[a, b] \subset \mathbb{R}$:

Theorem 5.9.1. (Weierstrass Theorem (1885).) *Polynomials are dense in* $C([a, b])$.

Proof. Evidently, it is enough to consider the case $[a, b] = [0, 1]$. Let $f \in C([0, 1])$, and let $g(x) = f(x) - (f(0) + (f(1) - f(0))x)$; then $g \in C(\mathbb{R})$ if we define $g(x) = 0$ for $x \in \mathbb{R} \setminus [0, 1]$. For $n \in \mathbb{N}$ let us define $k_n : \mathbb{R} \to [0, \infty[$ by

$$k_n(x) := \begin{cases} \frac{(1-x^2)^n}{\int_{-1}^1 (1-t^2)^n \, \mathrm{d}t}, & \text{when } |x| < 1, \\ 0, & \text{when } |x| \geq 1. \end{cases}$$

Then define $P_n := g * k_n$ (*convolution* of g and k_n), that is

$$\begin{aligned} P_n(x) &= \int_{-\infty}^{\infty} g(x-t)\ k_n(t)\ \mathrm{d}t = \int_{-\infty}^{\infty} g(t)\ k_n(x-t)\ \mathrm{d}t \\ &= \int_0^1 g(t)\ k_n(x-t)\ \mathrm{d}t, \end{aligned}$$

and from this last formula we see that P_n is a polynomial on $[0, 1]$. Notice that P_n is real-valued if f is real-valued. Take any $\varepsilon > 0$. Function g is uniformly continuous, so that there exists $\delta > 0$ such that

$$\forall x, y \in \mathbb{R} : \ |x - y| < \delta \Rightarrow |g(x) - g(y)| < \varepsilon.$$

Let $\|g\| = \max_{t\in[0,1]} |g(t)|$. Take $x \in [0,1]$. Then

$$\begin{aligned}
|P_n(x) - g(x)| &= \left| \int_{-\infty}^{\infty} g(x-t)\ k_n(t)\ \mathrm{d}t - g(x) \int_{-\infty}^{\infty} k_n(t)\ \mathrm{d}t \right| \\
&= \left| \int_{-1}^{1} (g(x-t) - g(x))\ k_n(t)\ \mathrm{d}t \right| \\
&\leq \int_{-1}^{1} |g(x-t) - g(x)|\ k_n(t)\ \mathrm{d}t \\
&\leq \int_{-1}^{-\delta} 2\|g\|\ k_n(t)\ \mathrm{d}t + \int_{-\delta}^{\delta} \varepsilon\ k_n(t)\ \mathrm{d}t + \int_{\delta}^{1} 2\|g\|\ k_n(t)\ \mathrm{d}t \\
&\leq 4\|g\| \int_{\delta}^{1} k_n(t)\ \mathrm{d}t + \varepsilon.
\end{aligned}$$

The reader may verify that $\int_\delta^1 k_n(t)\ \mathrm{d}t \to_{n\to\infty} 0$ for every $\delta > 0$. Hence $\|Q_n - f\| \to_{n\to\infty} 0$, where $Q_n(x) = P_n(x) + f(0) + (f(1) - f(0))x$. □

Exercise 5.9.2. Show that the last claim in the proof of the Weierstrass Theorem 5.9.1 is true.

For $f : X \to \mathbb{C}$ let us define $f^* : X \to \mathbb{C}$ by $f^*(x) := \overline{f(x)}$, and define $|f| : X \to \mathbb{C}$ by $|f|(x) := |f(x)|$. A subalgebra $\mathcal{A} \subset \mathcal{F}(X)$ is called *involutive* if $f^* \in \mathcal{A}$ whenever $f \in \mathcal{A}$. Notice that our definition of an algebra contains the existence of the unit element $\mathbf{1}$.

Theorem 5.9.3. (Stone–Weierstrass Theorem (1937).) *Let X be a compact space. Let $\mathcal{A} \subset C(X)$ be an involutive subalgebra separating the points of X. Then $\mathcal{A}$ is dense in $C(X)$.*

Proof. If $f \in \mathcal{A}$ then $f^* \in \mathcal{A}$, so that the real part $\Re f = \dfrac{f + f^*}{2}$ belongs to $\mathcal{A}$. Let us define

$$\mathcal{A}_{\mathbb{R}} := \{\Re f \mid f \in \mathcal{A}\};$$

this is a $\mathbb{R}$-subalgebra of the $\mathbb{R}$-algebra $C(X,\mathbb{R})$ of continuous real-valued functions on X. Then

$$\mathcal{A} = \{f + \mathrm{i}g \mid f, g \in \mathcal{A}_{\mathbb{R}}\},$$

so that $\mathcal{A}_{\mathbb{R}}$ separates the points of X. If we can show that $\mathcal{A}_{\mathbb{R}}$ is dense in $C(X,\mathbb{R})$ then $\mathcal{A}$ would be dense in $C(X)$.

First we have to show that $\overline{\mathcal{A}_{\mathbb{R}}}$ is closed under taking maximums and minimums. For $f, g \in C(X, \mathbb{R})$ we define

$$\max(f,g)(x) := \max(f(x), g(x)), \quad \min(f,g)(x) := \min(f(x), g(x)).$$

Notice that $\overline{\mathcal{A}_{\mathbb{R}}}$ is an algebra over the field $\mathbb{R}$. Since

$$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}, \quad \min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2},$$

it is enough to prove that $|h| \in \overline{\mathcal{A}_{\mathbb{R}}}$ whenever $h \in \overline{\mathcal{A}_{\mathbb{R}}}$. Let $h \in \overline{\mathcal{A}_{\mathbb{R}}}$. By the Weierstrass Theorem 5.9.1 there is a sequence of polynomials $P_n : \mathbb{R} \to \mathbb{R}$ such that

$$P_n(x) \xrightarrow[n\to\infty]{} |x|$$

uniformly on the interval $[-\|h\|, \|h\|]$. Thereby

$$\| |h| - P_n(h) \| \xrightarrow[n\to\infty]{} 0,$$

where $P_n(h)(x) := P_n(h(x))$. Since $P_n(h) \in \overline{\mathcal{A}_{\mathbb{R}}}$ for every n, this implies that $|h| \in \overline{\mathcal{A}_{\mathbb{R}}}$. Now we know that $\max(f,g), \min(f,g) \in \overline{\mathcal{A}_{\mathbb{R}}}$ whenever $f, g \in \overline{\mathcal{A}_{\mathbb{R}}}$.

Now we are ready to prove that $f \in C(X, \mathbb{R})$ can be approximated by elements of $\mathcal{A}_{\mathbb{R}}$. Take $\varepsilon > 0$ and $x, y \in X$, $x \neq y$. Since $\mathcal{A}_{\mathbb{R}}$ separates the points of X, we may pick $h \in \mathcal{A}_{\mathbb{R}}$ such that $h(x) \neq h(y)$. Let $g_{xx} = f(x)\mathbb{I}$, and let

$$g_{xy}(z) := \frac{h(z) - h(y)}{h(x) - h(y)}\ f(x) + \frac{h(z) - h(x)}{h(y) - h(x)}\ f(y).$$

Here $g_{xx}, g_{xy} \in \mathcal{A}_{\mathbb{R}}$, since $\mathcal{A}_{\mathbb{R}}$ is an algebra. Furthermore,

$$g_{xy}(x) = f(x), \quad g_{xy}(y) = f(y).$$

Due to the continuity of g_{xy}, there is an open set $V_{xy} \in \mathcal{V}(y)$ such that

$$z \in V_{xy} \ \Rightarrow \ f(z) - \varepsilon < g_{xy}(z).$$

Now $\{V_{xy} \mid y \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{V_{xy_j} \mid 1 \leq j \leq n\}$. Define

$$g_x := \max_{1 \leq j \leq n} g_{xy_j};$$

$g_x \in \overline{\mathcal{A}_\mathbb{R}}$, because $\overline{\mathcal{A}_\mathbb{R}}$ is closed under taking maximums. Moreover,

$$\forall z \in X : \ f(z) - \varepsilon < g_x(z).$$

Due to the continuity of g_x (and since $g_x(x) = f(x)$), there is an open set $U_x \in \mathcal{V}(x)$ such that

$$z \in U_x \ \Rightarrow \ g_x(z) < f(z) + \varepsilon.$$

Now $\{U_x \mid x \in X\}$ is an open cover of the compact space X, so that there is a finite subcover $\{U_{x_i} \mid 1 \leq i \leq m\}$. Define

$$g := \min_{1 \leq i \leq m} g_{x_i};$$

$g \in \overline{\mathcal{A}_\mathbb{R}}$, because $\overline{\mathcal{A}_\mathbb{R}}$ is closed under taking minimums. Moreover,

$$\forall z \in X : \ g(z) < f(z) + \varepsilon.$$

Thus

$$f(z) - \varepsilon < \min_{1 \leq i \leq m} g_{x_i}(z) = g(z) < f(z) + \varepsilon,$$

that is $|g(z) - f(z)| < \varepsilon$ for every $z \in X$, i.e. $\|g - f\| < \varepsilon$. Hence $\mathcal{A}_\mathbb{R}$ is dense in $C(X, \mathbb{R})$ implying that $\mathcal{A}$ is dense in $C(X)$. □

Remark 5.9.4. Notice that under the assumptions of the Stone–Weierstrass Theorem 5.9.3, the compact space is actually a compact Hausdorff space, since continuous functions separate the points.

Let us give a non-comprehensive list of sources that have had a positive impact in forming these lecture notes. I hope studying the earlier versions of the text has not caused permanent damage; I want to express my gratitude to Lauri Harhanen, Tapio Helin, Teemu Lukkari, Mathias Masson, and many other unnamed students. I hope the reader may find the following references useful.

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