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Abstract: *Three spheres theorem type is proved for the p-harmonic functions defined on the complement of k-balls in the Euclidean n-dimensional space.*

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1 Introduction

A classical theorem by J. Hadamard gives the following relation between the maximum absolute values of an analytic function on three concentric circles.

1.1 Theorem. *Let $R_1 < r_1 < r_2 < r_3 < R_2$ and let f be an analytic function in the annulus $\{z \in \mathbb{C} : R_1 < |z| < R_2\}$. Denote the maximum of $|f(z)|$ on the circle $|z| = r$ by $M(r)$. Then*

$$M(r_2)^{\log(r_3/r_1)} \le M(r_1)^{\log(r_3/r_2)} M(r_3)^{\log(r_2/r_1)}.$$

This result, known as the three circles theorem, was given by Hadamard without proof in 1896 [3]. For a discussion of the history of this result, see e.g. [8] and [5, pp. 323–325]. It is a natural question, what results of this type can be proved for other classes of functions. For example, a version of Hadamard's theorem can be proved for subharmonic functions in $\mathbb{R}^n$, $n \ge 2$, see [7, pp. 128–131].

Some generalizations of the three circles theorem will be studied here. For the formulation of our main result, Theorem 2.1, we recall some standard notation and definitions from the book [4]. We will consider solutions $v\colon \Omega \to \mathbb{R}$ of the *p-Laplace equation*

$$\operatorname{div}\big(|\nabla v|^{p-2}\nabla v\big) = 0, \qquad 1 < p < \infty, \tag{1.2}$$

on an open set $\Omega \subset \mathbb{R}^n$ in the sense that will be described shortly. When $p = 2$ equation (1.2) reduces to the Laplace equation $\Delta u = 0$, whose solutions, harmonic functions, are studied in the classical potential theory. When $p \ne 2$ equation (1.2) is nonlinear and degenerates at the zeros of the gradient of v. It follows that the solutions, *p-harmonic functions*, need not be in $C^2(\Omega)$ and the equation must be understood in the weak sense. A weak solution of (1.2) is a function v in the Sobolev space $W^{1,p}_{\text{loc}}(\Omega)$ such that

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle \, dm = 0 \tag{1.3}$$

for all $\varphi \in C_0^\infty(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the scalar product of vectors in $\mathbb{R}^n$, and m is the Lebesgue measure in $\mathbb{R}^n$.

It is easy to see that for all $\varphi \in C_0^\infty(\Omega)$ and $v \in C^2(\Omega)$,

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\varphi\rangle \, dm = -\int_\Omega \varphi \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) dm$$

and, consequently, each C^2-solution to (1.2) is a weak solution to (1.2).

Fix an integer k, $1 \le k \le n$ and a real number $t \ge 0$. The sets $B_k(t) = \{x \in \mathbb{R}^n : d_k(x) < t\}$ and $\Sigma_k(t) = \{x \in \mathbb{R}^n : d_k(x) = t\} = \partial B_k(t)$, where $d_k(x) = \Big(\sum\limits_{i=1}^k x_i^2\Big)^{1/2}$, are respectively called k-ball and k-sphere in $\mathbb{R}^n$. For $k = n$ the k-ball $B_k(t)$ coincides with the standard Euclidean ball $B^n(t)$ and the k-sphere $\Sigma_k(t)$ is the Euclidean sphere $S^{n-1}(t)$. In particular, the symbol $\Sigma_k(0)$ below denotes the k-sphere with the radius 0, i.e.

$$\Sigma_k(0) = \{x = (x_1, \dots, x_k, \dots, x_n) : x_1 = \dots = x_k = 0\}.$$

Let $0 < \alpha < \beta < \infty$ be fixed and let

$$D_{\alpha,\beta} = \{x \in \mathbb{R}^n : \alpha < d_k(x) < \beta\}.$$

For $k = 1$ the set $D_{\alpha,\beta}$ is the union of the two layers between two parallel hyperplanes. For $1 < k < n$ the boundary of the domain $D_{\alpha,\beta}$ consists of two coaxial cylindrical surfaces.

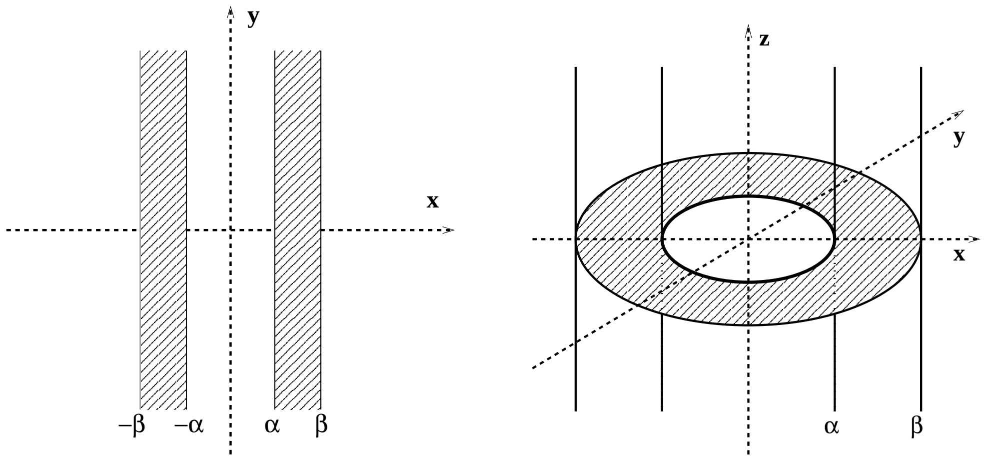

FIGURE: 1-annulus $D_{\alpha,\beta}$ in $\mathbb{R}^2$ (left) and 2-annulus $D_{\alpha,\beta}$ in $\mathbb{R}^3$ (right).

Let $v \in C^0(D_{r,R})$, and let $M(r) = \limsup_{z\to\Sigma_k(r)} v(z)$. Suppose that $M(R) \geq M(r)$. Consider the function

$$v_{r,R}(x) = \begin{cases} \frac{v(x)-M(r)}{M(R)-M(r)}, & \text{for } M(R) > M(r), \\ \infty, & \text{otherwise,} \end{cases}$$

for $r < R$. Clearly, $\limsup_{z\to\Sigma_k(r)} v_{r,R}(z) \leq 0$ and $\limsup_{z\to\Sigma_k(R)} v_{r,R}(z) \leq 1$. Let

$$\xi(r,t) = \int_r^t s^{(1-k)/(p-1)} ds, \text{ and } u_0^{k,p}(t) = \frac{\xi(r,t)}{\xi(r,R)}.$$

Let $u(x) = u_0^{k,p}\big(d_k(x)\big)$ for $x \in D_{r,R}$. It is clear (see Lemma 3.5) that u is a C^2-solution to (1.2). We have

$$u(x)|_{\Sigma_k(r)} \equiv 0, \;\; u(x)|_{\Sigma_k(R)} \equiv 1,$$

and

$$u(x) \geq v_{r,R}(x) \text{ if } x \in \Sigma_k(r) \text{ or } x \in \Sigma_k(R). \tag{1.4}$$

2 Main results

We will prove the following Hadamard type theorem for the p-harmonic functions defined on the complement of a k-ball. We use the method of proof from [6].

2.1 Theorem. *Let $1 < p < \infty$, $R > r > 0$ and let $v(x) \in W^{1,p}_{\mathrm{loc}}(D_{r,\infty})$ be a continuous weak solution of (1.2) such that*

$$\int_r^\infty dt\Big(\int_{\Sigma_k(t)} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)d\mathcal{H}^{n-1}\Big)^{-1} = \infty, \qquad (2.2)$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Then for all $t \in (r,R)$,

$$M(t) \le \big(M(R)-M(r)\big)u_0^{k,p}(t) + M(r). \qquad (2.3)$$

2.4 Corollary. *Let $1 < p < \infty$, $r > 0$ and let $v(x) \in W^{1,p}_{\mathrm{loc}}(D_{r,\infty})$, be a continuous weak solution of (1.2) such that*

$$\lim_{R\to\infty} \frac{1}{R^2}\int_{D_{r,R}} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)dm = 0. \qquad (2.5)$$

Then for all $t \in (r,\infty)$ the inequality (2.3) holds.

2.6 Corollary. *Let $1 < p < \infty$, $R > r > 0$ and let $v(x) \in W^{1,p}_{\mathrm{loc}}(D_{r,\infty})$ be a continuous weak solution of (1.2) such that*

$$\int_{D_{r,\infty}} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)dm \le M < \infty.$$

Then for all $t \in (r,R)$ the inequality (2.3) holds.

For the formulation of a result of S. Granlund [2], Theorem 2.7 below, we introduce some notation and terminology. Let $p > 1$, $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $F\colon \Omega \times \mathbb{R}^n \to \mathbb{R}$ be such that the following conditions hold.

1. There are constants $\beta > \alpha > 0$ such that for a.e. $x \in \Omega$
$$\alpha|z|^p \le F(x,z) \le \beta|z|^p.$$
2. For a.e. $x \in \Omega$ the function $z \mapsto F(x,z)$ is convex.
3. The function $x \mapsto F(x,\nabla u(x))$ is measurable for all $u \in W^{1,p}(\Omega)$.

Let

$$I(u) = \int_\Omega F\big(x,\nabla u(x)\big)\,dm.$$

A function $u \in W^{1,p}(\Omega)$ is a *subminimum* in Ω if $I(u) \le I(u-\eta)$ for all non-negative $\eta \in W^{1,p}_0(\Omega)$. Let

$$M(r) = \operatorname*{ess\,sup}_{x\in\overline{B}^n(r)} u(x), \qquad \overline{B}^n(r) \subset \Omega.$$

The following Hadamard type theorem was proved by S. Granlund in [2].

2.7 Theorem. *Let u be a subminimum of*

$$I(u) = \int_\Omega F\big(x, \nabla u(x)\big) dm,$$

$r_1 < r < r_2$, *and* $\overline{B}^n(r_2) \subset \Omega$. *Then u is bounded from above, and there is a constant*

$$\lambda = \lambda(n, p, r/r_1, r_2/r, \alpha/\beta),$$

$0 < \lambda < 1$ *such that*

$$M(r) \leq \lambda M(r_1) + (1-\lambda) M(r_2).$$

Since p-harmonic functions minimize (see e.g. [4, p. 59]) the integral

$$I(u) = \int_\Omega |\nabla u|^p \, dm,$$

Theorem 2.7 gives a special case of Theorem 2.1 with $k = n$.

3 Preliminaries

We start by recalling some basic properties of the Sobolev spaces from [4]. Let Ω be a nonempty open set in $\mathbb{R}^n$.

3.1 Lemma. *[4, Theorem 1.24] Let $u \in W_0^{1,p}(\Omega)$ and $v \in W^{1,p}(\Omega)$ be bounded. Then $uv \in W_0^{1,p}(\Omega)$.*

3.2 Lemma. *[4, Lemma 3.11] If $v \in W^{1,p}(\Omega)$ is a weak solution of (1.2) in Ω, then*

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \varphi \rangle \, dm = 0$$

for all $\varphi \in W_0^{1,p}(\Omega)$.

3.3 Theorem. *[1, p. 99] Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz mapping. Let $E \subset \mathbb{R}^n$ be an n-measurable set and $g\colon E \to \mathbb{R}$ be a nonnegative measurable function. Then*

$$\int\limits_E g(x) |\nabla f(x)| \, dx_1 \cdots dx_n = \int\limits_{\mathbb{R}} \Big(\sum_{x \in f^{-1}(y)} g(x) \Big) \, d\mathcal{H}^n(y). \tag{3.4}$$

3.5 Lemma. *Let $1 < p < \infty$, $0 < r < d_k(x)$ and fix an integer $1 \leq k \leq n$. Then*

$$u(x) = \int_r^{d_k(x)} s^{\frac{1-k}{p-1}} \, ds$$

is a solution of (1.2), i.e.

$$\sum_{i=1}^n \Big\{ \frac{\partial}{\partial x_i} \big(u_{x_i} [u_{x_1}^2 + \ldots + u_{x_n}^2]^{\frac{p-2}{2}} \big) \Big\} = 0.$$

Proof. We note that

$$\frac{\partial}{\partial x_i} d_k(x) = \frac{x_i}{d_k(x)},$$

and hence $u_{x_i} = x_i d_k(x)^{\frac{1-k}{p-1}-1}$. Then

$$\begin{aligned} u_{x_i}\big(u_{x_1}^2 + ... + u_{x_k}^2\big)^{\frac{p-2}{2}} &= x_i d_k(x)^{\frac{1-k}{p-1}-1}\Big[d_k(x)^{\frac{2(1-k)}{p-1}-2}\big(\sum_{j=1}^{k} x_j^2\big)\Big]^{\frac{p-2}{2}} \\ &= x_i d_k(x)^{\frac{1-k}{p-1}-1} d_k(x)^{\frac{(1-k)(p-2)}{p-1}} = x_i d_k(x)^{-k}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{i=1}^{k} \frac{\partial}{\partial x_i}\big(x_i d_k(x)^{-k}\big) &= \sum_{i=1}^{k} d_k(x)^{-k} - k \sum_{i=1}^{k} x_i^2 d_k(x)^{-k-2} \\ &= k d_k(x)^{-k} - k d_k(x)^{-k-2}\big(\sum_{i=1}^{k} x_i^2\big) = 0. \end{aligned}$$

□

Next we will prove two lemmas which are used later in the proof of Theorem 2.1.

3.6 Lemma. *Let $a > b > 0$, $p > 1$. Then*

$$C_1 \frac{a^{p-1} - b^{p-1}}{a-b} \le \frac{a^{p-1} + b^{p-1}}{a+b} \le C_2 \frac{a^{p-1} - b^{p-1}}{a-b}, \tag{3.7}$$

with some constants C_1, $C_2 > 0$.

Proof. We examine the function

$$g_1(x) = \frac{(x^{p-1}+1)(x-1)}{(x^{p-1}-1)(x+1)}, \qquad x > 1.$$

It is clear that

$$\lim_{x\to 1} g_1(x) = \frac{1}{p-1}, \qquad \lim_{x\to\infty} g_1(x) = 1. \tag{3.8}$$

It is sufficient to find positive bounds for $g_1(x)$ for $x > 1$. We will prove that the bounds are in fact given by (3.8). First we note that

$$\begin{cases} (p-2)(x^p - 1) + p(x - x^{p-1}) < 0, & \text{for } p \in (1,2), \\ (p-2)(x^p - 1) + p(x - x^{p-1}) = 0, & \text{for } p = 2, \\ (p-2)(x^p - 1) + p(x - x^{p-1}) > 0, & \text{for } p > 2, \end{cases}$$

and

$$\begin{cases} x - x^{p-1} < 0, & \text{for } p \in (1,2), \\ x - x^{p-1} = 0, & \text{for } p = 2, \\ x - x^{p-1} > 0, & \text{for } p > 2. \end{cases}$$

Hence

$$\begin{cases} g_1(x) \in \big(1, 1/(p-1)\big), & \text{for } p \in (1,2), \\ g_1(x) = 1, & \text{for } p = 2, \\ g_1(x) \in \big(1/(p-1), 1\big), & \text{for } p > 2. \end{cases}$$

□

3.9 Lemma. *Let $a > b > 0$. Then*

$$C_3\,(a^{p-2} + b^{p-2}) \le \frac{a^{p-1} - b^{p-1}}{a - b} \le C_4\,(a^{p-2} + b^{p-2}), \tag{3.10}$$

for $p \ge 2$, and

$$C_3\,(a^{2-p} + b^{2-p})^{-1} \le \frac{a^{p-1} - b^{p-1}}{a - b} \le C_4\,(a^{2-p} + b^{2-p})^{-1}, \tag{3.11}$$

for $p \in (1, 2]$ with some constants C_3, $C_4 > 0$.

Proof. The proof is similar to that of Lemma 3.6. First we study the function

$$g_2(x) = \frac{x^{p-1} - 1}{(x-1)(x^{p-2} + 1)}.$$

As in Lemma 3.6, it is sufficient for (3.10) to find positive bounds for $g_2(x)$ for $x > 0$. We note that $\lim_{x\to 1} g_2(x) = (p-1)/2$ and $\lim_{x\to\infty} g_2(x) = 1$. We obtain

$$\begin{cases} (p-3)(1 - x^{p-1}) + (p-1)x(1 - x^{p-3}) < 0, & \text{for } p \in (1,3), \\ (p-3)(1 - x^{p-1}) + (p-1)x(1 - x^{p-3}) = 0, & \text{for } p = 3, \\ (p-3)(1 - x^{p-1}) + (p-1)x(1 - x^{p-3}) > 0, & \text{for } p > 3, \end{cases}$$

and

$$\begin{cases} x(x^{p-3} - 1) < 0, & \text{for } p \in (1,3), \\ x(x^{p-3} - 1) = 0, & \text{for } p = 3, \\ x(x^{p-3} - 1) > 0, & \text{for } p > 3. \end{cases}$$

It follows that

$$\begin{cases} g_2(x) \in \big((p-1)/2, 1\big), & \text{for } p \in (1,3), \\ g_2(x) = 1, & \text{for } p = 3, \\ g_2(x) \in \big(1, (p-1)/2\big), & \text{for } p > 3. \end{cases}$$

To prove (3.11) we study the function

$$g_3(x) = \frac{(x^{p-1} - 1)(x^{2-p} + 1)}{x - 1}.$$

Now $\lim_{x\to 1} g_3(x) = 2(p-1)$ and $\lim_{x\to\infty} g_3(x) = 1$. Again, we have

$$\begin{cases} (-2p+3)(x-1) + (x^{p-1} - x^{2-p}) < 0, & \text{for } p \in (1, 3/2), \\ (-2p+3)(x-1) + (x^{p-1} - x^{2-p}) = 0, & \text{for } p = 3/2, \\ (-2p+3)(x-1) + (x^{p-1} - x^{2-p}) > 0, & \text{for } p > 3/2, \end{cases}$$

and

$$\begin{cases} x^{p-1} - x^{2-p} < 0, & \text{for } p \in (1, 3/2), \\ x^{p-1} - x^{2-p} = 0, & \text{for } p = 3/2, \\ x^{p-1} - x^{2-p} > 0, & \text{for } p > 3/2, \end{cases}$$

and thus

$$\begin{cases} g_3(x) \in \big(2(p-1), 1\big), & \text{for } p \in (1, 3/2), \\ g_3(x) = 1, & \text{for } p = 3/2, \\ g_3(x) \in \big(1, 2(p-1)\big), & \text{for } p > 3/2. \end{cases}$$

□

4 Proof of Theorem 2.1

Suppose the contrary, that is, there exists $x_0 \in D_{r,R}$ such that

$$v(x_0) > \big(M(R) - M(r)\big)u(x_0) + M(r), \tag{4.1}$$

or

$$v_{r,R}(x_0) > u(x_0).$$

Fix some $\varepsilon_0 > 0$, for which

$$v_{r,R}(x_0) - u(x_0) > \varepsilon_0.$$

Consider the set

$$U = \{x \in D_{r,R} : v_{r,R}(x) - u(x) > \varepsilon_0\} \neq \emptyset.$$

Choose a component O of U such that $x_0 \in O$. It is clear that $\overline{O} \cap \partial D_{r,R} = \emptyset$ and $\big(v_{r,R}(x) - u(x)\big)|_{\partial O} = 0$. Fix $\varepsilon_2 > \varepsilon_1 > 0$ and the balls $O_1 = B_k(x_0, \varepsilon_1)$, $O_2 = B_k(x_0, \varepsilon_2)$. Let $\varphi(x) = \eta(d_k(x))$ be a locally Lipschitz function with the properties:

$$\begin{cases} \varphi \equiv 1 & \text{for all } x \in O_1, \\ \varphi \equiv 0 & \text{for all } x \in D_{r,R} \setminus O_2. \end{cases} \tag{4.2}$$

Then the function $\psi = \big(v_{r,R}(x) - u(x)\big)\varphi^2$ has a support $\operatorname{supp}\psi \subset \overline{O}_2$ and by Lemma 3.1 $\psi \in W_0^{1,p}(\Omega)$. Since $v_{r,R}$ and u are generalized solutions of (1.2) we have by Lemma 3.2

$$\begin{aligned} &\int_{\operatorname{supp}\psi} \langle \nabla\psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm \\ &\quad = \int_{\operatorname{supp}\psi} \langle \nabla\psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R}\rangle \, dm - \int_{\operatorname{supp}\psi} \langle \nabla\psi, |\nabla u|^{p-2}\nabla u\rangle \, dm = 0. \end{aligned}$$

Next, we note that

$$\nabla\psi = \varphi^2(\nabla v_{r,R} - \nabla u) + 2\varphi(v_{r,R} - u)\nabla\varphi.$$

Thus, we may write

$$\begin{aligned} 0 &= \int_{\operatorname{supp}\psi} \langle \nabla\psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm \\ &= \int_{O\cap O_2} \langle \nabla\psi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm \\ &= \int_{O\cap O_2} \varphi^2 \langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm \\ &\quad + 2\int_{O\cap O_2} \varphi(v_{r,R} - u)\langle \nabla\varphi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm \end{aligned}$$

or

$$\int_{O\cap O_2} \varphi^2 \langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm$$
$$= -2\int_{O\cap O_2} \varphi(v_{r,R} - u)\langle \nabla\varphi, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm$$

or

$$\left| \int_{O\cap O_2} \varphi^2 \langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle \, dm\right|$$
$$\leq 2\int_{O\cap O_2} |\varphi||v_{r,R} - u||\nabla\varphi|\Big||\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\Big| dm. \quad (4.3)$$

Let

$$\Phi(\lambda) = |\nabla(\lambda v_{r,R} + (1-\lambda)u)|^{p-2}\nabla(\lambda v_{r,R} + (1-\lambda)u)$$

for $\lambda \in [0,1]$, and note that

$$\Phi(0) = |\nabla u|^{p-2}\nabla u \quad \text{and} \quad \Phi(1) = |\nabla v_{r,R}|^{p-2}\nabla v_{r,R}.$$

Now we write

$$|\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(\lambda)\, d\lambda$$
$$= \int_0^1 \Big[(\nabla v_{r,R} - \nabla u)\,|\nabla(\lambda v_{r,R} + (1-\lambda)u)|^{p-2} + (p-2)\nabla(\lambda v_{r,R} + (1-\lambda)u)$$
$$\cdot\, |\nabla(\lambda v_{r,R} + (1-\lambda)u)|^{p-4}\,\langle \nabla v_{r,R} - \nabla u, \nabla(\lambda v_{r,R} + (1-\lambda)u)\rangle\Big]\, d\lambda, \quad (4.4)$$

and obtain

$$\langle \nabla v_{r,R} - \nabla u,\; |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle$$
$$= |\nabla v_{r,R} - \nabla u|^2 \int_0^1 |\nabla(\lambda v_{r,R} + (1-\lambda)u)|^{p-2}\, d\lambda$$
$$+ (p-2)\int_0^1 |\nabla(\lambda v_{r,R} + (1-\lambda)u)|^{p-4}\langle \nabla v_{r,R} - \nabla u,\; \nabla(\lambda v_{r,R} + (1-\lambda)u)\rangle^2\, d\lambda. \quad (4.5)$$

If $p \geq 2$ then

$$\langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle$$
$$\geq |\nabla v_{r,R} - \nabla u|^2 \int_0^1 \big|\nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\big|^{p-2} d\lambda. \quad (4.6)$$

If $p < 2$, we have

$$|\nabla v_{r,R} - \nabla u|^2 \int_0^1 \left|\nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\right|^{p-2} d\lambda$$
$$+ (p-2) \int_0^1 \left|\nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\right|^{p-4} \langle \nabla v_{r,R} - \nabla u, \nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\rangle^2 \, d\lambda$$
$$\geq (p-1)|\nabla v_{r,R} - \nabla u|^2 \int_0^1 \left|\nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\right|^{p-2} d\lambda.$$

This together with (4.5) gives

$$\langle \nabla v_{r,R} - \nabla u, |\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\rangle$$
$$\geq (p-1)|\nabla v_{r,R} - \nabla u|^2 \int_0^1 \left|\nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\right|^{p-2} d\lambda, \quad 1 < p \leq 2. \quad (4.7)$$

It follows from (4.4) that for every $p > 1$,

$$\left||\nabla v_{r,R}|^{p-2}\nabla v_{r,R} - |\nabla u|^{p-2}\nabla u\right| \leq C_5 |\nabla v_{r,R} - \nabla u| \int_0^1 \left|\nabla\big(\lambda v_{r,R} + (1-\lambda)u\big)\right|^{p-2} d\lambda, \quad (4.8)$$

at every point where $v_{r,R}$ has differential. Here $C_5 = 1 + |p-2|$. Setting

$$I(p) = \int_0^1 \left|\nabla(\lambda v_{r,R} + (1-\lambda)u\right|^{p-2} d\lambda$$

and using (4.3), (4.6), (4.7) and (4.8) we obtain

$$\int_{O\cap O_2} \varphi^2 I(p) |\nabla v_{r,R} - \nabla u|^2 dm \leq C_6 \int_{O\cap O_2} I(p)|\varphi||v_{r,R} - u||\nabla\varphi||\nabla v_{r,R} - \nabla u| dm, \quad (4.9)$$

where $C_6 = 2C_5/\min\{1, p-1\}$.

We note that

$$|\nabla(\lambda v_{r,R} + (1-\lambda)u)|^2 = \lambda^2 \,|\nabla v_{r,R}|^2 + 2\lambda(1-\lambda)\langle \nabla v_{r,R}, \, \nabla u\rangle + (1-\lambda)^2 \,|\nabla u|^2,$$

and therefore

$$\big|\lambda|\nabla v_{r,R}| - (1-\lambda)\,|\nabla u|\big| \leq |\nabla(\lambda v_{r,R} + (1-\lambda)u)| \leq \lambda|\nabla v_{r,R}| + (1-\lambda)|\nabla u| \quad (4.10)$$

for an arbitrary $\lambda \in [0,1]$. Let $p \geq 2$. We suppose that $|\nabla v_{r,R}| > |\nabla u|$. Then by (4.10),

$$I(p) \leq \int_0^1 (\lambda(|\nabla v_{r,R}| - |\nabla u|) + |\nabla u|)^{p-2} \, d\lambda$$
$$= \frac{1}{|\nabla v_{r,R}| - |\nabla u|} \int_{|\nabla u|}^{|\nabla v_{r,R}|} s^{p-2} \, ds = \frac{1}{p-1} \frac{|\nabla v_{r,R}|^{p-1} - |\nabla u|^{p-1}}{|\nabla v_{r,R}| - |\nabla u|}. \quad (4.11)$$

Next by (4.10),

$$\begin{aligned} I(p) &\geq \int_0^1 \left|\lambda|\nabla v_{r,R}| - (1-\lambda)\,|\nabla u|\right|^{p-2} d\lambda \\ &= \int_0^1 \left|\lambda(|\nabla v_{r,R}| + |\nabla u|) - |\nabla u|\right|^{p-2} d\lambda \\ &= \int_s^1 (\lambda(|\nabla v_{r,R}| + |\nabla u|) - |\nabla u|)^{p-2}\ d\lambda \\ &\quad + \int_0^s (|\nabla u| - \lambda(|\nabla v_{r,R}| + |\nabla u|))^{p-2}\ d\lambda, \end{aligned}$$

where

$$s = \frac{|\nabla u|}{|\nabla v_{r,R}| + |\nabla u|}. \tag{4.12}$$

By computing both of the last two integrals, we obtain

$$I(p) \geq \frac{1}{p-1} \frac{|\nabla v_{r,R}|^{p-1} + |\nabla u|^{p-1}}{|\nabla v_{r,R}| + |\nabla u|}. \tag{4.13}$$

Let $1 < p < 2$. As above, we assume $|\nabla v_{r,R}| > |\nabla u|$. Then by (4.10),

$$\begin{aligned} I(p) &\leq \int_0^1 \left|\lambda|\nabla v_{r,R}| - (1-\lambda)\,|\nabla u|\right|^{2-p} d\lambda \\ &= \int_0^1 \left|\lambda(|\nabla v_{r,R}| + |\nabla u|) - |\nabla u|\right|^{2-p} d\lambda \\ &= \int_0^s \big(|\nabla u| - \lambda(|\nabla v_{r,R}| + |\nabla u|)\big)^{2-p} d\lambda \\ &\quad + \int_s^1 \big(\lambda(|\nabla v_{r,R}| + |\nabla u|) - |\nabla u|\big)^{2-p} d\lambda, \end{aligned}$$

where s is defined in (4.12), and hence

$$I(p) \leq \frac{1}{p-1} \frac{|\nabla v_{r,R}|^{p-1} + |\nabla u|^{p-1}}{|\nabla v_{r,R}| + |\nabla u|}. \tag{4.14}$$

By (4.11), it follows that

$$I(p) \geq \frac{1}{p-1} \frac{|\nabla v_{r,R}|^{p-1} - |\nabla u|^{p-1}}{|\nabla v_{r,R}| - |\nabla u|}. \tag{4.15}$$

Setting $a = |\nabla v_{r,R}|$ and $b = |\nabla u|$ in (3.7), (3.10) and (3.11), we can obtain by (4.11), (4.13), (4.14) and (4.15), for $p \geq 2$

$$C_7 \left(|\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2}\right) \leq I(p) \leq C_8 \left(|\nabla v_{r,R}|^{p-2} + |\nabla u|^{p-2}\right), \qquad (4.16)$$

or

$$C_7 \left(|\nabla v_{r,R}|^{2-p} + |\nabla u|^{2-p}\right)^{-1} \leq I(p) \leq C_8 \left(|\nabla v_{r,R}|^{2-p} + |\nabla u|^{2-p}\right)^{-1}, \quad (4.17)$$

$1 < p \leq 2$, with some constants C_7, $C_8 > 0$. The case $|\nabla v_{r,R}| < |\nabla u|$ is analogous. This may be written as

$$C_9\big(|\nabla v_{r,R}|^{|p-2|} - |\nabla u|^{|p-2|}\big) \leq I(p) \leq C_{10}\big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big), \quad (4.18)$$

where $C_9 = \min\{C_7, 1/C_8\}$ and $C_{10} = \max\{1/C_7, C_8\}$.

Thus by (4.9), (4.18) we find,

$$\begin{aligned}
&\int_{O\cap O_2} \varphi^2 |\nabla v_{r,R} - \nabla u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm \\
&\leq C_{11} \int_{O\cap O_2} |\varphi||v_{r,R} - u||\nabla\varphi||\nabla v_{r,R} - \nabla u| \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm \\
&\quad \leq C_{11}\bigg(\int_{O\cap O_2} |\nabla\varphi|^2 |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm\bigg)^{1/2} \\
&\qquad \cdot \bigg(\int_{O\cap O_2} \varphi^2 |\nabla v_{r,R} - \nabla u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm\bigg)^{1/2} \qquad (4.19)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{O\cap O_2} \varphi^2 |\nabla v_{r,R} - \nabla u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm \\
&\qquad\qquad \leq C_{11}^2 \int_{O\cap O_2} |\nabla\varphi|^2 |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm.
\end{aligned}$$

Remembering (4.2) we have

$$\begin{aligned}
&\int_{O\cap O_1} |\nabla v_{r,R} - \nabla u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm \\
&\qquad\qquad \leq C_{11}^2 \int_{D_{r,R}\cap(O_2\setminus\overline{O_1})} |\nabla\varphi|^2 |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm.
\end{aligned}$$

Because φ is constant on $\Sigma_k(t)$ and $|\nabla d_k| \equiv 1$, we have by Theorem 3.3

$$\begin{aligned}
&\int_{D_{r,R}\cap(O_2\setminus\overline{O_1})} |\nabla\varphi|^2 |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm \\
&\leq \int_{\{x:\varepsilon_1 < d_k(x) < \varepsilon_2\}} |\nabla\varphi|^2 |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) dm = \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) dt,
\end{aligned}$$

where

$$H(t) = \int_{\Sigma_k(t)} |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) d\mathcal{H}^{n-1}. \tag{4.20}$$

By Hölder's inequality

$$1 \le \int_{\varepsilon_1}^{\varepsilon_2} \eta' H(t)^{1/2} H(t)^{-1/2} dt \le \left(\int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) dt\right)^{1/2} \left(\int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) dt\right)^{1/2}.$$

It follows that

$$\left(\int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) dt\right)^{-1} \le \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) dt, \tag{4.21}$$

for all $\varphi(x) = \eta(d_k(x))$ satisfying (4.2).

We define a function $\hat{\eta}$ by the formula

$$\hat{\eta}(s) = \left(\int_{\varepsilon_1}^{s} H^{-1}(t) dt\right)\left(\int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) dt\right)^{-1}.$$

Now $\hat{\eta}(\varepsilon_1) = 0$ and $\hat{\eta}(\varepsilon_2) = 1$. Because

$$\hat{\eta}'(s) = \frac{1}{H(s)}\left(\int_{\varepsilon_1}^{\varepsilon_2} \frac{dt}{H(t)}\right)^{-1},$$

we have by (4.21)

$$\begin{aligned}\left(\int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) dt\right)^{-1} &\le \inf_{\varphi} \int_{\varepsilon_1}^{\varepsilon_2} \eta'^2 H(t) dt \\ &\le \int_{\varepsilon_1}^{\varepsilon_2} \hat{\eta}'^2 H(t) dt = \left(\int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) dt\right)^{-1}.\end{aligned}$$

Because

$$\begin{aligned}&\int_{\varepsilon_1}^{\varepsilon_2} H^{-1}(t) dt \\ &\quad = \int_{\varepsilon_1}^{\varepsilon_2} dt \left(\int_{\Sigma_k(t)} |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) d\mathcal{H}^{n-1}\right)^{-1} \to \infty,\end{aligned}$$

as $\varepsilon_2 \to \infty$, the claim follows. □

5 Proofs of the corollaries

Proof of Corollary 2.4

Let

$$H(t) = \int_{\Sigma_k(t)} |v_{r,R} - u|^2 \big(|\nabla v_{r,R}|^{|p-2|} + |\nabla u|^{|p-2|}\big) d\mathcal{H}^{n-1}. \tag{5.1}$$

By Hölder's inequality

$$(R-r)^2 = \Big(\int_r^R dt\Big)^2 = \Big(\int_r^R \frac{H^{-1/2}(t)}{H^{-1/2}(t)}dt\Big)^2$$
$$\leq \Big(\int_r^R H^{-1}(t)dt\Big)\Big(\int_r^R H(t)dt\Big).$$

Hence

$$(R-r)^2\Big(\int_r^R H^{-1}(t)dt\Big)^{-1} \leq \Big(\int_r^R H(t)dt\Big). \tag{5.2}$$

Now by (5.2) and Theorem 3.3

$$\left[\int_r^R dt\Big(\int_{\Sigma_k(t)} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)d\mathcal{H}^{n-1}\Big)^{-1}\right]^{-1}$$
$$\leq \frac{1}{(R-r)^2}\int_{D_{r,R}} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)dm \to 0,$$

as $R\to\infty$, proving the claim. □

Proof of Corollary 2.6

Since

$$\int_{D_{r,\infty}} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)dm = M < \infty,$$

we have for $R>r$,

$$\frac{1}{R^2}\int_{D_{r,R}} |v_{r,R}-u|^2\big(|\nabla v_{r,R}|^{|p-2|}+|\nabla u|^{|p-2|}\big)dm \leq \frac{M}{R^2} \to 0,$$

as $R\to\infty$. □

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