Institute of mathematics
Helsinki University of Technology

Arponen

Mat-1.3651 Numerical Linear Algebra (Numeerinen matriisilaskenta)

Final examination 05.05.2007

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Study programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KJO, KTA, KON, MAK, MAR, PUU, RAK, TFY, TIK, TLT, TUO, YHD.

Calculators are not allowed nor needed. Time for the exam is 3 hours. You can answer in Finnish if you wish.

1. Let $A = \begin{pmatrix} 2 & 1 & 1 \\ 8 & 4 & 8 \\ 6 & 7 & 9 \end{pmatrix}$. Use the Gaussian elimination with partial pivoting to compute P, L, and U s.t. $PA = LU$ where L is unit lower triangular, U upper triangular and P a permutation. (Hint: U should be integers now!)

2. Let $A \in \mathbb{C}^{m\times n}$ be full rank, $m \geq n$ and $b \in \mathbb{C}^m$. Describe (the main steps, no implementation details needed) the following three ways to solve the Least Squares problem: find $x \in \mathbb{C}^n$ s.t. the residual $\|b - Ax\|_2 =$ minimum.

   (a) Solving by normal equations,

   (b) Solving by QR,

   (c) Solving by SVD.

3. Let $A \in \mathbb{C}^{m\times m}$, $\lambda_1, \ldots, \lambda_m$ its eigenvalues, and

$$\begin{aligned} \rho(A) &:= \max_{j=1,\ldots,m} |\lambda_j| \quad \text{the spectral radius} \\ \alpha(A) &:= \max_{j=1,\ldots,m} \operatorname{Re}(\lambda_j) \quad \text{the spectral abscissa.} \end{aligned}$$

   Prove, using Schur decomposition, that

   (a) $\lim_{n\to\infty} \|A^n\| = 0$ if and only if $\rho(A) < 1$.

   (b) $\lim_{t\to\infty} \|e^{tA}\| = 0$ if and only if $\alpha(A) < 0$.

4. Let A be an invertible square matrix. Show that the singular matrix B which is closest to A in 2-norm, fulfils

$$\|A - B\|_2 = \frac{1}{\|A^{-1}\|}.$$

   Hint: SVD.

5. Describe

   (a) what is the Arnoldi iteration, and

   (b) how is it used in the GMRES iteration.