Helsinki University of Technology, Institute of Mathematics, Research Reports

Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja

Espoo 2006 A495

A posteriori error estimation for linear elliptic problems with mixed boundary conditions

Sergey Korotov

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Helsinki University of Technology, Institute of Mathematics, Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2006 A495

A posteriori error estimation for linear elliptic problems with mixed boundary conditions

Sergey Korotov

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Sergey Korotov: *A posteriori error estimation for linear elliptic problems with mixed boundary conditions*; Helsinki University of Technology, Institute of Mathematics, Research Reports A495 (2006).

Abstract: *The paper is devoted to verification of accuracy of approximate solutions obtained in computer simulations. This problem is strongly related to a posteriori error estimates, giving computable bounds for computational errors and detecting zones in the solution domain, where such errors are too large and certain mesh refinements should be performed. Mathematical model consisting of a linear elliptic equation with mixed Dirichlet/Neumann/Robin boundary conditions is considered in this work. We derive in a simple way an easily computable upper bound for the error, which is understood as difference between the exact solution of the model and its approximation measured in the corresponding energy norm. The estimate obtained is completely independent of the numerical technique used to obtain approximate solutions and can be made as close to the true error as resources of a concrete computer used for relevant computations allow. Several issues of practical realization of the approach are discussed and a representative numerical test is presented.*

AMS subject classifications: 65N15, 65N30

Keywords: reliable computations, a posteriori error estimation, error control in energy norm, differential equation of elliptic type, mixed boundary conditions

Correspondence

sergey.korotov@hut.fi

ISBN 951-22-8098-1
ISSN 0784-3143

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

Many physical phenomena can be described by means of mathematical models presenting boundary value problems of elliptic type [5, 14, 15]. Various numerical techniques (such as the finite difference method, the finite element method, the finite element volume, etc) are well developed for finding approximate solutions for such problems, see, e.g., [9].

However, in order to be practically meaningful, computer simulations always require an accuracy verification of computed approximations. Such a verification is the main purpose of a posteriori error estimation methods. Several approaches for deriving various a posteriori estimates for errors measured in global energy norm ([1], [2], [3], [16], [19], [20], [23]), or in terms of various local quantities ([4], [10], [12], [13], [20]) have been suggested (see also references in the above mentioned works).

Most of the estimates proposed so far strongly use the fact that the computed solutions are true finite element (FE) approximations which, in fact, rarely happens in real computations, e.g., due to quadrature rules, forcibly stopped iterative processes, various round-off errors, or even bugs in computer codes.

In this paper, on the base of a model linear elliptic problem, we present and test numerically a relatively simple technology for obtaining *computable guaranteed upper bound* needed for reliable control of the overall accuracy of computed approximations. This bound is valid for *any conforming approximations* independently of numerical method used to obtain them. The bound can be made *arbitrarily close* to the true error. In practical calculations this closeness only depends on resources of a concrete computer used. We shall also discuss main issues of a realization of the proposed error estimation in practice and present one representative numerical test.

2 Formulation of Problem

For definitions of functional spaces and finite element terminology used in the paper we refer, e.g., to monographs [15] and [9], respectively.

Consider the following model problem: Find a function u such that

$$-\mathrm{div}(A\nabla u) + cu = f \quad \text{in} \quad \Omega, \tag{1}$$

$$u = u_0 \quad \text{on} \quad \Gamma_D, \tag{2}$$

$$\nu^T \cdot A\nabla u = g \quad \text{on} \quad \Gamma_N, \tag{3}$$

$$\nu^T \cdot A\nabla u + \sigma u = h \quad \text{on} \quad \Gamma_R, \tag{4}$$

where Ω is a bounded domain in $\mathbf{R}^d$ with a Lipschitz continuous boundary $\partial\Omega$, such that $\overline{\partial\Omega} = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}_R$, $\mathrm{meas}_{d-1}\Gamma_D > 0$, ν is the outward normal to the boundary, $f \in L_2(\Omega)$, $u_0 \in H^1(\Omega)$, $g \in L_2(\Gamma_N)$, $h \in L_2(\Gamma_R)$,

$\sigma \in L_\infty(\Gamma_R)$, $c \in L_\infty(\Omega)$, the matrix of coefficients A is symmetric, with entries $a_{ij} \in L_\infty(\Omega)$, $i, j = 1, \ldots, d$, and is such that

$$C_2|\xi|^2 \geq A(x)\xi \cdot \xi \geq C_1|\xi|^2 \qquad \forall \xi \in \mathbf{R}^d \quad \forall x \in \Omega. \tag{5}$$

In addition, we assume that almost everywhere

$$c \geq 0 \quad \text{in} \quad \Omega, \quad \sigma \geq \sigma_0 > 0 \quad \text{on} \quad \Gamma_R, \tag{6}$$

and introduce the set

$$\Omega^c := \operatorname{supp} c = \{x \in \Omega \,|\, c(x) > 0\}. \tag{7}$$

The weak formulation of problem (1)–(4) reads: Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that

$$\int\limits_\Omega A\nabla u \cdot \nabla w \, dx + \int\limits_\Omega cuw \, dx + \int\limits_{\Gamma_R} \sigma uw \, ds \;=$$

$$= \int\limits_\Omega fw \, dx + \int\limits_{\Gamma_N} gw \, ds + \int\limits_{\Gamma_R} hw \, ds \qquad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{8}$$

where

$$H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_D\}. \tag{9}$$

Remark 2.1 *Conditions* (6) *on c and σ provide the existence and uniqueness of the weak solution u defined by* (8)*.*

Let us define bilinear form $a(\cdot,\cdot)$ and linear form $F(\cdot)$ as follows

$$a(v, w) := \int\limits_\Omega A\nabla v \cdot \nabla w \, dx + \int\limits_\Omega cvw \, dx + \int\limits_{\Gamma_R} \sigma vw \, ds, \quad v, w \in H^1(\Omega), \tag{10}$$

$$F(w) := \int\limits_\Omega fw \, dx + \int\limits_{\Gamma_N} gw \, ds + \int\limits_{\Gamma_R} hw \, ds, \quad w \in H^1(\Omega). \tag{11}$$

Then the weak formulation (8) can be rewritten in a short form: Find $u \in u_0 + H^1_{\Gamma_D}(\Omega)$ such that $\; a(u, w) = F(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega)$.

In what follows we shall need the Friedrichs inequality

$$\|w\|_{0,\Omega} \leq C_{\Omega,\Gamma_D} \|\nabla w\|_{0,\Omega} \quad \forall w \in H^1_{\Gamma_D}(\Omega), \tag{12}$$

and the inequality in the trace theorem

$$\|w\|_{0,\partial\Omega} \leq C_{\partial\Omega} \|w\|_{1,\Omega} \quad \forall w \in H^1(\Omega), \tag{13}$$

where C_{Ω,Γ_D} and $C_{\partial\Omega}$ are positive constants, depending only on Ω, Γ_D, and $\partial\Omega$. Proofs of the inequalities (12) and (13) can be found, e.g., in [18].

The so-called *energy functional* J of problem (8)

$$J(w) := \frac{1}{2}a(w,w) - F(w), \qquad w \in H^1(\Omega), \tag{14}$$

the corresponding *energy norm* is defined as $\sqrt{a(\cdot,\cdot)}$. It is well-known that problem (8) is equivalent to the problem of finding the minimizer of the energy functional J over the set $u_0 + H^1_{\Gamma_D}(\Omega)$ and such a minimizer is the solution of problem (8).

Let $\bar{u}$ be *any function* from $u_0 + H^1_{\Gamma_D}(\Omega)$ (e.g., computed by some numerical method) considered as an approximation of u. We can easily show that

$$a(u-\bar{u}, u-\bar{u}) = 2(J(\bar{u}) - J(u)), \tag{15}$$

which is, probably, the natural reason for measuring the overall accuracy of the approximations in terms of the energy norm (cf. [1], [2], [3], [23]). Thus, the main goal of our work is to show how to effectively estimate from above the value

$$a(u-\bar{u}, u-\bar{u}) = \int_\Omega A\nabla(u-\bar{u})\cdot\nabla(u-\bar{u})\,dx + \int_\Omega c(u-\bar{u})^2\,dx + \int_{\Gamma_R} \sigma(u-\bar{u})^2\,ds. \tag{16}$$

3 Estimation of Error

Let us stand $\|\!|y|\!\|_\Omega := \sqrt{\int_\Omega Ay\cdot y\,dx}$ for $y \in L_2(\Omega, \mathbf{R}^d)$ and $H_{N,R}(\Omega, \mathrm{div}) := \{y \in L_2(\Omega,\mathbf{R}^d) \mid \mathrm{div}\, y \in L_2(\Omega),\ \nu^T\cdot y \in L_2(\Gamma_N \cup \Gamma_R)\}$. In what follows, we shall also employ the denotation χ_S for a characteristic function of set S, i.e., $\chi_S(x) = 1$ if $x \in S$, and $\chi_S(x) = 0$ if $x \notin S$.

Theorem 3.1 *The following (upper) estimate for the error* (16) *holds*

$$a(u-\bar{u}, u-\bar{u}) \le \|\frac{1}{\sqrt{c}}(f + div\, y^* - c\bar{u})\|^2_{0,\Omega^c} + \|\frac{1}{\sqrt{\sigma}}(h - \sigma\bar{u} - \nu^T\cdot y^*)\|^2_{0,\Gamma_R}$$

$$+(1+\alpha)\|\!|A^{-1}y^* - \nabla\bar{u}|\!\|^2_\Omega + (1+\frac{1}{\alpha})(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|\chi_{\Omega\setminus\overline{\Omega}^c}(f + div\, y^* - c\bar{u})\|^2_{0,\Omega} \tag{17}$$

$$+(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C^2_{\Omega,\partial\Omega}\|g - \nu^T\cdot y^*\|^2_{0,\Gamma_N},$$

where α and β are arbitrary positive numbers and y^* is any function from $H_{N,R}(\Omega, \mathrm{div})$.

P r o o f : First of all, we notice that

$$a(u-\bar{u}, u-\bar{u}) = \|\!|\nabla(u-\bar{u})|\!\|^2_\Omega + \|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c} + \|\sqrt{\sigma}(u-\bar{u})\|^2_{0,\Gamma_R}, \tag{18}$$

see (7) for the definition of set Ω^c. Further, using the fact that $u - \bar{u} \in H^1_{\Gamma_D}(\Omega)$, integral identity (8), and the Green formula, we observe (for any function $y^* \in H_{N,R}(\Omega, \mathrm{div})$) that

$$a(u-\bar{u}, u-\bar{u}) = \int\limits_{\Omega} f(u-\bar{u})dx + \int\limits_{\Gamma_N} g(u-\bar{u})\,ds + \int\limits_{\Gamma_R} h(u-\bar{u})\,ds$$

$$-\int\limits_{\Omega} A\nabla\bar{u}\cdot\nabla\,(u-\bar{u})\,dx - \int\limits_{\Omega} c\bar{u}(u-\bar{u})\,dx - \int\limits_{\Gamma_R} \sigma\bar{u}(u-\bar{u})\,ds =$$

$$= \int\limits_{\Omega} (f - c\bar{u})(u-\bar{u})\ dx - \int\limits_{\Omega} (A\nabla\bar{u} - y^*)\cdot\nabla(u-\bar{u})\ dx - \int\limits_{\Omega} y^*\cdot\nabla(u-\bar{u})\ dx$$

$$+\int\limits_{\Gamma_N} g(u-\bar{u})\,ds + \int\limits_{\Gamma_R} (h-\sigma\bar{u})(u-\bar{u})\,ds = \tag{19}$$

$$= \int\limits_{\Omega} (f + \operatorname{div} y^* - c\bar{u})(u-\bar{u})\ dx - \int\limits_{\Omega} A(\nabla\bar{u} - A^{-1}y^*)\cdot\nabla(u-\bar{u})\ dx$$

$$+\int\limits_{\Gamma_N} g(u-\bar{u})\,ds + \int\limits_{\Gamma_R} (h-\sigma\bar{u})(u-\bar{u})\,ds - \int\limits_{\Gamma_N\cup\Gamma_R} \nu^T\cdot y^*(u-\bar{u})\,ds =$$

$$= \int\limits_{\Omega} A(A^{-1}y^* - \nabla\bar{u})\cdot\nabla(u-\bar{u})\ dx + \int\limits_{\Omega} (f + \operatorname{div} y^* - c\bar{u})(u-\bar{u})\ dx$$

$$+\int\limits_{\Gamma_N} (g - \nu^T\cdot y^*)(u-\bar{u})\,ds + \int\limits_{\Gamma_R} (h - \sigma\bar{u} - \nu^T\cdot y^*)(u-\bar{u})\,ds.$$

Now, the right-hand side of equality (19) (denoted as RHS of (19) in what follows) can be estimated, using the Caushy-Schwarz inequality, denotation (7), and trace inequality (13), from above as follows

$$\text{RHS of (19)} \le \|A^{-1}y^* - \nabla\bar{u}\|_{\Omega}\,\|\nabla(u-\bar{u})\|_{\Omega} + \|g - \nu^T\cdot y^*\|_{0,\Gamma_N}\|u-\bar{u}\|_{0,\Gamma_N}$$

$$+\int\limits_{\Omega} (f + \operatorname{div} y^* - c\bar{u})(u-\bar{u})\ dx + \int\limits_{\Gamma_R} (h - \sigma\bar{u} - \nu^T\cdot y^*)(u-\bar{u})\,ds \le$$

$$\le \|A^{-1}y^* - \nabla\bar{u}\|_{\Omega}\,\|\nabla(u-\bar{u})\|_{\Omega} + \|g - \nu^T\cdot y^*\|_{0,\Gamma_N} C_{\partial\Omega}\|u-\bar{u}\|_{1,\Omega} \tag{20}$$

$$+\int\limits_{\Omega^c} \frac{1}{\sqrt{c}}(f + \operatorname{div} y^* - c\bar{u})\,\sqrt{c}(u-\bar{u})\ dx + \int\limits_{\Omega\setminus\overline{\Omega}^c} (f + \operatorname{div} y^* - c\bar{u})\,(u-\bar{u})\ dx$$

$$+\int_{\Gamma_R} \frac{1}{\sqrt{\sigma}}(h-\sigma\bar{u}-\nu^T\cdot y^*)\sqrt{\sigma}(u-\bar{u})\,ds.$$

Further, using the ellipticity condition (5), Friedrichs inequality (12), and the Young inequality

$$2|a\,b| \le a^2+b^2, \tag{21}$$

we observe that

$$\text{RHS of (20)} \le \Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_\Omega + \frac{C_{\partial\Omega}\sqrt{1+C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)\|\!|\nabla(u-\bar{u})\|\!|_\Omega$$

$$+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+\int_\Omega \chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div}y^*-c\bar{u})\,(u-\bar{u})\,dx$$

$$+\frac{1}{2}\|\sqrt{\sigma}(u-\bar{u})\|^2_{0,\Gamma_R}+\frac{1}{2}\|\frac{1}{\sqrt{\sigma}}(h-\sigma\bar{u}-\nu^T\cdot y^*)\|^2_{0,\Gamma_R} \le$$

$$\le \Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_\Omega + C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)\|\!|\nabla(u-\bar{u})\|\!|_\Omega \tag{22}$$

$$+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+\|\chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div}y^*-c\bar{u})\|_{0,\Omega}\,\|u-\bar{u}\|_{0,\Omega}$$

$$+\frac{1}{2}\|\sqrt{\sigma}(u-\bar{u})\|^2_{0,\Gamma_R}+\frac{1}{2}\|\frac{1}{\sqrt{\sigma}}(h-\sigma\bar{u}-\nu^T\cdot y^*)\|^2_{0,\Gamma_R},$$

where $C_{\Omega,\partial\Omega} := \frac{C_{\partial\Omega}\sqrt{1+C^2_{\Omega,\Gamma_D}}}{\sqrt{C_1}}$.

Regrouping terms in RHS of (22) and using again the Young inequality (21), we get an estimate

$$\text{RHS of (22)} \le \Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_\Omega + C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}$$

$$+\frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|\chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div}y^*-c\bar{u})\|_{0,\Omega}\Big)\|\!|\nabla(u-\bar{u})\|\!|_\Omega+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}$$

$$+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\sqrt{\sigma}(u-\bar{u})\|^2_{0,\Gamma_R}+\frac{1}{2}\|\frac{1}{\sqrt{\sigma}}(h-\sigma\bar{u}-\nu^T\cdot y^*)\|^2_{0,\Gamma_R} \le$$

$$\le \frac{1}{2}\Big(\|\!|A^{-1}y^*-\nabla\bar{u})\|\!|_\Omega+C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}+\frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|\chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div}y^*-c\bar{u})\|_{0,\Omega}\Big)^2 \tag{23}$$

$$+\frac{1}{2}\|\!|\nabla(u-\bar{u})\|\!|^2_\Omega+\frac{1}{2}\|\sqrt{\sigma}(u-\bar{u})\|^2_{0,\Gamma_R}+\frac{1}{2}\|\sqrt{c}(u-\bar{u})\|^2_{0,\Omega^c}$$

$$+\frac{1}{2}\|\frac{1}{\sqrt{c}}(f+\operatorname{div}y^*-c\bar{u})\|^2_{0,\Omega^c}+\frac{1}{2}\|\frac{1}{\sqrt{\sigma}}(h-\sigma\bar{u}-\nu^T\cdot y^*)\|^2_{0,\Gamma_R}.$$

Using the final inequality in the above (resulting from (19)–(20) and (22)–(23)), multiplying it by two and regroupping, we immediately get that

$$\|\nabla(u-\bar{u})\|_\Omega^2 + \|\sqrt{c}(u-\bar{u})\|_{0,\Omega^c}^2 + \|\sqrt{\sigma}(u-\bar{u})\|_{0,\Gamma_R}^2 \leq$$

$$\leq \|\frac{1}{\sqrt{c}}(f+\operatorname{div} y^* - c\bar{u})\|_{0,\Omega^c}^2 + \|\frac{1}{\sqrt{\sigma}}(h-\sigma\bar{u}-\nu^T\cdot y^*)\|_{0,\Gamma_R}^2 \qquad (24)$$

$$+\Big(\|A^{-1}y^* - \nabla\bar{u}\|_\Omega + \frac{C_{\Omega,\Gamma_D}}{\sqrt{C_1}}\|\chi_{\Omega\setminus\overline{\Omega}^c}(f+\operatorname{div} y^* - c\bar{u})\|_{0,\Omega} + C_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|_{0,\Gamma_N}\Big)^2.$$

Now using two times obvious inequality $(a+b)^2 \leq (1+\lambda)a^2 + (1+\frac{1}{\lambda})b^2$ $(\lambda > 0)$ for the terms in the round brackets in the above inequality, we finally get (17). □

Remark 3.1 *We notice that in Theorem 3.1 we have not used any assumption that the function $\bar{u}$ is computed by the finite element method or by some another numerical technique. In fact, it is just any function from the set of admissible functions $u_0 + H^1_{\Gamma_D}(\Omega)$.*

Remark 3.2 *The estimate* (17) *is directly computable and is sharp. Really, if one takes $y^* = A\nabla u$, which obviously belongs to $H_{N,R}(\Omega, \operatorname{div})$, then the last two terms in the right-hand side of* (17) *vanish. Further, taking $\alpha = 0$, we finally observe that* (17) *holds as equality with such a choice of y^*.*

Remark 3.3 *The estimate* (17) *contains only two global constants, C_{Ω,Γ_D} and $C_{\partial\Omega}$, which do not depend on the computational process and must be computed only once when the problem* (1)–(4) *is posed. The other existing estimation techniques (of residual-type) involve many unknown constants, usually related to patches of computational meshes used. Those constants are very hard to compute (or even estimate from above) and their evaluation normally leads to a very big overestimation of the error even in simple cases (see* [8] *for examples). Moreover, those constants have to be always re-computed if we perform adaptive computations and change the computational mesh. On the contrary, the constants C_{Ω,Γ_D} and $C_{\partial\Omega}$ remain the same under any change in meshes.*

Remark 3.4 *The solution u minimizes the energy functional, i.e.,*

$$J(u) \leq J(w) \quad \forall w \in H^1_{\Gamma_D}(\Omega).$$

Using this fact, one can always try to get from (15) *a lower bound for the error as follows*

$$a(u-\bar{u}, u-\bar{u}) \geq 2(J(\bar{u}) - J(w)), \qquad (25)$$

where w is any function from $u_0 + H^1_{\Gamma_D}(\Omega)$. Two-sided (upper and lower) estimates of the error in global energy norm can be employed to get bounds for the error measured in terms of linear bounded functionals, see [10], [20] *for more detail and numerical tests on this subject.*

4 Various Choices of Boundary Conditions

Dirichlet boundary condition: In this case, $\Gamma_N = \emptyset$ and $\Gamma_R = \emptyset$, i.e., the second and fifth terms in the right-hand side of estimate (17) do not exist, and we get the following simpler variant of (17)

$$a(u-\bar{u}, u-\bar{u}) \leq \|\frac{1}{\sqrt{c}}(f + \operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega^c}$$

$$+(1+\alpha)|\!|\!| A^{-1}y^* - \nabla \bar{u} |\!|\!|^2_\Omega + (1+\frac{1}{\alpha})\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|\chi_{\Omega\backslash\overline{\Omega}^c}(f + \operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega}. \quad (26)$$

Remark 4.1 *In the case of pure Dirichlet boundary condition we need to estimate from above only one constant* C_{Ω,Γ_D}*.*

Remark 4.2 *If we assume, in adition, that* $c \equiv 0$ *(i.e.,* $\Omega^c = \emptyset$*), then* (17) *reduces to the estimate derived in* [19] *(via the duality theory) and in* [21] *(via the Helmholz decomposition of* $L_2(\Omega, \mathbf{R}^d)$*).*

Dirichlet/Neumann boundary condition: In this case, the set $\Gamma_R = \emptyset$, and estimate (17) takes the form

$$a(u-\bar{u}, u-\bar{u}) \leq \|\frac{1}{\sqrt{c}}(f + \operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega^c} + (1+\alpha)|\!|\!| A^{-1}y^* - \nabla \bar{u} |\!|\!|^2_\Omega$$

$$+(1+\frac{1}{\alpha})(1+\beta)\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|\chi_{\Omega\backslash\overline{\Omega}^c}(f+\operatorname{div} y^*-c\bar{u})\|^2_{0,\Omega}+(1+\frac{1}{\alpha})(1+\frac{1}{\beta})C^2_{\Omega,\partial\Omega}\|g-\nu^T\cdot y^*\|^2_{0,\Gamma_N}. \quad (27)$$

Remark 4.3 *For such mixed boundary conditions, one needs to estimate from above already two constants* C_{Ω,Γ_D} *and* $C_{\partial\Omega}$*.*

Remark 4.4 *For the case* $c \equiv 0$*,* (27) *reduces to the estimate obtained in* [22] *for this type of mixed boundary conditions again using tools of the duality theory.*

Dirichlet/Robin boundary condition: For such type of boundary conditions, estimate (17) takes the following form (due to $\Gamma_N = \emptyset$)

$$a(u-\bar{u}, u-\bar{u}) \leq \|\frac{1}{\sqrt{c}}(f + \operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega^c} + \|\frac{1}{\sqrt{\sigma}}(h - \sigma\bar{u} - \nu^T \cdot y^*)\|^2_{0,\Gamma_R}$$

$$+(1+\alpha)|\!|\!| A^{-1}y^* - \nabla \bar{u} |\!|\!|^2_\Omega + (1+\frac{1}{\alpha})\frac{C^2_{\Omega,\Gamma_D}}{C_1}\|\chi_{\Omega\backslash\overline{\Omega}^c}(f + \operatorname{div} y^* - c\bar{u})\|^2_{0,\Omega}. \quad (28)$$

Remark 4.5 *In this case one has to estimate only one constant, namely,* C_{Ω,Γ_D}*, similarly to the case of pure Dirichlet boundary condition.*

Remark 4.6 *In the case* $c \geq c_0 > 0$ *one need not estimate any constants at all.*

Remark 4.7 *To the author's knowledge such type of mixed boundary condition has never been analysed in the context of a posteriori error estimation so far.*

5 Practical Realization of Estimation Procedure

For convenience, we denote RHS of estimate (17) by the symbol $M_{up}(\bar{u}, y^*, \alpha, \beta)$, sometimes dropping the arguments in it and simply writing M_{up}.

Computation of constants C_{Ω,Γ_D} and $C_{\partial\Omega}$: It follows from the above considerations that for a complete error control we need a reliable estimation of two global constants only. The constant $C_{\Omega,\Gamma_D} = \frac{1}{\sqrt{\lambda_\Omega}}$, where λ_Ω is the smallest eigenvalue the Laplacian in Ω with homogeneous boundary conditions. Thus, computation of λ_Ω (or estimation of it from below) is sufficient. In the case of homogeneous Dirichlet boundary condition this task can be solved as proposed by S. Mikhlin in [17, p. 8] by enclosing the solution domain into a rectangular parallelepiped, for which we can easily obtain the value of the smallest eigenvalue which is smaller than λ_Ω. Namely,

$$C_{\Omega,\Gamma_D} \leq \frac{1}{\pi\sqrt{\frac{1}{a_1^2} + \cdots + \frac{1}{a_d^2}}}, \tag{29}$$

where $a_1, \ldots, a_d$ are lengths of the edges of the parallelepiped.

Suitable estimates of C_{Ω,Γ_D} for some conical domains are presented in [5]. On the contrary, estimation of the constant $C_{\partial\Omega}$ seems to be still an open problem for a general case. However, one trick on estimation of this constant for a quite special case is proposed in [22, Remark 3.3]. More sofisticated techniques for estimation of C_{Ω,Γ_D} and $C_{\partial\Omega}$, suitable for the purposes of a posteriori error analysis, and also another numerical tests, will be presented in our subsequent paper [6].

Minimization of estimate (17): If the approximation $\bar{u}$ is computed by the finite element method, then a "coarse" upper bound can be immediately computed using, e.g., value $y^* = G_{av}(\nabla\bar{u})$, where G_{av} is some commonly used *gradient averaging operator* [7, 11] and easy calculations of suitable values of the other parameters α and β. However, more sharp estimates require a real minimization of the upper bound with respect to the "free" variables y^*, α, β, which can be performed by a direct minimization of it or by finding the minimizer as a solution of the respective system of linear equations.

Mesh adaptation: Estimate (17) can be represented as integral over the solution domain Ω. Thus, let us write this integral as follows

$$M_{up} := \sum_{T \in \mathcal{T}^{(i)}} I_T,$$

where each contribution I_T is a value of the integral taken over a particular element T of the current mesh $\mathcal{T}^{(i)}$. To construct the next mesh $\mathcal{T}^{(i+1)}$ in order to obtain a more accurate approximation, we propose the following adaptive procedure. First, we find the maximum among all terms I_T's and, secondly, mark up those elements T's which have their contributions larger than the "user-given threshold" θ ($\theta \in [0, 1]$) times that maximum value. Refining the marked elements (and making the mesh conforming), we obtain $\mathcal{T}^{(i+1)}$.

6 Numerical Test

Consider problem (1)–(4) posed in a complicated planar domain Ω with a reentrant corner (see Fig. 1 (left)). Let A be the unit matrix, $c \equiv 0$, and $f \equiv 10$. For simplicity, we consider the problem with homogeneous Dirichlet boundary condition only, i.e., $\Gamma_D \equiv \partial\Omega$ and $u_0 \equiv 0$. In this case C_{Ω,Γ_D} can be estimated using Mikhlin's idea by the constant $\frac{\sqrt{2}}{\pi}$ (cf. (29)). The approximation $\bar{u}$ is computed by the finite element method on the mesh $\mathcal{T}$ having 92 nodes. We want to estimate the error in the energy norm, which is equal to $\|\nabla(u-\bar{u})\|_{0,\Omega}$ in our case. Since the exact solution u is unknown, we computed the "exact error" using the so-called *reference solution* (obtained by solving problem (1)–(4) on a very fine mesh, in our example – on 6 times globally refined mesh $\mathcal{T}$) instead of the exact solution u, which approximates the error $\|\nabla(u - \bar{u})\|_{0,\Omega}$ by the value 1.3234. The process of minimization of the coresponding estimate is illustrated in Fig. 1 (right): the square root from the upper bound, $\sqrt{M_{up}}$, is monotonically decreasing from 2.1547 to 1.5634.

It is a common practice to use the so-called effectivity index

$$I_{eff} := \frac{\sqrt{M_{up}(\bar{u}, y^*, \alpha, \beta)}}{\sqrt{a(u - \bar{u}, u - \bar{u})}},$$

which serves as an indicator of the quality of the obtained error estimate. More close it is to 1, the better estimate is. Such effectivity index I_{eff} decays from 1.62 to 1.18 during the minimization process in our test, which demonstrate a high effectivity of the approach proposed.

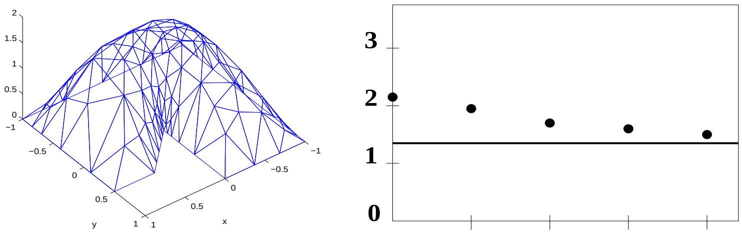

Figure 1: Approximation $\bar{u}$ computed by FEM on mesh $\mathcal{T}$ having 92 nodes (left). Minimization of the estimate $\sqrt{M_{up}}$ (black circles) versus the exact error $\|\nabla(u-\bar{u})\|_{0,\Omega}$ marked by the bold line (right).

Acknowledgement: The author is thankful to Mr. Antti Hannukainen for his help with the numerical test.

References

[1] M. Ainsworth and J. T. Oden. A Posteriori Error Estimation in Finite Element Analysis, John Wiley & Sons, Inc., 2000.

[2] I. Babuška and T. Strouboulis. The Finite Element Method and Its Reliability, Oxford University Press Inc., New York, 2001.

[3] W. Bangerth and R. Rannacher. Adaptive Finite Element Methods for Differential Equations, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 2003.

[4] R. Becker and R. Rannacher. A feed-back approach to error control in finite element methods: Basic approach and examples, East-West J. Numer. Math., **4**, 237–264 (1996).

[5] M. Borsuk and V. Kondratiev. Elliptic Boundary Value Problems of Second Order in Piecewise Smooth Domains. North-Holland Mathematical Library. Elsevier, 2006.

[6] M. Borsuk, A. Hannukainen and S. Korotov. Reliable a posteriori estimates for linear elliptic type boundary value problems (under preparation).

[7] J. Brandts and M. Křížek. Gradient superconvergence on uniform simplicial partitions of polytopes, IMA J. Numer. Anal., **23**, 489–505 (2003).

[8] C. Carstensen and S. A. Funken. Constants in Clément-interpolation error and residual based a posteriori error estimates in finite element methods, East-West J. Numer. Math. **8**, 153–175 (2000).

[9] Ph. G. Ciarlet. The Finite Element Method for Elliptic Problems. Studies in Mathematics and its Applications, vol. 4, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1978.

[10] A. Hannukainen and S. Korotov. Techniques for a posteriori error estimation in terms of linear functionals for elliptic type boundary value problems, Far East J. Appl. Math. **21**, 289–304 (2005).

[11] I. Hlaváček and M. Křížek. On a superconvergent finite element scheme for elliptic systems. I, II, III, Apl. Mat. **32** 131–154, 200–213, 276–289 (1987).

[12] S. Korotov. A posteriori error estimation of goal-oriented quantities for elliptic type BVPs, J. Comput. Appl. Math. (in press).

[13] S. Korotov, P. Neittaanmäki and S. Repin. A posteriori error estimation of goal-oriented quantities by the superconvergence patch recovery, J. Numer. Math. **11**, 33–59 (2003).

[14] M. Křížek and P. Neittaanmäki. Finite Element Approximation of Variational Problems and Applications. Pitman Monographs and Surveys in Pure and Applied Mathematics, 50. Longman Scientific & Technical, Harlow; copublished in USA with John Wiley & Sons, Inc., New York, 1990.

[15] O. A. Ladyzhenskaya, N. N. Uraltseva. Linear and Quasilinear Elliptic Equations. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis Academic Press, New York-London 1968.

[16] C. Lovadina, R. Stenberg. Energy norm a posteriori error estimates for mixed finite element methods, Math. Comp. (in press).

[17] S. G. Mikhlin. Constants in Some Inequalities of Analysis. A Wiley-Interscience Publication. John Wiley & Sons, Ltd., Chichester, 1986.

[18] J. Nečas. Les Méthodes Directes en Théorie des Équations Elliptiques. Academia, Prague, 1967.

[19] S. Repin. A posteriori error estimation for nonlinear variational problems by duality theory, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **243**, 201–214 (1997).

[20] S. Repin. Two-sided estimates of deviation from exact solutions of uniformly elliptic equations, Amer. Math. Soc. Transl. **209**, 143–171 (2003).

[21] S. Repin, S. Sauter and A. Smolianski. A posteriori error estimation for the Dirichlet problem with account of the error in the approximation of boundary conditions, Computing **70**, 205–233 (2003).

[22] S. Repin, S. Sauter and A. Smolianski. A posteriori error estimation for the Poisson equation with mixed Dirichlet/Neumann boundary conditions, J. Comput. Appl. Math. **164/165**, 601–612 (2004).

[23] R. Verfürth. A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques, Wiley-Teubner, 1996.

(continued from the back cover)

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A496 Jan Brandts , Sergey Korotov , Michal Krizek
Dissection of the path-simplex in $\mathbf{R}^n$ into n path-subsimplices
March 2005

A495 Sergey Korotov
A posteriori error estimation for linear elliptic problems with mixed boundary conditions
March 2005

A494 Antti Hannukainen , Sergey Korotov
Computational Technologies for Reliable Control of Global and Local Errors for Linear Elliptic Type Boundary Value Problems
February 2006

A493 Giovanni Formica , Stefania Fortino , Mikko Lyly
A $vartheta$ method–based numerical simulation of crack growth in linear elastic fracture
February 2006

A492 Beirao da Veiga Lourenco , Jarkko Niiranen , Rolf Stenberg
A posteriori error estimates for the plate bending Morley element
February 2006

ISBN 951-22-8098-1
ISSN 0784-3143