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Upper bounds for $\mathbb{R}$-linear resolvents

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Abstract. We discuss upper bounds for the resolvent of an $\mathbb{R}$-linear operator in $\mathbb{C}^d$.

Mathematics Subject Classification (2010). 47A10.

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1. Introduction

The resolvent

$$\lambda \mapsto (\lambda I - A)^{-1}$$

of a $d \times d$ complex matrix A is a matrix valued function with rational elements. Thus, in particular, all singularities are poles of at most order d, and the following lower and upper bound hold

$$\frac{1}{\text{dist}(\lambda, \sigma(A))} \leq \|(\lambda I - A)^{-1}\| \leq \frac{\|\lambda I - A\|^{d-1}}{\text{dist}(\lambda, \sigma(A))^d}, \tag{1.1}$$

where $\sigma(A)$ denotes the set of eigenvalues of A.

In order to represent conveniently $\mathbb{R}$-linear operators in $\mathbb{C}^d$ let us denote by τ the complex conjugation. Given two $d \times d$ complex matrices A, B we put

$$\mathcal{A} = A + B\tau \tag{1.2}$$

so that $\mathcal{A}$ maps a vector $x \in \mathbb{C}^d$ as

$$x \mapsto \mathcal{A}x = Ax + B\overline{x}. \tag{1.3}$$

We define the *spectrum* $\sigma(\mathcal{A})$ of $\mathcal{A}$ by

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid \mathcal{A}x = \lambda x \text{ for some } x \neq 0\}. \tag{1.4}$$

The spectrum consists of at most d curves on $\mathbb{C}$, it is compact but it can be empty [1], [5]. In what follows the norm is the induced operator norm

$$\|\mathcal{A}\| = \sup_{\|x\|=1} \|\mathcal{A}x\|.$$

The resolvent of $\mathcal{A}$ is the *real analytic function*

$$\lambda \mapsto \mathcal{R}(\lambda, \mathcal{A}) = (\lambda - \mathcal{A})^{-1} \tag{1.5}$$

defined outside the spectrum. For $|\lambda| > \|\mathcal{A}\|$ we have

$$\mathcal{R}(\lambda, \mathcal{A}) = \sum_{j=0}^{\infty} (\frac{1}{\lambda}\mathcal{A})^j \frac{1}{\lambda} \tag{1.6}$$

and thus in particular

$$\|\mathcal{R}(\lambda, \mathcal{A})\| \leq \frac{1}{|\lambda| - \|\mathcal{A}\|}. \tag{1.7}$$

From this one gets easily the analogue of the leftmost inequality in (1.1). However, unlike in the $\mathbb{C}$-linear case, the spectrum of an $\mathbb{R}$-linear operator can be empty. If we set the distance to an empty set to be infinite, then the lower bound trivially holds but simultaneously, it is clear that the analogue of the upper bound cannot hold. In [4] a lower bound was presented, based on a set $\delta(\mathcal{A})$ which is always nonempty, contains the spectrum and equals it in the $\mathbb{C}$-linear case. In this note we consider upper bounds.

It is in order to point out that while the real analytic resolvent satisfies

$$\lambda\mathcal{R}(\lambda, \mathcal{A}) - \mathcal{A}\mathcal{R}(\lambda, \mathcal{A}) = I \tag{1.8}$$

one can associate with $\mathcal{A}$ a *complex analytic* function, the *cosolvent* $\mathcal{C}(\lambda, \mathcal{A})$ as a solution of the Sylvester equation

$$\lambda\mathcal{C}(\lambda, \mathcal{A}) - \mathcal{C}(\lambda, \mathcal{A})\mathcal{A} = I \tag{1.9}$$

[4]. For $|\lambda| > \|\mathcal{A}\|$ we have

$$\mathcal{C}(\lambda, \mathcal{A}) = \sum_{j=0}^{\infty} \lambda^{-j-1}\mathcal{A}^j$$

and all singularities are poles. It is possible to give an analogue of the Cauchy integral in which the kernel is the cosolvent and not the resolvent.

Solution methods for $\mathbb{R}$-linear problems were discussed in [1] and eigenvalue problems in [5]. Additional material on real linear operators can be found e.g. in [5], [6], [7] and the references given there.

2. Additional preliminaries

Let $\mathcal{A} = A + B\tau$ be given. If

$$\mathcal{A}x = b, \tag{2.1}$$

that is,

$$Ax + B\overline{x} = b,$$

then also

$$\overline{B}x + \overline{A}\overline{x} = \overline{b}$$

so that

$$\begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \begin{pmatrix} x \\ \overline{x} \end{pmatrix} = \begin{pmatrix} b \\ \overline{b} \end{pmatrix}. \tag{2.2}$$

The matrix

$$\psi(\mathcal{A}) = \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} \in \mathbb{C}^{2d \times 2d} \tag{2.3}$$

is said to give the $\mathbb{C}$-*linear representation of* $\mathcal{A}$, see [4]. There holds $\|\mathcal{A}\| = \|\psi(\mathcal{A})\|$.

Lemma 2.1. *The equation* (2.1) *has a unique solution* $x \in \mathbb{C}^d$ *for every* $b \in \mathbb{C}^d$ *if and only if the matrix* $\psi(\mathcal{A})$ *is nonsingular.*

Proof. If $\psi(\mathcal{A})$ is nonsingular, then for every b there exists a unique

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

such that

$$\psi(\mathcal{A})y = \begin{pmatrix} b \\ \overline{b} \end{pmatrix}.$$

But then also

$$\psi(\mathcal{A}) \begin{pmatrix} y_1 - \overline{y_2} \\ \overline{y_1} - y_2 \end{pmatrix} = 0$$

and so $y_2 = \overline{y_1}$.

Reversely, if $\psi(\mathcal{A})$ is singular, there exists a nontrivial

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

such that

$$Ay_1 + By_2 = 0$$

and

$$A\overline{y_2} + B\overline{y_1} = 0$$

so that

$$A(y_1 + \overline{y_2}) + B(\overline{y_1} + y_2) = 0,$$

or $\mathcal{A}((y_1 + \overline{y_2}) = 0$. Thus $\mathcal{A}$ has a nontrivial kernel, except if $y_1 + \overline{y_2} = 0$. However, in that case we can set $x = iy_1$ to obtain $\mathcal{A}x = 0$. □

Definition 2.2. We set for given real linear operator $\mathcal{A}$ in $\mathbb{C}^d$

$$p(z,w) = det \begin{pmatrix} zI - A & -B \\ -\overline{B} & wI - \overline{A} \end{pmatrix} \tag{2.4}$$

and denote

$$\Sigma(\mathcal{A}) = \{(z,w) \in \mathbb{C}^2 \mid p(z,w) = 0\}.$$

Thus p is a polynomial of two complex variables of degree $2d$ of the form

$$p(z,w) = z^d w^d + r(z,w) \tag{2.5}$$

where r is a polynomial of degree at most $2d - 1$. Unless otherwise explicitly mentioned, we *measure the distances in* $\mathbb{C}^2$ *using the max-norm.*

Proposition 2.3. *The polynomial* p *does not depend on the coordinate system in* $\mathbb{C}^d$. *Moreover,*

$$\sigma(\mathcal{A}) = \{\lambda \in \mathbb{C} \mid (\lambda, \overline{\lambda}) \in \Sigma(\mathcal{A})\}. \tag{2.6}$$

For $|\lambda| > \|\mathcal{A}\|$ *we have*

$$\text{dist}(\ (\lambda, \overline{\lambda}), \Sigma(\mathcal{A})\ ) \geq |\lambda| - \|\mathcal{A}\|. \tag{2.7}$$

Proof. A coordinate change by a similarity matrix S leads to the $-$linear representation

$$\begin{pmatrix} S & 0 \\ 0 & \overline{S} \end{pmatrix} \begin{pmatrix} zI - A & -B \\ -\overline{B} & wI - \overline{A} \end{pmatrix} \begin{pmatrix} S^{-1} & 0 \\ 0 & \overline{S^{-1}} \end{pmatrix}$$

and so its determinant is independent of S.

The second claim is obvious.

The third one follows from inverting

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} \frac{1}{z} I & 0 \\ 0 & \frac{1}{w} I \end{pmatrix} \psi(\mathcal{A})$$

by the Neumann series which converges as ψ preserves the norm of $\mathcal{A}$ and the norm of the diagonal matrix equals the largest absolute value of the elements. It is clear that ψ preserves the norm if the operator is applied to vectors of the form $(x, \overline{x}) \in \mathbb{C}^{2d}$ where $x \in \mathbb{C}^d$. However, observe that the operator norm of $\psi(\mathcal{B})$ is obtained as square root of the largest eigenvalue of $\psi(\mathcal{B})^* \psi(\mathcal{B})$ and by Lemma 2.1 it suffices to work with vectors of the form $(x, \overline{x})$. □

Example 2.4. If

$$\mathcal{A} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \tau \tag{2.8}$$

then

$$(\lambda - \mathcal{A})^{-1} = \frac{1}{|\lambda|^2 + |\beta|^2} \begin{pmatrix} \overline{\lambda} & \beta\tau \\ -\beta\tau & \overline{\lambda} \end{pmatrix}$$

while $p(z, w) = (zw + |\beta|^2)^2$. Thus $\sigma(\mathcal{A}) = \emptyset$ whereas

$$\Sigma(\mathcal{A}) = \{(z, \frac{-|\beta|^2}{z}) \mid z \neq 0\}.$$

Finally, observe that the distance in the max-norm from the origin to $\Sigma(\mathcal{A})$ is $|\beta|$.

3. First upper bound

The idea of the upper bounds for the resolvent $(\lambda - \mathcal{A})^{-1}$ starts as follows. We consider $(\psi(\lambda - \mathcal{A}))^{-1}$ and apply the simple identity between the eigenvalues and singular values of matrices. Denoting by $s_j \geq s_{j+1}$ the singular values of $\psi(\lambda - \mathcal{A})$ we have

$$\frac{1}{s_{2d}} = \frac{\prod_{j=1}^{2d-1} s_j}{|\det \psi(\lambda - \mathcal{A})|}. \tag{3.1}$$

We arrive at the following bound.

Proposition 3.1. *For $\lambda \notin \sigma(\mathcal{A})$*

$$\|(\lambda - \mathcal{A})^{-1}\| \leq \frac{\|\lambda - \mathcal{A}\|^{2d-1}}{|p(\lambda, \overline{\lambda})|}. \tag{3.2}$$

Proof. We have for all j $s_j \le \|\psi(\lambda - \mathcal{A})\| = \|\lambda - \mathcal{A}\|$ and likewise

$$\frac{1}{s_{2d}} = \|(\lambda - \mathcal{A})^{-1}\|. \qquad \square$$

What remains is to bound $|p(\lambda, \overline{\lambda})|$ from below.

Lemma 3.2. *Let $\|\cdot\|$ be a norm in $\mathbb{C}^2$ and put $\gamma = \|(1,1)\|$. Then, in that norm, for all $(z,w) \in \mathbb{C}^2$*

$$|p(\lambda, \overline{\lambda})| \ge (\frac{1}{\gamma})^{2d} \ (\mathrm{dist}((z,w), \Sigma(\mathcal{A})))^{2d}. \tag{3.3}$$

Proof. Recall that $p(z,w) = (zw)^d + r(z,w)$ where r is at most of degree $2d-1$. Following [2] we consider points along the complex line $(z,w)+\zeta(1,1)$ and put

$$q(\zeta) = p((z,w) + \zeta(1,1))$$

so that q is a polynomial satisfying

$$q(\zeta) = \zeta^{2d} + \text{ lower order terms.}$$

Thus, denoting by ζ_j the zeros of q,

$$|p(z,w)| = |q(0)| = \prod_{j=1}^{2d} |\zeta_j|.$$

Since $(z,w) + \zeta_j(1,1) \in \Sigma(\mathcal{A})$ we have

$$|\zeta_j|\|(1,1)\| \ge \mathrm{dist}((z,w), \Sigma(\mathcal{A}))$$

and the claim follows. $\square$

We formulate our bound using max-norm in $\mathbb{C}^2$ as then $\gamma = 1$.

Theorem 3.3. *Let $\mathcal{A}$ be a real linear operator in $\mathbb{C}^d$. Then for all $\lambda \notin \sigma(\mathcal{A})$*

$$\|(\lambda - \mathcal{A})^{-1}\| \le \frac{\|(\lambda - \mathcal{A})\|^{2d-1}}{\mathrm{dist}((\lambda, \overline{\lambda}), \Sigma(\mathcal{A}))^{2d}}. \tag{3.4}$$

Example 3.4. In Example 2.4 the distance from the origin to $\Sigma(\mathcal{A})$ is $|\beta|$. Thus we obtain

$$\|\mathcal{A}^{-1}\| \le \frac{|\beta|^3}{(|\beta|)^4} = \frac{1}{|\beta|}$$

so the upper bound is at origin an equality.

4. Second upper bound

It is tempting to think that if $\sigma(\mathcal{A})$ is not empty, one could control the resolvent by the distance from λ to it in $\mathbb{C}$, in place of distance from $(\lambda, \overline{\lambda})$ to $\Sigma(\mathcal{A})$ in $\mathbb{C}^2$.

Example 4.1. Let $\mathcal{A}$ be the direct sum of complex scalar α and the operator in Example 2.4:

$$\mathcal{A} = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta\tau \\ 0 & -\beta\tau & 0 \end{pmatrix}.$$

If $|\alpha| > |\beta|$, then the norm of the resolvent bears no relation to the distance to $\sigma(\mathcal{A})$ near the origin.

Thus we may hope for a bound which holds near $\sigma(\mathcal{A})$.

Theorem 4.2. *For every real linear operator $\mathcal{A}$ in $\mathbb{C}^d$ there exists an open $U \subset \mathbb{C}$ such that*

$$\sigma(\mathcal{A}) \subset U$$

and for every $\lambda \in U \setminus \sigma(\mathcal{A})$ we have

$$\|(\lambda - \mathcal{A})^{-1}\| \leq \frac{\|\lambda - \mathcal{A}\|^{2d-1}}{\operatorname{dist}(\lambda, \sigma(\mathcal{A}))^{2d}}. \tag{4.1}$$

Proof. If the spectrum is empty U can be taken to be the empty set.

Otherwise, for each $\lambda \notin \sigma(\mathcal{A})$ there exists a closest $\lambda_0 \in \sigma(\mathcal{A})$. Likewise, $\overline{\lambda}_0$ is then a closest point to $\overline{\lambda}$ in the conjugate of the spectrum. Put $E = \{(z, \overline{z}) \mid z \in \mathbb{C}\}$ and denote

$$\Gamma = \Sigma(\mathcal{A}) \cap E.$$

Then the line

$$(\lambda_0, \overline{\lambda}_0) + \zeta(\lambda - \lambda_0, \overline{\lambda} - \overline{\lambda}_0)$$

is orthogonal to the tangent of the curve Γ at $(\lambda_0, \overline{\lambda}_0)$, or the point $(\lambda_0, \overline{\lambda}_0)$ is isolated, in which case there is nothing to be shown.

The technical part of the proof consists of showing that the line is normal not only to the curve Γ but to the whole tangent plane of $\Sigma(\mathcal{A})$. Assume this has been done. Since $\Gamma = \Sigma(\mathcal{A}) \cap E$ is compact there exists a $\delta > 0$ such that along each such line the point $(\lambda_0, \overline{\lambda}_0) \in \Gamma$ is a closest point to every point for $|\zeta| < \delta$. Thus, the claim follows from the previous theorem.

Let us prove the technical part. It follows from the definition of p that interchanging the roles of z and w causes its coefficients to be conjugated. Thus

$$p(z, w) = (zw)^d + \text{ lower order terms } = \sum_{j,k=0}^{d} (\alpha_{jk} z^j w^k + \overline{\alpha}_{jk} z^k w^j). \tag{4.2}$$

Let $w(z)$ denote one of the d roots such that $p(z, w(z)) = 0$. Denoting by ∂_j the partial derivative with respect to the j'th variable we have

$$\partial_1 p + \partial_2 p \ w' = 0.$$

If $\partial_2 p \neq 0$ we have at $(\lambda_0, \overline{\lambda}_0) \in \Gamma$

$$w'(\lambda_0) = -\frac{\partial_1 p}{\partial_2 p}.$$

However, for E we have from (4.2) that

$$\partial_2 p = \overline{\partial_1 p}. \tag{4.3}$$

Thus $w'(\lambda_0)$ is of modulus 1 and all we need its argument. If $\lambda - \lambda_0 = |\lambda - \lambda_0| e^{i\theta}$ and $\partial_1 p(\lambda_0, \overline{\lambda}_0) = \rho e^{i\varphi}$ then the orthogonality implies that either $\varphi = -\theta$ or $\varphi = \pi - \theta$. In either case

$$w'(\lambda_0) = -e^{2i\theta}.$$

The calculation above assumed that the partial derivatives do not vanish. This however can happen. But w' is even then regular and of modulus 1. In fact, let $z(w)$ be such that

$$p(z(w), w) = 0$$

and thus

$$z(w(z)) = z$$

which implies at $(z, w) = (\lambda_0, \overline{\lambda}_0)$

$$z'(\overline{\lambda}_0) w'(\lambda_0) = 1.$$

But interchanging their roles we have using (4.2)

$$z'(\overline{\lambda}_0) = \overline{w'(\lambda_0)}$$

and thus $|w'(\lambda_0)| = 1$.

Suppose we make a small perturbation Δz to λ_0 and denote $\lambda_1 = \lambda_0 + \Delta z$. Then the first component satisfies

$$|\lambda - \lambda_1| = |\ |\lambda - \lambda_0| - e^{-i\theta} \Delta z|$$

while the second component satisfies

$$|\overline{\lambda} - w(\lambda_1)| = |\ |\lambda - \lambda_0| + e^{-i\theta} \Delta z + O((\Delta z)^2)|.$$

Thus the maximum increases if $\Delta z \neq 0$. □

Remark 4.3. For *normal* operators $\mathcal{A}$ we have

$$\|(\lambda - \mathcal{A})^{-1}\| = \frac{1}{\operatorname{dist}(\lambda, \sigma(\mathcal{A}))}, \tag{4.4}$$

[4]. Here $\mathcal{A}$ is normal if there is a unitary matrix U such that $U\mathcal{A}U^*$ is a diagonal real linear operator. By Theorem 2.3 in [4] $\mathcal{A} = A + B\tau$ is normal if and only if A is normal and B, HB, KB are symmetric, where H and K denote the Hermitian and skew-Hermitian parts of A.

The equality (4.4) follows immediately by inverting the diagonal elements. In particular, if $\alpha, \beta \in \mathbb{C}$, then

$$\|(\lambda - \alpha - \beta\tau)^{-1}\| = |\frac{1}{|\lambda - \alpha| - |\beta|}|.$$

Observe however, that our method of proof in the bounds above give for circlets

$$\|(\lambda - \alpha - \beta\tau)^{-1}\| \leq \frac{\|\lambda - \alpha - \beta\tau)\|}{\operatorname{dist}(\lambda, \sigma(\alpha + \beta\tau))^2} = \frac{|\lambda - \alpha| + |\beta|}{(|\lambda - \alpha| - |\beta|)^2}.$$

Here the second power in the nominator appears inevitably due to the technique of the proof. Since $p(\lambda, \overline{\lambda}) = |\lambda - \alpha|^2 - |\beta|^2$ there actually is a common factor

$$\frac{\|\lambda - \alpha - \beta\tau\|}{|p(\lambda, \overline{\lambda})|} = |\frac{|\lambda - \alpha| + |\beta|}{|\lambda - \alpha|^2 - |\beta|^2}| = |\frac{1}{|\lambda - \alpha| - |\beta|}|$$

but we do not know how this effect could be utilized in the general case.

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