Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**
Stenberg/Juntunen

Exercise 4

Problem 1
Let $y = w(x)$ be a curve in xy-plane. Radius of curvature $\rho(x)$ is derived as the radius of the circle that fits best to $w(x)$ at point x. The conditions for this are that the circle and the curve have the same value $w(x)$, the same derivative $w'(x)$, and the same second derivative $w''(x)$ at the point x. Show that this gives

$$\rho(x) = \pm\frac{\left(1 + (w'(x))^2\right)^{3/2}}{w''(x)}.$$

Problem 2
Consider the rod problem : Find u such that

$$\begin{aligned} \frac{d}{dx}\left(EA\frac{du}{dx}\right) + f &= 0 \qquad\qquad 0 < x < L \\ u(0) &= 0 \\ EAu'(L) + ku(L) &= 0. \end{aligned}$$

Write the problem as a minimization and a variational problem. What boundary condition do you get in the limit $k \to \infty$ (by physical arguments)? Prove this a little bit more rigorously, let $|||\cdot|||_k$ (depends on $k > 0$) be the energy norm of the problem and let u_∞ be the candidate for the limit solution. Prove that it holds

$$|||u_\infty - u|||_k \leq k^{-1/2}|EAu'_\infty(L)|.$$

Problem 3
Consider the dynamics of a rod and show that the equation for the displacement $u(x,t)$ is

$$\frac{1}{c^2}\frac{d^2u}{dt^2} = \frac{d^2u}{dx^2},$$

with $c =$?. Solve with Fourier series and plot (animate) the solution for the following boundary conditions
a) $u(0,t) = 0$, $u(L,t) = 0$
b) $u(0,t) = 0$, $EA\frac{du}{dx}(L,t) = 0$
and the initial conditions $u(x,0) = f(x)$, $\frac{du}{dt}(x,0) = g(x)$. Functions f and g you can choose as you wish.

Problem 4 **(home exercise)**

As derived in the book (p. 75) the normal stress in the beam varies linearly with y;

$$\sigma = -Eu''y.$$

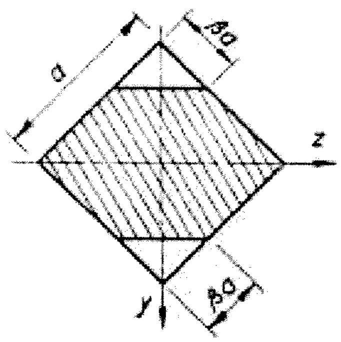
In addition, we showed that

$$M = EIu'',$$

which gives

$$\sigma = \frac{M}{I}y.$$

Consider now a beam with a square cross section and rotated 45° w.r.t. the x- and y-axis (see figure). Show that for a fixed moment the maximal normal stress decreases if you take away material from the corners. Otimize the beam by computing the β for which the maximal stress is smallest.



Problem 1

curve $y = w(x)$

circle $(x-x_0)^2 + (y-y_0)^2 = R^2$

$$\Longleftrightarrow \quad y_c(x) = \left(R^2 - (x-x_0)^2\right)^{1/2} + y_0$$

$$\begin{cases} y_c'(x) = \frac{1}{2}\left(R^2 - (x-x_0)^2\right)^{-1/2} 2(x-x_0)(-1) \\ y_c''(x) = -\frac{1}{4}\left(R^2 - (x-x_0)^2\right)^{-3/2} 2(x-x_0)\, 2(x-x_0) \\ \qquad - \frac{1}{2}\left(R^2 - (x-x_0)^2\right)^{-1/2} 2 \end{cases}$$

$$\begin{cases} y_c'(x) = -\left(R^2 - (x-x_0)^2\right)^{-1/2} (x-x_0) \\ y_c''(x) = -\left(R^2 - (x-x_0)^2\right)^{-1/2} \left[\left(R^2 - (x-x_0)^2\right)^{-1} (x-x_0)^2 + 1\right] \\ y_c(x) = \left(R^2 - (x-x_0)^2\right)^{1/2} + y_0 \end{cases}$$

First eq.

$$w(x) = y_c(x) = \left(R^2-(x-x_0)^2\right)^{1/2} + y_0$$

$$\Leftrightarrow \quad w(x) - y_0 = \left(R^2-(x-x_0)^2\right)^{1/2}$$

$$(w(x)-y_0)^2 = R^2-(x-x_0)^2$$

Second eq

$$\Rightarrow w'(x) = y_c'(x) = -\left(R^2-(x-x_0)^2\right)^{-1/2}(x-x_0)$$

$$= -\left((w(x)-y_0)^2\right)^{-1/2}(x-x_0)$$

$$= -(w(x)-y_0)^{-1}(x-x_0)$$

Third eq.

$$w''(x) = -\left(R^2-(x-x_0)^2\right)^{-1/2}\left[\left(R^2-(x-x_0)^2\right)^{-1}(x-x_0)^2+1\right]$$

$$= -(w(x)-y_0)^{-1}\left[(w(x)-y_0)^{-2}(x-x_0)^2+1\right]$$

$$= -(w(x)-y_0)^{-1}\left[w'(x)^2+1\right]$$

Combine 1° and 3°

$$R^2-(x-x_0)^2 = (w(x)-y_0)^2 = \frac{(w'(x)^2+1)^2}{w''(x)^2} \quad (\#)$$

Solve $(x-x_0)^2$ from '2'

$$w'(x) = -\left(R^2 - (x-x_0)^2\right)^{-1/2}(x-x_0)$$

$$w'(x)^2 = \left(R^2 - (x-x_0)^2\right)^{-1}(x-x_0)^2$$

$$(x-x_0)^2 = \left(R^2 - (x-x_0)^2\right) w'(x)^2$$

$$(x-x_0)^2 = R^2 w'(x)^2 - (x-x_0)^2 w'(x)^2$$

$$(x-x_0)^2\left(1 + w'(x)^2\right) = R^2 w'(x)^2$$

$$(x-x_0)^2 = \frac{R^2 w'(x)^2}{1 + w'(x)^2}$$

Substitute to (*)

$$R^2 - R^2 \frac{w'(x)^2}{1+w'(x)^2} = \frac{\left(w'(x)^2 + 1\right)^2}{w''(x)^2}$$

$$R^2\left(1 - \frac{w'(x)^2}{1+w'(x)^2}\right) = \frac{\left(w'(x)^2+1\right)^2}{w''(x)^2}$$

$$R^2\left(\frac{1}{1+w'(x)^2}\right) = \frac{\left(w'(x)^2+1\right)^2}{w''(x)^2}$$

$$R^2 = \frac{\left(w'(x)^2+1\right)^2\left(w'(x)^2+1\right)}{w''(x)^2} = \frac{\left(w'(x)+1\right)^3}{w''(x)^2}$$

Problem 2

$$-(EAu')' = f$$

$$u(0) = 0$$

$$EAu'(L) + ku(L) = 0$$

Solution and variation space:

$$K = \{ v \mid \|v\| < \infty , \ v(0) = 0 \}$$

$$\int_0^L fv\,dx = -\int_0^L (EAu')'v\,dx$$

$$= \int_0^L EAu'v'\,dx - \Big/_0^L EAu'v$$

$$\underbrace{\int_0^L fv\,dx}_{=:\,F(v)} = \underbrace{\int_0^L EAu'v'\,dx + ku(L)v(L)}_{=:\,D(u,v)}$$

Find $u \in K$ s.t. $D(u,v) = F(v) \quad \forall v \in K$

Find u s.t. $\min\limits_{v \in K} \frac{1}{2} D(v,v) - F(v)$

If $k \to \infty$, spring becomes very stiff

$\Rightarrow u(L) = 0$.

Limit solution u_∞ solves problem

$$\begin{cases} -(EA\,u_\infty')' = f \\ u_\infty(0) = u_\infty(L) = 0 \end{cases}$$

By definition $|||v|||_k^2 = D(v,v)$.

$$|||u_\infty - u|||_k^2 = D(u_\infty - u,\, u_\infty - u)$$

$$= \int_0^L EA\,(u_\infty - u)'^2\,dx + k\,(u_\infty(L) - u(L))^2$$

$$= -\int_0^L EA\,(u_\infty - u)''(u_\infty - u)\,dx$$

$$+ \Big/_0^L EA\,(u_\infty - u)'(u_\infty - u) + k\,(u_\infty(L) - u(L))^2$$

$$= -\int_0^L EA\,(u_\infty - u)''\,(u_\infty - u)\,dx$$

$$+ EA\,(u_\infty(L) - u(L))'\,(-u(L)) + k\,u(L)^2$$

$$= -\int_0^L \overbrace{(f - EA\,u'')}^{=0}(u_\infty - u)\,dx$$

$$+ \Big[-EA\,u_\infty'(L) + \underbrace{EA\,u'(L) + k\,u(L)}_{=0}\Big]\,u(L)$$

$$= -EA\,u_\infty'(L)\,u(L)$$

$$\le \frac{1}{\sqrt{k}}\,|EA\,u'(L)|\,\sqrt{k}\,|u(L)| \qquad (*)$$

Clearly

$$|||u-u_\infty|||_k^2 = D(u_\infty - u, u_\infty - u)$$

$$= \int_0^L \underbrace{EA\,(u_\infty - u)'^2}_{\geq 0} dx + k(\underbrace{u_\infty(L)}_{=0} - u(L))^2$$

$$\geq k\,u(L)^2$$

$$\Rightarrow \quad \sqrt{k}\,|u(L)| \leq |||u-u_\infty|||_k$$

Using this to (*)

$$|||u_\infty - u|||_k^2 \leq \frac{1}{\sqrt{k}} |EA\,u_\infty'(L)|\;|||u-u_\infty|||_k$$

$$\Leftrightarrow \quad |||u_\infty - u|||_k \leq \frac{1}{\sqrt{k}} |EA\,u_\infty'(L)|$$

Problem 3

In static case without external force, force equilibrium is

$$EA\,u''(x) = 0$$

In dynamic case this force is equal to force of acceleration i.e.

$$EA\,u''(x,t) = \rho A\,\ddot{u}(x,t)$$

where ρ is dencity/length.

$$\Rightarrow \quad \frac{E}{\rho}\frac{d^2u}{dx^2} = \frac{d^2u}{dt^2} \quad \Leftrightarrow \quad \frac{1}{c^2} = \frac{E}{\rho}.$$

Solution: Assume $u(x,t) = U(x)\,T(t)$.

$$\Rightarrow \quad \frac{1}{c^2}\,U''(x)\,T(t) = U(x)\,T''(t).$$

$$\Rightarrow \quad \frac{1}{c^2}\,\frac{U''(x)}{U(x)} = \frac{T''(t)}{T(t)} = \text{constant} = C_1 = -\lambda^2$$

We denote constant with $-\lambda^2$, since the constant needs to be negative for solution to exist.

$$\begin{cases} U''(x) = -\lambda^2 c^2 U(x) \\ T''(t) = -\lambda^2 T(t) \end{cases}$$

$$\Rightarrow \begin{cases} U(x) = A\sin(\lambda c x) + B\cos(\lambda c x) \\ T(t) = C\sin(\lambda t) + D\cos(\lambda t) \end{cases}$$

Case a)

$U(0) = 0 \Rightarrow B = 0$

$U(L) = 0 = A\sin(\lambda c L) \Rightarrow \lambda c L = n\pi$

$\Leftrightarrow \lambda = \dfrac{n\pi}{cL}$

Choose $A = 1$.

$$\begin{cases} U(x) = \sin\left(\frac{n\pi x}{L}\right) \\ T(t) = C_n \sin\left(\frac{n\pi t}{cL}\right) + D_n \cos\left(\frac{n\pi t}{cL}\right) \end{cases}$$

Initial conditions:

$$\begin{aligned} u(x,0) &= U(x)\,T(0) \\ &= \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[C_n \cdot 0 + D_n \cdot 1\right] \\ &= \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{L}\right) \\ &= f(x). \end{aligned}$$

Multiply with $\sin\left(\frac{k\pi x}{L}\right)$ and integrate over $[0, L]$.

$$\sum_{n=1}^{\infty} D_n \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$= \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$\Leftrightarrow \quad D_k \frac{L}{2} = \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$\Leftrightarrow \quad D_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$\frac{du(x,t)}{dt} = U(x)\, T'(t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) C_n \frac{n\pi}{cL} = g(x)$$

$$\Rightarrow \quad C_k \frac{k\pi}{cL} \cdot \frac{L}{2} = \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx$$

$$\Rightarrow \quad C_k = \frac{2c}{k\pi} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right) dx.$$

Case b)

$$U(0) = 0 \Rightarrow B = 0$$

$$U'(L) = 0 = \lambda c A \cos(\lambda c L) = 0$$

$$\Rightarrow \lambda c L = \pi\left(n - \tfrac{1}{2}\right) = \pi \frac{2n-1}{2}$$

$$\Leftrightarrow \lambda = \frac{(2n-1)\pi}{2cL}$$

Choose $A = 1$.

$$\begin{cases} U(x) = \sin\left(\frac{(2n-1)\pi x}{2L}\right) \\ T(t) = C \sin\left(\frac{(2n-1)\pi t}{2cL}\right) + D\cos\left(\frac{(2n-1)\pi t}{2cL}\right) \end{cases}$$

Initial conditions

$$u(x,0) = U(x)\,T(t)$$
$$= \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2L}\right) \cdot D_n = f(x)$$

$$D_k = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{(2k-1)\pi x}{2L}\right) dx$$

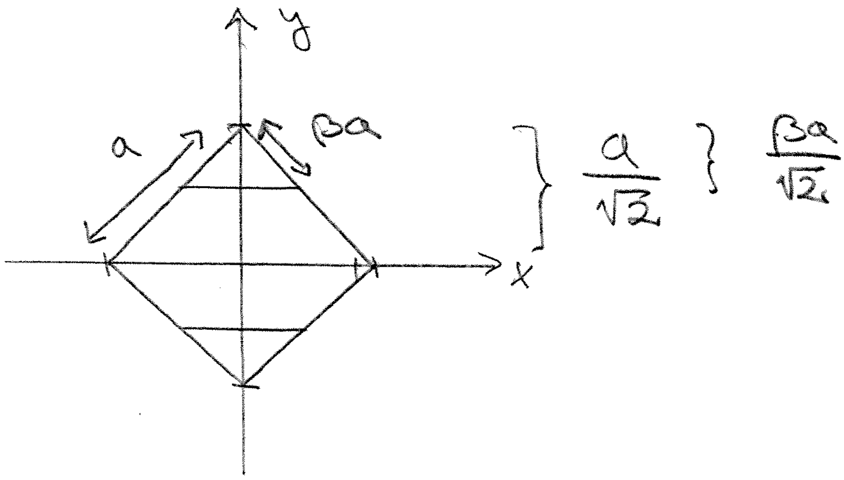
$$\dot{u}(x,t) = U(x)\, T'(t)$$

$$= \sum_{n=1}^{\infty} \sin\left(\frac{(2n-1)\pi x}{2L}\right) \cdot C_n \cdot \frac{(2n-1)\pi}{2cL}$$

$$\Rightarrow \quad C_k \frac{(2k-1)\pi}{2cL} \; \frac{L}{2} = \int_0^L g(x) \sin\left(\frac{(2k-1)\pi x}{2L}\right) dx$$

$$\Leftrightarrow \quad C_k = \frac{4c}{(2k-1)\pi} \int_0^L g(x) \sin\left(\frac{(2k-1)\pi x}{2L}\right) dx$$

Problem 4



Width as a function of y i.e. $w(y)$

$w(0) = \sqrt{2}\,a$ $\qquad$ w is linear

$w\left(\frac{a}{\sqrt{2}}\right) = 0$

$$\Rightarrow \quad w(y) = 2\left(\frac{a}{\sqrt{2}} - y\right)$$

$$I_z = \iint y^2 \, dS$$

$$I(\beta) = 2\int_0^{\frac{a}{\sqrt{2}}(1-\beta)} y^2 \cdot 2\left(\frac{a}{\sqrt{2}} - y\right) dy$$

$$= \int_0^{\frac{a}{\sqrt{2}}(1-\beta)} \frac{4a}{\sqrt{2}} y^2 - 4y^3 \, dy$$

$$= \Big/_0^{\frac{a}{\sqrt{2}}(1-\beta)} \frac{4a}{3\sqrt{2}} y^3 - y^4$$

$$= \frac{\cancel{4}a \cdot a^3}{3\sqrt{2}\,\cancel{\sqrt{2}^3}}(1-\beta)^3 - \frac{a^4}{\sqrt{2}^4}(1-\beta)^4$$

$$= \frac{a^4}{3}(1-\beta)^3 - \frac{a^4}{4}(1-\beta)^4$$

$\sigma = \frac{M}{I} y \Rightarrow$ clearly σ_{max} comes with y_{max}

$$y_{max}(\beta) = \frac{a}{\sqrt{2}}(1-\beta)$$

$$\frac{\sigma(\beta)}{M} = \frac{y(\beta)}{I(\beta)} = \frac{a}{\sqrt{2}}(1-\beta) \frac{1}{a^4\left(\frac{1}{3}(1-\beta)^3 - \frac{1}{4}(1-\beta)^4\right)}$$

$$= \frac{1}{\sqrt{2}a^3\left[\frac{1}{3}(1-\beta)^2 - \frac{1}{4}(1-\beta)^3\right]}$$

$$\frac{d}{d\beta}\frac{\sigma(\beta)}{M} = \frac{1}{\sqrt{2}a^3}(-1)\frac{1}{\left[\frac{1}{3}(1-\beta)^2 - \frac{1}{4}(1-\beta)^3\right]^2}\left[-\frac{2}{3}(1-\beta) + \frac{3}{4}(1-\beta\right.$$

$$= 0$$

$$\Rightarrow -\frac{2}{3}(1-\beta) + \frac{3}{4}(1-\beta)^2 = 0$$

$$\frac{3}{4}(1-\beta) = \frac{2}{3}$$

$$\beta = -\frac{8}{9} + 1 = +\frac{1}{9}$$