Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**
Stenberg/Juntunen

<u>Exercise 5</u>

<u>Problem 1</u>
How does the strain tensor $\mathbf{E}$,

$$\mathbf{E}_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\Big),$$

change under an orthogonal chance of coordinates $\mathbf{x}' = \mathbf{A}\mathbf{x}$? How is the case for the infinitesimal strain tensor $\boldsymbol{\varepsilon}$

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big).$$

<u>Problem 2</u> **(home exercise)**

a) Show that the physical meaning of $\sum_{k=1}^{3} \varepsilon_{kk} = \mathrm{tr}(\boldsymbol{\varepsilon})$ is the relative change in volume.

b) Assume that body is under hydrostatic pressure

$$\boldsymbol{\sigma} = -p\boldsymbol{I}, \qquad p = \text{constant} > 0$$

and show that

$$\sum_{k=1}^{3} \varepsilon_{kk} = -\frac{1}{\kappa}p, \qquad \kappa = \lambda + \frac{2}{3}\mu = \text{bulk modulus}.$$

c) Hence, the condition $\kappa > 0$ is natural. Present κ in terms of Youngs modulus E and the Poisson ratio ν. What is the condition for ν?

<u>Problem 3</u>
Let the characteristic polynomial for $\boldsymbol{\sigma}$ be

$$\det(\boldsymbol{\sigma} - \alpha\boldsymbol{I}) = -\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3.$$

Show that the invariants are

$$\begin{aligned}
I_1 &= \sum_{k=1}^{3} \sigma_{kk} = \mathrm{tr}(\boldsymbol{\sigma}),\\
I_2 &= \sum_{i,j=1}^{3} \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij})\\
&= \frac{1}{2}[(\mathrm{tr}(\boldsymbol{\sigma}))^2 - \mathrm{tr}(\boldsymbol{\sigma}^2)]\\
&= \begin{vmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{vmatrix} + \begin{vmatrix}\sigma_{11} & \sigma_{13}\\ \sigma_{13} & \sigma_{33}\end{vmatrix} + \begin{vmatrix}\sigma_{22} & \sigma_{23}\\ \sigma_{23} & \sigma_{33}\end{vmatrix},\\
I_3 &= \det(\boldsymbol{\sigma}).
\end{aligned}$$

Show that the invariants do not change under orthogonal coordinate transformations.

Problem 4

Let the strain state be a "pure shear", e.g.

$$\sigma = \begin{pmatrix} 0 & \sigma_{12} & 0 \\ \sigma_{12} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Compute the principal stresses and directions.

Problem 1

Olkoon (e_1, e_2, e_3) ja $(\tilde{e}_1, \tilde{e}_2, \tilde{e}_3)$ ortogonaaliset kannat. Annettu ehto $\tilde{x} = Ax$ sanoo että $\tilde{e}_i = \sum_k a_{ik} e_k$. Samoin pätee että $\tilde{u}_i = u \cdot \tilde{e}_i = u \cdot (\sum_k a_{ik} e_k)$

$$= \sum_k a_{ik}\, u \cdot e_k = \sum_k a_{ik} u_k$$

Näinpä

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}_j} = \sum_k a_{ik} \frac{\partial u_k}{\partial \tilde{x}_j} = \sum_{k,l} a_{ik} \frac{\partial u_k}{\partial x_l} \frac{\partial x_l}{\partial \tilde{x}_j}$$

Koska $\tilde{x} = Ax$, niin $x = A^T \tilde{x}$ eli

$$x_i = \sum_k a^T_{ik} \tilde{x}_k = \sum_k a_{ki} \tilde{x}_k$$

$$\Rightarrow \frac{\partial x_i}{\partial \tilde{x}_j} = a_{ji}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{x}_j} = \sum_{k,l} a_{ik} \frac{\partial u_k}{\partial x_l} a_{jl}$$

Toisin sanoen $\tilde{\nabla} \tilde{u} = A \nabla u A^T$.

Näinpä

$$\tilde{E} = \frac{1}{2}\left(\tilde{\nabla}\tilde{u} + \tilde{\nabla}\tilde{u}^T + \tilde{\nabla}\tilde{u}\tilde{\nabla}\tilde{u}^T\right)$$

$$= \frac{1}{2}\left(A\nabla u A^T + A\nabla u^T A^T + A\nabla u \underbrace{A^T A}_{=I} \nabla u A^T\right)$$

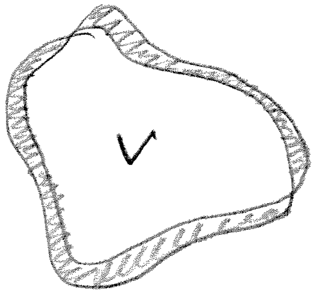
$$= A \; \frac{1}{2}\left(\nabla u + \nabla u^T + \nabla u \nabla u^T\right) A^T$$

$$= A E A^T$$

Lineaarinen ε muuntuu tietysti samoin.

Problem 2

a)



If body V undergoes small displacement u,

then the colored regios is $\int_{\partial V} u \cdot n \, dS$

$$\Rightarrow \frac{V - V^*}{V} = \frac{1}{|V|} \int_{\partial V} u \cdot n \, dS$$

$$= \frac{1}{|V|} \int_{V} \nabla \cdot u \, dS$$

Let $|V| \to 0$, then $\frac{V - V^*}{V} = \nabla \cdot u$

and $\nabla \cdot u = \sum_k \frac{\partial u_k}{\partial x_k} = tr(\varepsilon)$.

b) $\sigma = -pI$

by definition: $\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \left(\sum_k \varepsilon_{kk}\right) \delta_{ij}$

$\Rightarrow \quad \sigma_{ii} = -p = 2\mu \varepsilon_{ii} + \lambda \left(\sum_k \varepsilon_{kk}\right)$

$\Rightarrow \sum_l \sigma_{ll} = -3p = 2\mu \left(\sum_l \varepsilon_{ll}\right) + 3\lambda \left(\sum_k \varepsilon_{kk}\right)$

$\Leftrightarrow \quad -p = \left(\frac{2}{3}\mu + \lambda\right)\left(\sum_k \varepsilon_{kk}\right)$

$\Leftrightarrow \quad tr(\varepsilon) = \sum_k \varepsilon_{kk} = -\frac{1}{\underbrace{\frac{2}{3}\mu + \lambda}_{=: \frac{1}{K}}} p$

c) by definition:

$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad , \mu = \frac{E}{2(1+\nu)}$

$K = \frac{2}{3}\mu + \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} + \frac{E}{3(1+\nu)}$

$= E \quad \frac{3\nu + 1 - 2\nu}{3(1+\nu)(1-2\nu)} = E \quad \frac{1+\nu}{3(1+\nu)(1-2\nu)}$

$$K = E\,\frac{1}{3(1-2\nu)}$$

Condition $K > 0$

$$\Rightarrow \frac{1}{3(1-2\nu)} > 0 \quad \text{since } E > 0.$$

$$\Rightarrow 1-2\nu > 0 \Rightarrow \nu < \frac{1}{2}$$

At limit $\nu \to \frac{1}{2}$ material becomes incompressible.

Problem 3

$$\det(\sigma - \alpha I) = \begin{vmatrix} \sigma_{11}-\alpha & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\alpha & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\alpha \end{vmatrix}$$

$$= (\sigma_{11}-\alpha)\left[(\sigma_{22}-\alpha)(\sigma_{33}-\alpha) - \sigma_{23}\sigma_{32}\right]$$

$$- \sigma_{12}\left[\sigma_{21}(\sigma_{33}-\alpha) - \sigma_{23}\sigma_{31}\right]$$

$$+ \sigma_{13}\left[\sigma_{21}\sigma_{32} - (\sigma_{22}-\alpha)\sigma_{31}\right]$$

$$= (\sigma_{11}-\alpha)\left[\sigma_{22}\sigma_{33} - \alpha\sigma_{22} - \alpha\sigma_{33} + \alpha^2 + \sigma_{23}\sigma_{32}\right]$$

$$- \sigma_{12}\sigma_{21}\sigma_{33} + \alpha\sigma_{12}\sigma_{21} + \sigma_{12}\sigma_{23}\sigma_{31}$$

$$+ \sigma_{13}\sigma_{21}\sigma_{32} - \sigma_{13}\sigma_{22}\sigma_{31} + \alpha\sigma_{13}\sigma_{31}$$

$$= -\alpha^3 - \alpha\sigma_{22}\sigma_{33} + \alpha^2\sigma_{22} + \alpha^2\sigma_{33} - \alpha\sigma_{23}\sigma_{32}$$

$$+ \sigma_{11}\sigma_{22}\sigma_{33} - \alpha\sigma_{11}\sigma_{22} - \alpha\sigma_{11}\sigma_{33} + \alpha^2\sigma_{11}$$

$$+ \sigma_{11}\sigma_{23}\sigma_{32} - \sigma_{12}\sigma_{21}\sigma_{33} + \alpha\sigma_{12}\sigma_{21} + \sigma_{12}\sigma_{23}\sigma_{31}$$

$$+ \sigma_{13}\sigma_{21}\sigma_{32} - \sigma_{13}\sigma_{22}\sigma_{31} + \alpha\sigma_{13}\sigma_{31}$$

$$\det(\sigma - \alpha I) = -\alpha^3$$

$$+\alpha^2\left[\sigma_{22} + \sigma_{33} + \sigma_{11}\right]$$

$$+\alpha\left[\sigma_{23}\sigma_{32} - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{33} + \sigma_{12}\sigma_{21} + \sigma_{13}\sigma_{31}\right]$$

$$+\left[\sigma_{11}\sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{23}\sigma_{32} - \sigma_{12}\sigma_{21}\sigma_{33} + \sigma_{12}\sigma_{23}\sigma_{31}\right.$$
$$\left. + \sigma_{13}\sigma_{21}\sigma_{32} - \sigma_{13}\sigma_{22}\sigma_{31}\right]$$

Clearly $I_1 = \sum_k \sigma_{kk} = \mathrm{tr}(\sigma)$.

<u>I_2:</u>

$$\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} = \frac{1}{2}\sum_{\substack{i,j \\ i \neq j}} \sigma_{ii}\sigma_{jj}$$

$$\sigma_{23}\sigma_{32} - \sigma_{12}\sigma_{21} + \sigma_{13}\sigma_{31} = \frac{1}{2}\sum_{\substack{i,j \\ i \neq j}} \sigma_{ij}\sigma_{ji}$$

$$\Rightarrow I_2 = \frac{1}{2}\sum_{ij}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right).$$

Just as easily we see that

$I_3 = \det(\sigma)$ and that

$$I_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{2i} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \cdots \end{vmatrix} + \begin{vmatrix} \cdots \end{vmatrix}$$

What about $I_2 = \frac{1}{2}\left(tr(\sigma)^2 - tr(\sigma^2)\right)$?

$$tr(\sigma)^2 = (\sigma_{11} + \sigma_{22} + \sigma_{33})^2$$

$$= \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2$$

$$+ 2\sigma_{11}\sigma_{22} + 2\sigma_{11}\sigma_{33} + 2\sigma_{22}\sigma_{33}$$

$$\sigma^2 = \begin{bmatrix} \sigma_{11}^2 + \sigma_{12}\sigma_{21} + \sigma_{13}\sigma_{31} & \neq & \neq \\ \neq & \sigma_{12}\sigma_{2i} + \sigma_{22}^2 + \sigma_{23}\sigma_{32} & \neq \\ \neq & \neq & \sigma_{13}\sigma_{3i} + \sigma_{23}\sigma_{32} + \sigma_{33}^2 \end{bmatrix}$$

Now we see that

$$I_2 = \frac{1}{2}\left(tr(\sigma)^2 - tr(\sigma^2)\right).$$

Invariants do not change under

$x' = Ax$ where $AA^T = A^TA = I$??

$\underline{I_1}$

We know that $tr(AB) = tr(BA)$

and that $\tilde{\sigma} = A\sigma A^T$

$$\Rightarrow tr(\hat{\sigma}) = tr(A\sigma A^T) = tr(AA^T\sigma)$$
$$= tr(\sigma)$$

$\underline{I_2}$

$$tr(\hat{\sigma}^2) = tr(A\sigma A^T A\sigma A^T)$$
$$= tr(A\sigma^2 A^T) = tr(AA^T\sigma^2)$$
$$= tr(\sigma^2)$$

$\underline{I_3}$

We know that $det(AB) = det(A)\,det(B)$.

$$Det(A\sigma A^T) = Det(A)\,Det(\sigma)\,Det(A^T)$$
$$= Det(A)\,Det(A^T)\,Det(\sigma)$$
$$= Det(AA^T)\,Det(\sigma)$$
$$= Det(\sigma).$$

Problem 4

Principal stresses and directions are the eigenvalues and directions.

$$\mathrm{Det}(\sigma - \alpha I) = \begin{vmatrix} -\alpha & \sigma_{12} & 0 \\ \sigma_{12} & -\alpha & 0 \\ 0 & 0 & -\alpha \end{vmatrix}$$

$$= -\alpha \begin{vmatrix} -\alpha & \sigma_{12} \\ \sigma_{12} & -\alpha \end{vmatrix} = -\alpha(\alpha^2 - \sigma_{12}\sigma_{21})$$

$$= -\alpha^3 + \alpha(\sigma_{12}\sigma_{2i}) = 0$$

symm!

$$\Rightarrow \alpha = 0 \quad \text{or} \quad \alpha^2 = \sigma_{12}\sigma_{2i} = \sigma_{12}^2$$

$$\text{ie. } \alpha = \pm \sigma_{12}$$

$\underline{\alpha = 0}$: $V_1 = (0, 0, 1)^T$

$\underline{\alpha = -\sigma_{12}}$: $V_2 = (1, -1, 0)^T$

$\underline{\alpha = \sigma_{12}}$: $V_3 = (1, 1, 0)^T$