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Abstract: *In this paper we consider the quenching problem for the reaction diffusion equation $u_t - \Delta u = f(u)$ with Cauchy-Dirichlet data, in the case where the reaction term is singular at $u = 0$ in the sense that $\lim_{u\downarrow 0} f(u) = -\infty$. For $u > 0$ we take $f(u)$ to be smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$, $k = 0, 1, 2$. Furthermore we assume that the reaction term is weakly singular at the origin in the sense that: $|u^n f^{(n)}(u)| = o(|f(u)|)$, as $u \downarrow 0$, $n = 1$ and $n = 2$.*
We study the equation in $\Omega = \{x \in \mathcal{R}^N : |x| < R\}$ and in the case where the initial function is radial, i.e., $u_0(x) = u_0(r)$ and $u_0'(r) \geq 0$ $(r = |x|)$. We show that quenching occurs for sufficiently large Ω and prove the quenching rate estimate for the problem. We also study a refined asymptotic analysis of the solution near a quenching point for two examples of reaction terms. More precisely, we give an asymptotic expression for the solution with respect to space- and time variable in a backward space-time parabola near a quenching point.

AMS subject classifications: 35K20,35K55,35K57

Keywords: Reaction-diffusion equation, quenching, quenching rate, refined asymptotics

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1 Introduction

Let $\Omega = B_R(0) = \{x \in \mathcal{R}^N; |x| < R\}$. Consider the nonlinear diffusion problem

$$\begin{aligned} u_t - \Delta u = f(u), \qquad & x \in \Omega, \quad t > 0, \\ u(x,t) = 1, \qquad & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad & x \in \overline{\Omega}, \end{aligned} \tag{1.1}$$

where the initial function satisfies $0 < u_0(x) \leq 1$, $\Delta u_0(x) + f(u_0(x)) \leq 0$, $u_0(\pm R) = 1$, $u_0(x)$ is radial, i.e., $u_0(x) = u_0(r)$ and $u_0'(r) \geq 0$, $(r = |x|)$. We assume that the reaction term $f(u)$ is singular at the origin in the sense that $\lim_{u \downarrow 0} f(u) = -\infty$. For $u > 0$ we take $f(u)$ to be smooth and to satisfy $(-1)^k f^{(k)}(u) < 0$; $k = 0, 1, 2$.

Of special interest in the analysis of equation (1.1) has been the situation where the solution $u(x,t)$ approaches zero in finite time. The reaction term then tends to infinity and the smooth solution ceases to exist. This phenomenon is called quenching. We say that a is a quenching point and T is a quenching time for $u(x,t)$, if there exists a sequence $\{(x_n, t_n)\}$ with $x_n \to a$ and $t_n \uparrow T$, such that $u(x_n, t_n) \to 0$ as $n \to \infty$.

In most of the papers that deal with the quenching problem for the equation (1.1), the reaction term is a power singularity, i.e., $f(u) = -u^{-p}, \quad p > 0$. In this case it is well-known that for sufficiently large Ω quenching occurs in finite time [1, 2, 14, 16]. A feature of the quenching problem that has been extensively investigated is the qualitative behavior of solutions and in particular the asymptotic behavior of solutions in space and time near the quenching points. See the detailed review articles [15, 17].

In [19, 21] the equation was studied in the case where we have only a logarithmic singularity, i.e., $f(u) = \ln(\alpha u), \quad \alpha \in (0,1)$, for $\Omega = (-l, l) \subset R^1$. It was shown there that in spite of this weakening of the singularity, quenching occurs for sufficiently large l. Furthermore, it was proved that the set of quenching points is finite. This analysis has been extended to a more general class of weakly singular reaction terms in [20]. More precisely, it was assumed that

$$|u^n f^{(n)}(u)| = o(|f(u)|), \quad n = 1, 2, \tag{1.2}$$

as $u \downarrow 0$. Furthermore, we defined $\tilde{f}(s) = -e^s \cdot \frac{f(e^{-s})}{f'(e^{-s})}$, $s \in R$, and assumed that

$$\tilde{f}(s(1 + o(1))) = (1 + o(1))\tilde{f}(s), \tag{1.3}$$

as $s \to \infty$. This requirement means that for $a(s) \to 0$, as $s \to \infty$ there is $b(s) \to 0$, as $s \to \infty$ such that $\tilde{f}(s(1 + a(s))) = (1 + b(s))\tilde{f}(s)$, as $s \to \infty$. Note that (1.2) implies $\tilde{f}(s) \to \infty$, as $s \to \infty$. A more detailed discussion on (1.2) and (1.3) can be found in [20].

Of particular interest for the quenching problem (1.1) with power singularity has been the analysis of the local asymptotics of the solution as $t \uparrow T$ in a neighborhood of the quenching point. Especially, it has been shown that the quenching-rate satisfies

$$\lim_{t\uparrow T} u(x,t)(T-t)^{-1/(1+p)} = (1+p)^{1/(1+p)}, \tag{1.4}$$

uniformly for $|x-a| < C\sqrt{T-t}$. In one dimensional x-space this result was first established by Guo [9] for $p \geq 3$, and subsequently generalized to $p \geq 1$ by Fila and Hulshof [4]. For the weaker singularity $0 < p < 1$, (1.4) has been shown in [11]. The result (1.4) for higher dimensions has been obtained in [10] for the case $p > 1$ and in [5] for the case $p > 0$. These quenching-rate results have been refined in backward parabolas, see [6].

The main result in [19] concerns the asymptotic behavior of the solution in a neighborhood of a quenching point. It was proved there that the quenching-rate for the logarithmic singularity satisfies

$$\lim_{t\uparrow T} \Big(1 + \frac{1}{T-t}\int_0^{u(x,t)} \frac{d\tau}{f(\tau)}\Big) = 0 \tag{1.5}$$

uniformly, when $|x-a| < C\sqrt{T-t}$, for every $C \in (0,\infty)$. This Theorem was extended to nonlinearities satisfying (1.2) and (1.3) in [20], with symmetric initial data. Note that (1.5) reduces to (1.4), if we substitute $f(u) = -u^{-p}$ in (1.5).

In [21] the quenching-rate estimate (1.5) of [19] was refined in backward parabolas. More precisely, it was proved that (under certain assumptions) for any $C > 0$ and $\varepsilon > 0$ there exists t_0 such that

$$\begin{aligned} \sup_{|x|<C\sqrt{T-t}} &\Big|\frac{u(x,t)}{(T-t)(-\ln(T-t))} - 1 - \frac{(x^2/(T-t)-2)}{8\ln(-\ln(T-t))}\Big| = \\ &O\Big(\frac{\varepsilon}{\ln(-\ln(T-t))}\Big), \end{aligned} \tag{1.6}$$

as $t \geq t_0$. Note that the quenching point is $(0,T)$.

In this paper we prove the result (1.5) for the equation (1.1), where the conditions (1.2) and (1.3) hold and $x \in R^N$. This in done in Theorems 2.11 and 2.16 below. Furthermore we give two examples of refined asymptotics for nonlinearities of type (1.2) and (1.3). These results are formulated in Theorem 3.5.

2 Preliminary results and quenching rate estimates

We first show that for sufficiently large R quenching occurs.

Theorem 2.1. *Assume that $u(x,t)$ is the solution of the equation* (1.1)*, where the reaction term $f(u)$ satisfies* (1.2) *and* (1.3)*. Then for R sufficiently large, $u(x,t)$ quenches in finite time.*

Proof. By [1] it is sufficient to show that the corresponding stationary equation

$$\begin{aligned} \Delta u + f(u) = 0, \qquad & x \in \Omega, \\ u(x) = 1, \qquad & x \in \partial\Omega. \end{aligned} \tag{2.1}$$

does not have a solution $u \in (0,1]$ for sufficiently large x-domains. We assume that (2.1) has a solution, and show that this assumption yields a contradiction for sufficiently large R. We apply the idea of [13] below. Note that by [8] a solution of (2.1) is symmetric, and therefore we study the radial nonlinear Poisson equation with the boundary conditions $u(R) = 1$ and $u'(0) = 0$. Substituting $v(\rho) = u(r) - 1$ in (2.1), where $\rho = r/R$, we can derive that

$$\begin{aligned} v'' + \frac{N-1}{\rho} v' + R^2 f(v+1) = 0, \qquad & \rho \in (0,1), \\ v(1) = 0, \qquad v'(0) = 0. & \end{aligned} \tag{2.2}$$

The corresponding (linear) eigenvalue problem is

$$\begin{aligned} u_n'' + \frac{N-1}{\rho} u_n' = \lambda_n u_n, \qquad & \rho \in (0,1), \\ u_n(1) = 0, \qquad u_n'(0) = 0. & \end{aligned} \tag{2.3}$$

The eigenvalues of (2.3) are negative, i.e., $0 > \lambda_1 > \lambda_2 > ..$, and the corresponding eigenfunctions can be expressed by certain Bessel functions. Denote the radial Laplacian by $\Delta_\rho = \frac{\partial^2}{\partial \rho^2} + \frac{N-1}{\rho}\frac{\partial}{\partial \rho}$, and define the inner product on $L^2(0,1)$ with weight ρ^{N-1} by $\langle f, g \rangle = \int_0^1 f(\rho) g(\rho) \rho^{N-1} d\rho$. From (2.2) and the later part of (2.3) it follows that

$$\langle \Delta_\rho v, u_n \rangle + R^2 \langle f(v(\rho)+1), u_n \rangle = \langle v, \Delta_\rho u_n \rangle + R^2 \langle f(v(\rho)+1), u_n \rangle = 0.$$

Therefore, by (2.3), we get $\lambda_n \langle v, u_n \rangle + R^2 \langle f(v(\rho)+1), u_n \rangle = 0$, for all n. Take $n = 1$ to obtain that $\lambda_1 < 0$ and $u_1 > 0$, when $\rho \in (0,1)$. Then we conclude that

$$R^2 = \frac{-\lambda_1 \langle v, u_1 \rangle}{\langle f(v(\rho)+1), u_1 \rangle} \le M < \infty.$$

Note that $v \in (-1, 0]$, and then that the second term in this equation has an upper bound which is independent of R. Therefore the claim follows from this provided that we choose R large enough. □

We also recall two Theorems from [3] which are valid in the N-dimensional case. The first says that the quenching points are bounded away from the boundary. The second concerns the asymptotic behavior of the solution.

Theorem 2.2. [3] *The set of quenching points is a compact subset of* Ω.

Theorem 2.3. [3] *Assume that the initial function satisfies* $\Delta u_0(r) + f(u_0(r)) \le 0$ *and that quenching occurs at* $t = T$. *Then there exist positive constants* β, l_1 *and* t_1 *such that* (*a*) $u_t - \beta f(u) \leq 0$, *when* $r \in [0, l_1)$ *(the quenching points belong to this interval) and* $t \in [t_1, T)$.

(*b*) u_t *blows up, when* u *quenches.*

(*c*) $u_t(\underline{x}, t) - f(u(\underline{x}, t)) \geq 0$, *when* $t \in (0, T)$, *and* $\underline{x}$ *is a local minimum point of* $u(x, t)$ *with respect to* x.

We shall now study the local asymptotics of the solution $u(x, t)$ as the quenching point $(0, T)$ is approached. Note that for sufficiently large R and for the radial initial data we know that the solution quenches (at least) at $(0, T)$.

We assume in this paper that

$$\Delta u_0(r) + f(u_0(r)) \leq 0. \tag{2.4}$$

This condition guarantees that $u(r, t)$ is decreasing in time (for fixed $r \in [0, R)$).

Define new variables by

$$y = \frac{x}{\sqrt{T - t}}, \qquad s = -\ln(T - t).$$

Then define the function w in terms of these new variables by

$$w(y, s) = 1 + \frac{1}{T - t(s)} \int_0^{u(x(y,s), t(s))} \frac{d\tau}{f(\tau)} = 1 + \frac{1}{T - t} \int_0^{u(x,t)} \frac{d\tau}{f(\tau)}. \tag{2.5}$$

By differentiating (2.5) and using the equation (1.1), we obtain

$$w_s = \Delta w - \frac{1}{2} y \cdot \nabla w + w + F, \tag{2.6}$$

where $F = \frac{|\nabla u|^2}{(f(u))^2} f'(u)$.

Because the initial function $u_0(x)$ is radial, then by the maximum principle the solution $u(x, t)$ of (1.1) is also radial, and consequently the solution $w(y, s)$ of (2.6) is radial. Therefore we need the equation (2.6) in radially symmetric form. By defining $\rho = r/\sqrt{T - t}$, we can write the equation (2.6) in the form

$$\begin{gathered} w_s - w_{\rho\rho} + (\frac{\rho}{2} - \frac{N-1}{\rho}) w_\rho - w = F, \\ w_\rho(0, s) = 0, \quad w(Re^{\frac{1}{2}s}, s) = 1 + e^s \int_0^1 \frac{d\tau}{f(\tau)}, \\ w(\rho, -\ln(T)) = 1 + 1/T \int_0^{u_0(r)} \frac{d\tau}{f(\tau)}, \end{gathered} \tag{2.7}$$

where $(\rho, s) \in (0, Re^{\frac{1}{2}s}) \times (-\ln(T), \infty)$ and $F = \frac{u_r^2}{f(u)^2} f'(u)$.

We embark on the proof of Theorem 2.11. The main ideas are those of [19, 20]; however, significant alternations are necessary in the proofs. We first present a few technical lemmas with proofs in case the proofs of [19, 20] do not carry over.

At first we present the equations (2.8), (2.9) and the inequality (2.10), which are essential in what follows.

From (1.2) we can verify that

$$\int_0^u f(\tau)d\tau = uf(u) + o(|\int_0^u f(\tau)d\tau|) \tag{2.8}$$

and

$$\int_0^u \frac{d\tau}{f(\tau)} = \frac{u}{f(u)} + o(|\int_0^u \frac{d\tau}{f(\tau)}|), \tag{2.9}$$

as $u \to 0$.

By Theorem 2.3 (a) and (c) we have

$$-\int_0^{\underline{u}} \frac{d\tau}{f(\tau)} \leq T - t \leq -C\int_0^{\underline{u}} \frac{d\tau}{f(\tau)} \leq -C\int_0^u \frac{d\tau}{f(\tau)}, \tag{2.10}$$

for some $C > 0$.

Note that in the inequality (2.10) $u = u(x,t)$ is the solution of the equation (1.1) and $\underline{u} = u(0,t)$. But in (2.8) or (2.9) u need not be the solution.

Lemma 2.4. *Let $u(r,t)$ be the solution of (1.1) and $\underline{u} = u(0,t)$. Then $P(r,t) \stackrel{def}{=} \frac{1}{2}u_r^2 + \int_{\underline{u}}^u f(\tau)d\tau \leq 0$, when $(r,t) \in (0,R) \times (0,T)$.*

Proof. By (1.1) we derive that

$$P_r = u_r u_{rr} + u_r f(u) = u_r(u_t - \frac{N-1}{r}u_r).$$

Because by (2.4) $u_t < 0$ and $u_r > 0$, we get $P_r \leq 0$. Furthermore, $P(0,t) = 0$, and the claim follows. ☐

By this Lemma combined with (1.2) (where we take $n = 1$) and (2.8), we get

Lemma 2.5. *Let $u(r,t)$ be the radial solution of (1.1). Then*

(a) $u(r,t) \to 0$ uniformly, when $t \uparrow T$ and $r \leq C\sqrt{T-t}$.

(b) F is uniformly bounded, when $(r,t) \in [0,R] \times [0,T)$.

(c) $F \to 0$ uniformly, when $t \uparrow T$ and $r \leq C\sqrt{T-t}$.

Appropriate bounds for the radial solution $w(\rho,s)$ will be obtained in the sequel.

Lemma 2.6. *There exist positive constants c_1, c_2, c_3 and δ such that for all $s \geq -\ln T$,*

(a) $-c_1 \leq \Delta w(0,s) \leq 0$.

(b) $-c_2 \leq \Delta w(\rho,s)$, when $0 \leq \rho \leq Re^{\frac{1}{2}s}$.

(c) $-c_3\rho \le w_\rho(\rho, s) \le 0$, *when* $0 \le \rho \le Re^{\frac{1}{2}s}$.

(d) $0 \le w(0, s) \le 1 - \delta$.

(e) $-\frac{1}{2}c_3\rho^2 \le w(\rho, s) \le 1 - \delta$, *when* $0 \le \rho \le Re^{\frac{1}{2}s}$.

Proof. Items (a) and (b) can be obtained from the equation

$$\Delta w = \frac{\Delta u}{f(u)} - F,$$

by Lemma 2.5 and Theorem 2.3.

To conclude (c) we use the divergence theorem

$$w_\rho(\rho, s)m(\partial B_\rho) = \int_{\partial B_\rho} n \cdot \nabla w d\sigma_\rho = \int_{B_\rho} \Delta w dx \ge -c_2 m(B_\rho),$$

by item (b). Here m is Lebesgue measure and $B_\rho = \{x \in R^N \big| |x| < \rho\}$. Therefore

$$w_\rho(\rho, s) \ge -c_2 \frac{m(B_\rho)}{m(\partial B_\rho)} \ge -c_3\rho.$$

Items (d) and (e) can now be deduced as in the one dimensional case, see [19]. □

Lemma 2.7. *Let $l_1(\rho) = (1 - \frac{\rho^2}{2N})$ be the second order Laguerre-polynomial and $b(s)$ a bounded function. Then*

(a) $J(s) \stackrel{def}{=} \int_0^{Re^{\frac{1}{2}s}} w(\rho, s)e^{\frac{-\rho^2}{4}}\rho^{N-1}d\rho \to 0$, *when* $s \to \infty$.

(b) $\int_0^{Re^{\frac{1}{2}s}} (w(\rho, s) - b(s)l_1(\rho))e^{\frac{-\rho^2}{4}}\rho^{N-1}d\rho \to 0$, *when* $s \to \infty$.

Proof. Multiply the equation (2.7) by $\rho^{N-1}e^{\frac{-\rho^2}{4}}$ to obtain

$$\begin{aligned} J'(s) - J(s) =& \big((\frac{1}{2}Re^{\frac{1}{2}s}w(Re^{\frac{1}{2}s}, s) + w_\rho(Re^{\frac{1}{2}s}, s)\big)\times \\ & \times (Re^{\frac{1}{2}s})^{N-1}\exp\big(\frac{-(Re^{\frac{1}{2}s})^2}{4}\big) + \int_0^{Re^{\frac{1}{2}s}} F\rho^{N-1}e^{\frac{-\rho^2}{4}}d\rho. \end{aligned} \tag{2.11}$$

We can now conclude, by Lemmas 2.5 and 2.6, that $J'(s) - J(s) \to 0$. The claims (a) and (b) follow as in the one dimensional case.

□

Lemma 2.8. *There exist positive constants γ, δ_1 and δ_2 such that $u_{rt} - \gamma u_t \ge 0$, when $(r, t) \in [0, \delta_1) \times (T - \delta_2, T)$.*

Proof. Let $J = u_{rt} - \gamma u_t$, where the constant γ will be determined later. Then we get

$$J_t - \Delta_r J + (\frac{N-1}{r^2} - f'(u))J = -\frac{N-1}{r^2}\gamma u_t + f''(u)u_r u_t \ge 0,$$

for all $(r, t) \in (0, R) \times (0, T)$. Furthermore we see that $J(0, t) = -\gamma u_t(0, t) \ge 0$. By Theorem 2.2 we can choose γ sufficiently large such that $J \ge 0$ on the parabolic boundary of $(0, \delta_1) \times (T - \delta_2, T)$. An application of the maximum principle now yields the claim. □

By this Lemma we obtain

$$u_t(r,t) - u_t(0,t) = \int_0^r u_{tr}(\eta,t)d\eta \geq \gamma \int_0^r u_t(\eta,t)d\eta =$$

$$\gamma \int_0^r (u_{rr}(\eta,t) + \frac{N-1}{\eta}u_r(\eta,t) + f(u(\eta,t)))d\eta \geq \gamma(u_r(r,t) + rf(u(0,t))).$$

Because $u_t(0,t) - f(u(0,t)) \geq 0$ and $u_r > 0$, we deduce that

$$0 \leq -u_t(r,t) \leq -Cf(u(0,t)), \tag{2.12}$$

for some $C > 0$.

Lemma 2.9. *Let $w(\rho,s)$ be the solution of* (2.7)*. Then* $\limsup_{s\to\infty}(\Delta w)_\rho \leq 0$ *uniformly for bounded ρ.*

Proof. By differentiating the equation (2.6), we obtain $(\Delta w)_\rho = \sum_{i=1}^4 G_i$ where

$$G_1(r,t) = \sqrt{T-t}\frac{u_{rt}}{f(u)}, \quad G_2(r,t) = -3\sqrt{T-t}\frac{u_r u_t}{f(u)^2}f'(u),$$

$$G_3(r,t) = 2\sqrt{T-t}\big(\frac{u_r}{f(u)}(1+F)f'(u) + (\frac{u_r}{f(u)})^2\frac{N-1}{r}f'(u)\big),$$

$$G_4(r,t) = -\sqrt{T-t}\big(\frac{u_r}{f(u)}\big)^2 f''(u)u_r.$$

We shall prove that $\limsup_{t\uparrow T} G_1(r,t) \leq 0$, and that $G_i \to 0$ uniformly for bounded ρ, as $s \to \infty$ and $i = 2,3,4$.

1. and **2.** $i = 1,2$: Using the formulas (1.2), (2.9), (2.10) and (2.12), and Lemmas 2.5, 2.6 and 2.8, we can derive the claims as in [20].

3. $i = 3$: We estimate the two terms of G_3 separately. First we conclude by the definition and Lemma 2.6 that

$$|\sqrt{T-t}\frac{u_r}{f(u)}f'(u)| = |w_\rho(T-t)f'(u)| \leq C\rho(T-t)f'(u).$$

Because

$$T - t \leq C\frac{u}{-f(u)}, \tag{2.13}$$

we get from (1.2) that $\sqrt{T-t}\frac{u_r}{f(u)}f'(u) \to 0$. The rest follows from Lemma 2.5.

In the second step we estimate by the definition of w and ρ, and Lemma 2.6 that

$$\frac{u_r^2}{f(u)^2}\cdot\frac{N-1}{r}f'(u)\sqrt{T-t} = w_\rho^2\cdot\frac{N-1}{r}f'(u)(T-t)^{\frac{3}{2}} \leq$$

$$C\rho^2\frac{N-1}{\rho}f'(u)(T-t).$$

The claim follows from (2.13) and (1.2).

4. $i = 4$: An application of (2.13), Lemma 2.4 and (1.2) (when $n = 2$) we deduce the claim as in [20]. ∎

Lemma 2.10. *Let $w(\rho,s)$ be the solution of* (2.7)*. Then* $\lim_{s\to\infty} w_s(0,s) = 0$.

Proof. We first show as in [19] that there exists a positive constant M such that $(T-t)u_{tt} \le M$ in some neighborhood $N = (-a_1,a_1)\times\cdots\times(-a_N,a_N)\times(T-\delta,T)$ of $(0,T)$. Then by this fact and by a straightforward calculation we obtain that

$$\liminf_{s\to\infty}\big(w_{ss}(0,s) - w_s(0,s)\big) \ge 0. \tag{2.14}$$

Next we prove that $\liminf_{s\to\infty} w_s(0,s) \ge 0$. By Lemma 2.9 it follows that for every $\varepsilon > 0$ and $C > 0$ there exists $s^* \ge -\ln(T)$ such that $(\Delta w)_\rho < \varepsilon$, when $\rho < C$ and $s > s^*$. Integrating with respect to ρ we get $\Delta w(\rho,s) - \Delta w(0,s) < \rho\varepsilon$, when $\rho < C$ and $s > s^*$. Applying the divergence theorem, we obtain

$$w_\rho(\rho,s)m(\partial B_\rho) - \Delta w(0,s)m(B_\rho) < \rho\varepsilon m(B_\rho)$$

and therefore

$$w_\rho(\rho,s) - \Delta w(0,s)\frac{\rho}{N} < \varepsilon\frac{\rho^2}{N}.$$

Recalling the fact that $w_s(0,s) = \Delta w(0,s) + w(0,s)$, we get after integration that

$$\limsup_{s\to\infty}\big(w(\rho,s) + \Delta w(0,s)(1-\frac{\rho^2}{2N}) - w_s(0,s)\big) \le 0, \tag{2.15}$$

uniformly for bounded ρ.

The inequality (2.15) together with Lemmas 2.6 (a) and 2.7 (b) imply that $\liminf_{s\to\infty} w_s(0,s) \ge 0$. Finally this and (2.14) gives the claim. □

Theorem 2.11. *Let $w(\rho,s)$ be the solution of* (2.7)*. Then*

$$w(\rho,s) - w(0,s)\big(1-\frac{\rho^2}{2N}\big) \to 0,$$

as $s\to\infty$, uniformly for bounded ρ.

Proof. By Lemma 2.10 and (2.15) we get

$$\limsup_{s\to\infty}\big(w(\rho,s) - w(0,s)(1-\frac{\rho^2}{2N})\big) \le 0,$$

uniformly for bounded ρ. Combining this with Lemmas 2.6 and 2.7 (b), we conclude the claim. □

Our next goal is to prove Theorem 2.16. In order to do that, we study the equation (2.7) as a dynamical system in the space $L^q_{\rho,r}(R^N)$. Therefore we have to extend the domain of $u(x,t)$ to the entire R^N. This is done in Lemma 2.12 below. Note that we actually first cut a small piece from the x-domain, and then extend the solution $u(r,t)$, where $r \in [0, R-\varepsilon)$ for all $r > 0$. We do this extension in this manner because of Theorem 2.2, and because we are interested in a local asymptotics of the solution.

Lemma 2.12. *Let $\Omega = B_R(0)$ and let $u(x,t)$ be a positive, radially symmetric solution of* (1.1)*, which quenches at $(0,T)$. Let $t_0 \in (0,T)$. Then there exist $r_* \in (0,R)$, $\tilde{u}(x,t)$ and $g(x,t)$ such that $\tilde{u}(x,t) = u(x,t)$, when $|x| \le r_*$, $t_0 \le t < T$ and*

$$\tilde{u}_t - \Delta \tilde{u} = \begin{cases} f(u), & \text{when } |x| \le r_* \\ g(x,t), & \text{when } |x| > r_*. \end{cases}$$

Moreover there exist positive constants C_1 and C_2 such that $C_1 \le \tilde{u}(x,t) + |g(x,t)| \le C_2$, when $(x,t) \in R^N \setminus B_{r_*}(0) \times (t_0,T)$.

Proof. The idea of the proof follows that of [18] where certain blowup problems were studied. Because we know by Theorem 2.2 that the set of quenching points of $u(x,t)$ is a compact subset of $\Omega = B_R(0)$, we can find constans $M_1 > 0$ and $r_0 \in (0,R)$ such that

$$0 < M_1 \le u(x,t) \le 1, \tag{2.16}$$

when $t \in (0,T)$ and $r_0 \le |x| \le R$. Let $0 < r_0 < r_* < r_1 < r_2 < r_3 < R$. By classical regularity theory for parabolic equations, we have

$$u(x,t) + |\nabla u(x,t)| + |\Delta u(x,t)| \le M_2, \tag{2.17}$$

when $t \in (0,T)$ and $r_* \le |x| \le r_3$, for some positive constant M_2. Let now $\xi \in C^\infty(R^N, R)$ be a radially symmetric function such that

$$\begin{cases} \xi(x) = 1, & \text{if } |x| \le r_1, \\ \xi(x) = 0, & \text{if } |x| \ge r_2, \\ 0 < \xi(x) < 1, & \text{if } r_1 < |x| < r_2. \end{cases}$$

Define the extension $\tilde{u}(x,t)$ in $[t_0,T) \times R^N$ by

$$\tilde{u}(x,t) = \begin{cases} \xi(x)u(x,t) + 1 - \xi(x), & \text{when } |x| \le r_3, \\ 1, & \text{when } |x| > r_3. \end{cases}$$

We can verify that $\tilde{u}_t - \Delta \tilde{u} = \tilde{f}$, where $\tilde{f}$ equals

$$\begin{cases} f(u), & \text{if } |x| \le r_1, \\ \xi(x)f(u) - 2\nabla\xi(x) \cdot \nabla u(x,t) - u(x,t)\Delta\xi(x) + \Delta\xi(x), & \text{if } r_1 < |x| < r_3, \\ 0, & \text{if } |x| \ge r_3. \end{cases}$$

Finally we can see that $\tilde{u}(x,t)$ and $g(x,t)$ satisfy the desired properties. □

Let $\tilde{u}(x,t)$ be the extension of $u(x,t)$ to all $x \in R^N$ given by the proof of Lemma 2.12. Define

$$\tilde{w}(y,s) = 1 + \frac{1}{T-t} \int_0^{\tilde{u}(x,t)} \frac{d\tau}{f(\tau)}, \tag{2.18}$$

where y and s are as earlier. Differentiating (2.18), we obtain

$$\tilde{w}_s - \Delta \tilde{w} + \frac{y}{2} \cdot \nabla \tilde{w} - \tilde{w} = \tilde{F}, \tag{2.19}$$

where $\tilde{F} = \frac{|\nabla u|^2}{f(u)^2} f'(u) \equiv F$, when $|y| \le r_* e^{\frac{s}{2}}$; and for $|y| > r_* e^{\frac{s}{2}}$, $\tilde{F}$ is

$$\begin{cases} F, & \text{if } r_* e^{\frac{s}{2}} \le |y| \le r_1 e^{\frac{s}{2}}, \\ \frac{G}{f(\tilde{u})} - 1 + \frac{|\nabla \tilde{u}|^2}{f(\tilde{u})^2} f'(\tilde{u}), & \text{if } r_1 e^{\frac{s}{2}} < |y| < r_3 e^{\frac{s}{2}}, \\ -1, & \text{if } |y| \ge r_3 e^{\frac{s}{2}}, \end{cases}$$

where $G = \xi(x) f(u) - 2\nabla \xi(x) \cdot \nabla u(x,t) - u(x,t)\Delta\xi(x) + \Delta\xi(x)$. We can conclude that $\tilde{w}$ has same properties as w in Lemma 2.6 for $y \in R^N$, i.e.,$\rho \in R$.

Because the solution $\tilde{w}(y,s)$ is symmetric with respect to y, we will need the weighted spaces of radially symmetric functions:

$$L^q_{\rho,r}(R^N) = \{g \in L^q_\rho(R^N) : g(x) = g(|x|) \quad \text{for all} \quad x \in R^N\},$$

$$H^p_{\rho,r}(R^N) = \{g \in H^p_\rho(R^N) : g(x) = g(|x|) \quad \text{for all} \quad x \in R^N\},$$

where

$$L^q_\rho(R^N) = \{g \in L^q_{loc}(R^N) : \int_{R^N} |g(x)|^q \rho(x) dx < \infty\},$$

$$H^p_\rho(R^N) = \{g \in L^2_\rho(R^N) : D^\alpha g \in L^2_\rho(R^N), \quad \alpha = (\alpha_1, ..., \alpha_N), \quad |\alpha| \le p\},$$

with $\rho(x) = \exp\big(-\frac{|x|^2}{4}\big)$, $x \in R^N$ and $D^\alpha g = \frac{\partial^{|\alpha|} g}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_N} x_N}$, where $\alpha = (\alpha_1, ..., \alpha_N)$, α_i is a nonnegative integer for $1 \le i \le N$ and $|\alpha| = \sum_{i=1}^N \alpha_i$. We define the inner product in $L^2_{\rho,r}(R^N)$ by

$$\langle f, g \rangle = \int_{R^N} f(x) g(x) \rho(x) dx,$$

where $f, g \in L^2_{\rho,r}(R^N)$ and the norm in $L^q_\rho(R^N)$ by

$$\|f\|_{q,\rho} = \Big(\int_{R^N} |f(x)|^q \rho(x) dx\Big)^{1/q}.$$

The operator $\mathcal{L}$ with domain $H^2_{\rho,r}(R^N)$ is self-adjoint in $L^2_{\rho,r}(R^N)$, with eigenvalues $\lambda_n = 1 - n$, $(n = 0, 1, 2, ...)$ and corresponding eigenfunctions

$$L_n(x) = c_n (-1)^n \tilde{L}^{(\gamma)}_n \big(\frac{|x|^2}{4}\big),$$

with $\gamma = \frac{N-2}{2}$, where

$$c_n = c_n(N) = (4\pi)^{-N/4} \Big(\frac{\Gamma(N/2)}{n! \Gamma(n + N/2)}\Big)^{1/2},$$

and

$$\tilde{L}^{(\gamma)}_n(r) = e^r r^{-\gamma} \big(\frac{d}{dr}\big)^n (e^{-r} r^{n+\gamma})$$

is the standard n-degree Laguerre polynomial of order γ, so that $\|L_n\|_{2,\rho} = 1$ and Γ denotes the standard Euler's gamma function.

We have now presented the functional framework, which we need in the proof of Theorem 2.16. With this preliminary material at hand, we begin to prove it. Obviously, $\tilde{w}(y,s)$ may be expanded as a Fourier-Laguerre series:

$$\tilde{w}(y,s) = \sum_n b_n(s) L_n(y). \tag{2.20}$$

Lemma 2.13. *Let* $\tilde{b}_1(s) = -b_1(s)\frac{c_1 N}{2}$, *where* $c_1 = \frac{1}{(4\pi)^{\frac{N}{4}}} \cdot \left(\frac{\Gamma(N/2)}{\Gamma(1+N/2)}\right)^{1/2}$. *Then* $\tilde{b}_1(s) - w(0,s) \to 0$, *as* $s \to \infty$.

Proof. Set $\tilde{\phi}(\rho,s) = w(\rho,s) - w(0,s)\big(1 - \frac{\rho^2}{2N}\big)$. Then the projection to $L_1(\rho) = -c_1\big(\frac{N}{2} - \frac{\rho^2}{4}\big)$ yields

$$\langle \tilde{\phi}, L_1\rangle = \langle \sum b_n L_n, L_1\rangle - w(0,s)\langle \big(1 - \frac{\rho^2}{2N}\big), L_1\rangle = b_1(s) - w(0,s)\langle \hat{L}_1, L_1\rangle,$$

where $\hat{L}_1 = 1 - \frac{\rho^2}{2N}$. By Lemma 2.7 $\langle \tilde{\phi}, L_1\rangle \to 0$, and the claim follows from a straightforward calculation. □

The proofs of Lemmas 2.14 and 2.15 below are essentially the same as in the one-dimensional case. For the purposes we have in mind the function $g(s)$, defined by

$$\lim_{u\to 0} \frac{uf'(u)}{-f(u)} g(-\ln(u)) = 1. \tag{2.21}$$

This function satisfies

$$\lim_{s\to\infty} \int_K^s \frac{ds}{g(s)} = \infty. \tag{2.22}$$

The condition (1.3) on slow variation remains true if we replace $\tilde{f}$ by g. This is essential in Lemmas 2.14 and 2.15.

Lemma 2.14. *The inequalities* $0 \leq \; f'(u)(T-t)g(-\ln(T-t)) \leq M < \infty$ *hold on the set* $[-R,R] \times [0,T)$.

Proof. See [20]. □

Lemma 2.15. *For the solution* $u(r,t)$ *one has*

$$f'(u(r,t))(T-t)g(-\ln(T-t)) - \frac{1}{1 - w(0,s)\hat{L}_1(\rho)} \to 0$$

uniformly when $r \leq C\sqrt{T-t}$, *as* $t \uparrow T$.

Proof. We replace h_2 in [20] by $\hat{L}_1$, which is defined in the proof of Lemma 2.13. Otherwise the proof is the same as in [20]. □

Theorem 2.16. $w(0,s) \to 0$, *as* $s \to \infty$.

Proof. This Theorem can be obtained in the same way as the corresponding Theorem in the one-dimensional case in [20] provided minor changes are made. For the readers convenience we present the argument here.

We begin by projecting the equation $w_s = \mathcal{L}w + F$ to the subspace generated by the function $\hat{L}_1$. By the orthogonality and the self-adjointness of the base $\{L_n\}_{n=0}^{\infty}$, we can conlude that

$$Cg(s)b_1'(s) = \int_0^{\infty} (T-t)f'(u)g(-\ln(T-t))w_\rho^2 \hat{L}_1(\rho)\exp(-\tfrac{\rho^2}{4})\rho^{N-1}d\rho.$$

We write the right side as a sum of four term I_i; $i = 1, 2, 3, 4$; where

$$I_1 = \int_0^{\infty} (T-t)f'(u)g(s)(w_\rho^2 - w(0,s)^2\rho^2)\hat{L}_1(\rho)\exp(-\tfrac{\rho^2}{4})\rho^{N-1}d\rho,$$

$$I_2 = \int_0^{\infty} ((T-t)f'(u)g(s) - \tfrac{1}{1-w(0,s)\hat{L}_1(\rho)})w(0,s)^2\rho^2\hat{L}_1(\rho)\exp(-\tfrac{\rho^2}{4})\rho^{N-1}d\rho,$$

$$I_3 = \int_0^{\infty} \tfrac{1}{1-w(0,s)\hat{L}_1(\rho)}(w(0,s)^2 - \tilde{b}_1(s)^2)\rho^2\hat{L}_1(\rho)\exp(-\tfrac{\rho^2}{4})\rho^{N-1}d\rho,$$

and

$$I_4 = \int_0^{\infty} \tfrac{\tilde{b}_1(s)^2\rho^2}{1-w(0,s)\hat{L}_1(\rho)}\hat{L}_1(\rho)\exp(-\tfrac{\rho^2}{4})\rho^{N-1}d\rho.$$

By Theorem 2.11, Lemmas 2.13, 2.14 and 2.15 we may conclude that

$$\limsup_{s\to\infty}(g(s)b_1'(s) + c_3 b_1(s)^2) \le 0,$$

for some positive constant c_3. By this equation combined with (2.22) it follows that $b_1(s) \to 0$, as $s \to \infty$, and also $\tilde{b}_1(s) \to 0$, as $s \to \infty$. Finally, Lemma 2.13 gives the claim. □

3 Examples of refined asymptotics

In this Section we investigate the refined asymptotics of quenching for two examples of reaction terms in (1.1) that satisfy conditions (1.2) and (1.3). The first case is $f(u) = -|\ln(u)|^p$, and the second one is $f(u) = -|\ln(u)|^p - |\ln(u)|^q$, where $p \ge q + 1$.

The considerations below are strongly based on results in [21], where we have used the methods devoloped in [6, 7, 12, 22]. An essential tool in the analysis is the functional framework presented in the previous Section.

The first fundamental ingredient is Corollary 3.1. This result gives the quenching rate-estimate in the form which can be used after the change of variables (3.2). More precisely, (3.2) makes it possible for us to analyze the asymptotics of the solution with respect to space- and time variable in backward parabolas $|x| \le C\sqrt{T-t}$ of the quenching point $(0, T)$.

Corollary 3.1. *Let $u(r,t)$ be the solution of* (1.1)*, where $f(u) = -|\ln(u)|^p$, $p > 0$. Assume that* (2.4) *holds and that $u(r,t)$ quenches at $(0,T)$. Then*

$$\lim_{t \uparrow T} \frac{u(r,t)}{(T-t)(-\ln(T-t))^p} = 1,$$

uniformly when $r \leq C\sqrt{T-t}$.

Proof. We first show that

$$\frac{\ln(T-t)}{\ln(u)} \to 1, \tag{3.1}$$

as $t \uparrow T$ uniformly when $r \leq C\sqrt{T-t}$. By Theorems 2.11 and 2.16, and the formula (2.9) we can obtain that

$$\frac{u}{-u_t(T-t)} + \frac{u}{f(u)(T-t)} = \frac{u}{-f(u)(T-t)}\Big\{\frac{f(u)}{u_t} - 1\Big\} \to 0,$$

and

$$\frac{u}{f(u)(T-t)} \to -1,$$

as $t \uparrow T$ uniformly when $r \leq C\sqrt{T-t}$. Therefore, $\frac{u}{-u_t(T-t)} \to 1$. An application of l'Hospital's rule yields $\frac{\ln(T-t)}{\ln(u(0,t))} \to 1$ and $\frac{\ln(T-t)}{\ln(u(C\sqrt{T-t},t))} \to 1$, because $w_\rho \to 0$ by Theorems 2.11 and 2.16. We derive (3.1) from this and from the fact that $u_r > 0$.

Furthermore, $\frac{f'(u)u}{-f(u)} = \frac{p}{-\ln(u)}$; hence we can take $g(s) = s/p$ in (2.21). From Lemma 2.15 and Theorem 2.16 we get

$$\frac{p}{f'(u(r,t))(T-t)s} \to 1.$$

Finally from this and from (3.1) after some calculations the claim follows. □

Motivated by this Lemma we define

$$\phi(y,s) = \frac{\tilde{u}(x,t)}{(T-t)(-\ln(T-t))^p} - 1, \tag{3.2}$$

where $\tilde{u}(x,t)$ denotes the extended solution, which is defined in Lemma 2.12. From now on we simply denote $u = \tilde{u}$. Then we obtain by (1.1) that

$$\phi_s - \Delta\phi + \frac{y}{2} \cdot \nabla\phi - \phi = \frac{\tilde{f}}{s^p} + 1 - \frac{p}{s}(1+\phi), \tag{3.3}$$

where $\tilde{f}$ is defined as in Lemma 2.12. This equation may be written in the form

$$\phi_s - \Delta\phi + \frac{y}{2} \cdot \nabla\phi - \phi = G + H \cdot \chi_B, \tag{3.4}$$

where $B = \{|y| > r_1 e^{s/2}\}$,

$$G = -\Big(1 - \frac{p\ln(s)}{s} - \frac{\ln(1+\phi)}{s}\Big)^p + 1 - \frac{p}{s}(1+\phi),$$

and

$$H=\Big(1-\frac{p\ln(s)}{s}-\frac{\ln(1+\phi)}{s}\Big)^p+ \\ \frac{\xi(x)f(u)-2\nabla\xi(x)\cdot\nabla u(x,t)-u(x,t)\Delta\xi(x)+\Delta\xi(x)}{s^p}.$$

On B one has, by Lemma 2.12, that $\frac{C_1e^s}{s^p}\le\phi\le\frac{C_2e^s}{s^p}$. Thus: $|H\cdot\chi_B|\le Ce^{-\delta s}|\phi|$.

We study the term G. Write

$$G_1=\Big(1-\frac{p\ln(s)}{s}-\frac{\ln(1+\phi)}{s}\Big)^p.$$

An expansion of this yields

$$G_1=1-\frac{p}{s}(p\ln(s)+\ln(1+\phi))+\frac{p(p-1)}{2s^2}(p\ln(s)+\ln(1+\phi))^2+ \\ O\big[(\frac{p\ln(s)+\ln(1+\phi)}{s})^3\big].$$

Therefore we get

$$G=\frac{p}{s}(p\ln(s)-1)+\frac{p}{s}(\ln(1+\phi)-\phi)-\frac{p^3(p-1)(\ln(s))^2}{2s^2}(1+\frac{\ln(1+\phi)}{p\ln(s)})^2+ \\ O\big[(\frac{p\ln(s)+\ln(1+\phi)}{s})^3\big].$$

Finally,

$$G=\frac{p}{s}\big(p\ln(s)-1-\frac{p^2(p-1)(\ln(s))^2}{2s}\big)+ \\ \frac{p}{s}(\ln(1+\phi)-\phi)-\frac{p^2(p-1)\ln(s)}{s^2}\ln(1+\phi)-\frac{1}{2}p(p-1)\big(\frac{\ln(1+\phi)}{s}\big)^2+ \\ O\big[(\frac{p\ln(s)+\ln(1+\phi)}{s})^3\big]\equiv\sum_{i=1}^{5}\tilde{G}_i. \tag{3.5}$$

After these preliminary considerations we begin to study the refined asymptotics of $\phi(y,s)$ when $|y|\le C$. More precisely, we expand $\phi(y,s)$ in (3.6), and assume that (3.7) holds. Then we apply a dynamical system approach to the problem. Because the solution is symmetric, we study the situation in the space $L^q_{\rho,r}(R^N)$, which we have already presented in the previous Section. Actually the goal is to prove that the term $b_1(s)L_1(y)$ is dominant in (3.6), which will lead to the desired results.

Let

$$\phi(y,s)=\sum_{i=0}^{\infty}b_i(s)L_i(y), \tag{3.6}$$

where

$$|b_1(s)|\ge C\frac{(\ln(s))^2}{s}, \tag{3.7}$$

for some C.

Lemma 3.2. *Let $b_0(s)$ be as in* (3.6). *Then*

$$b_0(s) = -\frac{p^2(1+o(1))}{s}\big(4\pi\big)^{N/4}\ln(s),$$

as $s \to \infty$.

Proof. By projecting the equation (3.4) to the subspace generated by L_0, we get

$$b_0'(s) = b_0(s) + \langle G, L_0\rangle + \langle H \cdot \chi_B, L_0\rangle.$$

Splitting the term $\langle G, L_0\rangle$ as in (3.5) and using Corollary 3.1 we obtain

$$b_0'(s) = b_0(s) + \frac{(4\pi)^{N/4}p^2}{s}(1+o(1))\ln(s).$$

The claim follows from this as in the one dimensional case [21, Lemma 3.2]. □

Lemma 3.3. *Let $b_i(s)$ be defined by* (3.6). *Assume that* (3.7) *holds. Then*

$$\|\sum_{i=2}^{\infty} b_i(s)L_i(y)\|_{2,\rho} = o(|b_1(s)|).$$

Proof. We have now only one positive eigenvalue, i.e., $\lambda = 1$, which corresponds to $b_0(s)\times$ (constant) eigenfunction. Therefore the function $z(t)$ in [21] (see also [7]) is now zero, and we only need inequalities for $\|\phi_0(\underline{y}, s)\|_{2,\rho}$ and $\|\phi_-(\underline{y}, s)\|_{2,\rho}$ to conclude the claim.

(a) Projecting the equation (3.4) to L_1, we derive

$$b_1'(s) = \langle G, L_1\rangle + \langle H \cdot \chi_B, L_1\rangle.$$

First we get from the orthogonality that $\langle \tilde{G}_1, L_1\rangle = 0$. Then by (3.5) we deduce that: $|\langle \tilde{G}_2, L_1\rangle| \leq C\|\tilde{G}_2\|_2 \leq \frac{C}{s}\|\phi\|_2$. Correspondingly, $|\langle \tilde{G}_3, L_1\rangle| \leq \frac{C\ln(s)}{s^2}\|\phi\|_2$ and $|\langle \tilde{G}_4, L_1\rangle| \leq \frac{C}{s^2}\|\phi\|_2$. The term $\tilde{G}_5$ can be split in a part which is only dependent on s and a part which is dominated by $\frac{C}{s}\|\phi\|_2$. Hence we obtain $|\langle G, L_1\rangle| \leq \frac{C}{s}\|\phi\|_2$. On the other hand, it can be checked that $|\langle H \cdot \chi_B, L_1\rangle| \leq Ce^{-\delta s}\|\phi\|_2$. Then we estimate by (3.7) and by Lemma 3.2 that

$$\begin{aligned}\|\phi(\underline{y}, s)\|_{2,\rho} &\leq C\big(|b_0(s)| + |b_1(s)| + \|\phi_-(\underline{y}, s)\|_{2,\rho}\big) \leq \\ &\leq \frac{C}{\ln(s)}|b_1(s)| + |b_1(s)| + C\|\phi_-(\underline{y}, s)\|_{2,\rho}.\end{aligned}$$

After these preliminaries we can conclude that

$$||b_1(s)|_s| \leq \varepsilon\big(|b_1(s)| + \|\phi_-(\underline{y}, s)\|_{2,\rho}\big). \tag{3.8}$$

(b) Projecting the equation (3.4) to the negative eigenspace, multiplying by $\phi_-\rho$ and integrating, we get

$$\int_{R^N}(\frac{\partial}{\partial s}\phi_-)\phi_-\rho = \int_{R^N}(\mathcal{L}\phi_-)\phi_-\rho + \int_{R^N}(\pi_-(G + H \cdot \chi_B))\phi_-\rho.$$

Because the greatest negative eigenvalue is -1 and arguing as in the part (a), we can proceed as in [21] to conclude that

$$\frac{\partial}{\partial s}\|\phi_-(\underline{y},s)\|_{2,\rho} \le -\|\phi_-(\underline{y},s)\|_{2,\rho} + \varepsilon\big(\|\phi_-(\underline{y},s)\|_{2,\rho} + |b_1(s)|\big). \tag{3.9}$$

Now apply [7, Lemma 3.1] to Corollary 3.1, and recall (3.8) and (3.9) to obtain the claim. □

Lemma 3.4. *Let* $\phi(y,s)$ *be the solution of* (3.4)*. Assume that* (3.7) *holds. Then*

$$\|\phi(\underline{y},s)\|_{q,\rho} \le C\|\phi(\underline{y},s)\|_{2,\rho}.$$

Proof. This Lemma can be derived in two steps as in [21].

(1) We first demonstrate the following claim: Let $r > 1$ and $L > 0$. Then there exist $s_0^*(r)$ and $C(r,L) > 0$ such that

$$\|\phi(\underline{y},s+s_0)\|_{r,\rho} \le C\|\phi(\underline{y},s)\|_{2,\rho}, \tag{3.10}$$

for every $s > 0$ and $s_0 \in [s_0^*(r), s_0^*(r)+L]$.

We obtain by Kato's inequality ($\Delta g \cdot sgn(g) \le \Delta(|g|)$ in $\mathcal{D}'(R^N)$), and from (3.4) as in the previous Lemma that

$$\frac{\partial}{\partial s}|\phi(y,s)| \le \mathcal{A}|\phi(y,s)| + (1+c)|\phi(y,s)| + \tilde{G}_1(s),$$

where $\mathcal{A} = \Delta - \frac{y}{2}\cdot\nabla$. From this we can verify as in [21] that

$$\|\phi(\underline{y},s+s_0)\|_{r,\rho} \le C\|e^{s_0\mathcal{A}}|\phi(\underline{y},s)|\|_{r,\rho} + C\tilde{G}_1(s).$$

The right-hand side of this inequality can be estimated in two steps. First we use [22, Proposition 2.1] to obtain the corresponding inequality as [21, Lemma 3.5], and then apply this to the first term. The second term can be handled by Lemma 3.2. These two inequalities yield the claim (3.10).

(2) In the second step we prove that: Let $L > 0$. If (3.7) holds, then there exists $C = C(L)$ such that

$$\|\phi(\underline{y},s)\|_{2,\rho} \le C\|\phi(\underline{y},s+L)\|_{2,\rho}, \tag{3.11}$$

for all s.

We project the equation (3.3) to the subspace generated by L_1, and then notice that the necessary analysis may be carried out as in the one-dimensional case [21]. □

We shall now derive an ordinary differential equation for $b_1(s)$. This is done in the equation (3.15) below. By projecting the equation (3.3) to the subspace generated by L_1 we get by (3.5) that

$$b_1'(s) = \langle\sum_{i=2}^{5}\tilde{G}_i, L_1\rangle + \langle H\cdot\chi_B, L_1\rangle.$$

Here

$$\langle \tilde{G}_2, L_1\rangle = \frac{p}{2s}\langle -\phi^2 + O(|\phi|^3), L_1\rangle.$$

The terms $\tilde{G}_i$, when $i = 3, 4, 5$, can be estimated as follows. By Lemmas 3.2, 3.3, and by the relations (3.5) and (3.7) we get

$$\begin{aligned} |\langle \tilde{G}_3, L_1\rangle| &\leq C\Big(\frac{\ln(s)}{s^2}\Big)\|\phi(\underline{y}, s)\|_{2,\rho} \leq C\Big(\frac{\ln(s)}{s^2}\Big)\Big(C\frac{\ln(s)}{s} + C|b_1(s)|\Big) \leq \\ &\leq C\Big(\frac{(\ln(s))^2}{s^3}\Big) + C\Big(\frac{\ln(s)}{s^2}\Big)|b_1(s)| \leq \frac{C}{s^2}\Big(1 + C\ln(s)\Big)|b_1(s)|. \end{aligned} \tag{3.12}$$

Similarly,

$$|\langle \tilde{G}_4, L_1\rangle| \leq \frac{C}{s^2}O(1)|b_1(s)|, \tag{3.13}$$

and

$$|\langle \tilde{G}_5, L_1\rangle| \leq C\Big(\frac{\ln(s)}{s}\Big)^3 \leq C\frac{\ln(s)}{s^2}\cdot\frac{(\ln(s))^2}{s} \leq \frac{C\ln(s)}{s^2}|b_1(s)|. \tag{3.14}$$

Then we conclude that: $|\langle H\cdot\chi_B, L_1\rangle| \leq Ce^{-\delta s}\|\phi\|_2$.

Gathering these items we obtain

$$sb_1'(s) = -\frac{p}{2}\langle \phi^2, L_1\rangle + \langle O(|\phi|^3), L_1\rangle + O(1)\frac{\ln(s)}{s}b_1(s).$$

Next we estimate the term $\langle \phi^2, L_1\rangle$. Let: $\phi = b_0L_0 + b_1L_1 + \phi_r$. Then we get from the orthogonality that

$$\langle \phi^2, L_1\rangle = \langle b_1^2L_1^2, L_1\rangle + \langle \phi_r^2, L_1\rangle + 2b_0b_1\langle L_0L_1, L_1\rangle + 2b_1\langle L_1\phi_r, L_1\rangle \equiv \sum_{j=1}^{4} I_j.$$

We obtain that $\frac{p}{2}I_1 = c_1b_1(s)^2$, where $c_1 = \frac{p}{2}\langle L_1^2, L_1\rangle$ and $I_3 = -2\frac{p^2(1+o(1))\ln(s)}{s}b_1(s)$ from Lemma 3.2. The term $I_2 + I_4$ can be analyzed as in [21] to derive that $|I_2 + I_4| \leq Co(1)|b_1(s)|\max\Big\{C\frac{\ln(s)}{s}, |b_1(s)|\Big\}$.

On the other hand by Lemma 3.4 we get

$$|\int_R \phi^3(z)L_1(z)z^{N-1}e^{-z^2/4}dz| \leq C\|\phi\|_{6,\rho}^3 \leq C\|\phi\|_{2,\rho}^3.$$

These estimates yield

$$sb_1'(s) = -c_1b_1(s)^2(1+o(1)) + (2p^2(1+o(1)) + O(1))\frac{\ln(s)}{s}b_1(s) + O\big[(\frac{\ln(s)}{s})^3\big]. \tag{3.15}$$

We remark now that this equation is essentially the same as [21, (3.42)]. Therefore $b_1(s) = \frac{c^*+o(1)}{\ln(s)}$, as $s\to\infty$. We formulate this as

Theorem 3.5. *Let $f(u) = -|\ln(u)|^p$ in (1.1), where $p > 0$. Assume that the initial function $u_0(r)$ is symmetric and that (2.4) and (3.7) hold. Let $u(r,t)$ quench at $(0,T)$. Then for any $C > 0$ and $\varepsilon > 0$ there exists t_0 such that*

$$\begin{aligned}\sup_{r<C\sqrt{T-t}} \Big| \frac{u(r,t)}{(T-t)(-\ln(T-t))^p} - 1 - \frac{N(\frac{r^2}{4(T-t)} - \frac{N}{2})}{2p\ln(-\ln(T-t))} \Big| = \\ O\Big(\frac{\varepsilon}{\ln(-\ln(T-t))}\Big),\end{aligned} \tag{3.16}$$

when $t > t_0$.

To complete the proof of this Theorem we have to determine the constant c^* above. This is done in two steps. First we derive that

$$c_1 = \frac{p}{2}\langle L_1^2, L_1\rangle = \frac{p}{128}\int_{S_{N-1}} d\theta\cdot\big(4\pi\big)^{-3N/4}\cdot\Big(\frac{2}{N}\Big)^{3/2}\int_0^\infty \big(r^2-2N\big)^3 e^{-r^2/4} r^{N-1} dr.$$

Then we calculate the integrals and apply properties of the gamma function to conclude (3.16).

At the end of this Section we study the equation (1.1) when

$$f(u) = -|\ln(u)|^p - |\ln(u)|^q, \tag{3.17}$$

where $p \geq q + 1$.

First one can re-examine the proof of Corollary 3.1 to conclude that the claim of that Corollary holds for (3.17). Then we change the variables as in (3.2) and use the extension of Lemma 2.12. After some calculations we obtain that $\phi(y,s)$ satisfies

$$\phi_s - \Delta\phi + \frac{y}{2}\cdot\nabla\phi - \phi = G + H\cdot\chi_B, \tag{3.18}$$

where $B = \{|y| > r_1 e^{s/2}\}$,

$$\begin{aligned} G = -\Big(1 - \frac{1}{s}[p\ln(s) + \ln(1+\phi)]\Big)^p + 1 - \\ - \frac{p}{s}(1+\phi) - s^{q-p}\Big(1 - \frac{1}{s}[p\ln(s) + \ln(1+\phi)]\Big)^q, \end{aligned}$$

and

$$\begin{aligned} H = \Big(1 - \frac{1}{s}[p\ln(s) + \ln(1+\phi)]\Big)^p + s^{q-p}\Big(1 - \frac{1}{s}[p\ln(s) + \ln(1+\phi)]\Big)^q + \\ + \frac{\xi(x)f(u) - 2\nabla\xi(x)\cdot\nabla u(x,t) - u(x,t)\Delta\xi(x) + \Delta\xi(x)}{s^p}. \end{aligned}$$

An expansion of G now yields

$$G = \sum_{i=1}^{5}\tilde{G}_i + s^{q-p}\Big(1 - \frac{q}{s}[p\ln(s) + \ln(1+\phi)] + \frac{q(q-1)}{2s^2}[p\ln(s) + \ln(1+\phi)]^2 + \ldots\Big),$$

by (3.5). This equation can be written in the form $G = \sum_{i=1}^{5} \overline{G}_i$, where

$$\overline{G}_1 = \frac{p}{s}\big(p\ln(s) - 1 - \frac{p^2(p-1)(\ln(s))^2}{2s}\big) + s^{q-p}(1+o(1)),$$

$$\overline{G}_2 = \frac{p}{s}(\ln(1+\phi) - \phi), \quad \overline{G}_3 = -\frac{p^2(p-1)\ln(s)}{s^2}\ln(1+\phi)(1+o(1))$$

$$\overline{G}_4 = -\frac{1}{2}p(p-1)\big(\frac{\ln(1+\phi)}{s}\big)^2(1+o(1)), \quad \overline{G}_5 = O\big[\big(\frac{p\ln(s)+\ln(1+\phi)}{s}\big)^3.$$

We observe that $\overline{G}_2 = \tilde{G}_2$ and that $\overline{G}_i = (1+o(1))\tilde{G}_i$, when $i = 1, 3, 4, 5$ ($\tilde{G}_i$ defined in (3.5)). Then we can conclude that Lemmas 3.2-3.4 remain true for the reaction term (3.17) in (1.1). Because $\overline{G}_2 = \tilde{G}_2$, we obtain that the multiplier function $b_1(s)$ in (3.6) satifies the equation (3.15). Finally, by these remarks, we derive that the claim of Theorem 3.5 holds for the reaction terms (3.17).

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