TKK Matematiikan laitos Arponen/Pursiainen

Mat-1.3651 Numerical Linear Algebra, spring 2008
(Numeerinen matriisilaskenta, kevät 2008)

Exercise 9 (3.4.2008)

Please hand in the exercises marked with an asterisk (*) either to the assistant's folder in front of U313 or latest at the beginning of the exercise. In addition to that, hand in the exercises marked with [Comp. hand-in] in the *next* exercise session (10th April, that is).

* 1. (a) Let $A \in \mathbb{C}^{m\times m}$ be tridiagonal and hermitian, with all its sub- and superdiagonal entries nonzero. Prove that the eigenvalues of A are distinct. (Hint: show that for any $\lambda \in \mathbb{C}$, $\text{rank}(A - \lambda I) \geq m - 1$.)

(b) Let $A \in \mathbb{C}^{m\times m}$ be (upper-)Hessenberg, with all the entries on the 1st subdiagonal nonzero. Give an example that shows that the eigenvalues of A are not necessarily distinct.

(c) Let $A \in \mathbb{C}^{m\times m}$ be bidiagonal with nonzero entries (i.e. $a_{ii} \neq 0$, $a_{i,i+1} \neq 0$, and $a_{ij} = 0$ when $j \neq i, i+1$). Show that its singular values are distinct.

* 2. (a) Let $A \in \mathbb{C}^{m\times m}$ and (λ, x) an eigenpair of A. Now $\lambda \in \Lambda(A^T)$ hence $\exists u \in \mathbb{C}^m$ s.t. (λ, u) is an eigenpair for A^T and $u^T x = \lambda$. What is the connection between the eigenvalues and -vectors between A and $B := A - xu^T$?

(b) Let $A, B \in \mathbb{R}^{m\times m}$. What is the connection between the singular values and -vectors between matrices $A + \mathrm{i}B$ (i is the imaginary unit, $\mathrm{i}^2 = -1$) and $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$?

3. Given $A \in \mathbb{C}^{m\times m}$ with spectrum $\Lambda(A) \subset \mathbb{C}$ and $\epsilon > 0$, define the 2-norm *ϵ-pseudospectra of* A, denoted $\Lambda_\epsilon(A)$, to be the set of numbers $z \in \mathbb{C}$ satisfying any of the following conditions (the norms are 2-norms):

   (i) z is an eigenvalue of $A + \delta A$ for some δA with $\|\delta A\| \leq \epsilon$.

   (ii) $\exists u \in \mathbb{C}^m$ with $\|(A - zI)u\| \leq \epsilon$ and $\|u\| = 1$.

   (iii) The smallest singular value fulfils: $\sigma_m(zI - A) \leq \epsilon$.

   (iv) $\|(zI - A)^{-1}\| \geq 1/\epsilon$.

   Here the matrix $(zI - A)^{-1}$ is the *resolvent* of A at z. If $z \in \Lambda(A)$, the convention is $\|(zI - A)^{-1}\| = \infty$. Prove that these conditions are equivalent.

Please turn.

4. (2-dim Poisson.) Code Jacobi, Gauss-Seidel, and SOR methods. Experiment on the system[1] $Ax = b$ where

$$A = \begin{pmatrix} T_N + 2I_N & -I_N & & \\ -I_N & \ddots & \ddots & \\ & \ddots & \ddots & -I_N \\ & & -I_N & T_N + 2I_N \end{pmatrix}$$

where T_N as in the Exercise 8, I_N identity, and b is a random N^2-vector (`randn`). Experiment with $N = 10$ or 20, iterate about N times. Try different values for the relaxation parameter, but keep it $0 < \omega < 2$.

5. [Comp. hand-in] In Matlab, the default way to solve a linear system $Ax = b$, even in the least squares sense, is the "\" command: `x = A \ b`. Here you study its behaviour with the timing commands `tic, toc`. You will need at least `help slash` or `help mldivide`. In the following sequence of tests with random matrices, try to explain: (i) Why was the test carried out? (ii) Why did the result come out as it did (relative to the other timings)? If it is difficult to see differences, try a bigger m (a suitable m really depends on your hardware).

   (a) Choose `m=` something (e.g. m=100, 500, or 1000). `Z = randn(m,m); A = Z'*Z; b = randn(m,1);`
   `tic; x = A\b; toc;`

   (b) `tic; x = A\b; toc;`

   (c) `I = eye(m,m); emin = min(eig(A)); A2 = A-1.1*emin*I;`
   `tic; x = A2\b; toc;`

   (d) `A3 = A-1.01*emin*I;`
   `tic; x = A3\b; toc;`

   (e) `A4 = A-1.001*emin*I;`
   `tic; x = A4\b; toc;`

   (f) `A5 = triu(A);`
   `tic; x = A5\b; toc;`

   (g) `A6 = A5; A6(m,1) = A5(1,m);`
   `tic; x = A6\b; toc;`

[1] A is from a 2-dimensional Poisson system $-\Delta v = f(x, y)$ where $\Delta v := v_{xx} + v_{yy}$.