Helsinki University of Technology Institute of Mathematics Research Reports
Espoo 2010 A583

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Wolfgang Desch **Stig-Olof Londen**

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Wolfgang Desch **Stig-Olof Londen**

Aalto University
School of Science and Technology
Department of Mathematics and Systems Analysis

Wolfgang Desch, Stig-Olof Londen: *Semilinear stochastic integral equations in* L_p; Helsinki University of Technology Institute of Mathematics Research Reports A583 (2010).

Abstract: *We consider a semilinear parabolic stochastic integral equation*

$$u(t,\omega,x) = Ak_\alpha * u(t,\omega,x) + \sum_{k=1}^{\infty} k_\beta \star G^k(t,\omega,u(t,\omega,\cdot))(x) + k_\gamma * F(t,\omega,u(t,\omega,\cdot))(x) + u_0(\omega,x) + tu_1(\omega,x).$$

Here $t \in [0,T]$, ω *in a probability space* Ω, x *in a* σ*-finite measure space* B *with (positive) measure* Λ. *The kernels* $k_\mu(t)$ *are multiples of* $t^{\mu-1}$. *The operator* $A : \mathcal{D}A \subset L_p(B) \to L_p(B)$ *is such that* $(-A)$ *is a nonnegative operator. The convolution integrals* $k_\beta \star G^k$ *are stochastic convolutions with respect to independent scalar Wiener processes* w^k. $F : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to L_p(B)$ *and* $G : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to L_p(B, l_2)$ *are nonlinear with suitable Lipschitz conditions.*

We establish an L_p*-theory for this equation, including existence and uniqueness of solutions, and regularity results in terms of fractional powers of* $(-A)$ *and fractional derivatives in time.*

AMS subject classifications: 60H15, 60H20, 45N05

Keywords: semilinear stochastic integral equations, stochastic fractional differential equation, regularity, nonnegative operator, Volterra equation, singular kernel

Correspondence
Wolfgang Desch
Institut für Mathematik und Wissenschaftliches Rechnen
Karl-Franzens-Universität Graz
Heinrichstrasse 36
8010 Graz
Austria
georg.desch@uni-graz.at

Stig-Olof Londen
Aalto University
Department of Mathematics and Systems Analysis
P.O. Box 11100
FI-00076 Aalto
Finland
stig-olof.londen@tkk.fi

Received 2009-12-31
ISBN 978-952-60-3028-9 (print) ISSN 0784-3143 (print)
ISBN 978-952-60-3029-6 (PDF) ISSN 1797-5867 (PDF)

Aalto University
School of Science and Technology
Department of Mathematics and Systems Analysis
P.O. Box 11100, FI-00076 Aalto, Finland
email: math@tkk.fi `http://math.tkk.fi/`

SEMILINEAR STOCHASTIC INTEGRAL EQUATIONS IN L_p

WOLFGANG DESCH AND STIG-OLOF LONDEN

Abstract. We consider a semilinear parabolic stochastic integral equation

$$\begin{aligned} u(t,\omega,x) \quad &= \quad Ak_\alpha * u(t,\omega,x) + \sum_{k=1}^{\infty} k_\beta \star G^k(t,\omega,u(t,\omega,\cdot))(x) \\ &\qquad + k_\gamma * F(t,\omega,u(t,\omega,\cdot))(x) + u_0(\omega,x) + tu_1(\omega,x). \end{aligned}$$

Here $t \in [0,T]$, ω in a probability space Ω, x in a σ-finite measure space B with (positive) measure Λ. The kernels $k_\mu(t)$ are multiples of $t^{\mu-1}$. The operator $A : \mathcal{D}A \subset L_p(B) \to L_p(B)$ is such that $(-A)$ is a nonnegative operator. The convolution integrals $k_\beta \star G^k$ are stochastic convolutions with respect to independent scalar Wiener processes w^k. $F : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to L_p(B)$ and $G : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to L_p(B, l_2)$ are nonlinear with suitable Lipschitz conditions.

We establish an L_p-theory for this equation, including existence and uniqueness of solutions, and regularity results in terms of fractional powers of $(-A)$ and fractional derivatives in time.

Dedicated to Herbert Amann on the occasion of his 70th birthday.

1. Introduction

We consider the semilinear integral equation

$$\begin{aligned} u(t,\omega,x) = A\int_0^t & k_\alpha(t-s)u(s,\omega,x)\,ds \\ &+ \sum_{k=1}^{\infty}\int_0^t k_\beta(t-s)G^k(s,\omega,u(s,\omega,\cdot))(x)\,dw_s^k \\ &+ \int_0^t k_\gamma(t-s)F(s,\omega,u(s,\omega,\cdot))(x)\,ds + u_0(\omega,x) + tu_1(\omega,x). \end{aligned} \tag{1.1}$$

The real scalar valued solution $u(t,\omega,x)$ depends on $t \in [0,T]$, ω in a probability space Ω, and x in a measure space B. The convolution kernels k_μ are defined by

$$k_\mu(t) := \frac{1}{\Gamma(\mu)}\, t^{\mu-1}. \tag{1.2}$$

We assume $\alpha \in (0,2)$, $\beta > \frac{1}{2}$, and $\gamma > 0$. The operator $A : \mathcal{D}A \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ (with $2 \le p < \infty$) is such that $(-A)$ is a nonnegative linear operator (see Section 2 below). In particular we have in mind elliptic partial differential operators on a sufficiently smooth (bounded or unbounded) domain $B \subset \mathbb{R}^n$, but formally we require only that $(-A)$ is sectorial and the state space is an L_p-space on some measure space B. The processes w_s^k are scalar valued, independent Wiener processes. F and G^k are nonlinear and satisfy suitable Lipschitz estimates with respect to u. The functions u_0 and u_1 are given initial data. For the precise conditions, see Section 3.

1991 *Mathematics Subject Classification.* 60H15, 60H20, 45N05.

Key words and phrases. Semilinear stochastic integral equations, stochastic fractional differential equation, regularity, nonnegative operator, Volterra equation, singular kernel.

Our goal is to establish existence and uniqueness of solutions for the semilinear equation (1.1) in an L_p-framework with $p \in [2, \infty)$. Regularity results will be stated in terms of fractional powers of $-A$ (for spatial regularity) and fractional time integrals and derivatives as well as Hölder continuity (for time regularity).

Technically we rely primarily on results concerning a linear integral equation where the forcing terms F and G are replaced by functions independent of u, i.e., (5.1). In recent work [12] we have developed an L_p-theory for (5.1), albeit without the deterministic part and without the u_1-term. These results need, however, - for the purpose of analyzing (1.1) - to be extended and to be made more precise.

Our linear results build on an approach due to Krylov, developed for parabolic stochastic partial differential equations. This approach uses the Burkholder-Davis-Gundy inequality and estimates on the solution and on its spatial gradient. To analyze the integral equation (5.1) we combine Krylov's approach with transformation techniques and estimates involving both fractional powers of $-A$, and fractional time-derivatives (integrals) of the solution. Krylov's approach is very efficient in obtaining maximal regularity, however, it relies on a highly nontrivial Paley-Littlewood inequality [18]. A counterpart of this estimate can be given for general sectorial A by straightforward estimates on the Dunford integral, when we allow for an infinitesimal loss of regularity.

We also include results for the deterministic convolution and for the u_1-term. Obviously, no originality is claimed for these results.

To obtain result on the semilinear equation (1.1) we combine our linear theory with a standard contraction approach.

The paper is organized as follows: Before we can state our main results, we need to collect some facts about sectorial operators and fractional differentiation and integration in Section 2. Section 3 states the hypotheses and results for the semilinear equation. In Section 4 we provide the tools to define a stochastic integral and a stochastic convolution in L_p-spaces. The central part of this section is an application of the Burkholder-Davis-Gundy inequality to lift scalar valued Ito-integrals to stochastic integrals in L_p. This approach is adapted from [19]. Section 5 deals with the linear fractional differential equation. In the beginning we give the results on existence and regularity which are basic to obtain similar results on the semilinear equation. We construct the solution via the resolvent operator and a variation of parameters formula. The contribution of the initial data and of the forcing F, which enters as a Lebesgue integral, are well-known ([24], [33]). The contribution of the stochastic integral containing G is handled by a recent result [12]. We collect these results in a unified way to allow a comparison of the various requirements on regularity. In Section 6 we arrive at the proof of our main results on the semilinear equation by a standard contraction procedure. In Section 7 we make some comments on available maximal regularity results for the linear equation and their implications for the semilinear equation. Finally, in Section 8 we compare our results to some recent results on parabolic stochastic differential equations obtained recently using an abstract theory of stochastic integration in Banach spaces.

2. Fractional powers and fractional derivatives

In this paper $A : \mathcal{D}A \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ will be a linear operator such that $(-A)$ is nonnegative. Here $p \in [2, \infty)$, but fixed. Regularity in space will be expressed in terms of the fractional powers $(-A)^\theta$ of A. Regularity in time will be expressed in terms of fractional time derivatives $D_t^\eta f$. In corollaries we will also give regularity results in terms of the function spaces $h^\gamma_{0\to 0}([0,T];X)$, i.e., the little Hölder-continuous functions with $f(0) = 0$.

In this section we summarize briefly the definitions and some known results about nonnegative operators, their fractional powers, and about fractional integration and differentiation.

Let X be a complex Banach space and let $\mathcal{L}(X)$ be the space of bounded linear operators on X. Let B be a closed, linear map of $\mathcal{D}B \subset X$ into X. The operator $-B$ is said to be nonnegative if $\rho(B)$, the resolvent set of B, contains $(0,\infty)$, and

$$\sup_{\lambda>0} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} < \infty.$$

An operator is positive if it is nonnegative and, in addition, $0 \in \rho(B)$. For $\omega \in [0,\pi)$, we define

$$\Sigma_\omega := \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg\ \lambda| < \omega\}.$$

Recall that if $(-B)$ is nonnegative, then there exists a number $\eta \in (0,\pi)$ such that $\rho(B) \supset \Sigma_\eta$, and

$$\sup_{\lambda\in\Sigma_\eta} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{2.1}$$

The spectral angle of $(-B)$ is defined by

$$\phi_{(-B)} := \inf\{\omega \in (0,\pi] \mid \rho(B) \supset \Sigma_{\pi-\omega},\ \sup_{\lambda\in\Sigma_{\pi-\omega}} \|\lambda(\lambda I - B)^{-1}\|_{\mathcal{L}(X)} < \infty\}.$$

We will rely on the concept of fractional powers of $(-B)$: Let $(-B)$ be a densely defined nonnegative linear operator on X, and $\theta > 0$. If $(-B)$ is positive, then $(-B)^{-1}$ is a bounded operator, and $(-B)^{-\theta}$ can be defined by integral formulas [4, Ch. 3] or [20, Section 2.2.2]. As usual,

$$(-B)^\theta := ((-B)^{-\theta})^{-1}, \quad \theta > 0. \tag{2.2}$$

If $(-B)$ is nonnegative with $0 \in \sigma(-B)$, we proceed as in [4, Ch. 5]: Since $(-B+\epsilon I)$ is a positive operator if $\epsilon > 0$, its fractional power $(-B+\epsilon I)^\theta$ is well defined according to (2.2). We define

$$\mathcal{D}((-B)^\theta) := \{y \in \bigcap_{0<\epsilon\le\epsilon_0} \mathcal{D}((-B+\epsilon I)^\theta) \mid \lim_{\epsilon\to 0+} (-B+\epsilon I)^\theta y \text{ exists }\}, \tag{2.3}$$

$$(-B)^\theta y := \lim_{\epsilon\to 0+} (-B+\epsilon I)^\theta y \quad \text{for } y \in \mathcal{D}((-B)^\theta). \tag{2.4}$$

Lemma 2.1. *Let $-B$ be a nonnegative linear operator on a Banach space X with spectral angle $\phi_{(-B)}$, and let $\theta > 0$.*

1) *$(-B)^\theta$ is closed and $\overline{\mathcal{D}((-B)^\theta)} = \overline{\mathcal{D}(-B)}$.*
2) *Assume that $\theta\phi_{(-B)} < \pi$. Then $(-B)^\theta$ is nonnegative and has spectral angle $\theta\phi_{(-B)}$.*

Proof. For (1) see [4, p. 109, 142], also [7, Theorem 10]. For (2) see [4, p. 123]. □

Lemma 2.2. *Let $-B$ be a nonnegative linear operator on a Banach space X with spectral angle $\phi_{(-B)}$. Then for $\eta \in [0, \pi - \phi_{(-B)})$*

$$\sup_{|\arg\ \mu|\le\eta,\, \mu\ne 0} \|(-B)^\theta \mu^{1-\theta}(\mu I - B)^{-1}\|_{\mathcal{L}(X)} < \infty. \tag{2.5}$$

Proof. In case $\eta = 0$, see [4, Th. 6.1.1, p. 141]. The general case can be reduced to the case $\mu > 0$, [14, p.314]. See also [12, Lemma 3.3]. □

We turn now to fractional differentiation and integration in time:

Definition 2.3. Let X be a Banach space and $\alpha \in (0,1)$, let $u \in L_1((0,T);X)$ for some $T > 0$.

1) Fractional integration in time of order α is defined by $D_t^{-\alpha} u := \frac{1}{\Gamma(\alpha)} t^{\alpha-1} * u$.

2) We say that u has a fractional derivative of order $\alpha > 0$ provided $u = D_t^{-\alpha} f$, for some $f \in L_1((0,T);X)$. If this is the case, we write $D_t^\alpha u = f$.

Remark 2.4. Suppose that u has a fractional derivative of order $\alpha \in (0,1)$. Then $\frac{1}{\Gamma(1-\alpha)} t^{-\alpha} * u$ is differentiable a.e. and absolutely continuous with $D_t^\alpha u = \frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)} t^{-\alpha} * u\right)$.

For the equivalence of fractional derivatives in L_p and fractional powers of the realization of the derivative in L_p, we have the following Lemma.

Lemma 2.5. [8, Prop.2] *Let $p \in [1,\infty)$, X a Banach space and define*

$$\mathcal{D}L := \{u \in W^{1,p}((0,T);X) \mid u(0) = 0\}, \; Lu = u' \text{ for } u \in \mathcal{D}L.$$

Then, with $\beta \in (0,1)$,

$$L^\beta u \;=\; D_t^\beta u, \quad u \in \mathcal{D}(L^\beta), \tag{2.6}$$

where $\mathcal{D}(L^\beta)$ coincides with the set of functions u having a fractional derivative in L_p, i.e.,

$$\mathcal{D}(L^\beta) = \{u \in L_p((0,T);X) \mid \frac{1}{\Gamma(1-\beta)} t^{-\beta} * u \in W_0^{1,p}((0,T);X)\}.$$

In particular, D_t^β is closed.

We refer to [8] for further properties of the operator D_t^β.

3. The main result

Hypothesis 3.1. Let $(B,\mathcal{A},\Lambda)$ be a σ-finite measure space and fix $2 \le p < \infty$. Let $(-A) : \mathcal{D}A \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ be a nonnegative linear operator with spectral angle $\phi_{(-A)}$, and such that $\mathcal{D}A \cap L_1(B;\mathbb{R}) \cap L_\infty(B;\mathbb{R})$ is dense in $L_p(B;\mathbb{R})$.

Hypothesis 3.2. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space with an increasing, right continuous filtration $\{\mathcal{F}_t \mid t \ge 0\}$ satisfying $\mathcal{F}_t \subset \mathcal{F}$ for all $t \ge 0$. Let $\mathcal{P}$ denote the predictable σ-algebra on $[0,\infty) \times \Omega$ generated by $\{\mathcal{F}_t\}$, and assume that $\{w_s^k \mid k = 1,2,3,\dots\}$ is an independent family of (scalar valued) $\mathcal{F}_t$-adapted Wiener processes on $(\Omega,\mathcal{F},\mathbb{P})$.

Remark 3.3. On $[0,T] \times \Omega$, measurability will always be understood with respect to the predictable σ-algebra $\mathcal{P}$, and the product measure of the Lebesgue measure on $[0,T]$ and $\mathbb{P}$.

Hypothesis 3.4. For suitable $\theta \in [0,1)$ and $\epsilon \in [0,1)$, the function

$$F : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to \mathcal{D}(-A)^\epsilon$$

satisfies the following assumptions:

(a) For fixed $u \in \mathcal{D}(-A)^\theta$, the function $F(\cdot,\cdot,u)$ is measurable from $[0,T] \times \Omega$ into $\mathcal{D}(-A)^\epsilon$.

(b) There exists a constant $M_F > 0$, such that for all $t \in [0,T]$, and all $u_1, u_2 \in \mathcal{D}(-A)^\theta$ the following Lipschitz estimate holds

$$\|F(t,\omega,u_1) - F(t,\omega,u_2)\|_{\mathcal{D}(-A)^\epsilon} \le M_F \|u_1 - u_2\|_{\mathcal{D}(-A)^\theta} \quad \text{for a.e. } \omega \in \Omega. \tag{3.1}$$

(c) For $u = 0$ we have

$$\left[\int_\Omega \int_0^T \|F(t,\omega,0)\|_{\mathcal{D}(-A)^\epsilon}^p \, dt \, d\mathbb{P}\right]^{1/p} = M_{F,0} < \infty. \tag{3.2}$$

Hypothesis 3.5. For the same $\theta \in [0,1)$ as in Hypothesis 3.4, the function

$$G : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to L_p(B; l_2)$$
$$[G(t,\omega,u)](x) := \big(G^k(t,\omega,u)(x)\big)_{k=1}^\infty$$

satisfies the following assumptions:

(a) For fixed $u \in \mathcal{D}(-A)^\theta$, the function $G(\cdot,\cdot,u)$ is measurable from $[0,T]\times\Omega$ into $L_p(B;l_2)$.

(b) There exists a constant $M_G > 0$, such that for all $t \in [0,T]$, and all $u_1, u_2 \in \mathcal{D}(-A)^\theta$ the following Lipschitz estimate holds:

$$\|G(t,\omega,u_1) - G(t,\omega,u_2)\|_{L_p(B;l_2)} \le M_G \|u_1 - u_2\|_{\mathcal{D}(-A)^\theta} \quad \text{for a.e. } \omega \in \Omega. \tag{3.3}$$

(c) For $u = 0$ we have

$$\left[\int_\Omega \int_0^T \|G(t,\omega,0)\|^p_{L_p(B;l_2)}\, dt\, d\mathbb{P}\right]^{1/p} = M_{G,0} < \infty. \tag{3.4}$$

Theorem 3.6. *Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener processes $\big(w_s^k\big)_{k=1}^\infty$ be as in Hypothesis 3.2. Let $p \in [2,\infty)$, let the measure space $(B, \mathcal{A}, \Lambda)$ and the operator $A : \mathcal{D}A \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ satisfy Hypothesis 3.1. Let $\alpha \in (0,2)$, $\beta > \frac{1}{2}$ and $\gamma > 0$. Let $T > 0$ and assume that $F : [0,T]\times\Omega\times\mathcal{D}(-A)^\theta \to \mathcal{D}(-A)^\epsilon$ and $G : [0,T]\times\Omega\times\mathcal{D}(-A)^\theta \to L_p(B;l_2)$ satisfy Hypotheses 3.4 and 3.5 with suitable $\theta, \epsilon \in [0,1]$. Let $u_0 \in L_p(\Omega; \mathcal{D}(-A)^{\delta_0})$, $u_1 \in L_p(\Omega; \mathcal{D}(-A)^{\delta_1})$, with suitable $\delta_i \in [0,1]$, both u_i measurable with respect to $\mathcal{F}_0$. Suppose that the following inequalities hold:*

$$\alpha\theta < \gamma + \alpha\epsilon, \tag{3.5}$$

$$\frac{1}{2} + \alpha\theta < \beta, \tag{3.6}$$

$$\alpha\theta < \frac{1}{p} + \alpha\delta_0, \tag{3.7}$$

$$\alpha\theta < 1 + \frac{1}{p} + \alpha\delta_1. \tag{3.8}$$

Then there exists a unique function $u \in L_p([0,T]\times\Omega; \mathcal{D}(-A)^\theta)$ such that for almost all $t \in [0,T]$

$$\int_0^t k_\alpha(t-s) u(s,\omega,\cdot)\, ds \in \mathcal{D}A \quad \text{for a.e. } \omega \in \Omega,$$

and (1.1) *is satisfied for almost all $t \in [0,T]$ and almost all $\omega \in \Omega$.*

Theorem 3.7. *Let the assumptions of Theorem 3.6 hold. Moreover, assume that $\eta \in (-1,1)$, $\zeta \in [0,1]$ are such that*

$$\eta + \alpha\zeta < \gamma + \alpha\epsilon, \tag{3.9}$$

$$\frac{1}{2} + \eta + \alpha\zeta < \beta, \tag{3.10}$$

$$\eta + \alpha\zeta < \frac{1}{p} + \alpha\delta_0, \tag{3.11}$$

$$\eta + \alpha\zeta < 1 + \frac{1}{p} + \alpha\delta_1. \tag{3.12}$$

With the notation $1_{\{a>b\}} = 1$ if $a > b$ and $1_{\{a>b\}} = 0$ if $a \le b$, we put

$$v(t) = u(t) - 1_{\{\delta_0 > \zeta\}} u_0 - 1_{\{\delta_1 > \zeta\}} t u_1 - 1_{\{\epsilon > \zeta\}} \int_0^t k_\gamma(t-s) F(s,\omega,u(s))\, ds. \tag{3.13}$$

(a) *Then, if* $\eta > 0$*, the function* v*, considered as a Banach space valued function* $v : [0,T] \to L_p(\Omega; \mathcal{D}(-A)^\zeta)$*, has a fractional derivative of order* η*.*
(b) *If* $\eta < 0$*, the function* $v : [0,T] \to L_p(\Omega; L_p(B;\mathbb{R}))$ *has a fractional integral of order* $-\eta$*. Moreover,* $D_t^\eta v$ *takes values in* $L_p(\Omega; \mathcal{D}(-A)^\zeta)$*.*
(c) *If* $\eta = 0$*, of course, we denote* $D_t^0 v = v$*.*

In any case, there exists a constant M_u*, depending on* A*,* p*,* T*,* α*,* β*,* γ*,* δ_i*,* ϵ*,* ζ*,* η*,* θ*,* M_F*,* M_G *such that*

$$
\begin{aligned}
(3.14)\qquad & \|D_t^\eta v\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)} \\
\leq\ & M_u \left[\|u_0\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})} + \|u_1\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})} + M_{F,0} + M_{G,0} \right].
\end{aligned}
$$

Corollary 3.8. *Let the Assumptions of Theorem 3.6 hold. Let* $\zeta \in [0,1]$*. Let* u *be the solution of* (1.1) *and* v *be defined by* (3.13)*.*

(1) *Let* $p < q < \infty$ *be such that*

$$
\frac{1}{p} - \frac{1}{q} + \alpha\zeta < \gamma + \alpha\epsilon, \qquad \frac{1}{2} + \frac{1}{p} - \frac{1}{q} + \alpha\zeta < \beta,
$$
$$
\alpha\zeta - \frac{1}{q} < \alpha\delta_0, \qquad \alpha\zeta - \frac{1}{q} < 1 + \alpha\delta_1.
$$

Then $v \in L_q([0,T]; L_p(\Omega; \mathcal{D}(-A)^\zeta))$*.*

(2) *Let* $\mu \in (0, 1 - \frac{1}{p})$ *be such that*

$$
\frac{1}{p} + \mu + \alpha\zeta < \gamma + \alpha\epsilon, \qquad \frac{1}{2} + \frac{1}{p} + \mu + \alpha\zeta < \beta,
$$
$$
\mu + \alpha\zeta < \alpha\delta_0, \qquad \mu + \alpha\zeta < 1 + \alpha\delta_1.
$$

Then $v \in h^\mu_{0\to 0}([0,T]; L_p(\Omega; \mathcal{D}(-A)^\zeta))$*.*

Hypothesis 3.9. Let $F_1, F_2 : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to \mathcal{D}(-A)^\epsilon$ satisfy Hypothesis 3.4, $G_1, G_2 : [0,T] \times \Omega \times \mathcal{D}(-A)^\theta \to L_p(B; l_2)$ satisfy Hypothesis 3.5, and suppose that there are nonnegative functions $\mu_{\Delta F}$, $\mu_{\Delta G} \in L_p([0,T]\times\Omega;\mathbb{R})$ such that for all $t \in [0,T]$ and $u \in \mathcal{D}(-A)^\theta$, and almost all $\omega \in \Omega$

$$
\|F_1(t,\omega,u) - F_2(t,\omega,u)\|_{\mathcal{D}(-A)^\epsilon} \leq \mu_{\Delta F}(t,\omega), \tag{3.15}
$$
$$
\|G_1(t,\omega,u) - G_2(t,\omega,u)\|_{L_p(B;l_2)} \leq \mu_{\Delta G}(t,\omega). \tag{3.16}
$$

Remark 3.10. The standard example of F_i, G_i satisfying Hypothesis 3.9 is (for $i = 1, 2$):

$$
\begin{aligned}
F_i(t,\omega,u) &= F(t,\omega,u) + f_i(t,\omega), \\
G_i(t,\omega,u) &= G(t,\omega,u) + g_i(t,\omega),
\end{aligned}
$$

where F and G satisfy Hypotheses 3.4 and 3.5, respectively, and $f_i \in L_p([0,T] \times \Omega; \mathcal{D}(-A)^\epsilon)$, $g_i \in L_p([0,T]\times\Omega; L_p(B;l_2))$. Here we take

$$
\begin{aligned}
\mu_{\Delta F}(t,\omega) &= \|f_1(t,\omega) - f_2(t,\omega)\|_{\mathcal{D}(-A)^\epsilon}, \\
\mu_{\Delta G}(t,\omega) &= \|g_1(t,\omega) - g_2(t,\omega)\|_{L_p(B;l_2)}.
\end{aligned}
$$

Theorem 3.11. *Let the probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and the Wiener processes* w_s^k *be as in Hypothesis 3.2. Let* $p \in [2,\infty)$*, let the measure space* $(B, \mathcal{A}, \Lambda)$ *and the operator* $A : \mathcal{D}A \subset L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ *satisfy Hypothesis 3.1.*

Let $T > 0$*,* $\alpha \in (0,2)$*,* $\beta > \frac{1}{2}$*,* $\gamma > 0$*, and* δ_0*,* δ_1*,* $\epsilon \in [0,1]$ *be such that* (3.5)*,* (3.6)*,* (3.7)*, and* (3.8) *hold. Let* $\eta \in (-1,1)$ *and* $\zeta \in [0,1]$ *be such that* (3.9)*,* (3.10)*,* (3.11)*,* (3.12) *hold. Then there exists a constant* $M_{\Delta u} > 0$*, dependent on* $p, T, \alpha, \beta, \gamma, \delta_0, \delta_1, \epsilon, \zeta, M_F, M_G$*, such that the following Lipschitz estimate holds:*

Let F_1, F_2, G_1, G_2 satisfy Hypotheses 3.4, 3.5 and 3.9 with ϵ, θ as above. For $i = 1, 2$ let the initial data $u_{0,i} \in L_p(\Omega; \mathcal{D}(-A)^{\delta_0})$ and $u_{1,i} \in L_p(\Omega; \mathcal{D}(-A)^{\delta_1})$ be $\mathcal{F}_0$-measurable, and let $u_1(t,\omega,x)$, $u_2(t,\omega,x)$ be the solutions of (1.1) *with F, G, u_0, u_1 replaced by $F_i, G_i, u_{0,i}, u_{1,i}$. Let v_i be defined according to* (3.13) *with u replaced by u_i. Then*

$$\begin{aligned}
(3.17)\qquad & \|D_t^\eta v_1 - D_t^\eta v_2\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\varsigma)} \\
& \le\ M_{\Delta u}\Big[\|u_{0,1} - u_{0,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})} + \|u_{1,1} - u_{1,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})} \\
& \quad + \|\mu_{\Delta F}(t,\omega) + \mu_{\Delta G}(t,\omega)\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big].
\end{aligned}$$

4. Stochastic lemmas

Lemma 4.1 ([19],Theorem 3.10)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ satisfy Hypothesis 3.2. Let Y be a dense subspace of $L_p(B;\mathbb{R})$, $0 < T \le \infty$, and $g \in L_p([0,T]\times\Omega; L_p(B; l_2))$. Then there exists a sequence of functions $g_j \in L_p([0,T]\times\Omega; L_p(B;l_2))$ converging to g in $L_p([0,T]\times\Omega, L_p(B;l_2))$ such that each $g_j = (g_j^k)_{k=1}^\infty$ is of the form*

$$(4.1)\qquad g_j^k(t,\omega,x) = \begin{cases} \sum_{i=1}^j I_{\tau_{i-1}^j(\omega) < t \le \tau_i^j(\omega)}(t) g_{j,i}^k(x) & \text{if } k \le j, \\ 0 & \text{else,} \end{cases}$$

where $\tau_0^j \le \tau_1^j \le \cdots \tau_j^j$ are bounded stopping times with respect to the filtration $\mathcal{F}_t$, and $g_{j,i}^k \in Y$. (Here, for any set A, I_A denotes its indicator function.)

Remark 4.2. We will apply Lemma 4.1 with $Y = \mathcal{D}A \cap L_1(B;\mathbb{R}) \cap L_\infty(B;\mathbb{R})$.

Lemma 4.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener processes w_t^k be as in Hypothesis 3.2. Let $p \in [2,\infty)$. Let Y be a dense subspace of $L_p(B;\mathbb{R})$, let $T > 0$, and let $g_j \in L_p([0,T]\times\Omega; L_p(B;l_2))$ be of the simple structure given in* (4.1)*. For $t \in [0,T]$, let $V(t) : Y \to L_p(B;\mathbb{R})$ be a linear operator such that the function $t \mapsto V(t)y$ is in $L_2([0,T]; L_p(B;\mathbb{R}))$ for each $y \in Y$. Then there exists a constant M, depending only on p and T, such that for all $t \in (0,T]$*

$$\begin{aligned}
(4.2)\qquad & \int_B \int_\Omega \left| \sum_{k=1}^j \int_0^t [V(t-s) g_j^k(s,\omega)](x)\, dw_s^k \right|^p d\mathbb{P}(\omega)\, d\Lambda(x) \\
& \le\ M \int_B \int_\Omega \left(\int_0^t |[V(t-s) g_j(s,\omega)](x)|_{l_2}^2\, ds \right)^{\frac{p}{2}} d\mathbb{P}(\omega)\, d\Lambda(x).
\end{aligned}$$

Proof. First fix some $t \in (0,T]$. For $x \in B$, $r > 0$ we define

$$Y_j(r,\omega,x) = \sum_{k=1}^j \int_0^r [V(t-s) g_j^k(s,\omega)](x)\, dw_s^k.$$

By the elementary structure of g_j,

$$\int_0^r \left|[V(t-s) g_j^k(s,\omega)](x)\right|^2 ds < \infty$$

for allmost all $x \in B$, so that $Y_j(r,\omega,x)$ is well-defined as an Ito integral for such x, and it is a martingale. Since the Wiener processes w_s^k are independent, the quadratic variation of $Y_j(\cdot,\cdot,x)$ is

$$\sum_{k=1}^j \int_0^r \left|[V(t-s) g_j^k(s,\omega)](x)\right|^2 ds.$$

Now the Burkholder-Davis-Gundy inequality (see [17, p. 163]) yields for $r \in [0,t]$ and each $x \in B$,

$$\begin{aligned} &\int_\Omega \Big| \sum_{k=1}^{j} \int_0^r [V(t-s)g_j^k(s,\omega)](x)\ dw_s^k \Big|^p\, d\mathbb{P}(\omega) \qquad (4.3)\\ \le\ & M\int_\Omega \left(\int_0^r \sum_{k=1}^{j} |[V(t-s)g_j^k(s,\omega)](x)|^2\ ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega)\\ =\ & M\int_\Omega \left(\int_0^r |V(t-s)g_j(s,\omega)](x)|_{l_2}^2\ ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega). \end{aligned}$$

In (4.3), take $r = t$ and integrate over B:

$$\begin{aligned} &\int_B \int_\Omega \Big| \sum_{k=1}^{j} \int_0^t [V(t-s)g_j^k(s,\omega)](x)\ dw_s^k \Big|^p\, d\mathbb{P}(\omega)\, d\Lambda(x)\\ \le\ & M\int_B\int_\Omega \left(\int_0^t |[V(t-s)g_j(s,\omega)](x)|_{l_2}^2\ ds\right)^{\frac{p}{2}} d\mathbb{P}(\omega)\, d\Lambda(x). \end{aligned}$$

□

Lemma 4.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener processes w_t^k satisfy Hypothesis 3.2. Let $T > 0$, $2 \le p < \infty$, and $g \in L_p([0,T]\times\Omega; L_p(B;l_2))$, moreover, let $\{g_j\}$ be a sequence approximating g in the sense of Lemma 4.1. Let $\beta > \frac{1}{2}$, $\eta \in [0,1)$ such that $\beta - \eta > \frac{1}{2}$. Then the functions*

$$D_t^\eta \sum_{k=1}^{\infty} \int_0^t k_\beta(t-s) g_j^k(s,\omega,x)\, dw_s^k(\omega)$$

converge in $L_p([0,T]\times\Omega; L_p(B;\mathbb{R}))$, as $j\to\infty$.

Proof. Put $h_{i,j}^k := g_j^k - g_i^k$. The stochastic Fubini theorem implies that

$$D_t^{-\eta} \int_0^t k_{\beta-\eta}(t-s) h_{i,j}^k(s,\omega,x)\, dw_s^k = \int_0^t k_\beta(t-s) h_{i,j}^k(s,\omega,x)\, dw_s^k,$$

i.e., $$\int_0^t k_{\beta-\eta}(t-s) h_{i,j}^k(s,\omega,x)\, dw_s^k = D_t^\eta \int_0^t k_\beta(t-s) h_{i,j}^k(s,\omega,x)\, dw_s^k.$$

We use Lemma 4.3 and the fact that $k_{\beta-\eta}^2 \in L_1([0,T];\mathbb{R})$:

$$\begin{aligned} &\int_0^T \int_B \int_\Omega \Big| D_t^\eta \sum_{k=1}^{\infty} \int_0^t k_\beta(t-s) h_{i,j}^k(s,\omega,x)\, dw_s^k \Big|^p d\mathbb{P}(\omega)\, d\Lambda(x)\, dt\\ =\ & \int_0^T \int_B \int_\Omega \Big| \sum_{k=1}^{\infty} \int_0^t k_{\beta-\eta}(t-s) h_{i,j}^k(s,\omega,x)\, dw_s^k \Big|^p d\mathbb{P}(\omega)\, d\Lambda(x)\, dt\\ \le\ & M\int_0^T \int_B \int_\Omega \left(\int_0^t |k_{\beta-\eta}(t-s) h_{i,j}(s,\omega,x)|_{l_2}^2\, ds \right)^{\frac{p}{2}} d\mathbb{P}(\omega)\, d\Lambda(x)\, dt\\ \le\ & M\int_B\int_\Omega \left[\int_0^T k_{\beta-\eta}^2(s)\, ds\right]^{\frac{p}{2}} \left[\int_0^T |h_{i,j}(s,\omega,x)|_{l_2}^p\, ds \right] d\mathbb{P}(\omega)\, d\Lambda(x)\\ \le\ & M\|h_{i,j}\|^p_{L_p([0,T]\times\Omega;L_p(B;l_2))}. \end{aligned}$$

As $i,j\to\infty$, we have $h_{i,j}\to 0$ in $L_p([0,T]\times\Omega;L_p(B;l_2))$, thus $D_t^\eta \sum_{k=1}^\infty \int_0^t k_\beta(t-s)g_j^k(s,\omega,x)\, dw_s^k(\omega)$ is a Cauchy sequence in $L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$. □

Definition 4.5. Let $(\Omega,\mathcal{F},\mathbb{P})$ and the Wiener processes w_t^k satisfy Hypothesis 3.2. Let $T>0$, $2\le p<\infty$, and $g\in L_p([0,T]\times\Omega;L_p(B;l_2))$, moreover, let $\{g_j\}$ be a sequence approximating g in the sense of Lemma 4.1. Let $\beta>\frac{1}{2}$. Then we define

$$\begin{aligned}(k_\beta\star g)(t,\omega) &:= \sum_{k=1}^{\infty}\int_0^t k_\beta(t-s)g^k(s,\omega,x)\,dw_s^k(\omega)\\ &:= \lim_{j\to\infty}\sum_{k=1}^{\infty}\int_0^t k_\beta(t-s)g_j^k(s,\omega,x)\,dw_s^k(\omega).\end{aligned}$$

5. Linear theory

In this section we replace the semilinear inhomogeneity in (1.1) by inhomogeneities independent of u, so that we obtain a linear integral equation:

$$\begin{aligned}u(t,\omega,x) &= A\int_0^t k_\alpha(t-s)u(s,\omega,x)\,ds\\ &\quad+\sum_{k=1}^{\infty}\int_0^t k_\beta(t-s)g^k(s,\omega,x)\,dw_s^k\\ &\quad+\int_0^t k_\gamma(t-s)f(s,\omega,x)\,ds\ +\ u_0(\omega,x)+tu_1(\omega,x).\end{aligned}\tag{5.1}$$

We will prove the following propositions by a chain of Lemmas:

Proposition 5.1. *Let the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and the Wiener processes $\left(w_s^k\right)_{k=1}^{\infty}$ be as in Hypothesis 3.2. Let $p\in[2,\infty)$, let the measure space $(B,\mathcal{A},\Lambda)$ and the operator $A:\mathcal{D}A\subset L_p(B;\mathbb{R})\to L_p(B;\mathbb{R})$ satisfy Hypothesis 3.1. Assume that $T>0$ and let $f\in L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$, and $g\in L_p([0,T]\times\Omega;L_p(B,l_2))$. Let $u_0\in L_p(\Omega;L_p(B;\mathbb{R}))$ and $u_1\in L_p(\Omega;L_p(B;\mathbb{R}))$ be $\mathcal{F}_0$-measurable.*

Let $\alpha\in(0,2)$, $\beta>\frac{1}{2}$, $\gamma>0$. Then there exists a unique function $u\in L_p([0,T]\times\Omega;L_p(B,\mathbb{R}))$ such that for almost all $t\in[0,T]$

$$\int_0^t k_\alpha(t-s)u(s,\omega,\cdot)\,ds\in\mathcal{D}A\quad\text{for a.e. }\omega\in\Omega,$$

and (5.1) holds for almost all $\omega\in\Omega$ and almost all $t\in[0,T]$.

Proposition 5.2. *Let the assumptions of Proposition 5.1 hold. Suppose that $f\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^{\epsilon})$, $u_0\in L_p(\Omega;\mathcal{D}(-A)^{\delta_0})$ and $u_1\in L_p(\Omega;\mathcal{D}(-A)^{\delta_1})$ with suitable $\epsilon,\delta_0,\delta_1\in[0,1)$. Let u be as in Proposition 5.1. Let $\eta\in(-1,1)$, $\zeta\in[0,1]$ satisfy*

$$\eta+\alpha\zeta<\gamma+\alpha\epsilon,\tag{5.2}$$

$$\frac{1}{2}+\eta+\alpha\zeta<\beta,\tag{5.3}$$

$$\eta+\alpha\zeta<\frac{1}{p}+\alpha\delta_0,\tag{5.4}$$

$$\eta+\alpha\zeta<1+\frac{1}{p}+\alpha\delta_1.\tag{5.5}$$

With the notation $1_{\{a>b\}}=1$ if $a>b$ and $1_{\{a>b\}}=0$ else, we put

$$v(t)=u(t)-1_{\{\delta_0>\zeta\}}u_0-1_{\{\delta_1>\zeta\}}tu_1-1_{\{\epsilon>\zeta\}}\int_0^t k_\gamma(t-s)f(s)\,ds.$$

(a) *Then, if $\eta>0$, the function v, considered as a Banach space valued function $v:[0,T]\to L_p(\Omega;\mathcal{D}(-A)^{\zeta})$, has a fractional derivative of order η.*

(b) *If $\eta < 0$, the function $v : [0,T] \to L_p(\Omega; L_p(B;\mathbb{R}))$ has a fractional integral of order $-\eta$. Moreover, $D_t^\eta v$ takes values in $L_p(\Omega; \mathcal{D}(-A)^\zeta)$.*

(c) *If $\eta = 0$, clearly $D_t^0 v = v$.*

In either case, there exist constants M_{init}, $M_{T,\text{Leb}}$, and $M_{T,\text{Ito}}$ depending on p, T, α, β, γ, δ_0, δ_1, ϵ, ζ, η such that

$$(5.6) \qquad \begin{aligned} &\|D_t^\eta v(t)\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)} \\ \leq\ & M_{\text{init}}\big[\|u_0\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})} + \|u_1\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})}\big] \\ &\quad + M_{T,\text{Leb}}\|f\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon)} + M_{T,\text{Ito}}\|g\|_{L_p([0,T]\times\Omega;L_p(B,l_2))}. \end{aligned}$$

Moreover, the constants $M_{T,\text{Leb}}$ and $M_{T,\text{Ito}}$ can be made arbitrarily small by chosing the time interval $[0,T]$ sufficiently short.

The proof of the propositions above relies on the concept of a resolvent operator (see [24]), introduced by the following definition:

Definition 5.3. Let A satisfy Hypothesis 3.1, let $\alpha \in (0,2)$ and $\beta > 0$. For $t > 0$ we define the resolvent operator $S_{\alpha,\beta}(t) : L_p(B;\mathbb{R}) \to L_p(B;\mathbb{R})$ by

$$(5.7) \qquad S_{\alpha,\beta}(t)x := \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x\, d\lambda$$

along the contour

$$\Gamma_{\rho,\phi}(t) = \begin{cases} (t-\phi+\rho)e^{i\phi} & \text{for } t > \phi, \\ \rho e^{it} & \text{for } t \in (-\phi,\phi), \\ (-t-\phi+\rho)e^{-i\phi} & \text{for } t < -\phi, \end{cases}$$

with $\rho > 0$, $\phi > \frac{\pi}{2}$, $\alpha\phi + \phi_A < \pi$.

For $\beta = 1$, this definition coincides with the known notion of a resolvent operator, c.f. [24]. For $\beta > 1$, $S_{\alpha,\beta}$ could be obtained by fractional integration of $S_{\alpha,1}$.

Equation (5.1) is formally solved by the variation of parameters formula

$$(5.8) \qquad \begin{aligned} u(t) \;=\; & S_{\alpha,1}(t)u_0 + S_{\alpha,2}(t)u_1 \\ & + \int_0^t S_{\alpha,\gamma}(t-s)f(s)\,ds + \int_0^t \sum_{k=1}^\infty S_{\alpha,\beta}(t-s)g(s)\,dw_s^k. \end{aligned}$$

The task of the proof is to make sense of this formal expression in suitable function spaces, and to show that it gives a solution of (5.1). Moreover, the estimates claimed in Proposition 5.2 need to be verified. Since the equation is linear, all terms u_0, u_1, f, g can be treated separately. This is done in the following Lemmas 5.6, 5.7, and 5.9. Uniqueness can be proved by the standard reduction to a deterministic homogeneous equation with zero initial data, which has only the zero solution by the well-known theory of deterministic evolutionary integral equations (see [24]).

First we collect some basic facts about the resolvent operator:

Lemma 5.4. *Let A satisfy Hypothesis 3.1, let $\alpha \in (0,2)$ and $\beta > 0$. The resolvent operator defined above has the following properties:*

1) *For all $t > 0$ and all $\zeta \in [0,1]$, the operator $S_{\alpha,\beta}(t)$ is a bounded linear operator $L_p(B,\mathbb{R}) \to \mathcal{D}(-A)^\zeta$.*
2) *For all $x \in L_p(B;\mathbb{R})$, the function $t \mapsto S_{\alpha,\beta}(t)x$ can be extended analytically to some sector in the right half plane.*

3) *For all $x \in L_p(B;\mathbb{R})$ and all $t > 0$, we have*

$$\int_0^t k_\alpha(t-s)\|S_{\alpha,\beta}(s)x\|_{L_p(B;\mathbb{R})}\,ds < \infty,$$

$$\int_0^t k_\alpha(t-s)S_{\alpha,\beta}(s)x\,ds \in \mathcal{D}A,$$

$$S_{\alpha,\beta}(t)x = A\int_0^t k_\alpha(t-s)S_{\alpha,\beta}(s)x\,ds + k_\beta(t)x. \tag{5.9}$$

4) *Let $T > 0$, $\delta, \zeta \in [0,1]$, and $\eta \in (-1,1)$ such that*

$$\eta + \alpha\zeta < \beta + \alpha\delta. \tag{5.10}$$

Let $x \in \mathcal{D}(-A)^\delta$ and put

$$v(t) = \begin{cases} S_{\alpha,\beta}(t)x & \text{if } \delta \le \zeta, \\ S_{\alpha,\beta}(t)x - k_\beta(t)x & \text{if } \delta > \zeta. \end{cases}$$

(a) *Then, if $\eta > 0$, the function v, considered as a Banach-space valued function $v : [0,T] \to \mathcal{D}(-A)^\zeta$, admits a fractional derivative of order η.*
(b) *If $\eta < 0$, the function $v : [0,T] \mapsto L_p(B;\mathbb{R})$, has a fractional integral of order $-\eta$. Moreover, $D_t^\eta v$ takes values in $\mathcal{D}(-A)^\zeta$.*
(c) *If $\eta = 0$, we write $D_t^0 v = v$.*

In either case, there exists some $M > 0$ (dependent on $A, \alpha, \beta, \zeta, \delta, \eta$) such that for all $t \in (0,T]$ and all $x \in \mathcal{D}(-A)^\delta$,

$$\|D_t^\eta v(t)\|_{\mathcal{D}(-A)^\zeta} \le M t^{(\beta+\alpha\delta)-(\eta+\alpha\zeta)-1}\|x\|_{\mathcal{D}(-A)^\delta}. \tag{5.11}$$

Remark 5.5. In fact, if $x \in \mathcal{D}(-A)^\delta$ with $\delta \ge \zeta$ and $\beta > \eta$, the function $t \mapsto k_\beta(t)x$ admits a fractional derivative $D_t^\eta k_\beta x = k_{\beta-\eta}x$ in $\mathcal{D}(-A)^\zeta$. In this case, (5.10) holds, and both functions, $S_{\alpha,\beta}(t)x$ and $S_{\alpha,\beta}(t)x - k_\beta(t)x$ admit fractional derivatives of order η in $\mathcal{D}(-A)^\zeta$. On the other hand, evidently, if $\beta \le \eta$ or $x \notin \mathcal{D}(-A)^\zeta$, at most one of the two functions above can have a fractional derivative of order η in $\mathcal{D}(-A)^\zeta$.

Proof. All these results come out of standard estimates of the contour integral, along with the usual analyticity arguments. Since the estimate (5.11) is crucial in the sequel, we give a more detailed proof.

First we consider the case $\delta \le \zeta$ where we can utilize Lemma 2.2 with $\theta = 0$ for ρ in a suitable sector:

$$\|(\rho - A)^{-1}x\|_{\mathcal{D}(-A)^\zeta} \le M|\rho|^{\zeta-\delta-1}\|x\|_{\mathcal{D}(-A)^\delta}.$$

Formally, the Laplace transform of $D_t^\eta S_{\alpha,\beta}x$ is $\lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x$. We show that the contour integral

$$w(t) := \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x\,d\lambda$$

exists in $\mathcal{D}(-A)^\zeta$, if (5.10) holds.

$$\begin{aligned}
&\left\|\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x\,d\lambda\right\|_{\mathcal{D}(-A)^\zeta} \\
= &\left\|\int_{\Gamma_{t\rho,\phi}} e^{\mu}\,\left(\frac{\mu}{t}\right)^{\eta+\alpha-\beta}\,(\left(\frac{\mu}{t}\right)^\alpha - A)^{-1}x\,\frac{1}{t}d\mu\right\|_{\mathcal{D}(-A)^\zeta} \\
= &\; t^{\beta-\alpha-\eta-1}\left\|\int_{\Gamma_{1,\phi}} e^{\mu}\,\mu^{\eta+\alpha-\beta}\,(\left(\frac{\mu}{t}\right)^\alpha - A)^{-1}x\,d\mu\right\|_{\mathcal{D}(-A)^\zeta} \\
\le &\; t^{\beta-\alpha-\eta-1}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}\,|\mu|^{\alpha+\eta-\beta}\,\|(\left(\frac{\mu}{t}\right)^\alpha - A)^{-1}x\|_{\mathcal{D}(-A)^\zeta}\,|d\mu| \\
\le &\; t^{\beta-\alpha-\eta-1}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}|\mu|^{\alpha+\eta-\beta}\,M\left|\frac{\mu}{t}\right|^{\alpha(\zeta-\delta-1)}\|x\|_{\mathcal{D}(-A)^\delta}\,|d\mu| \\
= &\; t^{\beta-\eta-\alpha\zeta+\alpha\delta-1}M\|x\|_{\mathcal{D}(-A)^\delta}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}|\mu|^{\eta-\beta+\alpha(\zeta-\delta)}\,|d\mu|.
\end{aligned}$$

Because of (5.10), w is locally integrable and admits a Laplace transform. It requires some standard complex analysis, to show that $\hat{w}(\lambda) = \lambda^{\eta+\alpha-\beta}(\lambda^\alpha - A)^{-1}x$. Now we have to show that in fact $w = D_t^\eta S_{\alpha,\beta}x$. First consider the case $\eta > 0$: By the convolution theorem for Laplace transforms we have $[\widehat{D_t^{-\eta}w}](\lambda) = \lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x$, whence $w = D_t^\eta S_{\alpha,\beta}x$. In case $\eta < 0$, the convolution theorem yields $\widehat{D_t^\eta S_{\alpha,\beta}}x(\lambda) = \lambda^\eta\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x = \hat{w}(\lambda)$.

To handle the case $\delta > \zeta$, we will use Lemma 2.2 with $\theta = 1$:

$$\|A(\rho - A)^{-1}x\|_{\mathcal{D}(-A)^\zeta} \le M\rho^{\zeta-\delta}\|x\|_{\mathcal{D}(-A)^\delta}.$$

Notice first that $\hat{k}_\beta(\lambda) = \lambda^{-\beta}$, and

$$k_\beta(t) = \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{-\beta}\,d\lambda.$$

Therefore,

$$\begin{aligned}
S_{\alpha,\beta}(t)x - k_\beta(t)x &= \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,[\lambda^{\alpha-\beta}(\lambda^\alpha - A)^{-1}x - \lambda^{-\beta}x]\,d\lambda \\
&= \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{-\beta}A(\lambda^\alpha - A)^{-1}x\,d\lambda.
\end{aligned}$$

Now we estimate similarly as above

$$\begin{aligned}
&\|\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{\eta-\beta}A(\lambda^\alpha - A)^{-1}x\,d\lambda\|_{\mathcal{D}(-A)^\zeta} \\
\le &\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}\,\left|\frac{\mu}{t}\right|^{\eta-\beta}\,M\left|\frac{\mu}{t}\right|^{\alpha(\zeta-\delta)}\,\|x\|_{\mathcal{D}(-A)^\delta}\,\frac{1}{t}|d\mu| \\
= &\; M\|x\|_{\mathcal{D}(-A)^\delta}\,t^{-\eta+\beta-\alpha\zeta+\alpha\delta-1}\int_{\Gamma_{1,\phi}} e^{\Re(\mu)}|\mu|^{\eta-\beta+\alpha\zeta-\alpha\delta}\,|d\mu|.
\end{aligned}$$

Thus, for $t > 0$, the following integral exists in $\mathcal{D}(-A)^\zeta$:

$$\begin{aligned}
w_1(t) &:= \frac{1}{2\pi i}\int_{\Gamma_{\rho,\phi}} e^{\lambda t}\,\lambda^{\eta-\beta}A(\lambda^\alpha - A)^{-1}x\,d\lambda, \\
\|w_1(t)\|_{\mathcal{D}(-A)^\zeta} &\le Mt^{(\beta+\alpha\delta)-(\eta+\alpha\zeta)-1}.
\end{aligned}$$

In the end one verifies again that in fact $w_1(t) = D_t^\eta v(t)$. □

Lemma 5.6 (Contribution of the initial conditions u_0, u_1)**.** *Let A satisfy Hypothesis 3.1, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $p \in [2, \infty)$. Let $\alpha \in (0, 2)$, $0 < T < \infty$, and u_0, $u_1 \in L_p(\Omega; L_p(B; \mathbb{R}))$. We define $u(t) := S_{\alpha,1}(t)u_0 + S_{\alpha,2}(t)u_1$.*

1) *The function u exists in $L_\infty([0,T]; L_p(\Omega \times B; \mathbb{R}))$. For all $t > 0$ we have*

$$\int_0^t k_\alpha(t-s)u(s)\,ds \in \mathcal{D}A,$$
$$u(t) = A\int_0^t k_\alpha(t-s)u(s)\,ds + u_0 + tu_1.$$

2) *Moreover, suppose that $u_i \in L_p(\Omega; \mathcal{D}(-A)^{\delta_i})$ with some $\delta_i \in [0, 1]$. Let $\zeta \in [0, 1]$, $\eta \in (-1, 1)$ be such that*

$$\eta + \alpha\zeta < \frac{1}{p} + \alpha\delta_0, \tag{5.12}$$

$$\eta + \alpha\zeta < 1 + \frac{1}{p} + \alpha\delta_1, \tag{5.13}$$

Put

$$v(t) = u(t) - 1_{\zeta<\delta_0}u_0 - 1_{\zeta<\delta_1}tu_1.$$

Then v has a fractional derivative of order η (if $\eta < 0$: a fractional integral of order $-\eta$) which is in $L_p([0,T] \times \Omega; \mathcal{D}(-A)^\zeta)$ and satisfies

$$\begin{aligned} &\|D_t^\eta v(t,\omega)\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)} \\ \le\ & M\big[\|u_0\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})} + \|u_1\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})}\big] \end{aligned}$$

with a constant M depending on $p, A, T, \alpha, \delta_0, \delta_1, \zeta, \eta$.

Proof. This is a straightforward application of Lemma 5.4, applied pointwise for $\omega \in \Omega$, for the special cases $\beta = 1$ and $\beta = 2$. □

Lemma 5.7 (Contribution of f)**.** *Let A satisfy Hypothesis 3.1, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $p \in [2, \infty)$. Let $\alpha \in (0, 2)$, $\gamma > 0$, $0 < T < \infty$, and $f \in L_p([0,T] \times \Omega; L_p(B; \mathbb{R}))$.*

1) *For almost $t \in [0, T]$, the following integral exists in $L_p(B; \mathbb{R})$, pointwise for almost all $\omega \in \Omega$, as well as in $L_p(\Omega; L_p(B; \mathbb{R}))$:*

$$u(t,\omega) = \int_0^t S_{\alpha,\gamma}(t-s)f(s,\omega)\,ds. \tag{5.14}$$

Moreover, $u \in L_p([0,T] \times \Omega; L_p(B; \mathbb{R}))$, and for almost all $\omega \in \Omega$ and almost all $t \in [0, T]$,

$$\int_0^t k_\alpha(t-s)u(s,\omega)\,ds \in \mathcal{D}A,$$
$$u(t,\omega) = A\int_0^t k_\alpha(t-s)u(s,\omega)\,ds + \int_0^t k_\gamma(t-s)f(s,\omega)\,ds.$$

2) *Suppose, in addition, that $f \in L_p([0,T] \times \Omega; \mathcal{D}(-A)^\epsilon)$ with some $\epsilon \in [0, 1]$, let $\eta \in (-1, 1)$, $\zeta \in [0, 1]$ be such that*

$$\eta + \alpha\zeta < \gamma + \alpha\epsilon. \tag{5.15}$$

Put

$$v(t) = \begin{cases} u(t) & \text{if } \zeta \ge \epsilon, \\ u(t) - \int_0^t k_\gamma(t-s)f(s)\,ds & \text{if } \zeta < \epsilon. \end{cases}$$

Then, if $\eta > 0$, the function $t \mapsto v(t) \in L_p(\Omega; \mathcal{D}(-A)^\zeta)$ has a fractional derivative of order η in $L_p([0,T] \times \Omega; \mathcal{D}(-A)^\zeta)$. If $\eta < 0$, the function $t \mapsto v(t) \in L_p(\Omega; L_p(B; \mathbb{R}))$ has a fractional integral of order $-\eta$ with values *in $L_p(\Omega;\mathcal{D}(-A)^\zeta)$. If $\eta=0$, we define $D_t^\eta v = v$. In either case there exists a constant $M_{T,\text{Leb}}$ dependent on $A,T,p,\alpha,\gamma,\epsilon,\zeta,\eta$ such that*

$$\|D_t^\eta v\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)} \le M_{T,\text{Leb}}\|f\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon)}.$$

Moreover, the constant $M_{T,\text{Leb}}$ can be made arbitrarily small by taking the time interval $[0,T]$ sufficiently short.

Proof. The function $t\mapsto \int_0^t (t-s)^{\gamma-1}\|f(s)\|_{L_p(\Omega\times B,\mathbb{R})}\,ds$ is the convolution of an L_1-function and an L_p-function, therefore it is in $L_p([0,T],\mathbb{R})$. From (5.11) with $\delta=\zeta=\eta=0$ we obtain $\|S_{\alpha,\gamma}(t)\|_{L_p(B;\mathbb{R})\to L_p(B;\mathbb{R})}\le Mt^{\gamma-1}$. Consequently, the integral

$$u(t)=\int_0^t S_{\alpha,\gamma}(t-s)f(s)\,ds$$

exists as an integral in $L_p(\Omega\times B;\mathbb{R})$ for almost all t, and $u\in L_p([0,T]\times\Omega, L_p(B,\mathbb{R}))$. By standard arguments the integral (5.14) exists also in $L_p(B;\mathbb{R})$ for a.e. $\omega\in\Omega$ and a.e. $t\in[0,T]$. Now (5.9) implies (almost everywhere in Ω and $[0,T]$)

$$\begin{aligned} & u(t)-\int_0^t k_\gamma(t-s)f(s,\omega)\,ds \\ = & \int_0^t [S_{\alpha,\gamma}(t-s)f(s,\omega)-k_\gamma(t-s)f(s,\omega)]\,ds \\ = & \int_0^t A[\int_0^{t-s} k_\alpha(\sigma)S_{\alpha,\gamma}(t-s-\sigma)f(s,\omega)\,d\sigma]\,ds. \end{aligned}$$

We use the closedness of A and interchange the order of integrals to obtain

$$u(t)-\int_0^t k_\gamma(t-s)f(s,\omega)\,ds = A\int_0^t k_\alpha(\sigma)u(t-\sigma,\omega)\,d\sigma.$$

This proves Part (1) of the Lemma.

To prove Part (2), let η,ζ,ϵ be such that (5.15) holds. For shorthand put

$$V(t)x=\begin{cases} D_t^\eta S_{\alpha,\gamma}x & \text{if } \epsilon\le\zeta,\\ D_t^\eta[S_{\alpha,\gamma}(t)x-k_\gamma x] & \text{else.}\end{cases}$$

From (5.11) with β replaced by γ, and δ replaced by ϵ, we have

$$\|V(t)x\|_{\mathcal{D}(-A)^\zeta}\le Mt^{(\gamma+\alpha\epsilon)-(\eta+\alpha\zeta)-1}\|x\|_{\mathcal{D}(-A)^\epsilon}.$$

We obtain by a straightforward convolution argument that

$$\|\int_0^t V(t-s)f(s)\,ds\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)}\,ds \le M_{t,\text{Leb}}\|f\|_{L_p(\Omega,\mathcal{D}(-A)^\epsilon)}.$$

with

$$M_{T,\text{Leb}} = M\int_0^T t^{(\gamma+\alpha\epsilon)-(\eta+\alpha\zeta)-1}\,dt.$$

Clearly, $M_{T,\text{Leb}}$ converges to 0 as $T\to 0$. All we have to show is that in fact

$$D_t^\eta v(t)=\int_0^t V(t-s)f(s)\,ds.$$

We treat the case $\eta>0$, $\epsilon>\zeta$, the other cases are done similarly. The definition of $V(t)x$ yields

$$\int_0^t k_\eta(s)V(t-s)x\,dx = S_{\alpha,\gamma}(t)x-k_\gamma(t)x.$$

Fubini's Theorem implies

$$\int_0^t k_\eta(s) \int_0^{t-s} V(t-s-\sigma) f(\sigma)\, d\sigma\, ds = \int_0^t \int_0^{t-\sigma} k_\eta(s) V(t-\sigma-s) f(\sigma)\, ds\, d\sigma$$
$$= \int_0^t \left[S_{\alpha,\gamma}(t-\sigma) - k_\gamma(t-\sigma)\right] f(\sigma)\, d\sigma = v(t).$$

Thus $v(t)$, considered as a function with values in $\mathcal{D}(-A)^\zeta$, admits a fractional derivative of order η which is $V * f$. □

The following Lemma is the key to estimate the contribution of the stochastic integral:

Lemma 5.8. *Let A satisfy Hypothesis 3.1, $p \in [2,\infty)$. Let $\alpha \in (0,2)$, $\beta > \frac{1}{2}$, $\zeta \in [0,1]$ and $\eta \in (-1,1)$, such that (5.3) holds, i.e. $\frac{1}{2} + \eta + \alpha\zeta < \beta$. Let $T > 0$. Then there exists a constant $\tilde{M}_{T,\mathrm{Ito}} > 0$ depending on $A, p, T, \alpha, \beta, \eta, \zeta$ such that for all $h \in L_p([0,T]; L_p(B; l_2))$,*

$$\int_0^T \int_B \left(\int_0^t |(-A)^\zeta D_t^\eta S_{\alpha,\beta}(t-s) h(s,x)|_{l_2}^2\, ds \right)^{\frac{p}{2}} d\Lambda(x)\, dt$$
$$\leq \quad \tilde{M}_{T,\mathrm{Ito}} \int_0^T \int_B |h(s,x)|_{l_2}^p\, d\Lambda(x)\, ds.$$

Moreover, the constant $\tilde{M}_{T,\mathrm{Ito}}$ can be made arbitrarily small by taking the time interval $[0,T]$ sufficiently short.

Proof. Write $V(t) := (-A)^\zeta D_t^\eta S_{\alpha,\beta}(t)$ and notice that by (5.11) (with $\delta = 0$),

$$\|V(t)\|_{L_p(B,l_2)\to L_p(B,l_2)} \leq M t^{\beta-(\eta+\alpha\zeta)-1}.$$

First assume that $p > 2$. Notice that $\frac{p}{2}$ and $\frac{p}{p-2}$ are conjugate exponents. Take $f : [0,T] \times B \to \mathbb{R}^+$ such that $\int_0^T \int_B f^{\frac{p}{p-2}}(t,x)\, d\Lambda(x)\, dt = 1$. We estimate

$$\int_0^T \int_B f(t,x) \int_0^t |V(t-s)h(s,x)|_{l_2}^2\, ds\, d\Lambda(x)\, dt$$
$$= \quad \int_0^T \int_0^t \int_B f(t,x)\, |V(t-s)h(s,x)|_{l_2}^2\, d\Lambda(x)\, ds\, dt$$
$$\leq \quad \int_0^T \int_0^t \left[\int_B f(t,x)^{\frac{p}{p-2}}\, d\Lambda(x) \right]^{\frac{p-2}{p}} \left[\int_B |V(t-s)h(s,x)|_{l_2}^p\, d\Lambda(x) \right]^{\frac{2}{p}} ds\, dt$$
$$\leq \quad \int_0^T \|f(t,\cdot)\|_{L_{\frac{p}{p-2}}(B;\mathbb{R})} \int_0^t \|V(t-s)\|_{L_p(B;l_2)\to L_p(B;l_2)}^2 \|h(s,\cdot)\|_{L_p(B;l_2)}^2\, ds\, dt$$
$$\leq \quad \left[\int_0^T \|f(t,\cdot)\|_{L_{\frac{p}{p-2}}(B;\mathbb{R})}^{\frac{p}{p-2}}\, dt \right]^{\frac{p-2}{p}}$$
$$\left[\int_0^T \left| \int_0^t \|V(t-s)\|_{L_p(B;l_2)\to L_p(B;l_2)}^2 \|h(s,\cdot)\|_{L_p(B;l_2)}^2\, ds \right|^{\frac{p}{2}} dt \right]^{\frac{2}{p}}.$$

Thus

$$\left[\int_0^T \int_B \left(\int_0^t |V(t-s)h(s,x)|_{l_2}^2\, ds \right)^{\frac{p}{2}} d\Lambda(x)\, dt \right]^{\frac{2}{p}} \tag{5.16}$$
$$\leq \quad \left[\int_0^T \left| \int_0^t \|V(t-s)\|_{L_p(B;l_2)\to L_p(B;l_2)}^2 \|h(s,\cdot)\|_{L_p(B;l_2)}^2\, ds \right|^{\frac{p}{2}} dt \right]^{\frac{2}{p}}.$$

For $p = 2$, the estimate (5.16) is obvious. In either case, we obtain (by estimating the convolution with respect to s)

$$\left[\int_0^T \int_B \left(\int_0^t |V(t-s)h(s,x)|_{l_2}^2\, ds\right)^{\frac{p}{2}} d\Lambda(x)\, dt\right]^{\frac{2}{p}}$$

$$\leq \left(\int_0^T \|V(s)\|^2_{L_p(B;l_2)\to L_p(B;l_2)}\, ds\right)\left(\int_0^T \|h(s,.)\|^p_{L_p(B;l_2)}\, ds\right)^{\frac{2}{p}}$$

$$\leq M^2 \int_0^T s^{2(\beta-(\eta+\alpha\zeta)-1)}\, ds\, \|h\|^2_{L_p([0,T];L_p(B;l_2))}.$$

By (5.3) we infer that $s^{2(\beta-(\eta+\alpha\zeta)-1)}$ is integrable on $[0,T]$ so that

$$\tilde{M}_{T,\mathrm{Ito}} := \left[M^2 \int_0^T s^{2(\beta-(\eta+\alpha\zeta)-1)}\, ds\right]^{\frac{p}{2}}$$

is finite and converges to 0 as $T \to 0+$. □

Lemma 5.9 (Contribution of g)**.** *Let A satisfy Hypothesis 3.1, and let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the Wiener processes w_t^k be as in Hypothesis 3.2. Let $T > 0$, $2 \leq p < \infty$, $\alpha \in (0,2)$, and $\beta > \frac{1}{2}$. Let $g \in L_p([0,T] \times \Omega; L_p(B;l_2))$ and $\{g_j\}$ be a sequence approximating g in the sense of Lemma 4.1, where the values of g_j^k are in $\mathcal{D}A \cap L_1(B;\mathbb{R}) \cap L_\infty(B;\mathbb{R})$. Let $k_\beta \star g$ be given by Definition 4.5. For $j \in \mathbb{N}$ put*

$$u_j(t) = \sum_{k=1}^{j} \int_0^t S_{\alpha,\beta}(t-s) g_j^k(s)\, dw_s^k.$$

1) *The limit $u(t) = \lim_{j\to\infty} u_j(t)$ exists in $L_p([0,T]\times\Omega; L_p(B;\mathbb{R}))$. Moreover, for almost all $t \in [0,T]$ and almost all $\omega \in \Omega$,*

$$u(t,\omega) = A\int_0^t k_\alpha(t-s) u(s,\omega)\, ds + (k_\beta \star g)(t,\omega).$$

2) *Suppose $0 \leq \zeta \leq 1$ and $\eta \in (-1,1)$ are such that*

$$\eta + \alpha\zeta + \frac{1}{2} < \beta. \tag{5.17}$$

Then, if $\eta > 0$, the function $u : [0,T] \to L_p(\Omega, \mathcal{D}(-A)^\zeta)$ has a fractional derivative of order η. If $\eta < 0$, then $u : [0,T] \to L_p(\Omega; L_p(\Omega,\mathbb{R}))$ has a fractional integral of order $-\eta$ with values in $L_p(\Omega; \mathcal{D}(-A)^\zeta)$. If $\eta = 0$, we denote $D_t^0 u = u$. In either case there exists a constant $M_{T,\mathrm{Ito}}$ dependent on $A, p, \alpha, \beta, \eta, \zeta$ such that

$$\|D_t^\eta u\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)} \leq M_{T,\mathrm{Ito}} \|g\|_{L_p([0,T]\times\Omega;L_p(B;l_2))}.$$

Moreover, the constant $M_{T,\mathrm{Ito}}$ can be made arbitrarily small by choosing the time interval $[0,T]$ sufficiently short.

Proof. First, let $h \in L_p([0,T] \times \Omega; L_p(B;l_2))$ be of the elementary structure like the g_j in Lemma 4.1. Evidently, the following integral exists

$$\sum_{k=1}^{\infty} \int_0^t S_{\alpha,\beta}(t-s) h^k(s)\, dw_s^k = \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} \int_{\tau_{i-1}^k}^{\tau_i^k} S_{\alpha,\beta}(t-s) h_i^k\, dw_s^k$$

where τ_i^k are suitable stopping times, $h_i^k \in \mathcal{D}A \cap L_1(B;\mathbb{R}) \cap L_\infty(B;\mathbb{R})$, and the both sums are in fact only finite sums. For $\eta \in (-1,1)$, $\zeta \in [0,1]$, satisfying (5.17), put $V(t)x = (-A)^\zeta D_t^\eta S_{\alpha,\beta}(t)x$. We apply Lemma 4.3 and integrate for $t \in [0,T]$. Subsequently we apply Lemma 5.8:

$$(5.18)\qquad \int_0^T \int_\Omega \| \sum_{k=1}^\infty \int_0^t V(t-s)h^k(s)\, dw_s^k \|^p_{L_p(B;\mathbb{R})}\, d\mathbb{P}(\omega)\, dt$$
$$\leq M \int_0^T \int_B \int_\Omega \left(\int_0^t |[V(t-s)h(s,\omega)](x)|^2_{l_2}\, ds \right)^{\frac{p}{2}} d\mathbb{P}(\omega)\, d\Lambda(x)\, dt$$
$$\leq M\tilde{M}_{T,\mathrm{Ito}} \|h\|^p_{L_p([0,T]\times\Omega;L_p(B;l_2))}.$$

In particular, with $\zeta = \eta = 0$, and $h = g_j - g_m$, we have

$$\|u_j - u_m\|_{L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))} \leq M\|g_j - g_m\|_{L_p([0,T]\times\Omega;L_p(B;l_2))}$$

so that $\{u_j\}$ is a Cauchy sequence in $L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))$ and has a limit u. Without loss of generality, taking a subsequence, if necessary, we may assume that u_j converges also pointwise for almost all $t \in [0,T]$. Again we use the simple structure of g_j, in particular that $g_j^k(t,\omega) \in \mathcal{D}A$. From (5.9) and the Stochastic Fubini Theorem we obtain

$$\int_0^t [S_{\alpha,\beta}(t-\sigma)g_j^k(\sigma,\omega) - k_\beta(t-\sigma)g_j^k(\sigma,\omega)]\, dw_\sigma^k$$
$$= \int_0^t A[\int_0^{t-\sigma} k_\alpha(s)S_{\alpha,\beta}(t-\sigma-s)g_j^k(\sigma,\omega)\, ds]\, dw_\sigma^k$$
$$= A\int_0^t \int_0^{t-\sigma} k_\alpha(s)S_{\alpha,\beta}(t-\sigma-s)g_j^k(\sigma,\omega)\, ds\, dw_\sigma^k$$
$$= A\int_0^t k_\alpha(s) \int_0^{t-s} S_{\alpha,\beta}(t-\sigma-s)g_j^k(\sigma,\omega)\, dw_\sigma^k\, ds.$$

Taking the sum over $k = 1\cdots j$ we obtain

$$u_j(t,\omega) - k_\beta \star g_j(t,\omega) = A\int_0^t k_\alpha(s)u_j(t-s,\omega)\, ds.$$

Taking limits for $j \to \infty$ (pointwise a.e. in $[0,T]$), and using the closedness of A we have for almost all $t \in [0,T]$

$$u(t,\omega) - k_\beta \star g(t,\omega) = A\int_0^t k_\alpha(s)u(t-s,\omega)\, ds.$$

Thus Part (1) of the Lemma is proved.

To prove Part (2), let $\eta \in (-1,1)$ and $\zeta \in [0,1]$ satisfy (5.17). With $V(t)x = (-A)^\zeta D_t^\eta S_{\alpha,\beta}x$ and $h = g_j$, we obtain from (5.18),

$$\| \int_0^t \sum_{k=1}^j V(t-s)g_j^k(s)\, dw_s^k \|_{L_p([0,T]\times\Omega;L_p(B;\mathbb{R}))} \leq M_{T,\mathrm{Ito}}\|g_j\|_{L_p([0,T]\times\Omega;L_p(B;l_2))},$$

with a suitable constant $M_{T,\mathrm{Ito}}$ which converges to 0 as $T \to 0$. We have to show that in fact

$$\sum_{k=1}^j \int_0^t V(t-s)g_j^k(s)\, dw_s^k = (-A)^\zeta D_t^\eta u_j(t)$$

First let $\eta > 0$. By definition we know that

$$\int_0^t k_\eta(s)V(t-s)x\, ds = (-A)^\zeta S_{\alpha,\beta}(t)x.$$

Taking integrals and using the Stochastic Fubini Theorem, we obtain

$$\begin{aligned}(-A)^\zeta u_j(t) &= \sum_{k=1}^{j}\int_0^t (-A)^\zeta S_{\alpha,\beta}(t-\sigma)g_j^k(\sigma,\omega)\,dw_\sigma^k \\ &= \sum_{k=1}^{j}\int_0^t\int_0^{t-\sigma} k_\eta(s)V(t-\sigma-s)g_j^k(\sigma,\omega)\,ds\,dw_\sigma^k \\ &= \int_0^t k_\eta(s)\sum_{k=1}^{j}\int_0^{t-s} V(t-s-\sigma)g_j^k(\sigma,\omega)\,dw_\sigma^k\,ds\end{aligned}$$

Thus $(-A)^\zeta u_j$ has a fractional derivative of order η which is $V \star g_j$. Taking the limit for $j\to\infty$ we infer that $D_t^\eta(-A)^\zeta u = V\star g$. Now let $\eta<0$. Similarly as above, the Stochastic Fubini Theorem yields

$$\sum_{k=1}^{j}\int_0^t V(t-\sigma)g_j^k(\sigma)\,dw_\sigma^k = (-A)^\zeta\int_0^t k_{-\eta}(s)u_j(t-s)\,ds.$$

Again we take the limit for $j\to\infty$ and use the closedness of A, to see that the fractional integral $D_t^\eta u$ takes values in $\mathcal{D}(-A)^\zeta$ with $(-A)^\zeta D_t^\eta u = V\star g$. □

6. The semilinear equation

This section is devoted to the proof of Theorems 3.6, 3.7, 3.11, and Corollary 3.8.

Lemma 6.1. *Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space, let A satisfy Hypothesis 3.1, and let F and G satisfy Hypotheses 3.4 and 3.5. We define the operators*

$$\begin{aligned}&\mathcal{N}_F : L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)\to L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon),\\ &\mathcal{N}_G : L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)\to L_p([0,T]\times\Omega;L_p(B;l_2))\end{aligned}$$

by

$$[\mathcal{N}_F v](t,\omega) := F(t,\omega,v(t)),\quad [\mathcal{N}_G v](t,\omega) := G(t,\omega,v(t)).$$

1. *Then $\mathcal{N}_F$ and $\mathcal{N}_G$ are well defined and Lipschitz continuous with Lipschitz constants M_F, M_G, respectively.*
2. *Let F_1,F_2,G_1,G_2 satisfy Hypotheses 3.4, 3.5, and 3.9. Let $v\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$. Then*

$$\begin{aligned}&\|\mathcal{N}_{F_1}v-\mathcal{N}_{F_2}v\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon)}\le\|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})},\\ &\|\mathcal{N}_{G_1}v-\mathcal{N}_{G_2}v\|_{L_p([0,T]\times\Omega;L_p(B;l_2))}\le\|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}.\end{aligned}$$

Here the constants M_F, M_G and the functions $\mu_{\Delta F}$ and $\mu_{\Delta G}$ are as in Hypotheses 3.4, 3.5, and 3.9.

Proof. These are straightforward estimates. □

Lemma 6.2. *Let the assumptions of Theorem 3.6 hold, in addition assume that $\delta_0\le\theta$, $\delta_1\le\theta$. For $v\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$ let $\mathcal{T}_{[F,G,u_0,u_1]}v : [0,T]\times\Omega\to L_p(B;\mathbb{R})$ be the unique solution u of*

$$\begin{aligned}u(t,\omega) &= A\int_0^t k_\alpha(t-s)u(s,\omega)\,ds+u_0+tu_1\\ &\quad+\sum_{k=1}^{\infty}\int_0^t k_\beta(t-s)G_k(s,\omega,v(s))\,dw_s^k\\ &\quad+\int_0^t k_\gamma(t-s)F(s,\omega,v(s))\,ds.\end{aligned}$$

in the sense of Proposition 5.1.

(1) *Then* $\mathcal{T}_{[F,G,u_0,u_1]}$ *is well defined as a nonlinear operator from* $L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$ *into* $L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$. *Moreover,* $\mathcal{T}_{[F,G,u_0,u_1]}$ *is globally Lipschitz continuous with a Lipschitz constant dependent on* A, p, T, α, β, γ, ϵ, θ, M_F, M_G.

(2) *There exists an equivalent norm on* $L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$, *such that the Lipschitz constant of* $\mathcal{T}_{[F,G,u_0,u_1]}$ *is smaller than 1. This norm depends on* $T,p,A,\alpha,\beta,\gamma,\theta,\epsilon,M_F,M_G$.

(3) *There exists a constant* M, *depending on* $A,T,p,M_F,M_G,\alpha,\beta,\epsilon,\theta,\delta_0,\delta_1$, *such that the following Lipschitz estimate holds:*
If F_1,F_2,G_1,G_2 *satisfy Hypotheses 3.4, 3.5, and 3.9, if* $u_{0,1}$, $u_{0,2}$ *are in* $L_p(\Omega;\mathcal{D}(-A)^{\delta_0})$ *and* $u_{1,1},u_{1,2}\in L_p(\Omega;\mathcal{D}(-A)^{\delta_1})$, *measurable with respect to* $\mathcal{F}_0$, *then for any* $v\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$ *we have*

$$\begin{aligned}&\|\mathcal{T}_{[F_1,G_1,u_{0,1},u_{1,1}]}v-\mathcal{T}_{[F_2,G_2,u_{0,2},u_{1,2}]}v\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)}\\ \le\;& M\Big[\|u_{0,1}-u_{0,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})}+\|u_{1,1}-u_{1,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})}\\ &+\|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})}+\|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big].\end{aligned}$$

(4) $\mathcal{T}_{[F,G,u_0,u_1]}$ *has a unique fixed point* $u_{[F,G,u_0,u_1]}\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$. *Moreover, there exists a constant* M *dependent on* A, p, T, M_F, M_G, α, β, ϵ, θ, δ_0, δ_1 *such that the following Lipschitz estimate holds:*
If $u_{i,j}$, F_i, G_i *are as in (3), then*

$$\begin{aligned}&\|u_{[F_1,G_1,u_{0,1},u_{1,1}]}-u_{[F_2,G_2,u_{0,2},u_{1,2}]}\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)}\\ \le\;& M\Big[\|u_{0,1}-u_{0,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})}+\|u_{1,1}-u_{1,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})}\\ &+\|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})}+\|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big].\end{aligned}$$

Proof. We recall Proposition 5.2 with $\eta=0$ and ζ replaced by θ. Notice that the conditions (5.2), and (5.3), (5.4), (5.5) are satisfied. Let u solve

$$u=Ak_\alpha*u+k_\gamma*f+k_\beta\star g+u_0+tu_1,$$

Notice that with the present choice of coefficients the function v in (5.6) is simply u. Thus $u\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$ with

$$\begin{aligned}&\|u(t)\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)}\\ \le\;& M_{\text{init}}[\|u_0\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})}+\|u_1\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})}]\\ &+M_{T,\text{Leb}}\|f\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon)}+M_{T,\text{Ito}}\|g\|_{L_p([0,T]\times\Omega;L_p(B,l_2))}.\end{aligned}\tag{6.1}$$

Given $v\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$, we put $f=\mathcal{N}_Fv$ and $g=\mathcal{N}_Gv$ as in Lemma 6.1. Then $f\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon)$ and $g\in L_p([0,T]\times\Omega;L_p(B;l_2))$. Thus, by (6.1), $u=\mathcal{T}_{[F,G,u_0,u_1]}v\in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$. In particular for $v=0$ we have

$$\begin{aligned}&\|\mathcal{T}_{[F,G,u_0,u_1]}(0)\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)}\\ \le\;& M_{\text{init}}\big[\|u_0\|_{L_p(\Omega,\mathcal{D}(-A)^{\delta_0})}+\|u_1\|_{L_p(\Omega(\mathcal{D}(-A)^{\delta_1})}\big]+M_{T,\text{Leb}}M_{F,0}+M_{T,\text{Ito}}M_{G,0}.\end{aligned}$$

We could immediately get a Lipschitz estimate for $\mathcal{T}_{[F,G,u_0,u_1]}$ by (6.1), but we will get a better (contraction) estimate in an equivalent norm below.

To prove (2), we recall from Proposition 5.2 that $M_{T,\text{Leb}}$ and $M_{T,\text{Ito}}$ can be taken arbitrarily small, if the time intervals are sufficiently short. In particular, there exists $m\in\mathbb{N}$ such that

$$M_{T/m,\text{Leb}}M_F+M_{T/m,\text{Ito}}M_G<\frac{1}{4}.$$

With some $\kappa > 0$ to be specified below, we define for $v \in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$,

$$|||v||| := \sum_{q=1}^m \kappa^q \big[\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big]^{1/p}.$$

For $q = 1,\cdots,m$ we put

$$\begin{aligned} F_q(t,\omega,v) &:= I_{(q-1)T/m\le t<qT/m}(t)\,F(t,\omega,v(t)),\\ G_q(t,\omega,v) &:= I_{(q-1)T/m\le t<qT/m}(t)\,G(t,\omega,v(t)). \end{aligned}$$

If $v,\tilde v \in L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)$, then

$$\mathcal{T}_{[F,G,u_0,u_1]}v - \mathcal{T}_{[F,G,u_0,u_1]}\tilde v = \sum_{q=1}^m w_q,$$

where w_q solves

$$w_q = Ak_\alpha * w_q + k_\gamma * [F_q(v) - F_q(\tilde v)] + k_\beta \star [G_q(v) - G_q(\tilde v)].$$

Now $w_q = 0$ on $[0,\frac{T(q-1)}{m}]$. Lemma 6.1(1) and (6.1) imply

$$\begin{aligned} &\big(\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}\\ \le\;& M_{T/m,\mathrm{Leb}}M_F\big(\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|v(t,\omega)-\tilde v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}\\ &+ M_{T/m,\mathrm{Ito}}M_G\big(\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|v(t,\omega)-\tilde v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}\\ \le\;& \frac14\big(\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|v(t,\omega)-\tilde v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}. \end{aligned}$$

On the intervals $[\frac{(r-1)T}{m},\frac{rT}{m}]$ with $r > q$ we have the estimate

$$\begin{aligned} &\big(\int_{T(r-1)/m}^{Tr/m}\int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}\\ \le\;& \big(\int_0^T\int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}\\ \le\;& M\big(\int_{T(q-1)/m}^{Tq/m}\int_\Omega \|v(t,\omega)-\tilde v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big)^{1/p}. \end{aligned}$$

with $M = M_F M_{T,\mathrm{Leb}} + M_G M_{T,\mathrm{Ito}}$. We choose $\kappa \in (0,1)$ sufficiently small, such that $M\sum_{r=1}^\infty \kappa^r < \frac14$. We have therefore

$$\begin{aligned} &|||w_q||| = \sum_{r=q}^m \kappa^r\big[\int_{T(r-1)/m}^{Tr/m}\int_\Omega \|w_q(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big]^{1/p}\\ \le\;& [\frac14 + M\sum_{r=q+1}^m \kappa^{r-q}]\,\kappa^q\,\big[\int_{T(q-1)/m}^{Tq/m} \|v(t,\omega)-\tilde v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big]^{1/p}\\ \le\;& \frac12\,\kappa^q\,\big[\int_{T(q-1)/m}^{Tq/m} \|v(t,\omega)-\tilde v(t,\omega)\|^p_{\mathcal{D}(-A)^\theta}\,d\mathbb{P}(\omega)\,dt\big]^{1/p}. \end{aligned}$$

Summing for $q = 1, \cdots, m$ we obtain

$$|||\mathcal{T}_{[F,G,u_0,u_1]}v - \mathcal{T}_{[F,G,u_0,u_1]}\tilde{v}||| \leq \sum_{q=1}^{m} |||w_q|||$$

$$\leq \frac{1}{2}\sum_{q=1}^{m} \kappa^q \Big[\int_{T(q-1)/m}^{Tq/m} \|v(t,\omega) - \tilde{v}(t,\omega)\|^p_{\mathcal{D}(-A)^\theta} \, d\mathbb{P}(\omega)\, dt\Big]^{1/p}$$

$$= \frac{1}{2}|||v - \tilde{v}|||.$$

Part (3) is a straightforward application of (6.1) and Lemma 6.1 (2).

Finally, since for all F, G, u_0, u_1 the operator $\mathcal{T}_{[F,G,u_0,u_1]}$ is a strict contraction with Lipschitz constant $\frac{1}{2} < 1$ on $L_p([0,T] \times \Omega; \mathcal{D}(-A)^\theta)$ (with the norm $|||\cdot|||$), and since $\mathcal{T}_{[F,G,u_0,u_1]}v$ depends Lipschitz on F, G, u_0, u_1 by Part (3), the standard contraction arguments yield Part (4). □

We are now ready to finish the proofs of the main results:

Proof of Theorem 3.6. We may assume without loss of generality that δ_0, $\delta_1 \leq \theta$. (If any δ_i is greater that θ, it may be replaced by θ.) Obviously, the unique solution of (1.1) in $L_p([0,T] \times \Omega; \mathcal{D}(-A)^\theta)$ is exactly the unique fixed point of $\mathcal{T}_{[F,G,u_0,u_1]}$ constructed in Lemma 6.2. □

Proof of Theorem 3.7. Let u be the solution of (1.1), thus, with $f = \mathcal{N}_F u \in L_p([0,T] \times \Omega; \mathcal{D}(-A)^\epsilon)$ and $g = \mathcal{N}_G u \in L_p([0,T] \times \Omega; L_p(B; l_2))$ we have that u solves (5.1). Let v be defined by (3.13). Now, $\zeta, \eta, \delta_0, \delta_1, \epsilon$ satisfy the conditions of Proposition 5.2, which yields immediately the required additional regularity results. □

Proof of Corollary 3.8. To prove Part (1), choose η such that

$$\frac{1}{p} - \frac{1}{q} < \eta < 1,$$

and such that the conditions (3.9), (3.10), (3.11), and (3.12) from Theorem 3.7 are satisfied. Then $D_t^\eta v \in L_p([0,T]; L_p(\Omega; \mathcal{D}(-A)^\zeta))$. Notice that $q < \frac{p}{1-p\eta}$, so that we infer from [8, p. 421] that $v \in L_q([0,T]; L_p(\Omega; \mathcal{D}(-A)^\zeta))$.

To prove Part (2), put $\mu + \frac{1}{p} = \eta$. Consequently the conditions (3.9), (3.10), (3.11), and (3.12) from Theorem 3.7 hold. Then $D_t^\eta v \in L_p([0,T]; L_p(\Omega; \mathcal{D}(-A)^\zeta))$. Then by [8, p. 421] we infer that $v \in h_{0\to 0}^{\eta - p^{-1}}([0,T]; L_p(\Omega; \mathcal{D}(-A)^\zeta))$. □

Proof of Theorem 3.11. For $i = 1, 2$, let $u_{[F_i,G_i,u_{0,i},u_{1,i}]}$ be the solution of (1.1) with u_0 replaced by $u_{0,i}$, etc.. Let $v_{[F_i,G_i,u_{0,i},u_{1,i}]}$ be defined by (3.13) with the obvious modifications. From Lemma 6.2, Part (4) we have a Lipschitz estimate

$$\|u_{[F_1,G_1,u_{0,1},u_{1,1}]} - u_{[F_2,G_2,u_{0,2},u_{1,2}]}\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\theta)} \leq Md \text{ with}$$

$$d = \Big[\|u_{0,1} - u_{0,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_0})} + \|u_{1,1} - u_{1,2}\|_{L_p(\Omega;\mathcal{D}(-A)^{\delta_1})}$$

$$+ \ \|\mu_{\Delta F}\|_{L_p([0,T]\times\Omega;\mathbb{R})} + \|\mu_{\Delta G}\|_{L_p([0,T]\times\Omega;\mathbb{R})}\Big].$$

Now let $f_i = \mathcal{N}_F u_{[F_i,G_i,u_{0,i},u_{1,i}]}$ and $g_i = \mathcal{N}_G u_{[F_i,G_i,u_{0,i},u_{1,i}]}$. By Lemma 6.1(1) we have

$$\|f_1 - f_2\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\epsilon)} \leq M_F M d, \quad \|g_1 - g_2\|_{L_p([0,T]\times\Omega;L_p(B,l_2))} \leq M_G M d.$$

The difference $v = v_{[F_1,G_1,u_{0,1},u_{1,1}]} - v_{[F_2,G_2,u_{0,2},u_{1,2}]}$ solves (5.1) with u_0 replaced by $u_{0,1} - u_{0,2}$, etc.. Proposition 5.2 yields now

$$\|v_{[F_1,G_1,u_{0,1},u_{1,1}]} - v_{[F_2,G_2,u_{0,2},u_{1,2}]}\|_{L_p([0,T]\times\Omega;\mathcal{D}(-A)^\zeta)} \leq Md$$

with a suitable constant M. □

7. Maximal regularity considerations

In this section, we consider the case that $B = \mathbb{R}^n$ and $A = \Delta : W^{2,p}(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$, the Laplacian in $L_p(\mathbb{R}^n)$. In this case, a maximum regularity result can be proved. To keep the paper at a resonable size we concentrate on the stochastic part and confine ourselves to the equation

$$u(t,\omega,x) = \Delta \int_0^t k_\alpha(t-s)u(s,\omega,x)\,ds + \sum_{k=1}^\infty \int_0^t k_\beta(t-s)G^k(s,\omega,u(s,\omega,x))\,dw_s^k \tag{7.1}$$

and the linear equation

$$u(t,\omega,x) = \Delta \int_0^t k_\alpha(t-s)u(s,\omega,x)\,ds + \sum_{k=1}^\infty \int_0^t k_\beta(t-s)g^k(s,\omega,x)\,dw_s^k. \tag{7.2}$$

Notice that various results on maximal regularity with respect to deterministic forcing functions (see, e.g., [33]) and to inintial data (e.g., [8]) are available. These could be combined with the results given here and adapted to the semilinear case.

For (7.2) we obtain

Proposition 7.1 ([12], Theorem 4.14)**.** *For a positive integer n, let $\Delta : W^{2,p}(\mathbb{R}^n) \to L_p(\mathbb{R}^n)$ be the Laplacian with $1 < p < \infty$. Suppose that the probability space Ω and the Wiener processes w^k satisfy Hypothesis 3.2. Let $T > 0$, $\beta > \frac{1}{2}$, $\alpha \in (0,2)$, and $g \in L_p([0,T]\times\Omega; L_p(\mathbb{R}^n, l_2))$.*

(a) *Then there exists a unique function $u \in L_p([0,T]\times\Omega, L_p(\mathbb{R}^n))$ such that for almost all $t \in [0,T]$,*
$$\int_0^t k_\alpha(t-s)u(s)\,ds \in W^{2,p}(\mathbb{R}^n)$$
and (7.2) holds.

(b) *Moreover, if $\zeta \in [0,1]$ is such that*
$$\alpha\zeta + \frac{1}{2} \le \beta, \tag{7.3}$$
then $u \in L_p([0,T]\times\Omega, \mathcal{D}(-\Delta)^\zeta)$, and
$$\|u\|_{L_p([0,T]\times\Omega,\mathcal{D}(-\Delta)^\zeta)} \le M\|g\|_{L_p([0,T]\times\Omega,l_2)} \tag{7.4}$$
with a constant M dependent on $n, T, p, \alpha, \beta, \zeta$.

(c) *If strict inequality holds in (7.3), then M in (7.4) can be obtained arbitrarily small by taking sufficiently small T.*

Proof. Of course, if strict inequality holds in (7.3), then the assertions above are just a special case of Proposition 5.2 with $A = \Delta$, $u_0 = u_1 = 0$, and $\eta = 0$. But for such A and η, the assertion of Lemma 5.8 holds also if equality holds in (7.3), with the only exception that $\tilde{M}_{T,\mathrm{Ito}}$ cannot be made small by taking small T. See [11, Theorem 1.2]. (To prove this, the general estimates from Lemma 5.4 are replaced by a more sophisticated analysis of the resolvent kernel for the Laplacian, using the heat kernel and its self-similarity properties. This has been done for the heat equation by Krylov in [18], and generalized to the case of integral equations in [11].) Once Lemma 5.8 is established, the proof continues exactly as in Section 5. More details can be found in [12]. □

Since M in (7.4) cannot be controlled simply by taking short time intervals, we need a more sophisticated Lipschitz condition. (For the heat equation, compare [19, Assumption 5.6].)

Hypothesis 7.2. There exists some $\theta \in (0,1)$ such that

$$G : [0,T] \times \Omega \times \mathcal{D}(-\Delta)^\theta \to L_p(\mathbb{R}^n; l_2)$$
$$[G(t,\omega,u)](x) := \big(G^k(t,\omega,u)(x)\big)_{k=1}^\infty$$

satisfies the following assumptions:

(a) For fixed $u \in \mathcal{D}(-\Delta)^\theta$, the function $G(\cdot,\cdot,u)$ is measurable from $[0,T]\times\Omega$ into $L_p(\mathbb{R}^n; l_2)$.

(b) For each $\epsilon > 0$, there exists a constant $M_G(\epsilon) > 0$, such that for all $t \in [0,T]$, and all $u_1, u_2 \in \mathcal{D}(-\Delta)^\theta$ the following Lipschitz estimate holds:

$$\begin{aligned} &\|G(t,\omega,u_1) - G(t,\omega,u_2)\|_{L_p(\mathbb{R}^n;l_2)} \\ \leq\ &\big[\epsilon^p \|u_1-u_2\|^p_{\mathcal{D}(-\Delta)^\theta} + M_G(\epsilon)^p \|u_1-u_2\|^p_{L_p(\mathbb{R}^n)}\big]^{1/p} \quad \text{for } \omega\in\Omega \text{ a.e..} \end{aligned} \tag{7.5}$$

(c) For $u = 0$ we have

$$\left[\int_\Omega \int_0^T \|G(t,\omega,0)\|^p_{L_p(\mathbb{R}^n;l_2)}\, dt\, d\mathbb{P}\right]^{1/p} = M_{G,0} < \infty. \tag{7.6}$$

Theorem 7.3. *Let the probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *and the Wiener processes* $\big(w_s^k\big)_{k=1}^\infty$ *be as in Hypothesis 3.2. Let* $p \in [2,\infty)$*, and* Δ *be the Laplacian on* $L_p(\mathbb{R}^n)$*. Let* $\alpha \in (0,2)$*,* $\beta > \frac{1}{2}$*, and* $T > 0$*. Assume that* $G : [0,T]\times\Omega\times\mathcal{D}(-\Delta)^\theta \to L_p(\mathbb{R}^n;l_2)$ *satisfies Hypothesis 7.2 with suitable* $\theta\in(0,1)$*, such that*

$$\alpha\theta + \frac{1}{2} = \beta. \tag{7.7}$$

Then there exists a unique function $u \in L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)$ *such that for almost all* $t\in[0,T]$

$$\int_0^t k_\alpha(t-s) u(s,\omega,\cdot)\, ds \in \mathcal{D}\Delta \quad \textit{for a.e. } \omega\in\Omega,$$

and (7.1) *is satisfied for almost all* $\omega\in\Omega$*.*

Proof. We refine the contraction argument from Section 6. As in Lemma 6.1 we define for $v \in L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)$

$$\mathcal{N}_G(v) : \begin{cases} [0,T]\times\Omega & \to L_p(\mathbb{R}^n; l_2), \\ t\times\omega & \mapsto G(t,\omega,v(t,\omega)). \end{cases}$$

For $g \in L_p([0,T]\times\Omega; L_p(\mathbb{R}^n, l_2))$ we define $\mathcal{S}g := u \in L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)$, where u is the solution of (7.2) according to Proposition 7.1 with forcing function g. As in Section 6 the desired solution u is a fixed point of the operator $\mathcal{T} := \mathcal{S}\circ\mathcal{N}_G$ which maps $L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)$ into itself.

By (7.4) for $\zeta = 0$ and for $\zeta = \theta$ we infer

$$\|\mathcal{S}g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n))} \leq M_0(T)\|g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n;l_2))},$$
$$\|\mathcal{S}g\|_{L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)} \leq M_\theta\|g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n;l_2))},$$

with fixed M_θ, while $M_0(T)$ can be made arbitrarily small by taking T sufficiently small. We fix $\epsilon > 0$ such that $M_\theta\epsilon < \frac{1}{8}$ and choose the corresponding $M_G(\epsilon)$ according to Hypothesis 7.2. On $L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)$ we introduce the following equivalent norm

$$\begin{aligned} &\|v\|^p_{L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta),\text{equiv}} \\ :=\ &\int_0^T\int_\Omega \big[\epsilon^p\|v(t,\omega)\|^p_{\mathcal{D}(-\Delta)^\theta} + M_G^p(\epsilon)\|v(t,\omega)\|^p_{L_p(\mathbb{R}^n)}\big]\, d\mathbb{P}(\omega)\, dt. \end{aligned}$$

With respect to this norm, the nonlinear operator

$$\mathcal{N}_G : L_p([0,T] \times \Omega; \mathcal{D}(-\Delta)^\theta) \to L_p([0,T] \times \Omega; L_p(\mathbb{R}^n; l_2))$$

has Lipschitz constant 1 by Hypothesis 7.2. On the other hand

$$\|\mathcal{S}g\|_{L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta),\text{equiv}} \leq (\epsilon^p M_\theta^p + M_G(\epsilon)^p M_0(T)^p)^{1/p} \|g\|_{L_p([0,T]\times\Omega;L_p(\mathbb{R}^n;l_2))}.$$

We infer that $\mathcal{T}$ is Lipschitz on $L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta)$ with respect to the equivalent norm $\|\cdot\|_{L_p([0,T]\times\Omega;\mathcal{D}(-\Delta)^\theta),\text{equiv}}$, and if T is sufficiently small, so that $(\epsilon^p M_\theta^p + M_G(\epsilon)^p M_0(T)^p)^{1/p} < \frac{1}{4}$, then the Lipschitz constant of $\mathcal{T}$ is less than $\frac{1}{4}$. We can now proceed as in Lemma 6.2 (2) to construct an equivalent norm on $L_p([0,T] \times \Omega; \mathcal{D}(-A)^\theta)$ which makes $\mathcal{T}$ a strict contraction also for large T. □

8. Krylov's Approach Versus B-space Valued Stochastic Integration

At the center of the study of stochastic integral equations in Banach spaces is the problem of defining and estimating stochastic integrals, in particular stochastic convolutions, in Banach spaces. Krylov's approach, which is used in this paper, is elementary in the sense that stochastic integrals are taken pointwise, so they are classical Ito-integrals of scalar valued processes. The Burkholder-Davis-Gundy inequality provides the step from L_2-estimates to L_p. Of course, this can only be done for sufficiently "nice" integrands. The final step is to extend the results obtained for smooth initial data and elementary forcing terms to more general L_p-data by a completion argument.

On the other hand, the recent progress on stochastic integration in Banach spaces (see, e.g.[22]) provides a convenient tool to handle stochastic convolutions directly in the Banach space. While we do not know about applications of this method to integral equations, it has been used successfully to treat parabolic stochastic differential equations, e.g. [13], [32]. We expect that such results can be extended to integral equations. Clearly, this approach works in more general Banach spaces, while the more classical technique is confined to the special structure of L_p.

In [12] we compared our linear results with those obtained in [13], [32]. In the context of the present paper it appears interesting to make a similar brief comparison concerning semilinear equations.

First note - as mentioned above - that the results of [32] are more general than those presented here in the sense that equations in Banach spaces of type 2 - and even in UMD-spaces - are analyzed. Here we consider only L_p-spaces with $p \in [2,\infty)$. In addition, in [32], time-dependent operators $A(t)$ are considered. On the other hand, the aim of the present paper is to treat fractional differential equations and not only the differential equation case $\alpha = \beta = \gamma = 1$, considered in [13], [32].

With $\alpha = \beta = \gamma = 1$ our equation (1.1) reduces to the stochastic nonlinear differential equation

$$du(t) = Au(t)\,dt + G(t,\omega,u(t))\,dW_t + F(t,\omega,u(t))\,dt. \tag{8.1}$$

It is this case, where we can compare our results to the results obtained by the abstract integration theory. Note that in abstract notation, W_t is a cylindrical Wiener process in a separable Hilbert space H and that, for fixed u, $G \in L_p([0,T]\times\Omega;\gamma(H, L_p(B)))$ where $\gamma(H, L_p(B))$ denotes the space of γ-radonifying operators $H \to L_p(B)$. This is equivalent to writing the stochastic forcing in Krylov's notation

$$G = \sum_{k=1}^\infty G^k w_s^k.$$

with (for fixed u) $\{G^k\}_{k=1}^{\infty} \in L_p([0,T] \times \Omega; L_p(B, l_2))$ (use, e.g., [32, Proposition 3.2.3]).

In [32], Theorem 8.3.3 gives existence and uniqueness of solutions for (8.1) in the space $L^p(\Omega; C([0,T]; (X, \mathcal{D}A)_{a,1}))$. However, the crucial Lemma 8.3.1 in [32], which establishes the contraction, allows also (as a special case $r = p$) to consider the space $L_p([0,T] \times \Omega; (X, \mathcal{D}A)_{a,1})$. This compares best with our results. Note that the conditions (3.6), etc., are strict inequalities, so it makes no difference whether the results are stated in terms of $(L_p(B), \mathcal{D}A)_{a,1}$ or of $\mathcal{D}(-A)^a$. With the assumption that A is sectorial and independent of t, the Lipschitz conditions in [32] can be rewritten in our notation:

(F) For some $\theta_F \geq 0$, $a \in [0,1)$, $a + \theta_F < 1$,

$$\|(-A)^{-\theta_F}(F(t,\omega,x) - F(t,\omega,y))\|_{L_p(B)} \leq L_F \|x - y\|_{(L_p(B), \mathcal{D}A)_{a,1}}$$

for all $t \in [0,T]$, $\omega \in \Omega$; $x, y \in (L_p(B), \mathcal{D}A)_{a,1}$.

(G) For some $\theta_B \geq 0$, $a \in [0,1)$, $a + \theta_B < \frac{1}{2}$,

$$\|(-A)^{-\theta_B}(G(t,\omega,x) - G(t,\omega,y)\|_{L_p(B;l_2)} \leq L_G \|x - y\|_{(L_p(B);\mathcal{D}A)_{a,1}},$$

again for all $t \in [0,T]$ and $\omega \in \Omega$, $x, y \in (L_p(B), \mathcal{D}A)_{a,1}$.

In our paper, the range of G is specified to be $L_p(B; l_2)$, so that we are restricted to $\theta_B = 0$. (However, it would be easy to multiply the whole equation by A^{θ_B} to handle different values of θ_B. In this case, our conditions on ϵ, θ and δ_0 would need to be replaced by analogous conditions on $\epsilon + \theta_B$, $\theta + \theta_B$, $\delta_0 + \theta_B$.) The domain of F and G in our paper is $\mathcal{D}(-A)^\theta$, so our θ plays the role of a in (G) above, while our ϵ corresponds to $-\theta_F$ in (F) above. So the condition $\theta_F + a < 1$ translates exactly to our condition $-\epsilon + \theta < 1$ which is (3.5) for $\alpha = \gamma = 1$. Veraar's condition $a + \theta_B < \frac{1}{2}$ translates to $\theta + 0 < \frac{1}{2}$, which is (3.6) for $\alpha = \beta = 1$.

While [32] requires $u_0 \in L_p(\Omega; (L_p(B, \mathbb{R}); \mathcal{D}A)_{a,1})$ (i.e. $\delta_0 = \theta$ in our notation), our condition (3.7) requires slightly less, namely $\delta_0 > \theta - \frac{1}{p}$. Essentially, Krylov's method yields the same regularity as the abstract integration theory, but, as mentioned above, it is restricted to L_p-spaces instead of general UMD-spaces.

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W. D.: Institut für Mathematik und Wissenschaftliches Rechnen, Karl-Franzens-Universität Graz, Heinrichstrasse 36, 8010 Graz, Austria

E-mail address: `georg.desch@uni-graz.at`

S.-O. L.: Department of Mathematics and Systems Analysis, Helsinki University of Technology, FI–02015 TKK, Finland

E-mail address: `stig-olof.londen@tkk.fi`

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