Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**
Stenberg/Juntunen

Exercise 8

Problem 1
A scalar value function $u : \Omega \to \mathbb{R}$ is called subharmonic if

$$\Delta u \geq 0.$$

A well-known result from analysis is that a subharmonic function attains its maximum value on the boundary $\partial\Omega$. Consider the shear force vector

$$\tau = 2G\alpha \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1} \right)$$

in which ϕ is the stress function. Prove, using the theorem above, that $|\tau|$ attains its maximum value at the boundary $\partial\Omega$.

Problem 2
Compute approximately the torsional rigidity for the square $[0, a] \times [0, a]$ and the triangle

$$\{(x, y) \,|\, x \geq 0, y \geq 0, x + y \leq a\}$$

using the Galerkin method with only one basis function, i.e. the lowest polynomial in x and y that vanish on the boundary.
(Compare to the exact values. For the square see the previous exercise. For the triangle the exact value will be computed by Antti H.)

Problem 3 **(home exercise)**
Consider a thin tube with central radius R and thickness t and the same tube cut open. Derive the approximate torsional rigidities for both cases. Let the tube be loaded with the moment M. What are the maximal shear forces in the tube for the two cases?

Problem 1

$|\tau|$ is positive so it attains its maximum as $|\tau|^2$ attains its maximum.

$$\tau = 2G\alpha\left[\varphi^{(0,1)}, -\varphi^{(1,0)}\right]$$

$$|\tau|^2 = 4G^2\alpha^2 \underbrace{\left(\varphi^{(0,1)^2} + \varphi^{(1,0)^2}\right)}_{=:\,p}$$

$$\frac{\partial p}{\partial x_1} = 2\varphi^{(0,1)}\varphi^{(1,1)} + 2\varphi^{(1,0)}\varphi^{(2,0)}$$

$$\frac{\partial p}{\partial x_2} = 2\varphi^{(0,1)}\varphi^{(0,2)} + 2\varphi^{(1,0)}\varphi^{(1,1)}$$

$$\frac{\partial^2 p}{\partial x_1^2} = 2\varphi^{(1,1)^2} + 2\varphi^{(0,1)}\varphi^{(2,1)} + 2\varphi^{(2,0)^2} + 2\varphi^{(1,0)}\varphi^{(3,0)}$$

$$\frac{\partial^2 p}{\partial x_2^2} = 2\varphi^{(0,2)^2} + 2\varphi^{(0,1)}\varphi^{(0,3)} + 2\varphi^{(1,1)^2} + 2\varphi^{(1,0)}\varphi^{(1,2)}$$

$$\frac{1}{2}\Delta p = 2\varphi^{(1,1)^2} + \varphi^{(2,0)^2} + \varphi^{(0,2)^2}$$

$$+ \varphi^{(0,1)}\left\{\varphi^{(2,1)} + \varphi^{(0,3)}\right\}$$

$$+ \varphi^{(1,0)}\left\{\varphi^{(3,0)} + \varphi^{(1,2)}\right\}$$

$$= 2\varphi^{(1,1)^2} + \varphi^{(3,0)^2} + \varphi^{(0,2)^2}$$

$$+ \varphi^{(0,1)}\underbrace{\frac{\partial}{\partial x_2}\left\{\underbrace{\varphi^{(2,0)} + \varphi^{(0,2)}}_{=-1}\right\}}_{=0}$$

$$+ \varphi^{(1,0)}\frac{\partial}{\partial x_1}\left\{\varphi^{(2,0)} + \varphi^{(0,2)}\right\}$$

$$= 2\varphi^{(1,1)^2} + \varphi^{(2,0)^2} + \varphi^{(0,2)^2}$$

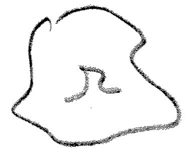
$\geq 0 \quad \Rightarrow$ subharmonic

$\Rightarrow |\vec{c}|^2$ attains its max at boundary

$\Rightarrow |\vec{c}|$ ——— 〃 ———

Problem 2

For domain Ω



we look for $\psi(x,y)$ s.t.

$\psi(x,y) = 0$ on $\partial\Omega$ and

solve problem $\begin{cases} \Delta\varphi = -1 & \Omega \\ \varphi = 0 & \partial\Omega \end{cases}$

$\Rightarrow$ Variational form.

$$\int_\Omega \nabla\varphi \cdot \nabla w \, dx = + \int_\Omega w \, dx \quad \forall w$$

Now we have only one basis function i.e. ψ and we say that

$$\varphi = c\psi \quad , c \in \mathbb{R}.$$

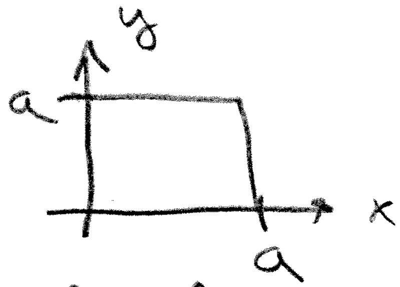
i.e. we want to solve coef c.

Clearly

$$c \int_\Omega \nabla\psi \cdot \nabla\psi \, dx = + \int_\Omega \psi \, dx.$$

$$\Leftrightarrow \quad c = \frac{+\int_\Omega \psi \, dx}{\int_\Omega \nabla\psi \cdot \nabla\psi \, dx}$$

For square



We need to now find the $\psi(x,y)$.

We assume $\psi = X(x)\, Y(y)$

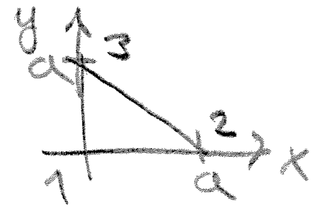
And Immediately see that

$Y = y(y-a)$ and $X = x(x-a)$

to have zero boundary.

$\Rightarrow \psi = xy(y-a)(x-a)$.

For triangle



the the usual basis is

$$\begin{cases} v_1 = 1 - \frac{x+y}{a} \\ v_2 = \frac{x}{a} \\ v_3 = \frac{y}{a} \end{cases}$$

and we set

$$\psi = v_1 v_2 v_3$$

$$= \frac{xy}{a^2} - \frac{(x+y)xy}{a^3}$$

The torsional rigidity is

$$J = 4 \int_\Omega \left(\frac{\partial \varphi}{\partial x_1}\right)^2 + \left(\frac{\partial \varphi}{\partial x_2}\right)^2 dx = 4 \int_\Omega |\nabla \varphi|^2 dx$$

So we have

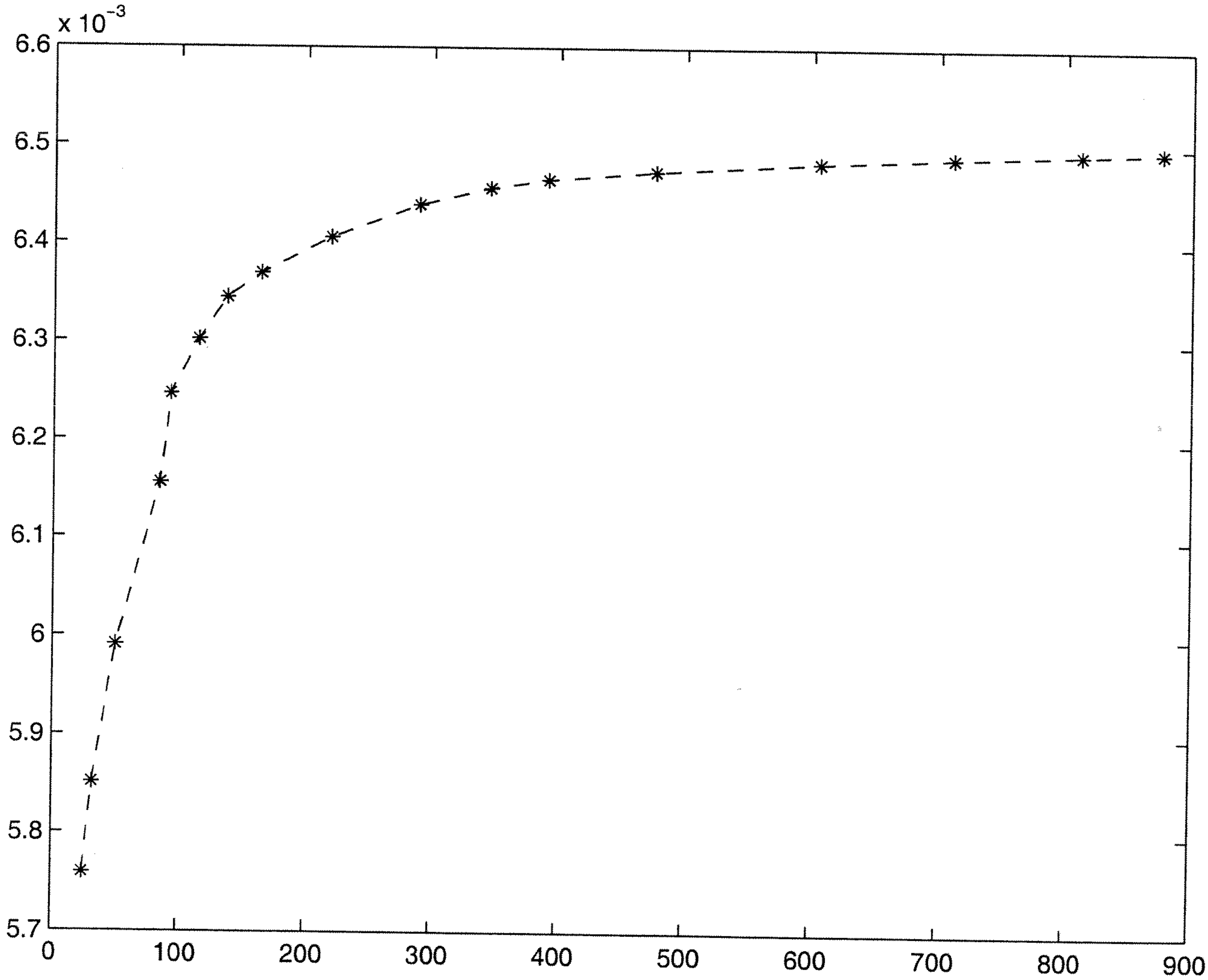
$$J = 4c^2 \int_\Omega |\nabla \psi|^2 dx.$$

where $$c = \frac{-\int_\Omega \psi \, dx}{\int_\Omega |\nabla \psi|^2 dx}$$

$$\Rightarrow J = 4 \frac{\left(\int_\Omega \psi \, dx\right)^2}{\left(\int_\Omega |\nabla \psi|^2 dx\right)^2} \left(\int_\Omega |\nabla \psi|^2 dx\right)$$

$$= \frac{4\left(\int_\Omega \psi \, dx\right)^2}{\int_\Omega |\nabla \psi|^2 dx}$$

$\frac{1}{4} J$



x 10^-3
6.6
6.5
6.4
6.3
6.2
6.1
6
5.9
5.8
5.7
0
100
200
300
400
500
600
700
800
900

```
In[58]:= ψ[x_, y_] = x * y * (x - a) * (y - a)

Out[58]= x (-a + x) y (-a + y)

In[59]:= gradψ = {D[ψ[x, y], x], D[ψ[x, y], y]}

Out[59]= {x y (-a + y) + (-a + x) y (-a + y), x (-a + x) y + x (-a + x) (-a + y)}

In[60]:= Jsq = 4 * (Integrate[ψ[x, y], {x, 0, a}, {y, 0, a}]) ^ 2 /
           Integrate[gradψ.gradψ, {x, 0, a}, {y, 0, a}]
```

Out[60]= $\frac{5\,a^4}{36}$

```
In[61]:= ψ[x_, y_] = x * y / a ^ 2 - (x + y) * x * y / a ^ 3
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Out[61]= $\frac{x\,y}{a^2} - \frac{x\,y\,(x+y)}{a^3}$

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In[62]:= gradψ = {D[ψ[x, y], x], D[ψ[x, y], y]}
```

Out[62]= $\left\{\frac{y}{a^2} - \frac{x\,y}{a^3} - \frac{y\,(x+y)}{a^3},\ \frac{x}{a^2} - \frac{x\,y}{a^3} - \frac{x\,(x+y)}{a^3}\right\}$

```
In[63]:= Jtri = 4 * (Integrate[ψ[x, y], {x, 0, a}, {y, 0, a - x}]) ^ 2 /
           Integrate[gradψ.gradψ, {x, 0, a}, {y, 0, a - x}]
```

Out[63]= $\frac{a^4}{40}$

```
In[64]:= N[Jtri /. a -> 1]

Out[64]= 0.025
```

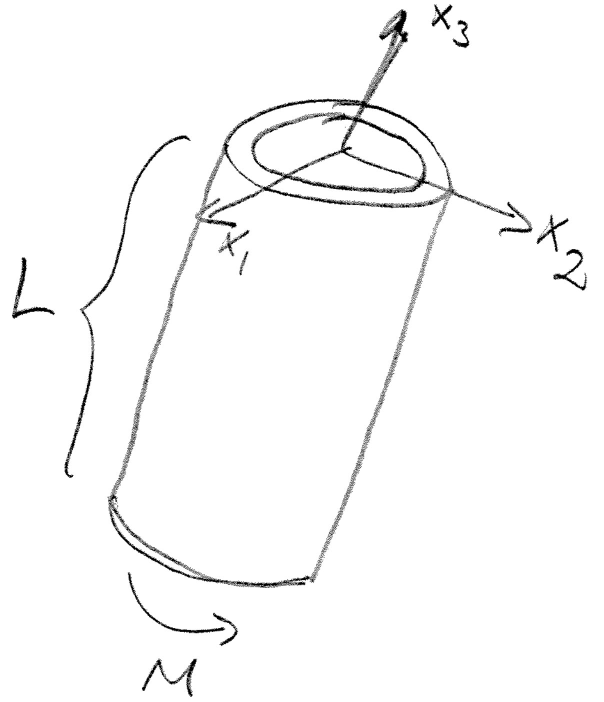
exact ≈ 0,0165

```
In[65]:= N[Jsq /. a → 1]

Out[65]= 0.138889
```

exact ≈ 2,24

Problem 3

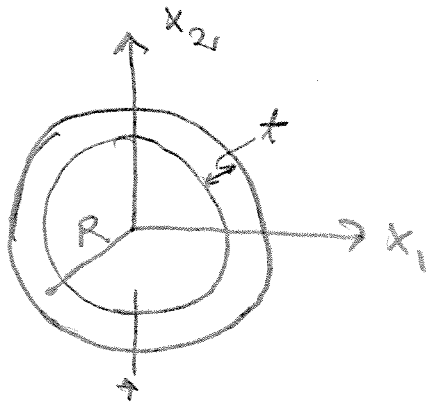


Stress at end is

$$\underline{\tau} = \underline{\underline{\sigma}} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{13} \\ \sigma_{23} \\ 0 \end{bmatrix}$$

According to our assumptions

$$\underline{\tau} = 2G\,\omega_3' \begin{bmatrix} \frac{\partial \varphi}{\partial x_2} \\ -\frac{\partial \varphi}{\partial x_1} \end{bmatrix}$$



Compute stress here!

We assume that φ varies only in t-direction.

$\Rightarrow$ At computing point $\frac{\partial \varphi}{\partial x_1} = 0$.

$$\Rightarrow \underline{\tau} = 2G\,\omega_3' \begin{bmatrix} \frac{\partial \varphi}{\partial x_2} \\ 0 \end{bmatrix}$$

Rotation angle ω is solved from

$$\begin{cases} GJ\omega'' = 0 & z \in (0, L) \\ GJ\omega' = M & z = L \end{cases}$$

$\Rightarrow$ ω is linear

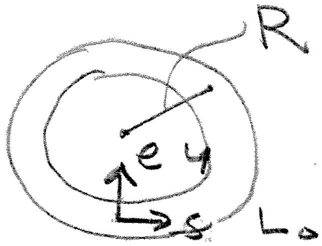
$\Rightarrow$ ω' is constant

and $\omega' = \dfrac{M}{GJ}$

So we have

$$\underline{\tau} = 2G\frac{M}{GJ}\begin{bmatrix} \frac{\partial \varphi}{\partial x_2} \\ 0 \end{bmatrix} = \frac{2M}{J}\begin{bmatrix} \frac{\partial \varphi}{\partial x_2} \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad |\underline{\tau}|^2 = \frac{4M^2}{J^2}\left(\frac{\partial \varphi}{\partial x_2}\right)^2$$



$$\varphi(e) = -\frac{1}{2}\left(e-\frac{t}{2}\right)\left(e+\frac{t}{2}\right) + \frac{H}{t}\left(e+\frac{t}{2}\right)$$

where
$$H = \frac{A}{\oint \frac{1}{t}\,ds} \underset{\substack{\uparrow \\ t=\text{constant}}}{=} \frac{\cancel{\pi} R^{\cancel{2}}}{\frac{1}{t}\,2\cancel{\pi}\cancel{R}} = \frac{Rt}{2}$$

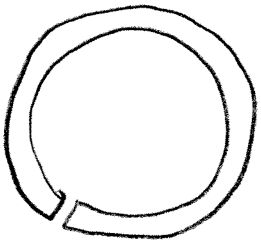
$$\frac{\partial\varphi}{\partial x_2} = \frac{\partial\varphi}{\partial e} = -\frac{1}{2}\left(e+\frac{t}{2}\right) - \frac{1}{2}\left(e-\frac{t}{2}\right) + \frac{H}{t}$$

$$= -e + \frac{H}{t} = -e + \frac{Rt}{2t} = -e + \frac{R}{2}$$

$$\approx \frac{R}{2} \quad \text{if} \quad t \ll R$$

$$J = 4HA = 4\cdot\frac{Rt}{2}\cdot\pi R^2 = 2\pi R^3 t$$

$$\Rightarrow |\tau|^2 = \frac{\cancel{4}M^2}{\cancel{4}\pi^2 R^{\cancel{6}\,4} t^2}\cdot\frac{\cancel{R^2}}{4} = \frac{M^2}{4\pi^2 R^4 t^2}$$



$$\varphi(e) = \frac{1}{2} e(t-e)$$

$$\frac{\partial \varphi}{\partial x_2} = \frac{\partial \varphi}{\partial e} = \frac{1}{2}(t-e) - \frac{1}{2}e = \frac{t}{2} - e$$

$$J = \frac{1}{3} bt^3 = \frac{1}{3}(2\pi R)t^3 = \frac{2\pi R t^3}{3}$$

$$|\tau|^2 = \frac{\cancel{4}M^2 \cdot 9}{\cancel{4}\pi^2 R^2 t^6} \cdot \left(\frac{t}{2} - e\right)^2$$

max reunalla, eli kun

$e=0$ tai $e=t$

$$\Rightarrow \left(\frac{t}{2} - e\right)^2 = \left(\frac{t}{2} - 0\right)^2 = \frac{t^2}{4}$$

$$\Rightarrow |\tau|^2 = \frac{9M^2}{\pi^2 R^2 t^6} \cdot \frac{t^2}{4} = \frac{9M^2}{4\pi^2 R^2 t^4}$$

$$\tau_m = \frac{M}{2\pi R^2 t}$$

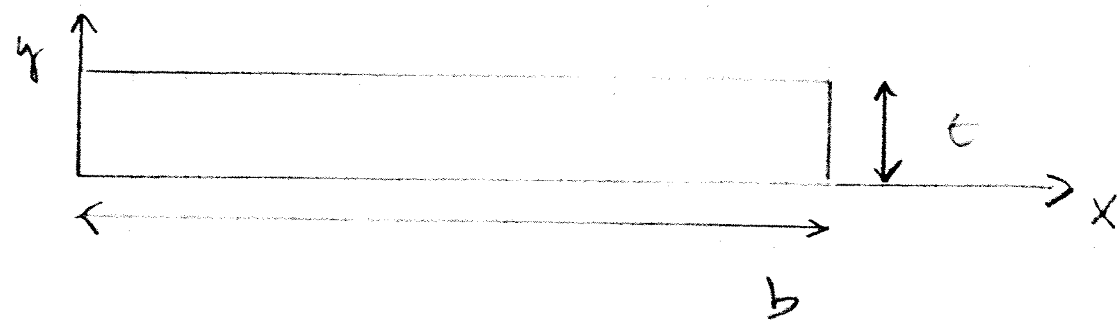
$$\tau_{m,cut} = \frac{3M}{2\pi R t^2}$$

$$\frac{\tau_m}{\tau_{m,cut}} = \frac{\cancel{M}}{\cancel{2\pi} R^2 \cancel{t}} \cdot \frac{\cancel{2\pi} R \cancel{t^2}}{3\cancel{M}} = \frac{R t^2}{R^2 t} = \frac{t}{R}$$

If $t \ll R$, then

$$\tau_m \ll \tau_{m,cut}$$

Föpplin kaava



Approksimoint $\psi = \psi(y) \Rightarrow$

$$\psi''(y) = -1.$$

$$\Rightarrow \quad \psi(y) = \frac{1}{2} y(t-y)$$

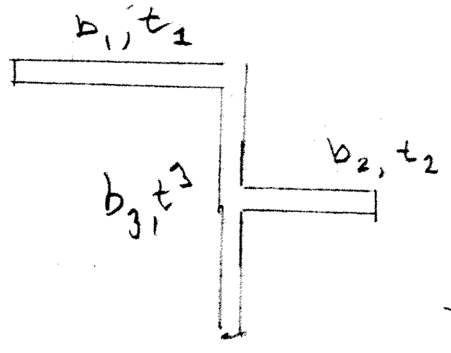
$$\int_A \psi \, dA = b \int_0^t \frac{1}{2} y(t-y)\, dy$$

$$= \frac{b}{2} \int_0^t (ty - y^2)\, dy = \frac{b}{2} \Big/_0^t \left(\frac{t}{2} y^2 - \frac{1}{3} y^3\right)$$

$$= \frac{bt^3}{2}\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{bt^3}{12} \quad \Rightarrow$$

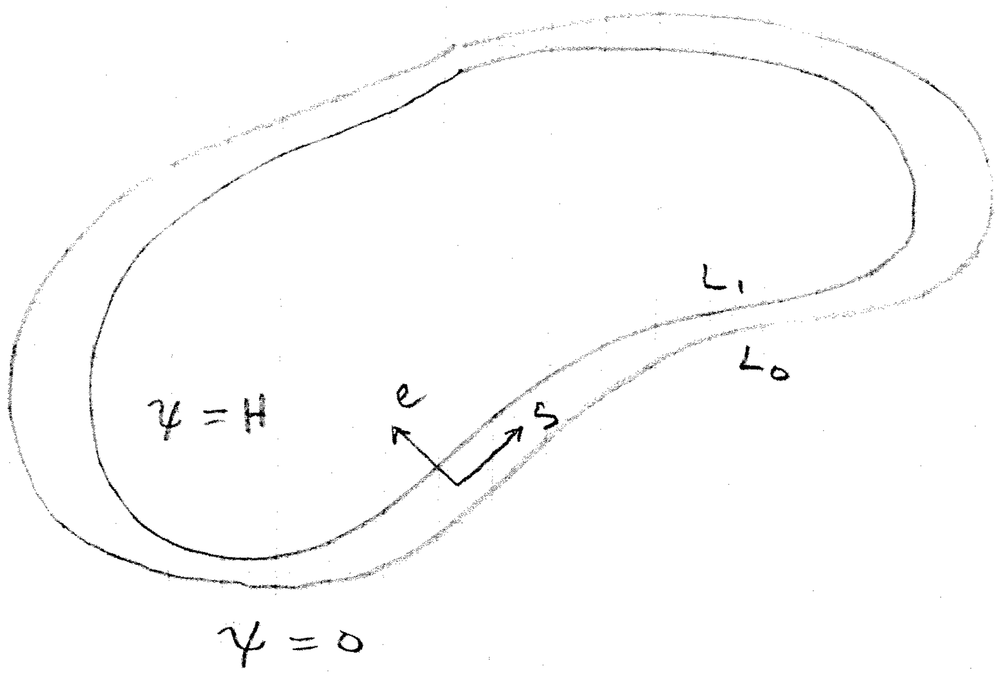
$$J = \frac{1}{3} bt^3$$

Sama menettely ohuille avoimille profiileille:



$$J = \frac{1}{3} \sum_{i=1}^{3} b_i t_i^3 \, .$$

Bredt'n kaava lieriöputkelle



$$\psi = \psi(e) \quad \Rightarrow$$

$$\psi''(e) = -1$$

$$\psi(e) = -\tfrac{1}{2}\left(e - t/2\right)\left(e + t/2\right) + \frac{H}{t}\left(e + t/2\right)$$

$$\left.\frac{\partial \psi}{\partial n}\right|_{L_0} = \psi'(t/2)$$

$$= \left.\left(-e + \frac{H}{t}\right)\right|_{e = t/2}$$

$$= -t/2 + \frac{H}{t} \approx \frac{H}{t}.$$

$$\oint_{L_1} \frac{\partial \psi}{\partial n} = A$$

$$H \oint_C \frac{ds}{t} = A.$$

$$H = \frac{A}{\oint_C \frac{ds}{t}}.$$

ja tästä

$$J \approx 4HA = \frac{4A^2}{\oint \frac{ds}{t}}.$$

Kun t = vakio saadaan

$$J = \frac{4A^2 t}{L}.$$

Tarkistus. Olkoon ympyrä

$$A = \pi R^2, \quad L = 2\pi R$$

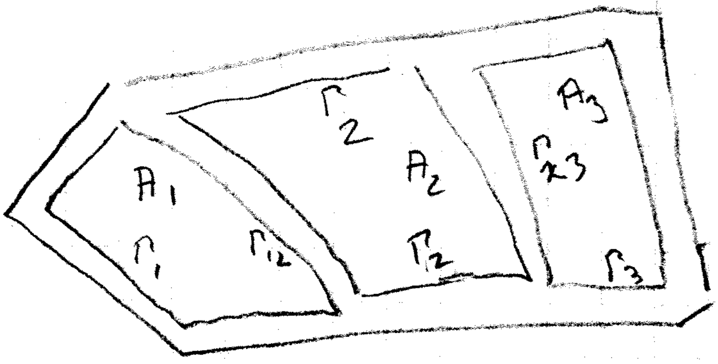
$$\Rightarrow \quad J = \frac{4\pi^2 R^4 t}{2\pi R} = 2\pi R^3 t.$$

Tarkan kaavan linearisointi

$$\frac{\pi}{2}\left(R^4 - (R-t)^4\right) \sim 2\pi R^3 t$$

$\square$.

Kotelo ja lokki



$$\int_{\Gamma_1} \frac{\psi_2}{t}\, ds + \int_{\Gamma_{12}} \frac{\psi_1 - \psi_2}{t}\, ds = A_1$$

$$\int_{\Gamma_2} \frac{\psi_2}{t}\, ds + \int_{\Gamma_{12}} \frac{\psi_2 - \psi_1}{t}\, ds + \int_{\Gamma_{13}} \frac{\psi_2 - \psi_3}{t}\, ds$$

$$= A_2$$

jne. Matriisimuodossa:

$$\begin{bmatrix} \int_{\Gamma_1 \cup \Gamma_{12}} \frac{ds}{t} & -\int_{\Gamma_{12}} \frac{ds}{t} & 0 \\ -\int_{\Gamma_{12}} \frac{ds}{t} & \int_{\Gamma_2 \cup \Gamma_{12} \cup \Gamma_{23}} \frac{ds}{t} & -\int_{\Gamma_{23}} \frac{ds}{t} \\ 0 & -\int_{\Gamma_{23}} \frac{ds}{t} & \int_{\Gamma_3 \cup \Gamma_{23}} \frac{ds}{t} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$$

<u>Huom.</u> Kerroinmatriisi on symmetrinen ja diagonaalisesti dominoiva. ⇒ Aina ratkaisu.

Nyt vääntövastus on

$$J = 4\left(\sum_{i=1}^{3} \gamma_i A_i\right).$$