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Sergey Korotov

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

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Abstract: *In this note we propose a nonstandard technique for constructing global a posteriori error estimates for the stationary convection-reaction-diffusion equation. In order to estimate the approximation error in appropriate weighted energy norms, which measures the overall quality of the approximations, the underlying bilinear form is decomposed into several terms which can be directly computed or easily estimated from above using elementary tools of functional analysis. Several auxiliary parameters are introduced to construct such a splitting and tune the resulting upper error bound. It is demonstrated how these parameters can be chosen in some natural and convenient for computations way so that the weighted energy norm of the error is almost recovered, which shows that the estimates proposed are, in fact, quasi-sharp. The presented methodology is completely independent of numerical techniques used to compute approximate solutions. In particular, it is applicable to approximations which fail to satisfy the Galerkin orthogonality, e.g., due to an inconsistent stabilization, flux limiting, low-order quadrature rules, round-off and iteration errors etc. Moreover, the only constant that appears in the proposed error estimates is of global nature and comes from the Friedrichs-Poincaré inequality.*

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Correspondence

sergey.korotov@hut.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

Mathematical models (and their numerical solution) consisting of convection-reaction-diffusion equations with various boundary conditions present a very important class of real-life problems [12, 16, 19]. The error incurred in the course of discretization and iterative solution of such problems is responsible for the defference between the computational results and the exact solutions of the models. A posteriori error estimates quantify this difference (also called the error) and constitute a modern reliable tool for constructing effective adaptive mesh refinement procedures aimed at reducing the error if necessary according to main computational goals. Currently, reliable error techniques are available, e.g., for finite element approximations of various elliptic problems (see [1, 3, 4, 5, 13, 21, 24, 25] and references therein). However, the derivation of reliable error estimates for convection-diffusion equations still represents a challenging open problem, although there exist a number of works on this topic written during the past two decades [9, 11, 26, 27].

A crucial limitation of many a posteriori error estimation techniques is the presence of a large set of so-called interpolation constants (see, e.g., [27]) which are extremely difficult to compute (or even to estimate, cf. [7] for a simpler elliptic problems), especially in the case of complex domains and unstructured computational meshes. The uncertainty involved in the computation of these constants may seriously reduce the actual reliability of the resulting estimates. Moreover, some popular methods rely on the existence of an equivalent minimization problem or assume the Galerkin orthogonality. For the residual to be orthogonal to the space of test functions, the discretization must be performed by a consistent (Petrov-)Galerkin method and the resulting algebraic equations must be solved exactly. These requirements are never satisfied in practice because of numerical quadrature, various round-off errors, slack tolerances for iterative solvers, programming bugs etc. The use of upwinding [2] or flux/slope limiters [17] in finite element codes may also violate the Galerkin orthogonality.

A very promising approach to error estimation was developed by S. Repin and his coauthors [21, 22, 23] during the last 10 years in the context of diffusion-type problems (and some other types of problems). Remarkably, it is applicable to any conforming approximation regardless of the numerical method used to compute it. The original derivation is based on rather sophisticated tools of functional analysis (duality theory, Helmholtz decomposition) but a simplified version was recently proposed and succesfully employed for reaction-diffusion problems in [10, 14].

The approach from [10, 14] with several modifications was further applied to convection-diffusion equations for the first time in the very recent work [15], where obtained estimates involve two free parameters for tuning the resulting bounds. However, the estimates from [15] may involve, in practice, a good resolution of the so-called adjoint problem, which can be sometimes unnecessary as this problem is of auxiliary nature in many situations.

In the present paper we describe another technique for constructing upper error estimtes for the same class of problems as in [15], which helps to get the upper bounds for the error which are valid for any admissible approximations without using any adjoint problems and which, in addition, hold in various global norms. The resulting upper bounds are shown be quasi-sharp in the sense that they almost reduce to the true error if the involved parameters are chosen in some quite natural and convenient for practical calculations way. Moreover, as in [15], there is just one global constant (due to the Friedrichs-Poincaré inequality) involved into the estimates, which depends on the geometry of the computational domain only and does not change during mesh adaptation.

2 Model problem

Consider the stationary convection-reaction-diffusion problem

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b}\cdot\nabla u + cu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 1$, is a bounded domain with a Lipschitz continuous boundary $\partial\Omega$. The constant diffusion coefficient ε, the velocity field $\mathbf{b}$, and the reaction rate c are supposed to satisfy the following conditions

$$\varepsilon > 0, \quad \mathbf{b} \in W^1_\infty(\Omega;\mathbb{R}^d), \quad c \in L_\infty(\Omega). \tag{2}$$

The weak formulation of the above problem reads: Find $u \in H^1_0(\Omega)$ such that

$$a(u,w) = F(w) \qquad \forall w \in H^1_0(\Omega), \tag{3}$$

where the bilinear form $a(\cdot,\cdot)$ and the linear functional $F(\cdot)$ are given by

$$a(v,w) = \int_\Omega \varepsilon\nabla v\cdot\nabla w\,dx + \int_\Omega \mathbf{b}\cdot\nabla v\, w\;dx + \int_\Omega cvw\;dx, \tag{4}$$

$$F(w) = \int_\Omega fw\;dx, \qquad u,w \in H^1_0(\Omega). \tag{5}$$

It is well known that the weak solution $u \in H^1_0(\Omega)$ of variational problem (3) exists and is unique provided that the following extra condition holds

$$\tilde{c}(x) := c(x) - \frac{1}{2}\nabla\cdot\mathbf{b}(x) \geq 0 \tag{6}$$

almost everywhere in Ω. Indeed, under this constraint, the bilinear form $a(\cdot,\cdot)$ is coercive

$$\begin{aligned} a(w,w) &= \int_\Omega \varepsilon\nabla w\cdot\nabla w\,dx + \int_\Omega (\mathbf{b}\cdot\nabla w)\,w\,dx + \int_\Omega cw^2\,dx \\ &= \varepsilon\int_\Omega |\nabla w|^2\,dx + \int_\Omega \left(c - \frac{1}{2}\nabla\cdot\mathbf{b}\right)w^2\,dx \geq C\|w\|^2_{1,\Omega}, \end{aligned} \tag{7}$$

where C is a positive constant and $\|\cdot\|_{1,\Omega}$ denotes the standard norm in $H^1(\Omega)$. And the coercivity of the bilinear form $a(\cdot,\cdot)$ implies the unique solvability of problem (3) due to the Lax-Milgram lemma (see, e.g., [8]).

Remark 1: Here, we mention that we have no special restrictions on the coefficients of the problem as done, e.g., in [26, 27] (condition (A3)).

3 Error estimation technique

Let $\bar{u}$ be some function from $H^1_0(\Omega)$ considered as an approximation of u. In our work, we do not specify how $\bar{u}$ has been computed, it is just an arbitrary function from an admissible class.

The natural global measure of the error $e := u - \bar{u}$ is the value $\|e\|_{1,\Omega}$ since both, the exact solution and the approximation belong to the same space $H^1_0(\Omega)$. However, in what follows, it is more convenient to analyse the estimation of the error in terms of suitable global weighted energy norm defined as follows

$$|||e|||^2_{\lambda,\mu,\Omega} := \lambda \int\limits_{\Omega} |\nabla e|^2 \; dx + \mu \int\limits_{\Omega} \tilde{c}\, e^2 \; dx, \tag{8}$$

where the weights λ and μ are nonnegative real numbers (will be defined later), such that $\lambda + \mu > 0$. It is clear that $|||e|||_{\lambda,\mu,\Omega}$ is equivalent to $\|\cdot\|_{1,\Omega}$ under the above conditions on λ and μ.

In particular, we also have

$$a(e,e) = |||e|||^2_{\varepsilon,1,\Omega}, \tag{9}$$

which is the reason for calling error measures defined in (8) weighted energy norms.

In order to construct a posteriori estimates for the error in certain weighted energy norms of type (8), we first observe that

$$\begin{aligned} a(e,e) &= a(u-\bar{u}, u-\bar{u}) = \\ = \varepsilon \int\limits_{\Omega} \nabla(u-\bar{u})\cdot\nabla(u-\bar{u}) \; dx &+ \int\limits_{\Omega} \mathbf{b}\cdot\nabla(u-\bar{u})(u-\bar{u}) \; dx + \int\limits_{\Omega} c(u-\bar{u})\,(u-\bar{u}) \; dx = \\ = \int\limits_{\Omega} f(u-\bar{u}) \; dx - \varepsilon \int\limits_{\Omega} \nabla\bar{u}\cdot\nabla(u-\bar{u}) \; dx &- \int\limits_{\Omega} \mathbf{b}\cdot\nabla\bar{u}\,(u-\bar{u}) \; dx - \int\limits_{\Omega} c\,\bar{u}\,(u-\bar{u}) \; dx, \end{aligned} \tag{10}$$

where the integral identity (3) with $w = u - \bar{u}$ has been used.

Further, we regroup terms in (10) and introduce a parametric vector-function $\mathbf{y}^\star \in H(\mathrm{div},\Omega)$ so that

$$a(u-\bar{u}, u-\bar{u}) =$$

$$= \int\limits_{\Omega} (f - \mathbf{b} \cdot \nabla \bar{u} - c\,\bar{u})(u - \bar{u})\;dx - \int\limits_{\Omega} (\varepsilon \nabla \bar{u} - \mathbf{y}^\star + \mathbf{y}^\star) \cdot \nabla (u - \bar{u})\;dx = \quad (11)$$

$$= \int\limits_{\Omega} (f - \mathbf{b} \cdot \nabla \bar{u} - c\,\bar{u})(u - \bar{u})\;dx - \int\limits_{\Omega} (\varepsilon \nabla \bar{u} - \mathbf{y}^\star) \cdot \nabla (u - \bar{u})\;dx - \int\limits_{\Omega} \mathbf{y}^\star \cdot \nabla (u - \bar{u})\;dx.$$

Using the Green formula for the last term in the right-hand side of (11), we finally get the following decomposition for the error in the standard energy norm

$$a(u - \bar{u}, u - \bar{u}) =$$

$$= \int\limits_{\Omega} (f - \mathbf{b} \cdot \nabla \bar{u} - c\,\bar{u} + \nabla \cdot \mathbf{y}^\star)(u - \bar{u})\;dx + \int\limits_{\Omega} (\mathbf{y}^\star - \varepsilon \nabla \bar{u}) \cdot \nabla (u - \bar{u})\;dx. \quad (12)$$

Now, we introduce another parameter, now being an arbitrary function v from the space $H_0^1(\Omega)$. Then we can represent (12) further as follows

$$a(u - \bar{u}, u - \bar{u}) =$$

$$= \int\limits_{\Omega} (f - \mathbf{b} \cdot \nabla \bar{u} - c\,\bar{u} + \nabla \cdot \mathbf{y}^\star - cv - \mathbf{b} \cdot \nabla v)(u - \bar{u})\;dx + \quad (13)$$

$$+ \int\limits_{\Omega} (\mathbf{y}^\star - \varepsilon \nabla \bar{u} + \varepsilon \nabla v) \cdot \nabla (u - \bar{u})\;dx +$$

$$+ \int\limits_{\Omega} \Big((c\,v + \mathbf{b} \cdot \nabla v)(u - \bar{u}) - \varepsilon \nabla v \cdot \nabla (u - \bar{u}) \Big)\;dx = \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3.$$

Thus, our further goal is to effectively estimate from above the foregoing three terms $\mathbf{T}_1$, $\mathbf{T}_2$, and $\mathbf{T}_3$.

First, we have

$$\mathbf{T}_1 \leq \|f - \mathbf{b} \cdot \nabla \bar{u} - c\,\bar{u} + \nabla \cdot \mathbf{y}^\star - cv - \mathbf{b} \cdot \nabla v\|_{0,\Omega} \|u - \bar{u}\|_{0,\Omega} \leq$$

$$\leq C_\Omega \|f - \mathbf{b} \cdot \nabla \bar{u} - c\,\bar{u} + \nabla \cdot \mathbf{y}^\star - cv - \mathbf{b} \cdot \nabla v\|_{0,\Omega} \|\nabla (u - \bar{u})\|_{0,\Omega},$$

where C_Ω is a constant from the Friedrichs-Poincaré inequality

$$\|w\|_{0,\Omega} \leq C_\Omega \|\nabla w\|_{0,\Omega} \quad (14)$$

valid for all functions $w \in H_0^1(\Omega)$, and $\|\cdot\|_{0,\Omega}$ denotes the standard norm in $L_2(\Omega)$. Similarly, we get the estimate

$$\mathbf{T}_2 \leq \|\mathbf{y}^\star - \varepsilon \nabla \bar{u} + \varepsilon \nabla v\|_{0,\Omega} \|\nabla (u - \bar{u})\|_{0,\Omega}.$$

In a view of a simple inequality for any $\alpha > 0$

$$pq \leq \frac{\alpha}{2} p^2 + \frac{1}{2\alpha} q^2, \qquad (p,\ q \geq 0), \quad (15)$$

we immediately get that

$$\mathbf{T}_1+\mathbf{T}_2 \leq \Big(\|\mathbf{y}^\star-\varepsilon\nabla\bar{u}+\varepsilon\nabla v\|_{0,\Omega}+C_\Omega\|f-\mathbf{b}\cdot\nabla\bar{u}-c\,\bar{u}+\nabla\cdot\mathbf{y}^\star-cv-\mathbf{b}\cdot\nabla v\|_{0,\Omega}\Big)\times$$

$$\times\|\nabla(u-\bar{u})\|_{0,\Omega} \leq \frac{\alpha}{2}\|\nabla(u-\bar{u})\|^2_{0,\Omega}+$$

$$\frac{1}{2\alpha}\Big(\|\mathbf{y}^\star-\varepsilon\nabla\bar{u}+\varepsilon\nabla v\|_{0,\Omega}+C_\Omega\|f-\mathbf{b}\cdot\nabla\bar{u}-c\,\bar{u}+\nabla\cdot\mathbf{y}^\star-cv-\mathbf{b}\cdot\nabla v\|_{0,\Omega}\Big)^2,$$

where α is any positive number. For the second term in the right-hand side of the above inequality we can employ the following inequality

$$(p+q)^2 \leq (1+\gamma)p^2+(1+\frac{1}{\gamma})q^2, \qquad (p,\ q\geq 0), \tag{16}$$

valid for any positive number γ. Finally, we get

$$\mathbf{T}_1+\mathbf{T}_2 \leq \frac{\alpha}{2}\|\nabla(u-\bar{u})\|^2_{0,\Omega}+\frac{1}{2\alpha}\Big((1+\gamma)\|\mathbf{y}^\star-\varepsilon\nabla\bar{u}+\varepsilon\nabla v\|^2_{0,\Omega}+ \tag{17}$$

$$+\Big(1+\frac{1}{\gamma}\Big)C^2_\Omega\|f-\mathbf{b}\cdot\nabla\bar{u}-c\,\bar{u}+\nabla\cdot\mathbf{y}^\star-cv-\mathbf{b}\cdot\nabla v\|^2_{0,\Omega}\Big).$$

Now, we shall estimate the third term $\mathbf{T}_3$. We observe that

$$\mathbf{T}_3 = \int\limits_\Omega\Big(c\,v(u-\bar{u})+\mathbf{b}\cdot\nabla v(u-\bar{u})+\varepsilon\nabla v\cdot\nabla\bar{u}-\varepsilon\nabla v\cdot\nabla u\Big)\,dx =$$

$$= \int\limits_\Omega\Big(c\,v(u-\bar{u})+\mathbf{b}\cdot\nabla v(u-\bar{u})+\varepsilon\nabla v\cdot\nabla\bar{u}-fv+\mathbf{b}\cdot\nabla uv+cuv\Big)\,dx =$$

$$= \int\limits_\Omega\Big(\mathbf{b}\cdot\nabla v\,(u-\bar{u})+2\,c\,v\,(u-\bar{u})+\mathbf{b}\cdot\nabla(u-\bar{u})\,v\Big)\,dx+$$

$$+\int\limits_\Omega\Big(-f\,v+\mathbf{b}\cdot\nabla\bar{u}\,v-c\,v\,(u-\bar{u})+\varepsilon\nabla v\cdot\nabla\bar{u}+cuv\Big)\,dx = \tag{18}$$

$$= \int\limits_\Omega\Big(\mathbf{b}\cdot\nabla\Big(v\,(u-\bar{u})\Big)+2\,c\,v\,(u-\bar{u})\Big)dx+\int\limits_\Omega\Big(\varepsilon\nabla v\cdot\nabla\bar{u}+\mathbf{b}\cdot\nabla\bar{u}\,v+c\,v\,\bar{u}-fv\Big)\,dx =$$

$$= \int\limits_\Omega\Big(\varepsilon\nabla v\cdot\nabla\bar{u}+\mathbf{b}\cdot\nabla\bar{u}\,v+c\,v\,\bar{u}-fv\Big)\,dx+2\int\limits_\Omega(c-\frac{1}{2}\nabla\cdot\mathbf{b})\,v\,(u-\bar{u})\,dx =$$

$$= \mathbf{T}_{3,1}(v,\bar{u})+\mathbf{T}_{3,2}(u,v,\bar{u}).$$

Obviously, the above term $\mathbf{T}_{3,1}$ is directly computable once we have the approximation $\bar{u}$ computed and fix the parameter v, but we should still estimate the term $\mathbf{T}_{3,2}$ containing the unknown exact solution u. For this purpose, employing the definition of $\tilde{c}$ in (6), we further apply the obvious estimate

$$\mathbf{T}_{3,2} \leq \frac{1}{\beta}\|\sqrt{\tilde{c}}(u-\bar{u})\|^2_{0,\Omega}+\beta\|\sqrt{\tilde{c}}\,v\|^2_{0,\Omega}, \tag{19}$$

where β is any positive number.

Now, combining (9) and (13) with estimates (17), (18), and (19), we prove the following theorem.

Theorem 1: Under conditions (2) and (6), for the solution u of the problem (3) and for an arbitrary function $\bar{u} \in H_0^1(\Omega)$, we have the following functional inequality

$$(\varepsilon - \frac{\alpha}{2})\|\nabla(u - \bar{u})\|_{0,\Omega}^2 + (1 - \frac{1}{\beta})\|\sqrt{\tilde{c}}(u - \bar{u})\|_{0,\Omega}^2 \leq$$

$$\leq \frac{1}{2\alpha}\Big((1+\gamma)\|\mathbf{y}^\star - \varepsilon\nabla\bar{u} + \varepsilon\nabla v\|_{0,\Omega}^2 + \tag{20}$$

$$+\Big(1 + \frac{1}{\gamma}\Big)C_\Omega^2\|f - \mathbf{b}\cdot\nabla\bar{u} - c\,\bar{u} + \nabla\cdot\mathbf{y}^\star - cv - \mathbf{b}\cdot\nabla v\|_{0,\Omega}^2\Big) +$$

$$+\int\limits_\Omega \Big(\varepsilon\nabla v\cdot\nabla\bar{u} + \mathbf{b}\cdot\nabla\bar{u}\,v + c\,v\,\bar{u} - fv\Big)\,dx + \beta\|\sqrt{\tilde{c}}\,v\|_{0,\Omega}^2,$$

which is valid for any positive numbers α, β, γ, and for any parameter-functions $\mathbf{y}^\star \in H(\mathrm{div}, \Omega)$ and $v \in H_0^1(\Omega)$.

Let us fix the numbers α and β and introduce a short denotation

$$\mathbf{EST}_{\alpha,\beta} := \mathbf{EST}_{\alpha,\beta}(\gamma, \mathbf{y}^\star, v, \bar{u})$$

for the functional on the right-hand side of the inequality (20). Now we formulate the main result of the paper.

Theorem 2: Let α and β be fixed positive numbers such that

$$2\varepsilon \geq \alpha > 0, \quad \beta \geq 1, \quad \varepsilon - \frac{\alpha}{2} + 1 - \frac{1}{\beta} > 0. \tag{21}$$

Then, in view of the inequality (20), we get the following a posteriori upper error estimates for the error in the weighted energy norms of type (8)

$$|||e|||_{\lambda,\mu,\Omega}^2 \leq \mathbf{EST}_{\alpha,\beta}(\gamma, \mathbf{y}^\star, v, \bar{u}), \tag{22}$$

where $\lambda := \varepsilon - \frac{\alpha}{2}$, $\mu := 1 - \frac{1}{\beta}$, and the parameters γ, $\mathbf{y}^\star$, and v are as defined in Theorem 1.

Remark 2: We see that under condition (21) in Theorem 2, we have $\lambda \geq 0$, $\mu \geq 0$, and $\lambda + \mu > 0$, i.e., the left-hand side of (22) really defines certain weighted norm of type (8). Taking different values of the numbers α and β within limits given in (21), we get the family of a posteriori error estimates in various weighted energy norms.

4 Final comments

4.1 On estimation of the constant C_Ω

We notice that the constant C_Ω that appears in the definition of the error functional $\mathbf{EST}_{\alpha,\beta}$ is global and depends solely on the geometry of the domain. Its usable upper estimate can be readily obtained by enclosing the domain Ω into a rectangular box as proposed by S. Mikhlin (see [18], p. 18)

$$C_\Omega \leq \frac{1}{\pi\sqrt{\frac{1}{a_1^2} + \cdots + \frac{1}{a_d^2}}}, \tag{23}$$

where $a_1, \ldots, a_d$ are the dimensions of the box. Note that C_Ω is independent of the mesh and needs to be evaluated only once for each particular domain.

4.2 On the quality of the proposed error estimates

Let us assume that we have minimized the upper bound $\mathbf{EST}_{\alpha,\beta}$, i.e., that we have found the optimal parametres γ_{opt}, $\mathbf{y}^\star_{opt}$, v_{opt}, and define

$$\underline{\mathbf{EST}}_{\alpha,\beta} := \mathbf{EST}(\gamma_{opt}, \mathbf{y}^\star_{opt}, v_{opt}, \bar{u}).$$

In what follows, we shall demonstrate that the number $\underline{\mathbf{EST}}_{\alpha,\beta}$ does not give a too pessimistic overestimation of the error in (22) in principle.

First, let us prescribe

$$\mathbf{y}^\star := \varepsilon \nabla u \qquad \& \qquad v := u - \bar{u}, \tag{24}$$

which is really correct in a view of properties of u and $\bar{u}$. Then, using (3) with $w = u - \bar{u}$ and the Green formula, we get

$$\underline{\mathbf{EST}}_{\alpha,\beta} \leq \frac{1}{2\alpha} 4\varepsilon^2 (1+\gamma) \|\nabla(u-\bar{u})\|^2_{0,\Omega} +$$

$$+ \int\limits_\Omega \Big(\varepsilon\nabla(u-\bar{u})\cdot\nabla\bar{u} + \mathbf{b}\cdot\nabla\bar{u}\,(u-\bar{u}) + c(u-\bar{u})\,\bar{u} - f(u-\bar{u}) \Big)\,dx + \beta\|\sqrt{\tilde{c}}\,(u-\bar{u})\|^2_{0,\Omega} =$$

$$= \frac{2(1+\gamma)\varepsilon^2}{\alpha} \|\nabla(u-\bar{u})\|^2_{0,\Omega} + \beta\|\sqrt{\tilde{c}}\,(u-\bar{u})\|^2_{0,\Omega}$$

$$- \int\limits_\Omega \Big(\varepsilon\nabla(u-\bar{u})\cdot\nabla(u-\bar{u}) + \mathbf{b}\cdot\nabla(u-\bar{u})\,(u-\bar{u}) + c\,(u-\bar{u})\,(u-\bar{u}) + f\,(u-\bar{u}) \Big)\,dx +$$

$$+ \int\limits_\Omega \Big(\varepsilon\nabla(u-\bar{u})\cdot\nabla u + \mathbf{b}\cdot\nabla u\,(u-\bar{u}) + c\,(u-\bar{u})\,u \Big)\,dx =$$

$$= \Big(\frac{2(1+\gamma)\varepsilon^2}{\alpha} - \varepsilon \Big) \|\nabla(u-\bar{u})\|^2_{0,\Omega} + (\beta-1)\|\sqrt{\tilde{c}}\,(u-\bar{u})\|^2_{0,\Omega}.$$

Thus, for the choice of parameters (24), from above we get the following inequalities for the value of $\underline{\mathbf{EST}}_{\alpha,\beta}$

$$|||e|||^2_{(\varepsilon-\frac{\alpha}{2}),(1-\frac{1}{\beta}),\Omega} \le \underline{\mathbf{EST}}_{\alpha,\beta} \le |||e|||^2_{\left(\frac{2(1+\gamma)\varepsilon^2}{\alpha}-\varepsilon\right),(\beta-1),\Omega}. \tag{25}$$

Now we shall present examples on how sharp the proposed estimates are, in principle, for several particular choices of weights in the estimates (25).

Example 1: Let $\alpha := \varepsilon$ and $\beta := 1$. Then we observe that

$$|||e|||^2_{\frac{\varepsilon}{2},0,\Omega} \le \underline{\mathbf{EST}}_{\alpha,\beta} \le |||e|||^2_{\varepsilon(1+2\gamma),0,\Omega}. \tag{26}$$

Since γ can be taken arbitrarily small, we see that in this case $\underline{\mathbf{EST}}_{\alpha,\beta}$ can overestimate the error (which is, in fact, the L_2-norm of the difference $u-\bar{u}$)

$$|||e|||^2_{\frac{\varepsilon}{2},0,\Omega} = \frac{\varepsilon}{2}\|\nabla(u-\bar{u})\|^2_{0,\Omega}$$

at most twice, which is quite aceptable for the error control in real computation.

Example 2: Let now $\alpha := \varepsilon$ and $\beta := 2$. In this case, we get the same result as in Example 1, but now for another global error, which is nothing else, but the halved standard energy norm of the error

$$\frac{\varepsilon}{2}\|\nabla(u-\bar{u})\|^2_{0,\Omega} + \frac{1}{2}\|\sqrt{\tilde{c}}(u-\bar{u})\|^2_{0,\Omega}.$$

Example 3: Let us now take $\alpha := 2\varepsilon$ and $\beta := 2$. Then we have

$$|||e|||^2_{0,\frac{1}{2},\Omega} \le \underline{\mathbf{EST}}_{\alpha,\beta} \le |||e|||^2_{\varepsilon\gamma,1,\Omega}. \tag{27}$$

Since γ can be taken arbitrarily small, we see that in this case $\underline{\mathbf{EST}}_{\alpha,\beta} = \underline{\mathbf{EST}}_{2\varepsilon,2}$ can overestimate the value $\frac{1}{2}\|\sqrt{\tilde{c}}(u-\bar{u})\|^2_{0,\Omega}$ at most twice, too. It is worth noting that the number $\|\sqrt{\tilde{c}}(u-\bar{u})\|_{0,\Omega}$ can also serve as a global measure for the error if, for example, we have

$$0 < c_0 \le \tilde{c}(x) \le c_1$$

valid almost everywhere in Ω with positive constants c_0 and c_1.

Remark 3: All three above examples demonstrate that the global a priori error estimates proposed in the paper are, in principle, quasi-sharp. Moreover, theoretically there still exists a possibility that for the better choice of parametres $\mathbf{y}^\star$ and v, different from that one propose in (24), we can prove that the estimates perform, in fact, even better.

Remark 4: Situations with another choices of weights α and β can be also considered if required by real calculations following some more specific goal.

4.3 On choice of parameters in computations

A very comprehensive analysis on how to choose the optimal values for various parameters involved in the definition of the estimates of type (20) is the subject of our subsequent paper, see also [15, Sect. 5] for some relevant ideas.

However, for completeness we propose here one simple way of usage of our estimates in the situations when computations are performed on a series of meshes, which is a quite typical situation in the engineering practice. Let us have a sequence of computational meshes $\mathcal{T}^{h_1}$, $\mathcal{T}^{h_2}$, ..., $\mathcal{T}^{h_k}$ $(k \geq 2)$ with corresponding computed approximate solutions u^{h_1}, u^{h_2}, ..., u^{h_k}.

Assume that we want to control the error $e_i = u - u^{h_i}$, $i = 1, \ldots, k$. Then as immediate candidates for the parameter v to be employed in our estimates we can take easily computable functions $v_{i,j} = u^{h_i} - u^{h_j}$, $i, j = 1, \ldots, k$, and use functions $\varepsilon G_{h_i}(\nabla u_{h_i})$, $i = 1, \ldots, k$ as values for the parameter $\mathbf{y}^\star$. Here, G_{h_i} denotes some gradient averaging operator (see [6, 20] and references therein) for concrete definitions.

4.4 On mesh adaptive strategy

It is also natural to ask how we could use our estimates for the mesh adaptation purposes. Here, we describe a general strategy for such a goal. The upper estimate (20) is, in fact, an integral over the solution domain Ω. Thus, we can represent the value of this integral as the following sum

$$\sum_{T \in \mathcal{T}^{(i)}} I_T,$$

where each contribution I_T is a value of the total domain integral taken over a particular element T of the current mesh $\mathcal{T}^{(i)}$. To construct the next mesh $\mathcal{T}^{(i+1)}$ in order to obtain a more accurate approximation, we could use the following scheme. First, we find the maximum among all terms $|I_T|$ and, secondly, mark up those elements T which have their contributions larger than the "user-given threshold" θ ($\theta \in [0, 1]$) times that maximum value. Refining the marked elements (and making the mesh conforming), we obtain the next mesh $\mathcal{T}^{(i+1)}$.

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