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MIXED FINITE ELEMENT METHODS FOR PROBLEMS WITH ROBIN BOUNDARY CONDITIONS

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Abstract. We derive new a-priori and a-posteriori error estimates for mixed finite element discretizations of second-order elliptic problems with general Robin boundary conditions, parameterized by a non-negative and piecewise constant function $\varepsilon \geq 0$. The estimates are robust over several orders of magnitude of ε, ranging from pure Dirichlet conditions to pure Neumann conditions. A series of numerical experiments is presented that verify our theoretical results.

1. Introduction. We consider the dual mixed finite element method for second order elliptic equations subject to general Robin boundary conditions. These we parameterize by $\varepsilon \geq 0$, with natural Dirichlet conditions for $\varepsilon = 0$ and essential Neumann conditions in the limit $\varepsilon \to \infty$. For the mixed method the Neumann condition is an essential condition and could be explicitly enforced. However, we prefer to see the method implemented in the same way for all possible boundary conditions and then the Neumann condition is obtained by penalization, i.e. choosing ε "large".

Let us recall that the situation for a primal (displacement) finite element method is the opposite, Neumann conditions are natural and Dirichlet essential, and the latter are penalized by choosing ε "small". For this case it is well known that the problem becomes ill-conditioned in two ways. The error estimates are not independent of ε and the stiffness matrix becomes ill-conditioned. We here remark that in [8] Nitsche's method was extended to general Robin boundary conditions yielding a primal formulation avoiding ill-conditioning.

The following question naturally arises now. Is the mixed method ill-conditioned near the Neumann limit? In this paper we will show that this is not the case. We will prove both a priori and a posteriori error estimates that are uniformly valid independent of the parameter ε. We also show that the stiffness matrix is well-conditioned. It seems that this has not earlier been reported in the literature. Robin conditions are treated in [12], but the robustness with respect to the parameter was not studied.

The outline of the paper is the following. In the next section we recall the mixed finite element method. In Section 3 we derive a-priori error estimates and prove an optimal bound for the error in the flux. In Section 4 we analyze the postprocessing method of [15, 14] which enhances the accuracy of the displacement variable. In Section 5, we introduce a residual-based a-posteriori error estimator and establish its reliability and efficiency. In Section 6 we consider the solution of the problem by hybridization and show that this leads to a well-conditioned linear system. A set of numerical examples are presented in Section 7 that verify the ε-robustness of our estimates. Finally, we end the paper with some concluding remarks in Section 8.

Throughout the paper, we use standard notation. We denote by C, C_1, C_2 etc. generic constants that are not necessarily identical at different places, but are always

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independent of ε and the mesh size.

2. Mixed finite element methods. In this section, we introduce two families of mixed finite element methods for the mixed form of Poisson's equation with Robin boundary conditions.

2.1. Model problem. We consider the following model problem:

$$\boldsymbol{\sigma} - \nabla u = \mathbf{0} \qquad \text{in } \Omega, \tag{2.1}$$
$$\operatorname{div} \boldsymbol{\sigma} + f = 0 \qquad \text{in } \Omega, \tag{2.2}$$

subject to the general Robin boundary conditions

$$\varepsilon \boldsymbol{\sigma} \cdot \boldsymbol{n} = u_0 - u + \varepsilon g \qquad \text{on } \partial\Omega. \tag{2.3}$$

Here, $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a bounded polygonal or polyhedral Lipschitz domain, f a given load, and u_0 and g are prescribed data on the boundary of Ω. The vector $\boldsymbol{n}$ denotes the unit outward normal vector on $\partial\Omega$. The boundary conditions (2.3) are parameterized by the non-negative function $\varepsilon \geq 0$. For simplicity, we assume ε to be piecewise constant on the boundary (with respect to the partition of $\partial\Omega$ induced by a triangulation of Ω). In the limiting case $\varepsilon = 0$, we obtain the Dirichlet boundary conditions

$$u = u_0 \qquad \text{on } \partial\Omega. \tag{2.4}$$

On the other hand, if $\varepsilon \to \infty$ everywhere on $\partial\Omega$, we recover the Neumann boundary conditions

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = g \qquad \text{on } \partial\Omega. \tag{2.5}$$

To cast (2.1)–(2.2) in weak form, we first note that $(\boldsymbol{\sigma}, u)$ satisfies

$$(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u) - \langle u, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} = 0 \qquad \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega), \tag{2.6}$$
$$(\operatorname{div} \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad \forall v \in L^2(\Omega). \tag{2.7}$$

Then we solve for u in the expression (2.3) for the boundary conditions and insert the result into (2.6). We find that

$$a_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u) = \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} \qquad \forall \boldsymbol{\tau} \in H(\operatorname{div}, \Omega), \tag{2.8}$$
$$(\operatorname{div} \boldsymbol{\sigma}, v) + (f, v) = 0 \qquad \forall v \in L^2(\Omega), \tag{2.9}$$

with $a_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau})$ defined by

$$a_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma}, \boldsymbol{\tau}) + \langle \varepsilon \boldsymbol{\sigma} \cdot \boldsymbol{n}, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega}.$$

Here, we denote by $(\cdot, \cdot)$ the standard L^2-inner product over Ω, and by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ the one over the boundary $\partial\Omega$. By introducing the bilinear form

$$\mathcal{B}_\varepsilon(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, v) = a_\varepsilon(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u) + (\operatorname{div} \boldsymbol{\sigma}, v),$$

we thus obtain the following weak form of (2.1)–(2.2): find $(\boldsymbol{\sigma}, u) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$ such that

$$\mathcal{B}_\varepsilon(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, v) + (f, v) = \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} \tag{2.10}$$

for all $(\boldsymbol{\tau}, v) \in H(\operatorname{div}, \Omega) \times L^2(\Omega)$.

The well-posedness of (2.10) follows from standard arguments of mixed finite element theory [4].

2.2. Mixed finite element discretization. In order to discretize the variational problem (2.10), let $\mathcal{K}_h$ be a regular and shape-regular partition of Ω into simplices. As usual, the diameter of an element K is denoted by h_K, and the global mesh size h is defined as $h = \max_{K\in\mathcal{K}_h} h_K$. We denote by $\mathcal{E}_h^0$ the set of all interior edges (faces) of $\mathcal{K}_h$, and by $\mathcal{E}_h^\partial$ the set of all boundary edges (faces). We write h_E for the diameter of an edge (face) E.

Mixed finite element discretization of (2.10) is based on finite element spaces $\boldsymbol{S}_h \times V_h \subset H(\mathrm{div},\Omega) \times L^2(\Omega)$ of piecewise polynomial functions with respect to $\mathcal{K}_h$. We will focus here on the Raviart-Thomas (RT) and Brezzi-Douglas-Marini (BDM) families of elements [11, 10, 3, 2, 4]. That is, for an approximation of order $k \geq 1$, the flux space $\boldsymbol{S}_h$ is taken as one of the following two spaces

$$
\begin{aligned}
\boldsymbol{S}_h^{RT} &= \{\, \boldsymbol{\sigma} \in H(\mathrm{div},\Omega) \,|\, \boldsymbol{\sigma}|_K \in [P_{k-1}(K)]^n \oplus \boldsymbol{x}\widetilde{P}_{k-1}(K),\ K \in \mathcal{K}_h \,\},\\
\boldsymbol{S}_h^{BDM} &= \{\, \boldsymbol{\sigma} \in H(\mathrm{div},\Omega) \,|\, \boldsymbol{\sigma}|_K \in [P_k(K)]^n,\ K \in \mathcal{K}_h \,\},
\end{aligned}
\tag{2.11}
$$

where $P_k(K)$ denotes the polynomials of total degree less or equal than k on K, and $\widetilde{P}_{k-1}(K)$ the homogeneous polynomials of degree $k-1$. For both choices of $\boldsymbol{S}_h$ above, the displacements are approximated in the multiplier space

$$
V_h = \{\, u \in L^2(\Omega) \,|\, u|_K \in P_{k-1}(K),\ K \in \mathcal{K}_h \,\}. \tag{2.12}
$$

The spaces are chosen such that the following equilibrium property holds:

$$
\mathrm{div}\ \boldsymbol{S}_h \subset V_h. \tag{2.13}
$$

The mixed finite element method now consists of finding $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{S}_h \times V_h$ such that

$$
\mathcal{B}_\varepsilon(\boldsymbol{\sigma}_h, u_h; \boldsymbol{\tau}, v) + (f, v) = \langle u_0 + \varepsilon g, \boldsymbol{\tau}\cdot \boldsymbol{n}\rangle_{\partial\Omega} \tag{2.14}
$$

for all $(\boldsymbol{\tau}, v) \in \boldsymbol{S}_h \times V_h$. We remark that, by the equilibrium condition (2.13), we have immediately the identity

$$
\mathrm{div}\ \boldsymbol{\sigma}_h = -P_h f, \tag{2.15}
$$

with P_h denoting the L^2-projection onto V_h.

3. A-priori error estimates. In this section, we derive a-priori error estimates for the method in (2.14).

3.1. Stability. We begin by introducing the jump of a piecewise smooth scalar function u. To that end, let $E = \partial K \cap \partial K'$ be an interior edge (face) shared by two elements K and K'. Then the jump of f over E is defined by

$$
[\![f]\!] = f|_K - f|_{K'}. \tag{3.1}
$$

Next, we recall the following well-known trace estimate: for an edge (face) E of an element K, there holds

$$
h_E \|\boldsymbol{\sigma}\|_{0,E}^2 \leq C \|\boldsymbol{\sigma}\|_{0,K}^2 \qquad \forall \boldsymbol{\sigma} \in \boldsymbol{S}_h. \tag{3.2}
$$

Stability will be measured in mesh-dependent norms. For the fluxes, we define

$$
|\!|\!|\boldsymbol{\sigma}|\!|\!|_{\varepsilon,h}^2 = \|\boldsymbol{\sigma}\|_0^2 + \sum_{E\in\mathcal{E}_h^\partial} (\varepsilon + h_E)\|\boldsymbol{\sigma}\cdot \boldsymbol{n}\|_{0,E}^2. \tag{3.3}
$$

Here, we denote by $\|\cdot\|_{0,D}$ the L^2-norm over a set D. In the case where $D=\Omega$, we simply write $\|\cdot\|_0$. For the displacement variables, we introduce the norm

$$\|u\|_{\varepsilon,h}^2 = \sum_{K\in\mathcal{K}_h} \|\nabla u\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h^0} \frac{1}{h_E}\|[\![u]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}_h^\partial} \frac{1}{\varepsilon+h_E}\|u\|_{0,E}^2. \tag{3.4}$$

The continuity of the bilinear forms in the above norms follows by straightforward estimation.

LEMMA 3.1. *We have*

$$a_\varepsilon(\boldsymbol{\sigma},\boldsymbol{\tau}) \le \|\boldsymbol{\sigma}\|_{\varepsilon,h}\|\boldsymbol{\tau}\|_{\varepsilon,h}, \qquad \boldsymbol{\sigma},\boldsymbol{\tau}\in\boldsymbol{S}_h, \tag{3.5}$$

$$(\mathrm{div}\,\boldsymbol{\sigma},u) \le C\|\boldsymbol{\tau}\|_{\varepsilon,h}\|u\|_{\varepsilon,h}, \qquad \boldsymbol{\sigma}\in\boldsymbol{S}_h,\ u\in V_h. \tag{3.6}$$

Furthermore, it holds

$$\mathcal{B}_\varepsilon(\boldsymbol{\sigma},u;\boldsymbol{\tau},v) \le C\big(\|\boldsymbol{\sigma}\|_{\varepsilon,h}+\|u\|_{\varepsilon,h}\big)\big(\|\boldsymbol{\tau}\|_{\varepsilon,h}+\|v\|_{\varepsilon,h}\big) \tag{3.7}$$

for all $\boldsymbol{\sigma},\boldsymbol{\tau}\in\boldsymbol{S}_h$ and $u,v\in V_h$.

Proof. The bound (3.5) is a simple consequence of the Cauchy-Schwarz inequality:

$$\begin{aligned} a_\varepsilon(\boldsymbol{\sigma},\boldsymbol{\tau}) &= (\boldsymbol{\sigma},\boldsymbol{\tau}) + \langle \varepsilon\boldsymbol{\sigma}\cdot\boldsymbol{n},\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial\Omega} \\ &= (\boldsymbol{\sigma},\boldsymbol{\tau}) + \sum_{E\in\mathcal{E}_h^\partial}\langle \varepsilon^{1/2}\boldsymbol{\sigma}\cdot\boldsymbol{n},\varepsilon^{1/2}\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_E \le \|\boldsymbol{\sigma}\|_{\varepsilon,h}\|\boldsymbol{\tau}\|_{\varepsilon,h}. \end{aligned}$$

To prove (3.6), we use partial integration, elementary manipulations and the Cauchy-Schwarz inequality to obtain

$$\begin{aligned} (\mathrm{div}\,\boldsymbol{\sigma},u) &= -\sum_{K\in\mathcal{K}_h}(\boldsymbol{\sigma},\nabla u)_K + \sum_{K\in\mathcal{K}_h}\langle\boldsymbol{\sigma}\cdot\boldsymbol{n}_{\partial K},u\rangle_{\partial K} \\ &\le \sum_{K\in\mathcal{K}_h}\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{0,K}\|\nabla u\|_{0,K} + \sum_{E\in\mathcal{E}_h^0} h_E^{\frac12}\|\boldsymbol{\sigma}\|_{0,E}h_E^{-\frac12}\|[\![u]\!]\|_{0,E} \\ &\quad + \sum_{E\in\mathcal{E}_h^\partial}(\varepsilon+h_E)^{\frac12}\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{0,E}(\varepsilon+h_E)^{-\frac12}\|u\|_{0,E}, \end{aligned}$$

with $\boldsymbol{n}_{\partial K}$ denoting the unit outward normal on ∂K. The trace estimate (3.2) and a repeated application of the Cauchy-Schwarz inequality then readily prove (3.6). Finally, the continuity bound (3.7) follows directly from (3.5) and (3.6). ☐

Next, we address the coercivity of the form a_ε.

LEMMA 3.2. *There is a constant $C>0$ such that*

$$a_\varepsilon(\boldsymbol{\sigma},\boldsymbol{\sigma}) \ge C\|\boldsymbol{\sigma}\|_{\varepsilon,h}^2 \qquad \forall\boldsymbol{\sigma}\in\boldsymbol{S}_h.$$

Proof. Since $a_\varepsilon(\boldsymbol{\sigma},\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}\|_0^2 + \sum_{E\in\mathcal{E}_h^\partial}\varepsilon\|\boldsymbol{\sigma}\cdot\boldsymbol{n}\|_{0,E}^2$, the trace estimate (3.2) readily yields the desired result. ☐

Finally, we prove the following inf-sup condition for the divergence form.

LEMMA 3.3. *There exists a constant $C>0$ such that*

$$\sup_{\boldsymbol{\sigma}\in\boldsymbol{S}_h}\frac{(\mathrm{div}\,\boldsymbol{\sigma},u)}{\|\boldsymbol{\sigma}\|_{\varepsilon,h}} \ge C\|u\|_{\varepsilon,h} \qquad \forall u\in V_h.$$

Proof. The proof is an extension of that of [9, Lemma 2.1]. Since $\boldsymbol{S}_h^{RT} \subset \boldsymbol{S}_h^{BDM}$, we only have to prove the condition in the Raviart-Thomas case. We recall that, on an element K, the local degrees of freedom for the RT family are given by the moments

$$\begin{aligned} \langle \boldsymbol{\sigma}\cdot \boldsymbol{n}_{\partial K}, z\rangle_E \qquad & \forall z \in P_{k-1}(E),\ E \subset \partial K, \\ (\boldsymbol{\sigma}, \boldsymbol{z})_K \qquad & \forall \boldsymbol{z} \in [P_{k-2}(K)]^n. \end{aligned}$$

Let now $u \in V_h$ be arbitrary. We then define $\boldsymbol{\sigma} \in \boldsymbol{S}_h^{RT}$ by setting on each element K:

$$\begin{aligned} \langle \boldsymbol{\sigma}\cdot \boldsymbol{n}_{\partial K}, z\rangle_E &= \frac{1}{h_E}\langle [\![u]\!], z\rangle_E \qquad && \forall z \in P_{k-1}(E),\ E \in \mathcal{E}_h^0,\ E \subset \partial K, \\ \langle \boldsymbol{\sigma}\cdot \boldsymbol{n}_{\partial K}, z\rangle_E &= \frac{1}{\varepsilon + h_E}\langle u, z\rangle_E \qquad && \forall z \in P_{k-1}(E),\ E \in \mathcal{E}_h^\partial,\ E \subset \partial K, \\ (\boldsymbol{\sigma}, \boldsymbol{z})_K &= -(\nabla u, \boldsymbol{z})_K && \forall \boldsymbol{z} \in [P_{k-2}(K)]^n. \end{aligned}$$

Choosing $z = [\![u]\!] \in P_{k-1}(E)$ and $\boldsymbol{z} = \nabla u \in [P_{k-2}(K)]^n$ gives

$$\begin{aligned} \langle \boldsymbol{\sigma}\cdot \boldsymbol{n}_{\partial K}, [\![u]\!]\rangle_E &= \frac{1}{h_E}\|[\![u]\!]\|_{0,E}^2, \qquad && E \in \mathcal{E}_h^0,\ E \subset \partial K, \\ \langle \boldsymbol{\sigma}\cdot \boldsymbol{n}_{\partial K}, [\![u]\!]\rangle_E &= \frac{1}{\varepsilon + h_E}\|u\|_{0,E}^2, && E \in \mathcal{E}_h^\partial,\ E \subset \partial K, \\ (\boldsymbol{\sigma}, \nabla u)_K &= -\|\nabla u\|_{0,K}^2. \end{aligned}$$

Then we employ partial integration over each element and apply the defining moments for $\boldsymbol{\sigma}$:

$$\begin{aligned} (\operatorname{div} \boldsymbol{\sigma}, u) &= \sum_{K\in\mathcal{K}_h} -(\boldsymbol{\sigma}, \nabla u)_K + \sum_{K\in\mathcal{K}_h} \langle \boldsymbol{\sigma}\cdot \boldsymbol{n}_{\partial K}, u\rangle_{\partial K} \\ &= \sum_{K\in\mathcal{K}_h} \|\nabla u\|_{0,K}^2 + \sum_{E\in\mathcal{E}_h^0} \frac{1}{h_E}\|[\![u]\!]\|_{0,E}^2 + \sum_{E\in\mathcal{E}_h^\partial} \frac{1}{\varepsilon + h_E}\|u\|_{0,E}^2 \\ &= |\!|\!| u |\!|\!|_{\varepsilon,h}^2. \end{aligned} \tag{3.8}$$

Moreover, an explicit inspection of the degrees of freedom readily yields

$$|\!|\!| \boldsymbol{\sigma} |\!|\!|_{\varepsilon,h} \le C |\!|\!| u |\!|\!|_{\varepsilon,h}. \tag{3.9}$$

Identity (3.8) and the bound (3.9) give the desired inf-sup condition. ☐

By combining continuity (Lemma 3.1), coercivity (Lemma 3.2) and the inf-sup condition (Lemma 3.3), we readily obtain the following stability result.

LEMMA 3.4. *There is a constant $C > 0$ such that*

$$\sup_{(\boldsymbol{\tau},v)\in \boldsymbol{S}_h\times V_h} \frac{\mathcal{B}(\boldsymbol{\sigma}, u; \boldsymbol{\tau}, v)}{|\!|\!| \boldsymbol{\tau} |\!|\!|_{\varepsilon,h} + |\!|\!| v |\!|\!|_{\varepsilon,h}} \ge C(|\!|\!| \boldsymbol{\sigma} |\!|\!|_{\varepsilon,h} + |\!|\!| u |\!|\!|_{\varepsilon,h}) \qquad \forall (\boldsymbol{\sigma}, u) \in \boldsymbol{S}_h \times V_h.$$

3.2. Error estimates. We are now ready to derive a-priori error estimates. To that end, let $(\boldsymbol{\sigma}, u)$ be the solution of (2.10), and $(\boldsymbol{\sigma}_h, u_h)$ the mixed finite element approximation of (2.14).

Let $\boldsymbol{R}_h : H(\mathrm{div}, \Omega) \to \boldsymbol{S}_h$ be the RT or BDM interpolation operator [4]. It satisfies

$$(\mathrm{div}\,(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), v) = 0 \qquad \forall v \in V_h, \tag{3.10}$$

as well as the commuting diagram property

$$\mathrm{div}\ \boldsymbol{R}_h\boldsymbol{\sigma} = P_h\,\mathrm{div}\ \boldsymbol{\sigma}, \tag{3.11}$$

see, e.g., [4]. Moreover, we note that the equilibrium property (2.13) implies

$$(\mathrm{div}\,\boldsymbol{\tau}, u - P_h u) = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{S}_h. \tag{3.12}$$

REMARK 3.5. *In order for $\boldsymbol{R}_h\boldsymbol{\sigma}$ to be well-defined, some extra regularity is required for $\boldsymbol{\sigma}$; see [4]. Since the regularity assumptions of Theorem 3.7 below are more than sufficient, we neglect this issue in the following.*

PROPOSITION 3.6. *There is a constant $C > 0$ such that*

$$\|\!|\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}|\!\|_{\varepsilon,h} + \|\!|u_h - P_h u|\!\|_{\varepsilon,h} \le C\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0.$$

Proof. By the stability result in Lemma 3.4 there exists $(\boldsymbol{\tau}, v) \in \boldsymbol{S}_h \times V_h$ such that $\|\!|\boldsymbol{\tau}|\!\|_{\varepsilon,h} + \|\!|v|\!\|_{\varepsilon,h} \le C$ and

$$\|\!|\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}|\!\|_{\varepsilon,h} + \|\!|u_h - P_h u|\!\|_{\varepsilon,h} \le \mathcal{B}_\varepsilon(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}, u_h - P_h u; \boldsymbol{\tau}, v).$$

Using the consistency of the mixed method and properties (3.10), (3.12), we obtain

$$\begin{aligned}
&\mathcal{B}_\varepsilon(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}, u_h - P_h u; \boldsymbol{\tau}, v)\\
&\quad = a_\varepsilon(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u_h - P_h u) + (\mathrm{div}\ (\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}), v)\\
&\quad = a_\varepsilon(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u - P_h u) + (\mathrm{div}\ (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), v)\\
&\quad = (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + \sum_{E \in \mathcal{E}_h^\partial} \varepsilon\langle(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_E.
\end{aligned}$$

Then the defining moments for RT or BDM interpolation yield (noting that ε is edgewise (facewise) constant)

$$\sum_{E \in \mathcal{E}_h^\partial} \varepsilon\langle(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_E = 0, \tag{3.13}$$

so that

$$\mathcal{B}_\varepsilon(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}, u_h - P_h u; \boldsymbol{\tau}, v) = (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}).$$

Thus, we conclude that

$$\|\!|\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}|\!\|_{\varepsilon,h} + \|\!|u_h - P_h u|\!\|_{\varepsilon,h} \le \|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0\|\boldsymbol{\tau}\|_0 \le C\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0,$$

which completes the proof. ☐

In the sequel, we denote by $\|\cdot\|_k$ the standard Sobolev norm of order k. The following theorem is the main result of this section.

THEOREM 3.7. *We have the approximation bound*

$$\|\!|\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}\|\!|_{\varepsilon,h} + \|\!|P_h u - u_h\|\!|_{\varepsilon,h} \leq \begin{cases} Ch^k\|\boldsymbol{\sigma}\|_k & \text{for RT elements,} \\ Ch^{k+1}\|\boldsymbol{\sigma}\|_{k+1} & \text{for BDM elements.} \end{cases} \tag{3.14}$$

Moreover, we have the following optimal a-priori error estimate for the L^2-error in the flux:

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 \leq \begin{cases} Ch^k\|\boldsymbol{\sigma}\|_k & \text{for RT elements,} \\ Ch^{k+1}\|\boldsymbol{\sigma}\|_{k+1} & \text{for BDM elements.} \end{cases} \tag{3.15}$$

Proof. The bound (3.14) is an immediate consequence of Proposition 3.6 and the approximation properties of $\boldsymbol{R}_h$, see, e.g., [4]. The error estimate (3.15) follows readily from the triangle inequality, the consistency bound in Proposition 3.6 and the approximation properties of $\boldsymbol{R}_h$. ☐

REMARK 3.8. *We point out that the quantity $\|\!|P_h u - u_h\|\!|_{\varepsilon,h}$ in (3.14) is superconvergent. As in [9], this fact allows us to enhance the displacement approximation via local postprocessing; see Section 4 below. We further emphasize that the constant C in the error bound (3.15) is independent of ε. Hence, the L^2-norm error estimate for the fluxes is ε-robust.*

4. Postprocessing. In this section, we introduce a local postprocessing for the displacement and prove an optimal error estimate in the postprocessed displacement.

4.1. Postprocessing method. Let u_h be the displacement obtained by the mixed method (2.14). The postprocessed displacement u_h^* is sought in the augmented space $V_h^* \supset V_h$ defined as

$$V_h^* = \begin{cases} \{\, u^* \in L^2(\Omega) \,|\, u^*|_K \in P_k(K),\ K \in \mathcal{K}_h \,\} & \text{for RT elements,} \\ \{\, u^* \in L^2(\Omega) \,|\, u^*|_K \in P_{k+1}(K),\ K \in \mathcal{K}_h \,\} & \text{for BDM elements.} \end{cases} \tag{4.1}$$

The postprocessed displacement u_h^* is defined on each element K by the conditions

$$P_h u_h^* = u_h, \tag{4.2}$$

$$(\nabla u_h^*, \nabla v)_K = (\boldsymbol{\sigma}_h, \nabla v)_K \qquad \forall v \in (I - P_h)V_h^*|_K, \tag{4.3}$$

where we recall that P_h is the L^2-projection onto V_h.

In order to analyze the error in the postprocessed displacement u_h^*, we introduce the modified bilinear form

$$\mathcal{B}^*_{\varepsilon,h}(\boldsymbol{\sigma}, u^*; \boldsymbol{\tau}, v^*) = \mathcal{B}_\varepsilon(\boldsymbol{\sigma}, u^*; \boldsymbol{\tau}, v^*) + \sum_{K\in\mathcal{K}_h} (\nabla u^* - \boldsymbol{\sigma}, \nabla(I - P_h)v^*)_K. \tag{4.4}$$

Then we will consider the modified variational problem: find $(\boldsymbol{\sigma}_h, u_h^*) \in \boldsymbol{S}_h \times V_h^*$ such that

$$\mathcal{B}^*_{\varepsilon,h}(\boldsymbol{\sigma}_h, u_h^*; \boldsymbol{\tau}, v^*) + (P_h f, v) = \langle u_0 + \varepsilon g, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial\Omega} \tag{4.5}$$

for all $(\boldsymbol{\tau}, v^*) \in \boldsymbol{S}_h \times V_h^*$. The following proposition relates the solution of the modified problem (4.5) to that of the original problem (2.14). Its proof is exactly the same as the one for the standard mixed methods considered in [9, Lemma 2.4].

PROPOSITION 4.1. *Let $(\boldsymbol{\sigma}_h, u_h^*) \in \boldsymbol{S}_h \times V_h^*$ be the solution of problem* (4.5) *and set $u_h = P_h u_h^*$. Then $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{S}_h \times V_h$ is the solution of the original problem* (2.14). *Conversely, if $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{S}_h \times V_h$ is the solution of the original problem* (2.14) *and u_h^* is the postprocessed displacement obtained from u_h, then $(\sigma_h, u_h^*) \in \boldsymbol{S}_h \times V_h^*$ is the solution of problem* (4.5).

In order to show the stability of the modified method (4.5), we shall first state the following useful result whose proof is nearly identical to that of [9, Lemma 2.5].

LEMMA 4.2. *There exist constants $C_1 > 0$, $C_2 > 0$ such that for every $u^* \in V_h^*$ there holds*

$$\|u^*\|_{\varepsilon,h} \leq \|P_h u^*\|_{\varepsilon,h} + \|(I - P_h)u^*\|_{\varepsilon,h} \leq C_2 \|u^*\|_{\varepsilon,h}, \tag{4.6}$$

as well as

$$C_1 \|u^*\|_{\varepsilon,h} \leq \|P_h u^*\|_{\varepsilon,h} + \left(\sum_{K \in \mathcal{K}_h} \|\nabla (I - P_h) u^*\|_{0,K}^2 \right)^{1/2} \leq C_2 \|u^*\|_{\varepsilon,h}. \tag{4.7}$$

Since $(I - P_h)u^*$ is L^2-orthogonal to constant functions, there exists a third constant $C_3 > 0$ such that

$$\|(I - P_h)u^*\|_{\varepsilon,h} \leq C_3 \left(\sum_{K \in \mathcal{K}_h} \|\nabla (I - P_h) u^*\|_{0,K}^2 \right)^{1/2}. \tag{4.8}$$

With exactly the same arguments as in [9, Lemma 2.6], we then have the following inf-sup stability result for the modified bilinear form $\mathcal{B}_{\varepsilon,h}^*$.

PROPOSITION 4.3. *There exists a constant $C > 0$ such that*

$$\sup_{(\boldsymbol{\tau}, v^*) \in \boldsymbol{S}_h \times Q_h^*} \frac{\mathcal{B}_{\varepsilon,h}^*(\boldsymbol{\sigma}, u^*; \boldsymbol{\tau}, v^*)}{\|\boldsymbol{\tau}\|_{\varepsilon,h} + \|v^*\|_{\varepsilon,h}} \geq C(\|\boldsymbol{\sigma}\|_{\varepsilon,h} + \|u^*\|_{\varepsilon,h}) \qquad \forall (\boldsymbol{\sigma}, u^*) \in \boldsymbol{S}_h \times V_h^*. \tag{4.9}$$

4.2. Error in the postprocessed displacement. Now we state and prove a-priori error estimates for the postprocessed displacement u_h^*. As before, let $(\boldsymbol{\sigma}, u)$ be the solution of (2.10), and $(\boldsymbol{\sigma}_h, u_h^*)$ the postprocessed finite element approximation of (4.5). We now have the following result.

THEOREM 4.4. *There holds:*

$$\|\boldsymbol{\sigma}_h - \boldsymbol{R}_h \boldsymbol{\sigma}\|_{\varepsilon,h} + \|u - u_h^*\|_{\varepsilon,h} \leq C \|\boldsymbol{\sigma} - \boldsymbol{R}_h \boldsymbol{\sigma}\|_0 + \inf_{u^* \in V_h^*} \|u - u^*\|_{\varepsilon,h}$$

As a consequence, we have the ε-robust error estimate

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|u - u_h^*\|_{\varepsilon,h} \leq \begin{cases} Ch^k \|u\|_{k+1} & \text{for RT elements,} \\ Ch^{k+1} \|u\|_{k+2} & \text{for BDM elements.} \end{cases}$$

Proof. Let $u^* \in V_h^*$. From Proposition 4.3 it follows that there is a tuple $(\boldsymbol{\tau}, v^*) \in \boldsymbol{S}_h \times V_h^*$ such that $\|\boldsymbol{\tau}\|_{\varepsilon,h} + \|v^*\|_{\varepsilon,h} \leq C$ and

$$\|\boldsymbol{\sigma}_h - \boldsymbol{R}_h \boldsymbol{\sigma}\|_{\varepsilon,h} + \|u_h^* - u^*\|_{\varepsilon,h} \leq \mathcal{B}_{\varepsilon,h}^*(\boldsymbol{\sigma}_h - \boldsymbol{R}_h \boldsymbol{\sigma}, u_h^* - u^*; \boldsymbol{\tau}, v^*).$$

From the definition of the method (4.5), we then have

$$
\begin{aligned}
&\mathcal{B}^*_{\varepsilon,h}(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}, u_h^* - u^*; \boldsymbol{\tau}, v^*)\\
&\qquad = \mathcal{B}^*_{\varepsilon,h}(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, u - u^*; \boldsymbol{\tau}, v^*) + (f - P_h f, v^*)\\
&\qquad = a_\varepsilon(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u - u^*) + (\mathrm{div}\ (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}_h), v^*) + (f - P_h f, v^*)\\
&\qquad\qquad + \sum_{K\in\mathcal{K}_h} (\nabla(u - u^*) - (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), \nabla(I - P_h)v^*)_K.
\end{aligned}
$$

Due to the commuting diagram property (3.11) and the equation (2.2), div $\boldsymbol{\sigma} = -f$, there holds

$$
(\mathrm{div}\ (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), v^*) = (\mathrm{div}\ \boldsymbol{\sigma} - P_h \mathrm{div}\ \boldsymbol{\sigma}, v^*) = (-f + P_h f, v^*),
$$

so that

$$
\begin{aligned}
\mathcal{B}^*_{\varepsilon,h}(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma}, u_h^* - u^*; \boldsymbol{\tau}, v^*) &= a_\varepsilon(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u - u^*)\\
&\quad + \sum_{K\in\mathcal{K}_h} (\nabla(u - u^*) - (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), \nabla(I - P_h)v^*)_K.
\end{aligned}
$$

As in the proof of Proposition 3.6, we use (3.13) and get

$$
a_\varepsilon(\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) = (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}, \boldsymbol{\tau}) \le C\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0.
$$

Moreover, by integration by parts as in the continuity proof of Lemma 3.1, we have

$$
(\mathrm{div}\ \boldsymbol{\tau}, u - u^*) \le C|\!|\!|\boldsymbol{\tau}|\!|\!|_{\varepsilon,h}|\!|\!|u - u^*|\!|\!|_{\varepsilon,h} \le C|\!|\!|u - u^*|\!|\!|_{\varepsilon,h}.
$$

Furthermore, by Lemma 4.2 the last term can be bounded by

$$
\begin{aligned}
&\sum_{K\in\mathcal{K}_h} (\nabla(u - u^*) - (\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}), \nabla(I - P_h)v^*)_K\\
&\qquad\qquad \le C(\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0 + |\!|\!|u - u^*|\!|\!|_{\varepsilon,h})|\!|\!|v^*|\!|\!|_{\varepsilon,h}\\
&\qquad\qquad \le C(\|\boldsymbol{\sigma} - \boldsymbol{R}_h\boldsymbol{\sigma}\|_0 + |\!|\!|u - u^*|\!|\!|_{\varepsilon,h}).
\end{aligned}
$$

Since $v^* \in V_h^*$ was arbitrary, the first assertion is proved.

The error estimate is now an immediate consequence of the first bound, the triangle inequality and standard approximation properties. ☐

5. A-posteriori estimates.

We now derive a residual-based a-posteriori estimator for the postprocessed solution $(\boldsymbol{\sigma}_h, u_h^*)$. We point out that using the postprocessed solution is vital for obtaining an estimator whose local residual terms are properly matched with respect to their convergence properties [9].

5.1. Error estimator.

For an element K, we define the local error indicators

$$
\eta_{1,K}^2 = \|\nabla u_h^* - \boldsymbol{\sigma}_h\|_{0,K}^2, \qquad \eta_{2,K}^2 = h_K^2\|f - P_h f\|_{0,K}^2.
$$

For an interior edge (face) $E \in \mathcal{E}_h^0$, we introduce the jump indicator

$$
\eta_E^2 = h_E^{-1}\|[\![u_h^*]\!]\|_{0,E}^2.
$$

For a boundary edge (face) $E \in \mathcal{E}_h^\partial$, we define

$$
\eta_E^2 = \frac{1}{\varepsilon + h_E}\|\varepsilon(\boldsymbol{\sigma}_h \cdot \boldsymbol{n} - g_h) + u_h^* - u_0\|_{0,E}^2
$$

where g_h is the L^2-projection of g onto $P_{k-1}(E)$ for RT elements and onto $P_k(E)$ for BDM elements. To also take into account the approximation of g, we introduce the set

$$\mathcal{E}_{h,+}^{\partial} = \left\{ E \in \mathcal{E}_h^{\partial} \,|\, \varepsilon|_E > 0 \right\}. \tag{5.1}$$

of all boundary edges (faces) E with a non-vanishing $\varepsilon|_E$. For a boundary edge (face) $E \in \mathcal{E}_{h,+}^{\partial}$, we then introduce the indicator related to the approximation of g by setting

$$\eta_{g,E}^2 = h_E \|g - g_h\|_{0,E}^2.$$

Summing up these local indicators, our error estimator is given by

$$\eta = \left(\sum_{K \in \mathcal{K}_h} \left(\eta_{1,K}^2 + \eta_{2,K}^2\right) + \sum_{E \in \mathcal{E}_h^0} \eta_E^2 + \sum_{E \in \mathcal{E}_h^{\partial}} \eta_E^2 + \sum_{E \in \mathcal{E}_{h,+}^{\partial}} \eta_{g,E}^2 \right)^{\frac{1}{2}}. \tag{5.2}$$

REMARK 5.1. *Note that, for $\varepsilon = 0$, the indicator $\eta_{g,E}$ can be omitted in the definition of η. The resulting estimator then coincides with the ones derived in the papers [9, 13] for homogeneous and inhomogeneous Dirichlet boundary conditions, respectively.*

The efficiency of the indicators $\eta_{1,K}$ and η_E is given by the following lower bounds (note that the indicators $\eta_{2,K}$ and $\eta_{E,g}$ are data approximation terms). We denote by by Π_h the L^2-projection on the boundary edges, onto $P_{k-1}(E)$ for RT elements and onto $P_k(E)$ for BDM elements.

PROPOSITION 5.2. *There holds:*

$$\begin{aligned}
\eta_{1,K} &\leq \|\nabla(u - u_h^*)\|_{0,K} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,K}, && K \in \mathcal{K}_h,\\
\eta_E &\leq \frac{\|[\![u - u_h^*]\!]\|_{0,E}}{\sqrt{h_E}}, && E \in \mathcal{E}_h^0,\\
\eta_E &\leq \frac{\varepsilon\|(\boldsymbol{\sigma}_h - \boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}\|_{0,E} + \|u - u_h^*\|_{0,E}}{\sqrt{\varepsilon + h_E}} + \min\{R_1(u_0), R_2(\boldsymbol{\sigma}, g)\} && E \in \mathcal{E}_{h,+}^{\partial},\\
\eta_E &\leq \frac{\|u - u_h^*\|_{0,E}}{\sqrt{h_E}} && E \in \mathcal{E}_h^{\partial} \setminus \mathcal{E}_{h,+}^{\partial}.
\end{aligned}$$

Here the terms R_1 and R_2 are defined as

$$\begin{aligned}
R_1(u_0) &= \frac{\|u_0 - \Pi_h u_0\|_{0,E}}{\sqrt{\varepsilon + h_E}},\\
R_2(\boldsymbol{\sigma}, g) &= \sqrt{\varepsilon}\|(\sigma - \boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}\|_{0,E} + \sqrt{\varepsilon}\|g - g_h\|_{0,E}.
\end{aligned}$$

Proof. The first bound follows simply from the triangle inequality, the second from the fact that $[\![u]\!] = 0$ over $E \in \mathcal{E}_h^0$. To prove the third assertion, we note that, by the defining moments for $\boldsymbol{R}_h$, we have

$$\boldsymbol{R}_h\boldsymbol{\sigma}\cdot\boldsymbol{n} = \Pi_h(\boldsymbol{\sigma}\cdot\boldsymbol{n})$$

on all boundary edges $E \in \mathcal{E}_{h,+}^{\partial}$. Thus, applying the L^2-projection Π_h to the boundary conditions in (2.3) we see that

$$\varepsilon(\boldsymbol{R}_h\boldsymbol{\sigma}\cdot\boldsymbol{n} - g_h) = \Pi_h(u_0 - u).$$

Therefore, the function to be evaluated in the residual η_E can be written as

$$\begin{aligned}\varepsilon(\boldsymbol{\sigma}_h\cdot\boldsymbol{n}-g_h)+u_h^*-u_0 &= \varepsilon(\boldsymbol{\sigma}_h\cdot\boldsymbol{n}-g_h)+u_h^*-u_0-\varepsilon(\boldsymbol{R}_h\boldsymbol{\sigma}\cdot\boldsymbol{n}-g_h)+\Pi_h(u_0-u)\\ &= \varepsilon(\boldsymbol{\sigma}_h-\boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}-(u_0-\Pi_h u_0)-(u-u_h^*).\end{aligned}$$

Taking the L^2-norm and applying the triangle inequality yields the estimate with $R_1(u_0)$ for any edge $\eta_E\in\mathcal{E}_{h,+}^{\partial}$ with non-vanishing $\varepsilon|_E$. On the other hand, we can directly insert the exact boundary condition into the term to be estimated, yielding

$$\begin{aligned}\varepsilon(\boldsymbol{\sigma}_h\cdot\boldsymbol{n}-g_h)+u_h^*-u_0 &= \varepsilon(\boldsymbol{\sigma}_h\cdot\boldsymbol{n}-g_h)+u_h^*-\varepsilon(\boldsymbol{\sigma}\cdot\boldsymbol{n}-g)-u\\ &= \varepsilon(\boldsymbol{\sigma}_h-\boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}+(u_h^*-u)-\varepsilon(\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}+\varepsilon(g-g_h).\end{aligned}$$

Using the triangle inequality and the relation $\varepsilon/\sqrt{\varepsilon+h}\le\sqrt{\varepsilon}$ gives the third assertion with the term $R_2(\sigma,g)$. Finally, if $\varepsilon|_E=0$ for a boundary edge E, we have $u=u_0$ on E, which yields immediately the fourth estimate. ☐

REMARK 5.3. *In conjunction with our a-priori results, Proposition 5.2 indicates that all the indicators converge with at least the same order and are thus properly matched. In particular, due to Proposition 3.6 it holds*

$$\frac{\varepsilon\|(\boldsymbol{\sigma}_h-\boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}\|_{0,E}}{\sqrt{\varepsilon+h_E}}\le\sqrt{\varepsilon}\|(\boldsymbol{\sigma}_h-\boldsymbol{R}_h\boldsymbol{\sigma})\cdot\boldsymbol{n}\|_{0,E}\le C\|\boldsymbol{\sigma}-\boldsymbol{R}_h\boldsymbol{\sigma}\|_0$$

and thus the term converges independently of ε. Similarly, for the terms inside the minimum we have optimal convergence rate for R_1 for $h_E<\varepsilon$ and for R_2 for the case $h_E\ge\varepsilon$. Thus, the boundary estimator is fully ε-robust. This is confirmed numerically in Section 7 below.

5.2. Reliability. To derive an upper bound for the a-posteriori estimator η in (5.2), we denote by $(\widetilde{\boldsymbol{\sigma}},\widetilde{u})$ the solution of the perturbed problem where we replace g by g_h:

$$\mathcal{B}_\varepsilon(\widetilde{\boldsymbol{\sigma}},\widetilde{u};\boldsymbol{\tau},v)+(f,v)=\langle u_0+\varepsilon g_h,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial\Omega} \tag{5.3}$$

for all $(\boldsymbol{\tau},v)\in H(\mathrm{div},\Omega)\times L^2(\Omega)$. Since

$$\langle g_h,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_E=\langle g,\boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_E\qquad\forall\boldsymbol{\tau}\in\boldsymbol{S}_h,\ E\in\mathcal{E}_h^{\partial},$$

it is clear that the finite element approximations $(\boldsymbol{\sigma}_h,u_h^*)$ are in fact also approximations to $(\widetilde{\boldsymbol{\sigma}},\widetilde{u})$.

We will make use of the following saturation assumption [8, 9]: Let $\mathcal{K}_{h/2}$ be a uniformly refined subtriangulation of $\mathcal{K}_h$, obtained by dividing each simplex $K\in\mathcal{K}_h$ into 2^n elements. We denote by $\boldsymbol{\sigma}_{h/2}$ and $u_{h/2}^*$ the flux and postprocessed displacement obtained on the finer mesh $\mathcal{K}_{h/2}$. The saturation assumption can now be formulated as follows.

ASSUMPTION 5.4 (Saturation assumption). *There exists a constant $\beta<1$ such that*

$$\|\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_{h/2}\|_0+|\!|\!|\widetilde{u}-u_{h/2}^*|\!|\!|_{\varepsilon,h/2}\le\beta\left(\|\widetilde{\boldsymbol{\sigma}}-\boldsymbol{\sigma}_h\|_0+|\!|\!|\widetilde{u}-u_h^*|\!|\!|_{\varepsilon,h}\right).$$

The following result establishes the reliability of the estimator η.

THEOREM 5.5. *Suppose that Assumption 5.4 holds. Then there exists a constant $C > 0$ such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\!|u - u_h^*|\!\|_{\varepsilon,h} \leq C\eta. \tag{5.4}$$

Proof. We proceed in several steps.

Step 1: Let $(\boldsymbol{\sigma}, u)$ and $(\widetilde{\boldsymbol{\sigma}}, \widetilde{u})$ denote the solutions of (2.10) and (5.3), respectively. The difference $u - \widetilde{u}$ in the displacement then satisfies the second-order equation:

$$\begin{aligned} -\Delta(u - \widetilde{u}) &= 0 && \text{in } \Omega, \\ \varepsilon\nabla(u - \widetilde{u}) \cdot \boldsymbol{n} + (u - \widetilde{u}) &= \varepsilon(g - g_h) && \text{on } \partial\Omega. \end{aligned}$$

Multiplying this equation by $u - \widetilde{u}$ and integrating by parts the left-hand side, we obtain

$$\|\nabla(u - \widetilde{u})\|_0^2 - \sum_{E \in \mathcal{E}_{h,+}^{\partial}} \langle \nabla(u - \widetilde{u}) \cdot \boldsymbol{n}, u - \widetilde{u} \rangle_E = 0.$$

Using the boundary condition, we conclude that

$$\|\nabla(u - \widetilde{u})\|_0^2 + \sum_{E \in \mathcal{E}_{h,+}^{\partial}} \frac{1}{\varepsilon} \|u - \widetilde{u}\|_{0,E}^2 = \sum_{E \in \mathcal{E}_{h,+}^{\partial}} \langle g - g_h, u - \widetilde{u} \rangle_E.$$

Let P_0 be the L^2-projection onto the piecewise constants. For any edge $E \in \mathcal{E}_{h,+}^{\partial}$ with $E \subset \partial K$, we now use the definition of g_h and standard approximation results to get

$$\langle g - g_h, u - \widetilde{u} \rangle_E = \langle g - g_h, u - \widetilde{u} - P_0(u - \widetilde{u}) \rangle_E \leq C h_E^{\frac{1}{2}} \|g - g_h\|_{0,E} \|\nabla(u - \widetilde{u})\|_{0,K}.$$

We thus readily obtain

$$\|\nabla(u - \widetilde{u})\|_0^2 + \sum_{E \in \mathcal{E}_{h,+}^{\partial}} \frac{1}{\varepsilon} \|u - \widetilde{u}\|_{0,E}^2 \leq C \sum_{E \in \mathcal{E}_{h,+}^{\partial}} \eta_{g,E}^2.$$

The definition of the norm $\|\!|\cdot|\!\|_{\varepsilon,h}$, the inequality $(\varepsilon|_E + h_E)^{-1} \leq \varepsilon|_E^{-1}$ for all $E \in \mathcal{E}_{h,+}^{\partial}$, and the fact that $u - \widetilde{u}|_E = 0$ on all edges (faces) with $\varepsilon|_E = 0$ yield

$$\|\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}\|_0 + \|\!|u - \widetilde{u}|\!\|_{\varepsilon,h} \leq C\eta. \tag{5.5}$$

Step 2: From the triangle inequality and the bound (5.5), we obtain

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + \|\!|u - u_h^*|\!\|_{\varepsilon,h} \leq C\eta + \|\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h\|_0 + \|\!|\widetilde{u} - u_h^*|\!\|_{\varepsilon,h}.$$

It is thus sufficient to bound the error of the finite element approximation $(\boldsymbol{\sigma}_h, u_h^*)$ to the perturbed solution $(\widetilde{\boldsymbol{\sigma}}, \widetilde{u})$ in (5.3). From Assumption 5.4, we conclude that

$$\|\widetilde{\boldsymbol{\sigma}} - \boldsymbol{\sigma}_h\|_0 + \|\!|\widetilde{u} - u_h^*|\!\|_{\varepsilon,h} \leq \frac{1}{1-\beta} \left(\|\boldsymbol{\sigma}_{h/2} - \boldsymbol{\sigma}_h\|_0 + \|\!|u_{h/2}^* - u_h^*|\!\|_{\varepsilon,h/2} \right).$$

Thus, it remains to prove that there is a constant $C > 0$ such that

$$\left(\|\boldsymbol{\sigma}_{h/2} - \boldsymbol{\sigma}_h\|_0 + \|\!|u_{h/2}^* - u_h^*|\!\|_{\varepsilon,h/2} \right) \leq C\,\eta. \tag{5.6}$$

Step 3: We show (5.6). To that end, we employ the inf-sup condition in Proposition 4.3 over the finer spaces and conclude that there is $(\boldsymbol{\tau}, v^*) \in \boldsymbol{S}_{h/2} \times Q_{h/2}$ such that

$$\|\boldsymbol{\tau}\|_{\varepsilon,h/2} + \|v^*\|_{\varepsilon,h/2} \leq C \tag{5.7}$$

and

$$C(\|\boldsymbol{\sigma}_{h/2} - \boldsymbol{\sigma}_h\|_0 + \|u^*_{h/2} - u^*_h\|_{\varepsilon,h/2}) \leq \mathcal{B}^*_{\varepsilon,h/2}(\boldsymbol{\sigma}_{h/2} - \boldsymbol{\sigma}_h, u^*_{h/2} - u^*_h; \boldsymbol{\tau}, v^*).$$

From linearity and the definition of the postprocessed method, we obtain

$$\begin{aligned}
&\mathcal{B}^*_{\varepsilon,h/2}(\boldsymbol{\sigma}_{h/2} - \boldsymbol{\sigma}_h, u^*_{h/2} - u^*_h; \boldsymbol{\tau}, v^*) \\
&\quad = -(P_{h/2}f, v^*) + \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} - \mathcal{B}^*_{\varepsilon,h/2}(\boldsymbol{\sigma}_h, u^*_h; \boldsymbol{\tau}, v^*) \\
&\quad = -(P_{h/2}f, v^\star) + \langle u_0 + \varepsilon g, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} \\
&\qquad - (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - \langle \varepsilon \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} - (\text{div } \boldsymbol{\tau}, u^*_h) - (\text{div } \boldsymbol{\sigma}_h, v^*) \\
&\qquad - \sum_{K \in \mathcal{K}_{h/2}} (\nabla u^*_h - \boldsymbol{\sigma}_h, \nabla(I - P_{h/2})v^*)_K.
\end{aligned}$$

To simplify this identity, we use that div $\boldsymbol{\sigma}_h = -P_h f$, see (2.15). Moreover, we integrate by parts the term (div $\boldsymbol{\tau}, u^*_h$) over the elements $K \in \mathcal{K}_h$:

$$-(\text{div } \boldsymbol{\tau}, u^*_h) = \sum_{K \in \mathcal{K}_h} \Big((\nabla u^*_h, \boldsymbol{\tau})_K - \langle \boldsymbol{\tau} \cdot \boldsymbol{n}_{\partial K}, u^*_h \rangle_{\partial K} \Big).$$

Rearranging the terms, we conclude that

$$C(\|\boldsymbol{\sigma}_{h/2} - \boldsymbol{\sigma}_h\|_0 + \|u^*_{h/2} - u^*_h\|_{\varepsilon,h/2}) \leq T_1 + T_2 + T_3 + T_4 + T_5, \tag{5.8}$$

where

$$\begin{aligned}
T_1 &= -\sum_{K \in \mathcal{K}_h} (\boldsymbol{\sigma}_h - \nabla u^*_h, \boldsymbol{\tau}), \\
T_2 &= -\langle \varepsilon(\boldsymbol{\sigma}_h \cdot \boldsymbol{n} - g_h) + u^*_h - u_0, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega}, \\
T_3 &= -(P_{h/2}f - P_h f, v^*), \\
T_4 &= -\sum_{K \in \mathcal{K}_h} \langle \boldsymbol{\tau} \cdot \boldsymbol{n}_K, u^*_h \rangle_{\partial K \setminus \partial\Omega}, \\
T_5 &= -\sum_{K \in \mathcal{K}_{h/2}} (\nabla u^*_h - \boldsymbol{\sigma}_h, \nabla(I - P_{h/2})v^*)_K.
\end{aligned}$$

By (5.7), the term T_1 can be bounded by

$$T_1 \leq \big(\sum_{K \in \mathcal{K}_h} \eta^2_{1,K} \big)^{\frac{1}{2}} \|\boldsymbol{\tau}\|_0 \leq C\eta.$$

To bound T_2, we use the Cauchy-Schwarz inequality and the fact that ε is piecewise constant. We obtain

$$\begin{aligned}
T_2 &\leq \big(\sum_{E \in \mathcal{E}^\partial_h} \eta^2_E \big)^{\frac{1}{2}} \big(\sum_{E \in \mathcal{E}^\partial_h} (\varepsilon + h_E) \|\boldsymbol{\tau} \cdot \boldsymbol{n}\|^2_{0,E} \big)^{\frac{1}{2}} \\
&\leq C\eta \big(\sum_{E \in \mathcal{E}^\partial_{h/2}} (\varepsilon + h_E) \|\boldsymbol{\tau} \cdot \boldsymbol{n}\|^2_{0,E} \big)^{\frac{1}{2}} \leq C\eta.
\end{aligned}$$

To estimate T_3, we use exactly the same arguments as in equations (3.19)–(3.21) of [9] to get

$$T_3 \le C \sum_{K\in\mathcal{K}_h} \big(h_K^2 \|f - P_h f\|_{0,K}^2\big)^{\frac{1}{2}} \le C\eta.$$

The term T_4 can be rewritten as

$$T_4 = \sum_{E\in\mathcal{E}_h^0} \langle \boldsymbol{\tau}\cdot\boldsymbol{n}, [\![u_h^*]\!]\rangle_E.$$

Using the Cauchy-Schwarz inequality and the polynomial trace inequality (3.2) over the finer mesh $\mathcal{K}_{h/2}$, it can then be bounded by

$$\begin{aligned} T_4 &\le \Big(\sum_{E\in\mathcal{E}_h^0} h_E\|\boldsymbol{\tau}\|_{0,E}^2\Big)^{\frac{1}{2}}\Big(\sum_{E\in\mathcal{E}_h^0} h_E^{-1}\|[\![u_h^*]\!]\|_{0,E}^2\Big)^{\frac{1}{2}} \\ &\le C\eta\Big(\sum_{E\in\mathcal{E}_{h/2}^0} h_E\|\boldsymbol{\tau}\|_{0,E}^2\Big)^{\frac{1}{2}} \\ &\le C\eta\|\boldsymbol{\tau}\|_0 \le C\eta. \end{aligned}$$

Finally, due to (5.7) and (4.7), we get

$$T_5 \le \Big(\sum_{K\in\mathcal{K}_h} \eta_{1,K}^2\Big)^{\frac{1}{2}}\Big(\sum_{K\in\mathcal{K}_{h/2}} \|\nabla(I - P_{h/2})v^*\|_{0,K}^2\Big)^{\frac{1}{2}} \tag{5.9}$$

$$\le C\eta|\!|\!|v^*|\!|\!|_{\varepsilon,h/2} \le C\eta. \tag{5.10}$$

Referring to (5.6), (5.8) and the above bounds for T_1 through T_5 completes the proof. ◻

6. A remark on hybridization. It is a well-known procedure for mixed finite elements to simplify the solution of the algebraic system resulting from equations (2.6) –(2.7) by introducing a Lagrange multiplier to enforce the normal continuity of the flux variable $\boldsymbol{\sigma}_h$ [1, 5]. We introduce the multiplier spaces

$$M_h^{BDM} = \{\, m\in L^2(E)\,|\, m\in P_k(E),\ E\in\mathcal{E}_h^0,\quad m|_E = 0,\ E\in\mathcal{E}_h^\partial\,\}. \tag{6.1}$$

$$M_h^{RT} = \{\, m\in L^2(E)\,|\, m\in P_{k-1}(E),\ E\in\mathcal{E}_h^0,\quad m|_E = 0,\ E\in\mathcal{E}_h^\partial\,\}. \tag{6.2}$$

Now it can be easily shown, that the normal continuity of the flux $\boldsymbol{\sigma}_h$ is equivalent to the requirement

$$\sum_{K\in\mathcal{K}_h} \langle \boldsymbol{\sigma}_h\cdot\boldsymbol{n}, p\rangle_{\partial K} = 0, \quad \forall p\in M_h. \tag{6.3}$$

Thus, the original finite element problem (2.14) can be formulated as follows [6]: find $(\boldsymbol{\sigma}_h, u_h, m_h)\in \boldsymbol{S}_h\times V_h\times M_h$ such that

$$a_\varepsilon(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + (\mathrm{div}\ \boldsymbol{\tau}, u_h) + \sum_{K\in\mathcal{K}_h}\langle\boldsymbol{\tau}\cdot\boldsymbol{n}, m_h\rangle_{\partial K} = \langle u_0 + \varepsilon g, \boldsymbol{\tau}\cdot\boldsymbol{n}\rangle_{\partial\Omega}, \tag{6.4}$$

$$(\mathrm{div}\ \boldsymbol{\sigma}_h, v) + (f, v) = 0, \tag{6.5}$$

$$\sum_{K\in\mathcal{K}_h}\langle\boldsymbol{\sigma}_h\cdot\boldsymbol{n}, p\rangle_{\partial K} = 0, \tag{6.6}$$

for all $(\boldsymbol{\tau}, v, p) \in \boldsymbol{S}_h \times V_h \times M_h$. Due to the property (6.3), the pair $(\boldsymbol{\sigma}_h, u_h)$ retrieved from equations (6.4)–(6.6) coincides with the solution of the original discrete problem (2.14). Thus we can apply the same postprocessing scheme as proposed earlier, even if we use hybridization to solve the initial system. The corresponding algebraic system is of the form

$$\begin{aligned} (A + \varepsilon\tilde{A})\sigma + Bu + Cm &= \varepsilon g + u_0 \\ B^T\sigma &= -f \\ C^T\sigma &= 0, \end{aligned}$$

where $(\boldsymbol{\sigma}, u, m)$ are the coefficient vectors of the solution. We can solve for $\boldsymbol{\sigma}$ easily using element-by-element inversion of the block diagonal matrix $A+\varepsilon\tilde{A}$. On elements in the interior of the domain, we have

$$\sigma = A^{-1}(-Bu - Cm), \tag{6.7}$$

and on elements with at least one edge on the boundary

$$\sigma = (A + \varepsilon\tilde{A})^{-1}(\varepsilon g + u_0 - Bu - Cm) \tag{6.8}$$

Thus, we only get an ε-dependent problem on the elements touching the boundary of the domain. Furthermore, the matrix $\tilde{A}$ corresponding to the part $\langle \boldsymbol{\sigma}\cdot \boldsymbol{n}, \boldsymbol{\tau}\cdot \boldsymbol{n}\rangle_E$ has nonzero components only corresponding to the boundary degrees of freedom. Since these degrees of freedom do not couple to the interelement Lagrange multipliers, one has no ε-dependence in the condition number of the final linear system for the variables (u, m). In addition, the resulting linear system for (u, m) will be symmetric and positive definite.

7. Numerical results. In this section, we present a series of numerical tests. The main focus is on showing that the proposed mixed finite element method is ε-robust, both in the sense of the a-priori estimates in Theorem 4.4 and the a-posteriori estimates in Theorem 5.5. We use two test cases, the first with a smooth solution and the second with a singular one. For the constant in estimate of Theorem 5.5 we choose $C = 1$. Furthermore, the data approximation terms are neglected, thus the estimator is smaller than the actual error in all of the results presented. This is particularly clearly visible in the test case with non-smooth boundary data.

7.1. Smooth solution. In the first test case, we use a smooth solution to retrieve the convergence rates predicted by our theoretical results. We consider the rectangular domain $\Omega = (0, 1)^2$ and choose the load in problem (2.1)–(2.2) so that the displacement $u(x, y)$ is given by the smooth function

$$u(x, y) = -\sin(x)\sinh(y) + C, \tag{7.1}$$

and the flux by $\boldsymbol{\sigma} = \nabla u$. The constant C is chosen so as to ensure a zero-mean displacement, i.e., we take $C = -(\cos(1) - 1)(\cosh(1) - 1)$. The boundary data are computed from u and $\boldsymbol{\sigma}$ by setting $u_0 = u$ and $g = \boldsymbol{\sigma}\cdot \boldsymbol{n}$. We then enforce the Robin boundary conditions (2.3) for several values of ε. We test the proposed mixed method both for first-order ($k = 1$) and second-order ($k = 2$) BDM elements. We use uniformly refined triangular meshes of mesh size $h \propto 1/N^2$, N being the number of degrees of freedom of the discretization.

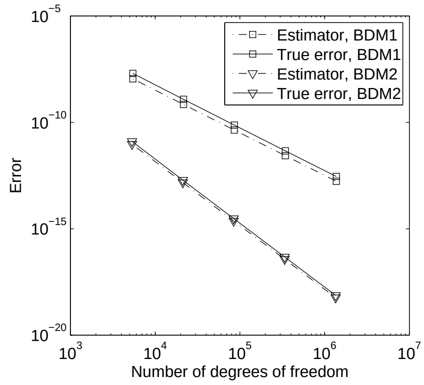

FIG. 7.1. *Convergence of the smooth solution with $\varepsilon = 0$.*

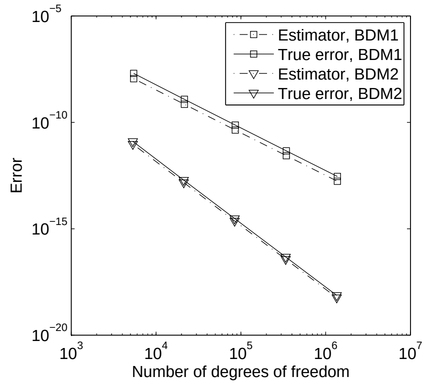

FIG. 7.3. *Convergence of the smooth solution with $\varepsilon = 10^4$.*

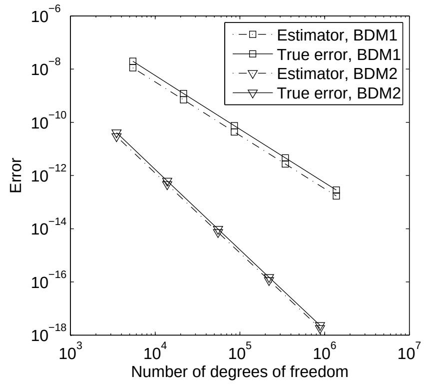

FIG. 7.2. *Convergence of the smooth solution with $\varepsilon = 10^{-2}$.*

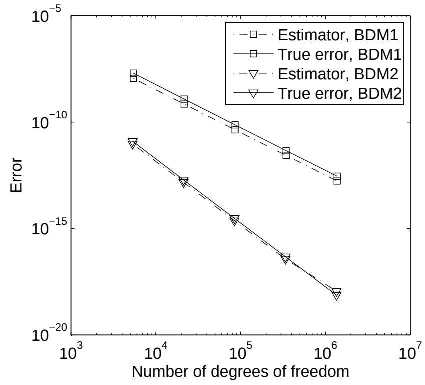

FIG. 7.4. *Convergence of the smooth solution with $\varepsilon = 10^{12}$.*

In Figures 7.1 through 7.4, we plot the errors $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_0 + |||u - u_h^*|||_{\varepsilon,h}$ and the values of the a-posteriori estimator η in (5.2) with respect to the number of degrees of freedom N for $\varepsilon = 0$, 10^{-2}, 10^4, 10^{12}, respectively. The slopes in the logarithmic scale in the figures are half of the actual convergence rates. In all the curves, we see convergence of order $k+1$ in the mesh size, in agreement with Theorem 4.4. Moreover, the curves clearly confirm the reliability and efficiency of the estimator η; see Theorem 5.5 and Proposition 5.2. Notably, we also get optimal order of convergence for the estimator η in the test case with $\varepsilon = 10^{-2}$, where the situation $\varepsilon = h$ is encountered.

Evidently, the convergence is completely independent of the value of ε, and the magnitude of the errors does not depend on ε either. In particular, the performance of the a posteriori estimator truly is ε-independent.

7.2. Singular solution. In the second example, we consider again the domain $(0,1)^2$. In polar coordinates (r,Θ) about the origin, the displacement is chosen to be

$$u(r,\Theta) = r^\beta \sin(\beta\Theta) + C, \tag{7.2}$$

and the flux is $\boldsymbol{\sigma} = \nabla u$. Here, the parameter β defines the exact regularity of u and can be used to model singular behavior at the origin. The constant C is once again defined such that u will have zero mean value. For this solution, we have $u \in H^{1+\beta}(\Omega)$ and subsequently $\boldsymbol{\sigma} \in [H^{\beta}(\Omega)]^n$; see [7]. The boundary data are computed from u and $\boldsymbol{\sigma}$ as before.

In the following test, we set $\beta = 0.67$, corresponding to a highly singular displacement. We use only the lowest-order ($k = 1$) BDM elements, since raising the polynomial degree would give no advantage due to the insufficient regularity of the solution. We compute the solutions on adaptively refined meshes for various values of ε. To create the mesh sequences, we have chosen to refine all elements in which the elemental indicator exceeds 30 percent of the maximal value. Contribution from an individual edge estimators is divided evenly to the elements sharing the edge or face. To ensure sufficient refinement, we further halve the refinement threshold until at least 10 percent of all elements are refined. For comparison, the computations are also performed using uniform mesh refinement with an equivalent number of degrees of freedom. In Figures 7.5 through 7.8, we show the errors $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_0+\|u-u_h^*\|_{\varepsilon,h}$ and the values of the estimator η obtained for this problem with the values $\varepsilon = 0, 1, 10^4, 10^8$.

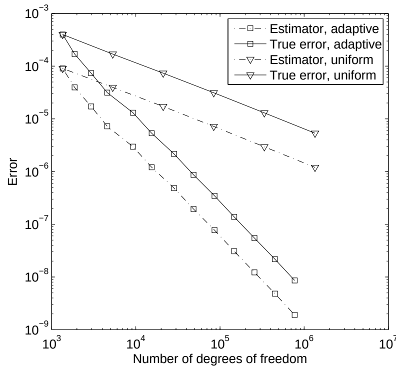

Fig. 7.5. *Convergence of the singular solution with $\varepsilon = 0$.*

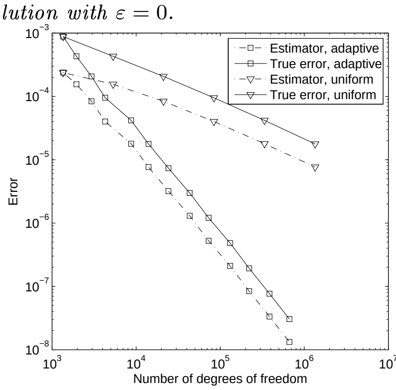

Fig. 7.6. *Convergence of the singular solution with $\varepsilon = 1$.*

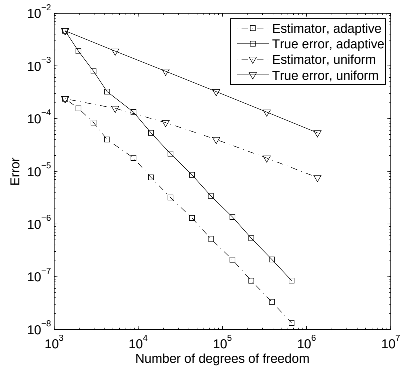

Fig. 7.7. *Convergence of the singular solution with $\varepsilon = 10^4$.*

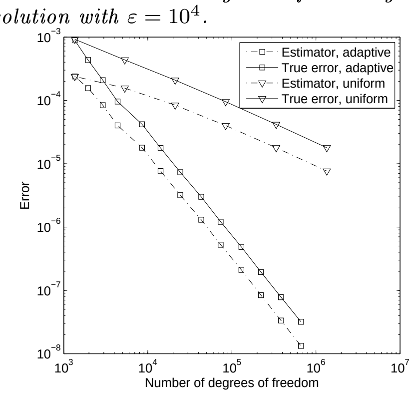

Fig. 7.8. *Convergence of the singular solution with $\varepsilon = 10^8$.*

In the case of uniform mesh refinement, the convergence rates are now limited by the regularity of the solution. They are of the order h^{β} as expected. On the other hand, the adaptive mesh refinement strategy clearly is able to retrieve the convergence to some extent. More importantly, even for the irregular boundary data considered here, both the a-priori and a-posteriori estimates are observed to remain fully ε-independent, thus confirming the theoretical results.

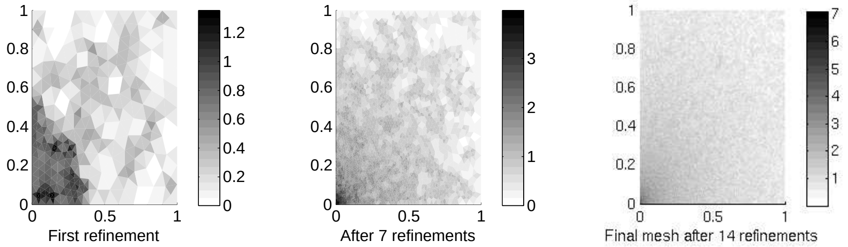

FIG. 7.9. *Mesh density after one step, in the middle of the refinement procedure, and on the final mesh with* $\varepsilon = 0$.

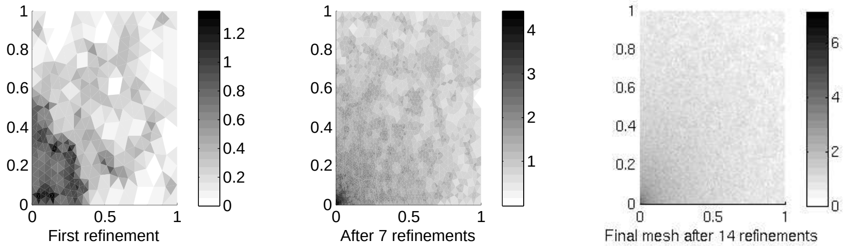

FIG. 7.10. *Mesh density after one step, in the middle of the refinement procedure, and on the final mesh with* $\varepsilon = 1$.

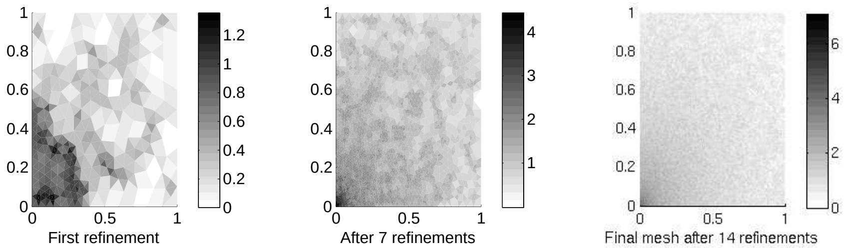

FIG. 7.11. *Mesh density after one step, in the middle of the refinement procedure, and on the final mesh with* $\varepsilon = 10^4$.

The singularity of the solution in (7.2) lies in the origin and we expect the adaptive meshes to be strongly refined into this corner. To ascertain the ε-independence of the adaptive refinement procedure, we plot the mesh densities for different values of the parameter ε at different stages of the refinement procedure. The plots are normalized with respect to the largest element size such that the largest element size equals unity, and are shown using a logarithmic scale.

From Figures 7.9 through 7.11, we see that our adaptive mesh refinement procedure clearly yields equivalent mesh density distributions for different values of ε. This is what is desired for this problem, since the refinement should only be driven by the irregularity of the solution. These numerical tests indicate that the residual-based estimators introduced in this paper are indeed appropriate for controlling adaptive refinement.

8. Conclusions. In this paper, we have analyzed mixed finite element methods for the dual mixed form of the Poisson problem with general Robin-type boundary conditions. We have extended a well-known postprocessing technique to this case. As a consequence, we obtain optimal error estimates where the convergence rates for the broken H^1-errors in the postprocessed displacements correctly match the ones for the L^2-errors in the fluxes. Furthermore, the postprocessing allows us to design a properly functioning residual-based error indicator. A key feature of our analysis is that both the a-priori and a-posteriori error estimates are fully ε-robust. The numerical results verify our theoretical results. They show the optimality and sharpness of our estimates and confirm their robustness. They also show that the error indicator can be employed to drive adaptive refinement for all values of ε.

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