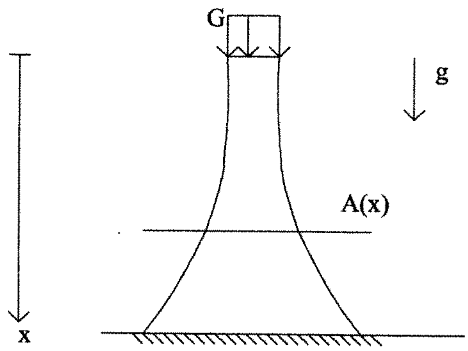
Mat-5.3741 Theory of Elasticity (5 cp) L **Spring 2007**
Stenberg/Juntunen

Exercise 3

Problem 1

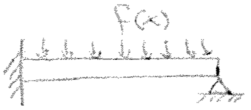
A vertical bar, with varying cross section $A(x)$, is loaded by its own weight (density ρ) and by weight G at the top. What should $A(x)$ be so that stress $\sigma(x)$ will be independent of x?



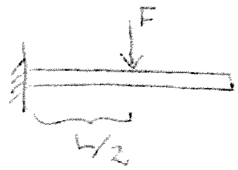
Problem 2

Write the total energy, variational form and boundary value problem for the following beam problems ~~(A, A, L):~~ (E, I, L)

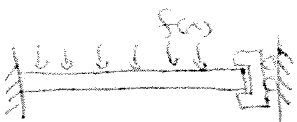
a)



b)



c)



slides up/down freely
rotation not allowed

Problem 3 **(home exercise)**

Find out the variational forms and boundary value problems for the following energies:

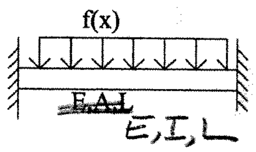
a) $J(v) = \frac{1}{2}\int_0^L EI(v''(x))^2\,dx - Fv(L/2) - Mv'(L/2)$
$K = \{v \mid \|v\| < \infty, v(0) = 0\}$

b) $J(v) = \frac{1}{2}\int_0^L EI(v''(x))^2\,dx + k(v(L))^2 - \int_0^L fv\,dx$
$K = \{v \mid \|v\| < \infty, v(0) = 0\}$

c) $J(v) = \frac{1}{2}\int_0^L EI(v''(x))^2\,dx + c\int_0^L (v(x))^2\,dx - \int_0^L fv\,dx$
$K = \{v \mid \|v\| < \infty\}$.

Problem 4

Consider a beam clamped at both ends.



Find out the Greens function for the solution, i.e. the function $K(x, y)$ such that the solution is $u(x) = \int_0^L K(x, y)f(y)\,dx$.

Problem 1

We want that $\sigma(x) = \text{constant}$.

$\Rightarrow \sigma(x) = E\varepsilon(x) = Eu'(x) = \text{constant}$

$\Rightarrow u(x) = ax + b.$

Since $u(L) = 0 \Rightarrow u(x) = a(x-L)$

The problem is

$$\begin{cases} -(EAu')' = \rho g A \\ u(L) = 0 \\ -EAu'(0) = G \end{cases}$$

BC: $-EA(0)\,u'(0) = G$

$\Leftrightarrow -EA(0)\,a = G$

$\Leftrightarrow A(0) = -\dfrac{G}{Ea}$

DY: $-\dfrac{d}{dx}\left(EA\dfrac{d}{dx}(a(x-L))\right) = \rho g A$

$-\dfrac{d}{dx}(EAa) = \rho g A$

$-Ea\dfrac{dA}{dx} = \rho g A$

$$\Rightarrow \int \frac{1}{A}\, dA = \int -\frac{\rho g}{Ea}\, dx$$

$$\Rightarrow \ln(A) = -\frac{\rho g}{Ea} x + \tilde{C}$$

$$\Rightarrow A(x) = C \exp\left(-\frac{\rho g}{Ea} x\right)$$

$$\text{BC: } A(0) = C = -\frac{G}{Ea}$$

$$\Rightarrow A(x) = -\frac{G}{Ea} \exp\left(-\frac{\rho g}{Ea} x\right)$$

DY again:

$$-(EAu')' = -EA'u' - \underbrace{EAu''}_{=0}$$

$$\Rightarrow -EA'u' = \rho g A$$

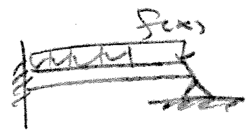
$$\Rightarrow -E\left(-\frac{\rho g}{Ea}\right) A = \rho g A$$

$$\Rightarrow \frac{1}{a} = 1$$

$$\Rightarrow a = 1.$$

$$\Rightarrow \begin{aligned} u(x) &= x - L \\ A(x) &= -\frac{G}{E} \exp\left(-\frac{\rho g}{E} x\right) \end{aligned}$$

Problem 2

a) 

$$-(EIu'')'' = f$$

$$u(0) = u'(0) = 0 \qquad (D)$$

$$u(L) = 0$$

$$EIu''(L) = 0$$

$$K = \{v \mid \|v\| < \infty,\ v(0) = v'(0) = v(L) = 0\}.$$

$$\int (EIu'')''v \, dx = \int fv \, dx$$

$$= -\int_0^L (EIu'')'v' \, dx + \Big|_0^L (EIu'')' \underbrace{v}_{=0} \, dx$$

$$= \int_0^L EIu''v'' \, dx - \Big|_0^L \underbrace{EIu''}_{=0,\ x=L} \underbrace{v'}_{=0,\ x=0} \, dx$$

$$= \int_0^L EIu''v'' \, dx$$

Find $u \in K$ s.t.

$$\underbrace{\int_0^L EIu''v'' \, dx}_{=: D(u,v)} = \underbrace{\int_0^L fv \, dx}_{=: F(v)} \quad \forall v \in K \quad (V)$$

$$J(v) := \frac{1}{2} D(v,v) - F(v)$$

Find $u \in K$ s.t.

$$\min_{v \in K} J(v) . \qquad (M)$$

b) Internal energy:

$$\frac{1}{2} \int_0^L EI \, v''^2 \, dx$$

External energy:

$$F v(\tfrac{L}{2}) .$$

$$J(v) = \frac{1}{2} \int_0^L EI \, v''^2 \, dx - F v(\tfrac{L}{2}) .$$

$$K = \{ v \mid \|v\| < \infty , \; v(0) = v'(0) = 0 . \}$$

$$\min_{v \in K} J(v) \qquad (M)$$

If u is minimum, then

$$\frac{J(u+tv)}{dt} \Big|_{t=0} = 0 .$$

$$\frac{d}{dt}\left(J(u+tv)\right)_{|t=0}$$

$$= \frac{d}{dt}\left[\frac{1}{2}\int_0^L EI\,(u+tv)''^{2}\,dx - F\left(u(\tfrac{L}{2})+tv(\tfrac{L}{2})\right)\right]_{|t}$$

$$= \int_0^L EI\,(u+tv)''\,v''\,dx - Fv(\tfrac{L}{2})\Big]_{|t=0}$$

$$= \int_0^L EI u'' v''\,dx - Fv(\tfrac{L}{2})$$

Find $u \in K$ s.t.

$$\int_0^L EI\, u''v''\,dx = Fv(\tfrac{L}{2}) \qquad \forall v \in K \quad (V)$$

$$Fv(\tfrac{L}{2}) = \int_0^L EI u'' v''\,dx$$

$$= \int_0^{L/2} EI u'' v''\,dx + \int_{L/2}^L (EI u'')\,v''\,dx$$

$$= -\int_0^{L/2} (EI u'')'\,v'\,dx + \Big|_0^{L/2} EI u'' v'$$

$$- \int_{L/2}^L (EI u'')'\,v'\,dx + \Big|_{L/2}^L EI u'' v'$$

$$= \int_0^{L/2} (EIu'')'' v\,dx + \Big/_0^{L/2} EIu''v' - \Big/_0^{L/2} (EIu'')'v$$

$$+ \int_{L/2}^{L} (EIu'')'' v\,dx + \Big/_{L/2}^{L} EIu''v' - \Big/_{L/2}^{L} (EIu'')'v$$

$$\left[v(0) = v'(0) = 0 \right]$$

$$= \int_0^{L} (EIu'')'' v\,dx + EIu''v'\Big|_{x=\frac{L}{2}^-} - (EIu'')'v\Big|_{x=\frac{L}{2}}$$

$$+ EIu''v'\Big|_{x=L} - EIu''v'\Big|_{x=\frac{L}{2}^+}$$

$$- (EIu'')'v\Big|_{x=L} + (EIu'')'v\Big|_{x=\frac{L}{2}^+}$$

$$= Fv\left(\tfrac{L}{2}\right)$$

$$\Longleftrightarrow \int_0^{L} (EIu'')'' v\,dx + EIu''v'\Big|_{x=L} - (EIu'')'v\Big|_{x=L}$$

$$+ EIu''v'\Big|_{x=\frac{L}{2}^-} - EIu''v'\Big|_{x=\frac{L}{2}^+}$$

$$- (EIu'')'v\Big|_{x=\frac{L}{2}^-} + (EIu'')'v\Big|_{x=\frac{L}{2}^-} - Fv\left(\tfrac{L}{2}\right)$$

$$= 0$$

1) $EIu''(L) = 0 \Rightarrow u''(L) = 0$

2) $(EIu'')'_{|x=L} = 0$ (D)

3) EIu'' is continuous at $x = \frac{L}{2}$.

4) $-(EIu'')'_{|x=\frac{L}{2}^-} + (EIu'')'_{|x=\frac{L}{2}^+} - Fv(\frac{L}{2}) = 0$

jump in shear force

5) $(EIu'')'' = 0 \quad 0 < x < L$.

6) $u(0) = u'(0) = 0$, since $u \in K$.

c) $$\begin{cases} (EIu'')'' = f \\ u(0) = u'(0) = 0 \\ u'(L) = 0 \\ (EIu'')'_{|x=L} = 0 \end{cases} \quad (D)$$

$K = \{ v \mid \|v\| < \infty,\ u(0) = u'(0) = u'(L) = 0 \}$.

$$\int_0^L f v \, dx = \int_0^L (EIu'')'' v \, dx$$

$$= - \int_0^L (EIu'')' v' \, dx + \Big/_0^L (EIu'')' v$$

$$= \int_0^L EIu''v'' \, dx + \Big/_0^L (EIu'')' v - \Big/_0^L EIu'' v'$$

$[v(0) = v'(0) = v'(L) = 0]$

$$= \int_0^L EIu''v'' \, dx + \underbrace{(EIu'')'}_{=0} v \Big|_{x=L} .$$

$$= \int_0^L EIu''v'' \, dx$$

Find $u \in K$ s.t.

$$\underbrace{\int_0^L EIu''v'' \, dx}_{=: D(u,v)} = \underbrace{\int_0^L f v \, dx}_{=: F(v)} \quad \forall v \in K. \qquad (V)$$

$$J(v) := \frac{1}{2} D(v,v) - F(v)$$

Find u s.t.

$$\min_{v \in K} J(v)$$

Problem 3

a) $J(v) = \frac{1}{2}\int_0^L EI v''^2 dx - Fv(\frac{L}{2}) - Mv'(\frac{L}{2})$

$K = \{v \mid \|v\| < \infty,\ v(0) = 0\}$

If u is minimum, then

$$\frac{J(u+tv)}{dt}\Big|_{t=0} = 0 .$$

$$\frac{d}{dt}\Big[\frac{1}{2}\int_0^L EI\,(u+tv)''^2 dx - F(u(\tfrac{L}{2}) + tv(\tfrac{L}{2})) - M(u'(\tfrac{L}{2}) + tv'(\tfrac{L}{2}))\Big]\Big|_{t=0}$$

$$= \Big[\int_0^L EI\,(u+tv)'' v'' dx - Fv(\tfrac{L}{2}) - Mv'(\tfrac{L}{2})\Big]_{t=0}$$

$$= \underbrace{\int_0^L EI\, u'' v'' dx}_{=:D(u,v)} - \underbrace{Fv(\tfrac{L}{2}) - Mv'(\tfrac{L}{2})}_{=:G(v)}$$

Find $u \in K$ s.t.

$$D(u,v) = G(v) \quad \forall v \in K \qquad (V) .$$

$$Fv\left(\tfrac{L}{2}\right) - Mv'\left(\tfrac{L}{2}\right)$$

$$= \int_0^L EIu''v''\,dx$$

$$= -\int_0^{L/2} (EIu'')'v'\,dx + \Big|_0^{L/2} EIu''v'$$

$$-\int_{L/2}^{L} (EIu'')'v'\,dx + \Big|_{L/2}^{L} EIu''v'$$

$$= +\int_0^{L/2} (EIu'')''v\,dx + \Big|_0^{L/2} EIu''v' - \Big|_0^{L/2} (EIu'')'v$$

$$+\int_{L/2}^{L} (EIu'')''v\,dx + \Big|_{L/2}^{L} EIu''v' - \Big|_{L/2}^{L} (EIu'')'v$$

1) $EIu''(L) = 0$, $EIu''(0) = 0$

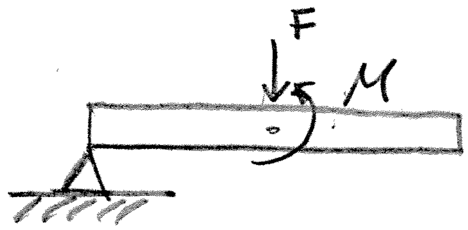
$(EIu'')'\big|_{x=L} = 0$

2) $EIu''\big|_{x=\frac{L}{2}+} - EIu''\big|_{x=\frac{L}{2}-} + M = 0$

$-(EIu'')'\big|_{x=\frac{L}{2}+} + (EIu'')'\big|_{x=\frac{L}{2}-} + F = 0$

3) $(EIu'')'' = 0 \qquad 0 < x < L$

4) $u(0) = 0$ since $u \in K$.



b)

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^L EI\,(u+tv)''^2\,dx + k\,(u(L)+tv(L))^2 - \int_0^L f(u+tv)\,dx\right]_{|t=0}$$

$$=\left[\int_0^L EI\,(u+tv)''\,v''\,dx + 2k\,(u(L)+tv(L))\,v(L) - \int_0^L fv\,dx\right]_{|t=0}$$

$$=\underbrace{\int_0^L EI\,u''v''\,dx + 2k\,u(L)v(L)}_{=:\,D(u,v)} - \underbrace{\int_0^L fv\,dx}_{=:\,G(v)}$$

Find $u \in K$ s.t.

$$D(u,v) = G(v) \quad \forall v \in K.$$

$$\int_0^L f v \, dx - 2k\, u(L) v(L)$$

$$= \int_0^L (EI u'') v'' \, dx$$

$$= -\int_0^L (EI u'')' v' \, dx + \Big|_0^L EI u'' v'$$

$$= \int_0^L (EI u'')'' v \, dx + \Big|_0^L EI u'' v' - \Big|_0^L (EI u'')' v$$

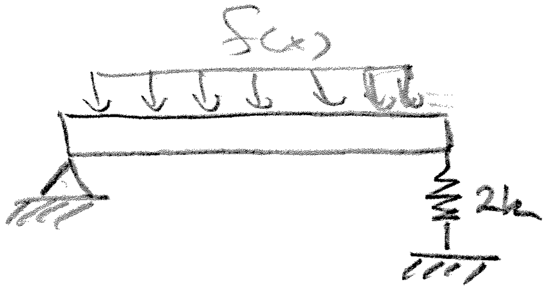
1) $EI u''(0) = 0$

$EI u''(L) = 0$

$u(0) = 0$ since $u \in K$

2) $2k\, u(L) = (EI u'')'\big|_{x=L}$

3) $(EI u'')'' = f \quad 0 < x < L$.



c) $$\frac{d}{dt}\left[\frac{1}{2}\int_0^L EI(u+tv)''^2\,dx + c\int_0^L (u+tv)^2\,dx - \int_0^L f(u+tv)\,dx\right]_{|t=0}$$

$$= \left[\int_0^L EI(u+tv)''v''\,dx + 2c\int_0^L (u+tv)v\,dx - \int_0^L fv\,dx\right]_{|t=0}$$

$$= \underbrace{\int_0^L EIu''v''\,dx + 2c\int_0^L uv\,dx}_{=:D(u,v)} - \underbrace{\int_0^L fv\,dx}_{=:G(v)}.$$

Find $u \in K$ s.t.

$$D(u,v) = G(v) \quad \forall v \in K.$$

$$\int_0^L fv\,dx - 2c\int_0^L uv\,dx$$

$$= \int_0^L EIu''v''\,dx$$

$$= -\int_0^L (EIu'')'v'\,dx + \Big/_0^L EIu''v'$$

$$= \int_0^L (EIu'')''v\,dx + \Big/_0^L EIu''v' - \Big/_0^L (EIu'')'v$$

1) $EIu''(0) = EIu''(L) = 0$

$(EIu'')'\big|_{x=0} = 0$

$(EIu'')'\big|_{x=L} = 0$

2) $(EIu'')'' + 2cu = f \quad 0 < x < L.$

Problem 4

Assume E, I are constants.

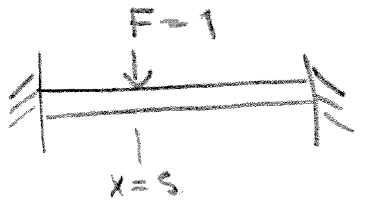
Define: $EIu''''(x) = \mathcal{L}u$.

Green's function is defined as

$$\mathcal{L}K(x,s) = \delta(x-s).$$

+ boundary conditions.

So we need to solve



On $0<x<s$ and on $s<x<L$ $u''''(x) = 0$

$$\Rightarrow \quad u(x) = \begin{cases} a_1x^3 + b_1x^2 + c_1x + d_1 & 0<x<s \\ a_2x^3 + b_2x^2 + c_2x + d_2 & s<x<L \end{cases}$$

Boundary conditions:

$$u(0) = u'(0) = u(L) = u'(L) = 0.$$

- $u(0) = 0 \Rightarrow d_1 = 0$

- $u'(x) = \begin{cases} 3a_1x^2 + 2b_1x + c_1 \\ 3a_2x^2 + 2b_2x + c_2 \end{cases}$

$u'(0) = 0 \Rightarrow c_1 = 0$

$$u'(L) = 0 = 3a_2L^2 + 2b_2L + c_2$$

$$c_2 = -3a_2L^2 - 2b_2L$$

$$u(x) = \begin{cases} a_1x^3 + b_2x^2 \\ a_2x^3 + b_2x^2 - (3a_2L^2 + 2b_2L)x + d_2 \end{cases}$$

$$u(L) = 0 = a_2L^3 + b_2L^2 - 3a_2L^3 - 2b_2L^2 + d_2$$
$$= -2a_2L^3 - b_2L^2 + d_2$$

$$\Rightarrow d_2 = 2a_2L^3 + b_2L^2$$

$$u(x) = \begin{cases} a_1x^3 + b_2x^2 \\ a_2x^3 + b_2x^2 - (3a_2L^2 + 2b_2L)x + 2a_2L^3 + b_2L^2 \end{cases}$$

Compatibility:

- u is continuous at $x = s$.
  $u(s_-) = u(s_+)$
- derivative is continuous
  $u'(s_-) = u'(s_+)$
- momentum is continuous
  $u''(s_-) = u''(s_+)$
- jump in shear force
  $EI\left(-u'''(s_-) + u'''(s_+)\right) = 1$

Mathematica tells that

$$u(x) = \begin{cases} \dfrac{1}{2EIL^2}\left(-(-L^2s+2Ls^2-s^3)x^2 - \dfrac{(L^3-3Ls^2+2s^3)x^3}{3L}\right) & 0<s<x \\[2ex] -\dfrac{(-3Ls^2+2s^3)}{6EIL^3}x^3 - \dfrac{(2Ls^2-s^3)}{2EIL^2}x^2 + \dfrac{s^2}{2EI}x - \dfrac{s^3}{6EI} & s<x<L \end{cases}$$

$$=: K(x,s)$$

And, like we showed before, solution to $\mathcal{L}u = f(x)$ is now $u(x) = \int_0^L K(x,s) f(s)\, ds$.