Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2003 A458

TRANSLATION-INVARIANT OPERATORS ON SPACES OF VECTOR-VALUED FUNCTIONS

Tuomas Hytönen

$$|m(\xi)|\,,\ |\xi|\left|\frac{\partial m}{\partial \xi_i}(\xi)\right|,\ |\xi|^2\left|\frac{\partial^2 m}{\partial \xi_j \partial \xi_k}(\xi)\right| \leq c \qquad \forall\ 1 \leq i,j,k \leq 3,\ j \neq k,\ \forall\ \xi \in \mathbb{R}^3 \setminus \{0\}$$
$$\implies \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} m(\xi)\hat{f}(\xi) e^{\mathbf{i}2\pi x\cdot\xi}\,\mathrm{d}\xi \right|^p \mathrm{d}x \leq C \int_{\mathbb{R}^3} |f(x)|^p\,\mathrm{d}x \qquad \forall\ p \in\,]1,\infty[$$

Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 2003 A458

TRANSLATION-INVARIANT OPERATORS ON SPACES OF VECTOR-VALUED FUNCTIONS

Tuomas Hytönen

Dissertation for the degree of Doctor of Science in Technology to be presented, with due permission of the Department of Engineering Physics and Mathematics, for public examination and debate in auditorium E of Helsinki University of Technology on April 25th, 2003, at 12 o'clock noon.

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics

Tuomas Hytönen: *Translation-invariant operators on spaces of vector-valued functions*; Helsinki University of Technology Institute of Mathematics Research Reports A458 (2003).

AMS subject classifications: 42B15, 42B20, 46E40 (Primary);
34G10, 42B30, 42B35, 46B09 (Secondary).

tuomas.hytonen@hut.fi

ISBN 951-22-6456-0
ISSN 0784-3143
Printed by Otamedia
Espoo, 2003

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
Box 1100, 02015 Otaniemi, Finland
email:math@hut.fi http://www.math.hut.fi/

HELSINKI UNIVERSITY OF TECHNOLOGY BOX 1000, FIN-02015 OTANIEMI http://www.hut.fi	ABSTRACT OF DOCTORAL DISSERTATION
Author	Tuomas Pentinpoika Hytönen
Name of the dissertation *Translation-invariant operators on spaces of vector-valued functions*	
Date of manuscript 24.3.2003	Date of dissertation 25.4.2003
Form of dissertation	Monograph
Department Laboratory Field of research Opponent Instructor and supervisor	Engineering Physics and Mathematics Institute of Mathematics Fourier analysis, functional analysis Prof. Peter Sjögren (Göteborg) Prof. Stig-Olof Londen

Abstract: The treatise deals with translation-invariant operators on various function spaces (including Besov, Lebesgue–Bôchner, and Hardy), where the range space of the functions is a possibly infinite-dimensional Banach space X. The operators are treated both in the convolution form $Tf = k * f$ and in the multiplier form in the frequency representation, $\widehat{Tf} = m\hat{f}$, where the kernel k and the multiplier m are allowed to take values in $\mathcal{L}(X)$ (bounded linear operators on X). Several applications, most notably the theory of evolution equations, give rise to non-trivial instances of such operators.

Verifying the boundedness of operators of this kind has been a long-standing problem whose intimate connection with certain randomized inequalities (the notion of "R-boundedness" which generalizes classical square-function estimates) has been discovered only recently. The related techniques, which are exploited and developed further in the present work, have proved to be very useful in generalizing various theorems, so far only known in a Hilbert space setting, to the more general framework of UMD Banach spaces.

The main results here provide various sufficient conditions (with partial converse statements) for verifying the boundedness of operators T as described above. The treatment of these operators on the Hardy spaces of vector-valued functions is new as such, while on the Besov and Bôchner spaces the convolution point-of-view taken here complements the multiplier approach followed by various other authors. Although general enough to deal with the vector-valued situation, the methods also improve on some classical theorems even in the scalar-valued case: In particular, it is shown that the derivative condition

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq C \qquad \forall \alpha \in \{|\alpha|_\infty \leq 1\} \cap \{|\alpha|_1 \leq \lfloor n/2 \rfloor + 1\}$$

is sufficient for m to be a Fourier multiplier on $L^p(\mathbb{R}^n)$, $p \in]1, \infty[$—the set of required derivatives constitutes the intersection of the ones in the classical theorems of S. G. Mihlin and L. Hörmander.

Keywords: Operator-valued Fourier multiplier; singular convolution integral; Lebesgue–Bôchner, Hardy, and Besov spaces of vector-valued functions; Mihlin's theorem; Hörmander's integral condition; R-boundedness; UMD space; Fourier embedding; evolution equation; maximal regularity.

UDC 517.9	Number of pages 194
ISBN (printed) 951-22-6456-0	ISBN (pdf) 951-22-6457-9
ISBN (others)	ISSN 0784-3143
Publisher Helsinki University of Technology, Institute of Mathematics	
The dissertation can be read at http://lib.hut.fi/Diss/	

This dissertation consists of the introductory Chapter 0, four main Chapters 1–4, and an epilogue in Chapter 5. Out of the main chapters, Chapters 1 and 4 are entirely based on my individual work, whereas Chapters 2 and 3 represent the output of joint research with Prof. Lutz Weis (Karlsruhe). The main theorems of both Chapters 2 and 3 were first conjectured, and their initial proofs sketched, by Lutz Weis, whereas substantial details were worked out by myself.

Contents

Preface

Writing this dissertation was an exciting adventure. Not only did it take me to various abstract spaces of vector-valued functions, but also to such exotic corners of the world as Karlsruhe in Baden-Württenberg and Columbia in South-Carolina; and although my work dealt with operators which are translation-invariant, my own journey was far from being that, and each geographic location entailed some new people and ideas which helped bring this work to its completion.

Of course, I owe my advisor STIG-OLOF LONDEN a lot for leading me to such an interesting path to walk. The beginning of this journey must be traced back to our first discussions on my Master's thesis [**44**] in the late Summer 2000: the topic he proposed and I would study, *R-boundedness and multiplier theorems*, and the thesis I wrote on this in Spring 2001, turned out to be my ticket to the wide world—something my advisor perhaps could foresee, but I certainly did not.

At two conferences around these themes (one in Blaubeuren, Germany, and the other in Delft, the Netherlands) in Summer 2001, I had the chance to meet many of the experts in the field, and with my Master's thesis at hand, an excuse to talk with some of them. These included PHILIPPE CLÉMENT who had already proposed the Hardy space framework of Chapter 1 for my research during the time of my finishing of my Master's degree, and also LUTZ WEIS, whose operator-valued Mihlin theorem [**87**] was the culmination of my thesis, and who would become an even more prominent character in my doctoral work later on.

I spent much of the Summer and Autumn of 2001 working with the problems in Chapter 1, and I am indebted to PHILIPPE CLÉMENT for numerous fruitful discussions and suggestions during this period, and to STIG-OLOF LONDEN for many helpful conversations, and for proof-reading my manuscripts. JUHA KINNUNEN was also always ready to discuss mathematics, and often pointed me to useful references, also at later phases of my work.

There is no doupt that the single event most influential in shaping this dissertation was my invitation, by LUTZ WEIS, to spend some months in Karlsruhe in the coming year 2002. By the time of my departure on March 1st, I had completed the proof of the Main Theorem 1.6 of Chapter 1, and was ready to consider a new challenge. This would be the problem of boundedness of singular integrals with operator-valued kernel which I would study with LUTZ WEIS, first on the Lebesgue–Bôchner and then on the Besov spaces of vector-valued functions. Chapters 2 and 3 represent the output of this joint research carried out during my visit to Karlsruhe from March to July in 2002. Our collaboration was most successful and inspiring, and continued in Columbia, South-Carolina, in the last two months of the year, around a topic falling outside the scope of the present work.

During my stay both in Karlsruhe and later Columbia, I had many mathematical discussions with, and lots of assistance in practical matters from MARIA GIRARDI and CORNELIA KAISER. I am also most grateful to MARIA GIRARDI and LUTZ WEIS for kindly giving me the access to some still unfinished manuscripts of their papers.

Back home from Karlsruhe in August, I started developing some ideas on new kinds of Fourier multiplier theorems—these considerations, building on and tieing together the results of Chapters 1–3, but also introducing some new ideas which I found very fascinating to discover, make up the last Chapter 4. During my writing of this chapter in August–October 2002, my advisor's careful proof-reading helped locate numerous typographical bugs and notational inconsistencies, and improved the style of the presentation substantially.

The "honest work" on this dissertation was essentially finished when I left for Columbia for two months on October 24th; during my stay there, I completed some remaining details and wrote the introductory Chapter 0.

In addition to the ones already mentioned, I wish to thank GIOVANNI DORE for answering (in less than an hour!) my e-mail enquiry on maximal regularity; STEFAN GEISS for sharing his knowledge on the UMD-spaces; NIGEL KALTON for offering his insights on "$p < 1$" at a couple of occasions; PEER KUNSTMANN for several discussions on Hardy spaces in Karlsruhe; IRENA LASIECKA for pointing me to the interesting potential applications of the theory developed in Chapter 1 and for inviting me to give a talk on this topic at the *Workshop on Nonlinear Wave Equations* in Charlottesville in December 2002; and HANS-OLAV TYLLI for bringing a useful reference to my knowledge. I am also grateful to KARI ASTALA and EERO SAKSMAN for refereeing this dissertation and for locating several misprints and making constructive suggestions to improve the final form of the work. My special thanks are addressed to all those I forgot to mention.

During this research, I was financially supported by the Marie Curie Fellowship of the European Union while in Karlsruhe, and by the Magnus Ehrnrooth Foundation starting from my return.

Giving a proper account of the contribution of my close ones to the life I live, therefore also to the finishing of this dissertation, is beyond the level of the present treatment.

Otaniemi, 24.3.2003 T. H.

CHAPTER 0

Historical introduction and overview

1. Introduction

Numerous interesting operators of mathematical analysis, as well as of applied fields, possess a translation-invariant structure. This is not surprising in view of the somewhat heuristic but very general principle of "relativity": The intrinsic phenomena in a given space (whether mathematical or physical) are independent of the particular choice of a coordinate representation, and in particular of the choice of the origin. This principle governs such mathematical transformations as the (in fact prototypical) harmonic conjugation, time-evolution operators of autonomous differential equations, and even the laws of physics, which are generally believed to remain invariant under the action of spatial translations.

Thus the sources of translation-invariant operators T are abundant, but what do these operators actually look like, say, on the Lebesgue spaces $L^p(\mathbb{R}^n)$? In his classical paper [**43**], L. HÖRMANDER showed that those T which, in addition, are linear and bounded, are always representable in the form $Tf = k * f$ of a *singular convolution operator*, or equivalently, in the form $\widehat{Tf} = m\hat{f}$ of a *Fourier multiplier transformation*. (Here the hat $\hat{}$ denotes the Fourier transform, k is a tempered distribution, and $m = \hat{k}$; the two formulae above are valid, at least, for all f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$, a dense subspace of $L^p(\mathbb{R}^n)$ whenever $p \in [1, \infty[$.)

The difficult question is the converse: to determine whether or not a particular *kernel* k, or the corresponding *multiplier* m, gives rise to a bounded operator in the way described above. This remains a challenging problem even in the classical situation considered in [**43**]; several sufficient conditions, meeting the needs of multiple applications, have been obtained by various authors, but a complete answer is only known for $p \in \{1, 2, \infty\}$. (See e.g. [**76, 78**].)

This dissertation examines the question of boundedness of translation-invariant operators on several spaces of *vector-valued* functions, where a vector refers to a point of a possibly infinite-dimensional Banach space X. The task of developing a reasonable theory in this setting, preferably allowing for *operator-valued* multipliers and kernels, has been studied since the 60's and constitutes a highly non-trivial generalization of the classical scalar-valued results. More precisely, *some* of the scalar-valued results do generalize to the vector-valued situation as soon as absolute values are replaced by norms; some turn out to be simply false in infinite-dimensional spaces; but there are various central results which call for generalization but require considerable new effort and possibly completely new ideas to go through in the more general setting. Some of the essential tools for doing this have been discovered only very recently.

The call for the vector-valued extensions is also related to significant applications, in addition to being of purely theoretical interest. In fact, the vector-valued extension of the theory was motivated, from the beginning, by the needs of the functional analytic approach to partial differential equations. One of the basic questions in this field, the so-called *maximal regularity* problem for the abstract evolution equation

$$\dot{u}(t) + Au(t) = f(t) \qquad (t \in [0, \infty[), \qquad u(0) = 0 \tag{1.1}$$

is to determine, for a given operator A, whether the solution map $f \mapsto Au$ maps $L^p(\mathbb{R}_+; X)$ into itself. This map is translation-invariant in the positive direction, in the sense that $u(\cdot - h)$ is the solution of (1.1) with $f(\cdot - h)$ in place of f, whenever $h \geq 0$, and we extend f and u to the negative half-line by zero-fill. Moreover, $f \mapsto Au$ can be extended, in a natural way, to an operator on $L^p(\mathbb{R}; X)$ (instead of $L^p(\mathbb{R}_+; X)$), which is translation-invariant in both directions.

In typical applications, the Banach space X could be $L^q(U)$ (for some $q \in]1, \infty[$, and U a domain in $\mathbb{R}^n$, say), and A the realization of some differential operator $\sum a_\alpha(x) D^\alpha$, with appropriate boundary conditions, on this space. The equation (1.1) would hence be short-hand for

$$\begin{aligned} \frac{\partial u}{\partial t} + \sum_\alpha a_\alpha(x) \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} &= f(t, x) \qquad && t \in [0, \infty[\,, \; x = (x_1, \ldots, x_n) \in U, \\ u(0, x) &= 0 && x = (x_1, \ldots, x_n) \in U, \end{aligned}$$

with boundary conditions.

Even more importantly, (1.1) could represent the linearization of some non-linear differential equation: A common line of attack on non-linear problems is via fixed-point arguments, in which the maximal regularity property of the linearized problem plays a decisive rôle.

When $-A$ is the generator of an analytic semigroup e^{-tA}, the variation-of-constants formula is valid for the solution u of (1.1), and, moreover, Au is given in the form

$$Au(t) = \int_0^t A e^{-sA} f(t - s) \, \mathrm{d}s, \tag{1.2}$$

at least for $f \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}_+; \mathcal{D}(A))$, in which case it is legitimate to bring the unbounded operator A inside the integral. This is recognized as the convolution of f with the operator-valued kernel $k(t) := Ae^{-tA} \chi_{]0,\infty[}(t)$; it is a bounded operator for each fixed t, but the kernel is singular at the origin, having norm proportional to t^{-1} as $t \downarrow 0$. One can also write (1.2) in the frequency representation as a Fourier multiplier transformation

$$\widehat{Au}(\xi) = A(\mathbf{i}2\pi\xi + A)^{-1} \hat{f}(\xi) =: m(\xi) \hat{f}(\xi), \tag{1.3}$$

with an operator-valued multiplier m. [A heuristic derivation of this formula is simply to take the Fourier transform of both sides (1.1); for a rigorous argument, see Prop. 3.7 of Chapter 1.]

To appreciate the non-trivial nature of the vector-valued theory of translation-invariant operators required by the maximal regularity problem, note that its solution, except in Hilbert spaces, remained open for almost 40 years since the 60's, when pioneering work was made by L. de Simon [**74**] and by P. E. Sobolevskij [**75**]. The problem attracted considerable interest over the decades, and important partial solutions were given by T. Coulhon and D. Lamberton [**24**], by G. Dore and A. Venni [**29**], by M. Hieber and J. Prüss [**42**], and by Lamberton [**56**], but a fairly complete answer was only obtained in the turn of the millennium by N. Kalton and G. Lancien [**49**] and by L. Weis [**87**]. Moreover, this solution did not come out of void but was built on several fundamental ideas developed in the decades in between; in particular, it required some deep insights into the geometry of Banach spaces due to J. Bourgain [**10, 12**], D. L. Burkholder [**13, 14, 15**], and J. Lindenstrauss and L. Tzafriri [**58**].

Agreeing that translation-invariant operators on spaces of vector-valued functions are interesting, the specific problems in the field to be studied in this dissertation are the following:

- *The extension of* Weis*' maximal regularity results, as well as the more general theory of translation-invariant operators, from the L^p-setting ($p \in]1,\infty[$) down to the real-variable Hardy spaces H^p ($p \in]0,1]$).* While the Hardy spaces are theoretically interesting as the natural continuation of the L^p-scale for exponents $p \leq 1$, there is also a call for such a theory from certain applications. To this call, Chapter 1 seems to provide the first answers.
- *A description of sufficient conditions for the $L^p(\mathbb{R}^n; X)$-boundedness of an operator $f \mapsto Tf = k * f$ in terms of the convolution kernel k.* This approach taken in Chapter 2 complements the operator-valued Fourier multiplier theorems recently proved in [**2, 22, 25, 36, 40, 80, 87**], where the operators are written in the multiplier form $\widehat{Tf} = m\hat{f}$. While the two forms are equivalent, some operators in applications appear more naturally in the convolution form, and thus it is useful to have a device for checking their boundedness directly in this representation.
- *The investigation of singular, operator-valued convolution integrals on Besov spaces of vector-valued functions.* The Besov spaces provide a useful substitute for the L^p-scale in situations where the structure of the underlying Banach space X is not good enough to allow the boundedness of interesting translation-invariant operators on $L^p(\mathbb{R}^n; X)$. While the multiplier theory in this setting has been treated in [**1, 35, 85**], the purpose of Chapter 3 is to provide the alternative convolution-integral point of view to this problem.

- *A return to the multiplier set-up, via the convolution point of view, to derive Fourier multiplier theorems which improve on several known results even in the scalar case.* The convolution theory developed in the previous chapters is combined with some new Fourier-embedding theorems in Chapter 4; these allow the deduction of very sharp sufficient conditions for multipliers on all the function spaces mentioned above.

As indicated above, one chapter is devoted to each of the topics described. For the convenience of the reader who is mainly interested in a particular topic, I have tried to make the different chapters relatively self-contained. Only Chapter 4 relies substantially on the other chapters, but it can also be read independently, provided one is ready to accept the results quoted from the earlier chapters. Most of the cross-references refer to different parts of the same chapter; hence "(2.3)" refers to the equation labelled (2.3) inside the same chapter where the reference is made. Otherwise the chapter is indicated explicitly, e.g. "Theorem 4.21 of Chapter 2".

In the remaining part of this introductory chapter, I first sketch a historical perspective to the theory of translation-invariant operators and some of its applications. The account given is not meant to be representative of the whole of this field, which is much wider, but rather of those developments which are closely related to the present work. After the historical overwiev, the main content of each of the Chapters 1–4 will be explained.

2. A brief history of translation-invariant operators

Classical theory from M. Riesz to Hörmander. Translation-invariant linear operators on $L^p(\mathbb{R}^n)$ are abundant; e.g., every linear differential operator $\sum a_\alpha D^\alpha$ falls into this category. But such operators are neither continuous nor even defined on the whole space $L^p(\mathbb{R}^n)$. On the other hand, it is well-known that every integrable function k induces, via

$$Tf(x) := \int_{\mathbb{R}^n} k(x-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n} k(y)f(x-y)\,\mathrm{d}y, \qquad \text{a.e. } x \in \mathbb{R}^n, \tag{2.1}$$

a linear translation-invariant operator $f \in L^p(\mathbb{R}^n) \mapsto Tf \in L^p(\mathbb{R}^n)$ which is also continuous, and in fact $\|T\|_{\mathcal{L}(L^p(\mathbb{R}^n))} \leq \|k\|_{L^1(\mathbb{R}^n)}$ for all $p \in [1,\infty]$.

However, the restriction of the considerations to the integrable kernels k is neither satisfactory in view of many of the applications one would like to consider, nor necessary to have the operator T in (2.1) well-defined and bounded on $L^p(\mathbb{R}^n)$, at least for $p \in \,]1,\infty[$. Interesting problems lie between the first-mentioned differential operators, which are obviously unbounded, and the convolution operators, which are clearly bounded.

Probably the oldest and certainly the best-known instance of such an intermediate case is the operator of harmonic conjugation which maps a given function u, harmonic in the unit disc, to another harmonic funtion v (unique up to an additive constant which may be fixed by requiring $v(0) = 0$, say) such that $u+iv$ is analytic. A related problem which attracted the attention of complex analysts of the early 20th century was whether the p-integral norm of v could in some way be controlled in terms of that of u.

It is well-known today (see e.g. [**51**]) that this problem concerning functions on the disc can be transformed to an equivalent problem related to the boundary values on the unit circle $\mathbb{T}$, and in fact, it can be cast into a form with close resemblance with (2.1): The question is whether the periodic Hilbert transform

$$H_\pi f(t) := \text{p.v.-}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t-\theta)}{\tan(\theta/2)}\,\mathrm{d}\theta = \lim_{\epsilon\downarrow 0}\frac{1}{2\pi}\left(\int_{-\pi}^{-\epsilon}+\int_{\epsilon}^{\pi}\right)\frac{f(t-\theta)}{\tan(\theta/2)}\,\mathrm{d}\theta \tag{2.2}$$

(where the limit is easily seen to exist at least for all continuously differentiable f) can be extended to a bounded linear operator on $L^p(\mathbb{T}) \approx L^p(-\pi,\pi)$. If f is the boundary function of the harmonic function u, then $H_\pi f$ is the boundary function of the harmonic conjugate v of u.

Another equivalent formulation of the problem, foreshadowing the more general Fourier multiplier transformations, is obtained by considering the action of H_π on trigonometric polynomials. In this dense subspace of $L^p(\mathbb{T})$, $p < \infty$, the conjugation essentially changes some of the signs of the trigonometric coefficients:

$$\sum_{-n}^{n} c_k e^{\mathbf{i}kt} \mapsto \mathbf{i}\sum_{-n}^{-1} c_k e^{\mathbf{i}kt} - \mathbf{i}\sum_{1}^{n} c_k e^{\mathbf{i}kt}. \tag{2.3}$$

From this observation and the orthogonality of the trigonometric polynomials, the boundedness of H_π on $L^2(\mathbb{T})$ follows at once. This boundedness phenomenon of translation-invariant operators on L^2 is in no way connected with the particular properties of the Hilbert transform; rather, it is characteristic of all of the scalar-valued theory of these operators. The boundedness or unboundedness of such operators on L^2 can immediately be seen from the boundedness or unboundedness of the corresponding multiplier m, owing to the isometry on L^2 of the Fourier transform, which simultaneously diagonalizes all the translation-invariant operators.

Even in this scalar case, the boundedness of H_π on $L^p(\mathbb{T})$ for $p \neq 2$ turned out to be a more difficult problem. It was eventually answered in the affirmative, for $p \in\,]1,\infty[$, by M. Riesz [**70**], who applied tricky methods of complex analysis which hardly suggested the significant generalizations of his result in the following decades.

Nevertheless, the generalizations were to come. Between the publication of the pioneering paper of M. Riesz and the work of Hörmander [**43**] in 1960, the theory of translation-invariant operators progressed along two distinct main lines:

J. Marcinkiewicz [**60**] considered in 1939 a problem generalizing (2.3) in that the multiplication of the trigonometric coefficients by different signs was replaced by a more general transformation

$$\sum_{-n}^{n} c_k e^{\mathbf{i}kt} \mapsto \sum_{-n}^{n} \lambda_k c_k e^{\mathbf{i}kt},$$

with $\lambda_k \in \mathbb{C}$. MARCINKIEWICZ also considered the n-dimensional generalizations of such operators and introduced the name *multipliers of Fourier series* (multiplicateurs de séries de Fourier) to describe them.

He showed that a sufficient condition for the sequence (λ_k) to induce a bounded operator on $L^p(\mathbb{T})$, for $p \in]1, \infty[$, is the boundedness of the sequence, combined with the uniform boundedness of its variation on the dyadic intervals, i.e.,

$$|\lambda_k| \leq \kappa, \qquad \sum_{2^m \leq |k| \leq 2^{m+1}} |\lambda_k - \lambda_{k+1}| \leq \kappa.$$

In fact, the theorem of MARCINKIEWICZ for the one-dimensional situation was an almost direct consequence of certain quadratic estimates, due to J. LITTLEWOOD, R. PALEY and A. ZYGMUND, foreshadowing the notion of unconditional decompositions which would be used 70 years later to extend these result to the vector-valued setting. Most part of MARCINKIEWICZ' paper [**60**] is concerned with generalizing the multiplier theorem to the two-dimensional torus $\mathbb{T}^2$ (as a model for the general n-dimensional case which is then reached similarly). It is worth noting that the Rademacher functions and their basic properties, also vital to the recent operator-valued results, are already present in a decisive rôle in his work.

In 1956, S. G. MIHLIN [**62, 63**] used transference techniques, building on the theorem of MARCINKIEWICZ, to treat the analogous multiplier problem for functions on $\mathbb{R}^n$. In this setting, he showed that the derivative condition

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq \kappa \qquad \text{for all } \alpha \in \{0,1\}^n, \tag{2.4}$$

is sufficient for the boundedness of the operator $f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ on $L^p(\mathbb{R}^n)$ for $p \in]1, \infty[$. This condition has turned out to be extremely useful in connection with multipliers arising from partial differential equations.

P. I. LIZORKIN was later able to relax the condition (2.4) to

$$|\xi^\alpha D^\alpha m(\xi)| \leq \kappa \qquad \text{for all } \alpha \in \{0,1\}^n; \tag{2.5}$$

references to this and related work can be found in TRIEBEL's book [**82**]. A further weakening of this kind of conditions (with the uniformity in ξ replaced by L^1 averages over "dyadic blocks") is given in STEIN's book [**76**].

A rather different-looking programme for generalizing M. RIESZ' result was initiated by A. P. CALDERÓN and A. ZYGMUND [**16**] in 1952. Taking the singular integral representation (2.2) of the Hilbert transform, or rather, its non-periodic version

$$Hf(t) := \text{p.v.-}\int_{-\infty}^{\infty} \frac{1}{\pi s} f(t-s)\,\mathrm{d}s,$$

as the prototype of the operators to be treated, Calderón and Zygmund considered more general n-dimensional singular integrals of the form

$$Tf(t) = \text{p.v.-}\int_{\mathbb{R}^n} \frac{\Omega(s^0)}{|s|^n} f(t-s)\,\mathrm{d}s = \lim_{\epsilon\downarrow 0}\int_{|s|>\epsilon} \frac{\Omega(s^0)}{|s|^n} f(t-s)\,\mathrm{d}s, \tag{2.6}$$

where $s^0 := s/|s|$ and Ω is an integrable function on the unit sphere S^{n-1} of $\mathbb{R}^n$ satisfying the cancellation and continuity conditions

$$\int_{S^{n-1}} \Omega(u)\,\mathrm{d}\sigma(u) = 0, \qquad \int_0^1 \sup_{\substack{u,v\in S^{n-1}\\ |u-v|\leq t}} |\Omega(u)-\Omega(v)|\,\frac{\mathrm{d}t}{t} < \infty.$$

Clearly the Hilbert transform H is a special case with $n = 1$ and $\Omega(\pm 1) = \pm 1/\pi$.

The boundedness on $L^p(\mathbb{R}^n)$ of the operator defined in (2.6) was obtained under the abovementioned conditions on Ω. In later works, the homogeneous convolution kernels $\Omega(s^0)/|s|^n$ were further replaced by the more general form $k(s)$.

In order to attack these problems, Calderón and Zygmund introduced the so-called *real method*, as opposed to the complex analytic roots of singular integrals. This method, based on a clever decomposition (now known as the Calderón–Zygmund decomposition) of a function into its "good" and "bad" parts, opened the way for wide-ranging generalizations of the theory far beyond its complex analytic origin. The Calderón–Zygmund theory, developed to multiple directions by these two authors and many others, has become a well-established branch of mathematical analysis, and the Chicago school of analysis, founded by Zygmund, has produced several prominent mathematicians devoted to these topics. [A proper account of these developments would take us too far away from our main line, and the interested reader is referred, e.g., to the introduction of [**4**] for a history of the Chicago school.]

After the various pioneering contributions in the 50's, the theories of singular convolution integrals and Fourier multipliers had already reached a certain maturity by the year 1960 when Hörmander published his elegant paper [**43**]. In this work, he performed a comprehensive analysis of the boundedness of translation-invariant operators on $L^p(\mathbb{R}^n)$, adopting a general point of view (exploiting, in particular, L. Schwartz' theory of distributions [**73**]) from which many of the results of the previous decade could be derived in a unified manner.

Hörmander showed that his celebrated integral condition $k \in K^1$, termed the "almost L^1 functions" [a terminology which did not become as popular as did the condition itself] and defined by the requirement

$$\int_{|t|>2|s|} |k(t) - k(t-s)|\;\mathrm{d}t \leq \kappa \qquad \text{for all } s \neq 0, \tag{2.7}$$

is a sufficient condition for $k*$ to be bounded on all $L^p(\mathbb{R}^n)$, $p \in\,]1,\infty[$, provided it is *a priori* known to be bounded on one $L^{\tilde{p}}(\mathbb{R}^n)$ with $\tilde{p}$ in the same range.

Using this integral condition and the equivalence of convolution and multiplier operators, Hörmander also derived a variant of the theorem of Mihlin where he could replace the uniform boundedness by quadratic averages over annuli and moreover reduce the order of the highest required derivative from n to $\lfloor n/2 \rfloor + 1$. More precisely, Hörmander's multiplier condition, leading to the same conclusion as (2.4) was

$$\left(\frac{1}{r^n} \int_{r<|\xi|<2r} |D^\alpha m(\xi)|^2 \, \mathrm{d}\xi \right)^{1/2} \leq \kappa r^{-|\alpha|} \tag{2.8}$$
$$\text{for } r \in]0,\infty[,\ \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq \lfloor n/2 \rfloor + 1.$$

The unification in Hörmander's work of the various aspects of the theory of translation-invariant operators can be seen as the culmination of the classical epoch in the development of the theory.

[The reader should we warned of the fact that there is considerable confusion in the literature concerning the attribution of the various multiplier theorems to the original authors. Sometimes, the names of Marcinkiewicz and Mihlin are mixed, and often a result mostly resembling that of Hörmander appears under the title "Hörmander–Mihlin multiplier theorem". This is somewhat misleading, since Hörmander's multiplier theorem does *not* include the original result of Mihlin (except when $n = 1$), as is apparent from the fact that Hörmander's condition involves, when $n > 1$, estimates for the derivatives $\partial^2 m/\partial x_i^2$, but Mihlin's condition does not. This theme will be developed further in Chapter 4.]

Vector-valued generalizations, the abstract Cauchy problem, and the geometry of Banach spaces. The possibility of generalizing the theory of translation-invariant operators to the vector-valued situation, as well as its applications to the functional analytic approach to partial differential equations, were recognised already in the early 60's.

J. Schwartz [**72**] in 1961, and A. Benedek, A. P. Calderón and R. Panzone [**5**] in 1962 observed that Hörmander's integral condition, combined with the *a priori* boundedness of a convolution operator on one $L^{\tilde{p}}(\mathbb{R}^n; X)$-space, is sufficient for the boundedness on $L^p(\mathbb{R}^n; X)$, for all $p \in]1,\infty[$, even for general Banach spaces X and kernels taking values in $\mathcal{L}(X)$. J. Schwartz used these techniques to extend results of Calderón and Zygmund for singular integrals to functions taking values in an L^q space ($q \in]1,\infty[$), and Marcinkiewicz' and Mihlin's multiplier theorems (with $n = 1$) to Hilbert space valued functions.

In 1964, L. de Simon [**74**] applied the theory of singular integrals to the abstract Cauchy problem (1.1) on a Hilbert space X, proving that the solution u satisfies the maximal regularity property

$$\|\dot{u}\|_{L^p(\mathbb{R}^n;X)} + \|Au\|_{L^p(\mathbb{R}^n;X)} \leq C \|f\|_{L^p(\mathbb{R}^n;X)} \tag{2.9}$$

if and only if $-A$ is the generator of a bounded analytic semigroup on X.

In the same year, P. E. Sobolevskij [**75**] showed that $-A$ generating a bounded analytic semigroup is *necessary* for maximal L^p-regularity also on a general Banach space X; moreover, using the result of Benedek, Calderón and Panzone he showed that it is also sufficient for the maximal L^p-regularity on *all* $p \in]1,\infty[$, *provided* the maximal $L^{\tilde{p}}$-regularity for *one* $\tilde{p} \in]1,\infty[$ is known *a priori*. (A more recent proof of this result which does not rely on the general theory of singular integrals but works directly with the special properties of the variation-of-constants formula can be found in P. Cannarsa and V. Vespri [**17**].)

It is worth observing that the multiplier $m(\xi) = A(\mathbf{i}2\pi\xi + A)^{-1}$ related to the maximal regularity problem, for $-A$ the generator of a bounded analytic semigroup, satisfies exactly the estimates required in Mihlin's theorem. In fact, it follows from the well-known spectral characterization of generators (see e.g. [**30**]) that the function $m(\xi)$ from (1.3) is bounded on $\mathbb{R} \setminus \{0\}$ for such A; moreover, the derivative satisfies

$$\xi m'(\xi) = \xi(-\mathbf{i}2\pi)A(\mathbf{i}2\pi\xi + A)^{-2} = -m(\xi)(I - m(\xi)), \tag{2.10}$$

which is bounded on $\mathbb{R} \setminus \{0\}$ if m is. Thus, *if* Mihlin's theorem were true for operator-valued multipliers on a particular space X, then every negative generator A would have maximal L^p-regularity, but such a theorem was only known for a Hilbert space X. Sobolevskij actually conjectured that the *a priori* regularity could be dropped from his assumptions, on a general Banach space X, but this remained open for a long period of time.

The stalemate situation with the abstract Cauchy problem, the things being settled in the Hilbert space setting but essentially open in the more general framework, was characteristic of the entire vector-valued theory of singular integrals. It was hardly more difficult than in the scalar-valued setting to *extend* an operator boundedly to the whole scale of the spaces $L^p(\mathbb{R}^n; X)$, as soon as the boundedness was guaranteed on one such space; in fact, as pointed out by J. Schwartz and by Benedek *et al.*, the arguments establishing these assertions were essentially repetitions of the real method of Calderón and Zygmund with absolute values replaced by norms.

In the scalar-valued setting, the extension procedure is usually sufficient, since the boundedness on $L^2(\mathbb{R}^n)$ is a matter of checking the boundedness of the multiplier m, and this same technique can be carried to the Hilbert space framework, but hardly further; in the lack of a notion of orthogonality and Plancherel's theorem, the space $L^2(\mathbb{R}^n; X)$, for a non-Hilbert space X, is just as bad as any of the spaces $L^p(\mathbb{R}^n; X)$.

For $X = L^q(S, \Sigma, \mu)$, $q \in]1,\infty[$, the space $L^q(\mathbb{R}^n; X)$ can, to a certain extent, take the rôle of the starting point of boundedness considerations of an operator on the spaces $L^p(\mathbb{R}^n; X)$, as already observed by J. Schwartz. In fact, if k is a scalar-valued kernel which induces a bounded convolution operator on $L^q(\mathbb{R}^n)$ (of scalar-valued functions), and $F \in L^q(\mathbb{R}^n, L^q(S)) \approx L^q(\mathbb{R}^n \times$

S), then one merely needs to write $F(x,s)$ in place of $f(x)$ in the inequality $\int \left|\int k(x-y)f(y)\,\mathrm{d}y\right|^q \mathrm{d}x \le C\int |f(x)|^q \,\mathrm{d}x$, integrate over $s \in S$ and apply Fubini's theorem to get

$$\int \left\| \int k(x-y)F(y,\cdot)\,\mathrm{d}y \right\|_{L^q(S)}^q \mathrm{d}x \le C \int \|F(x,\cdot)\|_{L^q(S)}^q \,\mathrm{d}x. \tag{2.11}$$

This is exactly the boundedness of $k*$ on $L^q(\mathbb{R}^n; L^q(S))$, and if the kernel k is appropriate, one can take this as the *a priori* estimate required in proving the boundedness of $k*$ on all $L^p(\mathbb{R}^n; L^q(S))$, $p \in\,]1,\infty[$. However, this trick is heavily based on the structure of the space $L^q(S)$ and does not suggest a generalization to more general Banach spaces X. While the trick is simple, it is truly a trick, and does not give an indication of what exactly is the property of the $L^q(S)$ spaces which makes the result hold. Moreover, even for $X = L^q(S)$, the case where $k(x) \in \mathcal{L}(X)$ falls beyond the reach of this approach.

In the beginning of the 80's, the fundamental connection with the multiplier problem and the geometric structure of the underlying space X was finally discovered. It was D. L. Burkholder [**13, 14, 15**] who was able to present a geometric (and superficially simple-looking) condition which, when satisfied by X, guarantees the boundedness of certain singular integrals of X-valued functions, including the prototype example given by the Hilbert transform. This geometric condition was the so-called *ζ-convexity*, i.e., the existence of a biconvex function $\zeta : X \times X \to \mathbb{R}$ satisfying

$$\zeta(0,0) > 0 \qquad \text{and} \qquad \zeta(x,y) \le |x-y|_X \ \text{ for } |x|_X = |y|_X = 1.$$

Burkholder also showed the equivalence of the ζ-convexity of X with the *unconditionality of martingale difference sequences* on $L^p(\Omega, (\mathfrak{A}_i)_{i=0}^\infty, \mathbb{P}; X)$ (where $p \in\,]1,\infty[$, Ω is an arbitrary probability space with a filtration $(\mathfrak{A}_i)_{i=0}^\infty$, i.e., an increasing sequence of σ-algebras on Ω, and $\mathbb{P}$ is a probability measure on $\sigma(\cup_{i=0}^\infty \mathfrak{A}_i)$). This means the inequality

$$\mathbb{E}\left|\sum_{i=0}^\infty \epsilon_i d_i\right|_X^p \le C\,\mathbb{E}\left|\sum_{i=0}^\infty d_i\right|_X^p$$

holding with a fixed $C < \infty$, whenever $(\epsilon_i)_{i=0}^\infty$ is a sequence of signs ± 1 and $(d_i)_{i=0}^\infty$ is a martingale difference sequence on $L^p(\Omega; X)$, $p \in\,]1,\infty[$, adapted to $(\mathfrak{A}_i)_{i=0}^\infty$ [which means that $d_i \in L^p(\mathfrak{A}_i; X)$, and the conditional expectations satisfy $\mathbb{E}[d_{i+1}|\mathfrak{A}_i] = 0$, for all $i \in \mathbb{N}$]. This is the *UMD condition* which is most often used as the defining property and as the name of the ζ-convex spaces in the current literature. The condition is independent of $p \in\,]1,\infty[$, i.e., holds for all such p provided it holds for one.

J. Bourgain [**10**] completed the results of Burkholder by showing that the boundedness of the Hilbert transform on $L^p(\mathbb{T}; X)$ (which is equivalent to its boundedness on $L^p(\mathbb{R}; X)$ by straightforward transference-arguments), conversely, implies the UMD property for X. Thus the class of Banach spaces X

for which the prototype singular integral p.v.-$1/\pi x*$, or equivalently, the simplest non-trivial multiplier $-\mathbf{i}\,\mathrm{sgn}(\xi)$, would induce a bounded operator on $L^p(\mathbb{R}^n; X)$ was *characterized* in terms of the UMD property by these two authors.

The UMD condition is satisfied by the $L^q(S)$ spaces for $q \in]1, \infty[$ (a proof of which is essentially contained above around Eq. (2.11), taking the characterization by means of the boundedness of the Hilbert transform for granted) but also by several other interesting examples such as the Schatten ideals $S^q := \{A \in \mathcal{L}(\mathcal{H}) : \|A\|_{S^q} = (\mathrm{tr}(A^*A)^{q/2})^{1/q} < \infty\}$, again with $q \in]1, \infty[$—this was observed by Bourgain [**10**]. Moreover, the UMD condition is inherited from X to X', to $L^p(S, \Sigma, \mu; X)$, $p \in]1, \infty[$, and to (closed) subspaces and quotient spaces of the original space. [These are based on standard extension techniques.] A useful review of UMD-spaces, containing the above properties and more, is given by J. L. Rubio de Francia [**71**]. Detailed proofs of some of the results mentioned above can be found in my Master's thesis [**44**].

Bourgain went on in [**12**] (see also [**11**]) to show that the already established boundedness of the Hilbert transform could further be used as a tool in proving the boundedness of a much richer class of Fourier multiplier and singular integral transformations on UMD-spaces, thus promoting these spaces to a central rôle, not only from the point-of-view of geometricians of Banach spaces but of all those working with vector-valued integral operators in numerous applications. More precisely, Bourgain translated the UMD property into the Littlewood–Paley-type estimate

$$\|f\|_{L^p(\mathbb{T};X)} \sim \left(\mathbb{E}\left\|\sum \varepsilon_j f_j\right\|^p_{L^p(\mathbb{T};X)}\right)^{1/p}, \qquad \text{where} \tag{2.12}$$

$$f_j(t) := \sum_{2^{j-1} \le |k| < 2^j} \hat{f}(k) e^{\mathbf{i}kt} \text{ for } j = 1, 2, \ldots, \qquad f_0(t) := \hat{f}(0),$$

where $\sim$ denotes the boundedness of the ratio of the two quantities, from above and from below, by positive numbers independent of the particular choice of $f \in L^p(\mathbb{T}; X)$, and $(\varepsilon_j)_{j=0}^\infty$ is the Rademacher system of independent random variables with $\mathbb{P}(\varepsilon_j = +1) = \mathbb{P}(\varepsilon_j = -1) = 1/2$.

While this result reduces, for $X = \mathbb{C}$, to the very same quadratic estimates that were used by Marcinkiewicz in proving his multiplier theorem, the only way of formulating this inequality in the general vector-valued setting is with the help of the Rademacher functions. The usefulness of these functions, even in scalar-valued analysis, lies in the fact that the many quadratic estimates of harmonic analysis admit an equivalent *linear* formulation in terms of the Rademacher means. In the general situation, this becomes even more essential, as the quadratic estimates become nonsense but the linearized versions remain valid in a much wider vector-valued framework. Using these ideas and the decomposition at hand, Bourgain established a UMD-valued analogue of Marcinkiewicz' theorem on $L^p(\mathbb{T}; X)$.

Multiplier theorems for vector-valued functions in n variables were considered by T. R. McConnell [**61**] and by F. Zimmermann [**89**]. McConnell used a different approach from Bourgain's, applying heavy probabilistic machinery and obtaining also an independent proof of some of Bourgain's results for $n = 1$. Zimmermann, on the other hand, built on the methods developed by Bourgain but observed two different ways of generalizing from one to n dimensions. The results from the two approaches, only one of which could be carried out in general UMD-spaces, revealed further connections between vector-valued multiplier theorems and the properties of the underlying Banach spaces: Zimmermann showed that whereas the theorem of Mihlin, with assumption (2.4), holds true with $L^p(\mathbb{R}^n)$ replaced by $L^p(\mathbb{R}^n; X)$, for an arbitrary UMD-space X, the theorem of Lizorkin, with assumption (2.5), does not hold in general unless additional assumptions on the space are imposed. In particular, Zimmermann showed that Lizorkin's theorem fails on the Schatten ideals S^p when $p \neq 2$.

The notion of the UMD condition and its equivalent characterizations also helped in understanding the properties of the Hardy spaces of vector-valued functions. Several results in this direction were proved by O. Blasco in a series of papers, of which [**8**] contains the most interesting results for the present point-of-view. Blasco showed, in particular, that exactly under the UMD-property of the space X do the various classical characterizations of the Hardy space $H^1(\mathbb{T})$ agree to give the same space $H^1(\mathbb{T}; X)$. This result will play a rôle in Chapter 1.

All this proved, the theory of scalar-valued Fourier multipliers on the Bôchner spaces had reached a rather satisfactory form by the end of the 80's. However, the case of operator-valued multipliers, essential for the treatment of the abstract Cauchy problem, did not seem to have come any closer to a positive solution. The main progress in this direction since the 60's was an unpublished result of G. Pisier from 1978 which, from the point of view of applications, was strongly in the negative: *If* Mihlin's theorem holds true for *all* operator-valued multipliers on $L^p(\mathbb{R}^n; X)$, then X is necessarily isomorphic to a Hilbert space. (The result, with proof, can be found in a recent paper by W. Arendt and S. Bu [**3**].)

Of course, this did not yet imply an equally negative answer to the problem of maximal regularity, where the relevant multiplier (1.3) has a very special structure, and so might conceivably induce a bounded operator even if the general Mihlin theorem fails. But even for multipliers of this specific type, T. Coulhon and D. Lamberton [**24**] were able to reduce the class of admissible Banach spaces, once the interrelation between the UMD condition and vector-valued singular integrals was recognized: These authors demonstrated that the negative generator of the Poisson semigroup on $X = L^2(\mathbb{R}; E)$ does not have maximal regularity if E is not a UMD space. (See also C. Le Merdy [**57**].) This showed that Sobolevskij's conjecture failed in the full generality; however, the possibility of its validity on all UMD spaces was still left open.

Positive partial results in this direction were proved in 1987 by G. Dore and A. Venni [**29**] and by Lamberton [**56**]. Dore and Venni showed that the maximal regularity property is satisfied by the Cauchy problem on a UMD space provided that the operator A admits *bounded imaginary powers* (BIP). (C. Le Merdy [**57**] has later shown that the BIP alone do not suffice, i.e., he constructed an operator on a non-UMD space with BIP but without maximal regularity.) Lamberton, on the other hand, proved maximal regularity for negative generators of contraction semigroups on L^p spaces. Further sufficient conditions were obtained ten years later by M. Hieber and J. Prüss [**42**] for negative generators of semigroups representable by heat kernels in terms of kernel estimates, but the complete solution of the problem still had to wait.

Translation-invariant operators on different function spaces. Even though the L^p theory of translation-invariant operators forms without doubt the heart of the matter, such operators are of interest and have been considered by several authors on various other function spaces, too.

The real variable theory of the Hardy spaces H^p, and in particular the paper [**31**] of C. Fefferman and E. M. Stein, opened the way for the extension of the results from the reflexive L^p spaces (i.e., those with $p \in]1, \infty[$) to the Hardy spaces H^p with $p \in]0, 1]$. The extent to which the theory of singular integrals carries over to this setting, together with the fact that the spaces H^p agree with the corresponding L^p spaces for $p \in]1, \infty[$, has given rise to the idea that the Hardy spaces H^p, rather than the Lebesgue spaces L^p, form the "right" continuation to $p \in]0, 1]$ of the reflexive L^p scale. (Cf. e.g. Stein [**77**].)

The extension of the theory to the H^p spaces has proven to be very interesting theoretically, but there are also significant applications. E.g., I. Chueshov and I. Lasiecka [**18**] have recently used the h^1 space (a local version, introduced by D. Goldberg [**39**], of H^1) as a framework for the treatment of non-linear equations of elasticity. This also motivates the study of abstract evolution equations in the H^1 setting in Chapter 1.

One can further consider *weighted* L^p and H^p spaces. The fundamental work of B. Muckenhoupt [**64**] in characterizing, in terms of the celebrated A_p condition, the weights w for which the Hardy–Littlewood maximal function is bounded on $L^p(\mathbb{R}^n, w(x)\mathrm{d}x)$ paved the way for the understanding of the continuity properties of singular integrals on the weighted spaces. See in particular the paper of R. R. Coifman and C. Fefferman [**23**] and the references cited there for more on these developments.

Although weighted spaces will not be treated in the present work, it turns out that some of the techniques developed to handle the weighted situation are also useful in the vector-valued setting. In particular, the methods of D. S. Kurtz and R. L. Wheeden [**54**] (in treating weighted L^p spaces), which were further elaborated by J.-O. Strömberg and A. Torchinsky [**81**] (in the context of weighted H^p spaces) are of interest, since they do not blindly rely on the use of Plancherel's theorem but exploits the more general Hausdorff–Young inequality

$$\|\hat{f}\|_{p'} \leq C \|f\|_p \tag{2.13}$$

for different values of $p \in [1, 2]$.

This inequality is always true with $p = 1$ for $f \in L^p(\mathbb{R}^n; X)$, X being an arbitrary Banach space. The estimate (2.13) is not true, in general, for any larger value of p; however, *if* it holds for a given $p \in [1, 2]$ (with $C < \infty$ independent of f), the space X is said to have *Fourier-type p*. This notion is due to J. Peetre [**65**] who also showed that every $L^q(\mu)$ space (with μ a σ-finite measure) has Fourier-type $\min(q, q')$, and in general no higher. Thus there are many interesting Banach spaces having a *non-trivial Fourier-type*, i.e., (2.13) valid for some $p > 1$. On the other hand, the Hausdorff–Young ineqality with $p = 2$, which is essentially Plancherel's theorem, is valid if (easy to see) and only if (a deeper result due to S. Kwapień [**55**]) the underlying space is isomorphic to a Hilbert space. These ideas will play an important rôle in Chapter 1; they were first exploited in connection with the vector-valued multiplier problem by L. Weis [**85**], as will be explained in more detail below.

While the development of the theory of translation-invariant operators on H^p could be considered an *extension* of the corresponding L^p theory, the related results on the Besov spaces $B^{s,p}_q$ may be regarded, in a certain sense, as a *substitute* for it. This statement is particularly true in the vector-valued setting: As the land of operator-valued multipliers on $L^p(\mathbb{R}^n; X)$ seemed to be left barren by the result of Pisier, the Besov spaces $B^{s,p}_q(\mathbb{R}^n; X)$ provided a more promising path. This path was followed by H. Amann [**1**] and independently by L. Weis [**85**] in the late 90's.

The Besov spaces can be defined as follows (among various other equivalent characterizations): For $f \in \mathcal{S}'(\mathbb{R}^n; X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n); X)$ (the space of X-valued tempered distributions), the quantity

$$\|f\|_{s,p;q} := \left\| \left(2^{\mu s} \|\varphi_\mu * f\|_p \right)_{\mu=0}^\infty \right\|_{\ell^q}$$

is required to be finite, in order that $f \in B^{s,p}_q(\mathbb{R}^n; X)$, and $\|\cdot\|_{s,p;q}$ is the norm on this space (making it a Banach space whenever $s \in \mathbb{R}$ and $p, q \in [1, \infty]$; the quasi-Banach spaces obtained when p or q lies in $]0, 1[$ will not be considered here). The functions φ_μ appearing here constitute a *resolution of unity* in the frequency domain, i.e., $\hat{\varphi}_0 \in \mathcal{S}(\mathbb{R}^n)$ is radial and decreasing, with range $[0, 1]$, equal to unity in $\bar{B}(0, \frac{1}{2})$ and vanishing outside $\bar{B}(0, 1)$, and $\hat{\varphi}_\mu := \hat{\varphi}_0(2^{-\mu}\cdot) - \hat{\varphi}_0(2^{-\mu+1}\cdot)$ for $\mu = 1, 2, \ldots$, so that $\sum_{\mu=0}^\infty \hat{\varphi}_\mu(\xi) \equiv 1$. More information on Besov spaces can be found, e.g., in the book of Triebel [**83**]; unlike in the L^p setting, the theory of these spaces is essentially the same in the vector-valued setting, and the reader will notice that the statements and proofs in [**83**] regarding Besov spaces generalize to the vector-valued setting essentially by replacing absolute values by norms.

[The reader should be warned that many texts, including [**83**], use the notation φ_μ for what is here called $\hat{\varphi}_\mu$. The motivation for the present notation is the consistency in denoting objects "living" in the frequency representation by symbols with the hat ˆ. The only exception to this rule is the conventional notation m for Fourier multipliers, even though they manifestly are objects of the frequency domain.]

While the above definition does seem difficult at the first sight, as it is, the complications in the definition will pay off as simplifications of the theorems. In fact, the dyadic decomposition, which in the Bôchner spaces was a deep result of BOURGAIN and quite non-trivial even for the classical Lebesgue spaces, is now built into the very definition of the spaces $B_q^{s,p}(\mathbb{R}^n;X)$. Moreover, the present decomposition is much easier to handle , as the different parts of the decomposition only contribute to the norm $\|f\|_{s,p;q}$ in terms of their absolute size, whereas in the Bôchner space setting they were coupled in a subtle manner in terms of the Rademacher means.

Making use of this defining property of the Besov spaces, AMANN was able to establish an analogue of MIHLIN's theorem for operator-valued multipliers on $B_q^{s,p}(\mathbb{R}^n;X)$, with no restrictions on the underlying Banach space X. His sufficient condition was the set of estimates

$$(1+|\xi|)^{|\alpha|}\left\|D^\alpha m(\xi)\right\|_{\mathcal{L}(X)} \leq \kappa \qquad \text{for all } |\alpha| \leq n+1. \tag{2.14}$$

While this, clearly, was not a solution to the original L^p multiplier problem, AMANN pointed out that it was not too far away from one. In fact, combining the multiplier theorem on the Besov scale with appropriate embedding results between Besov spaces and more classical ones, AMANN derived boundedness results for operators between the classical spaces, and also demonstrated the usefulness of this approach in numerous applications in his comprehensive paper [**1**]. However, as he pointed out, the information on the precise smoothness of the original function that was lost in the embeddings was just enough to lose the possibility of obtaining *maximal* regularity results in this way.

An independent study of the Besov space multipliers was carried out by L. WEIS in connection with semigroup stability problems [**85**]. (An expanded study along these lines was recently produced by M. GIRARDI and WEIS [**35**].) Whereas the multiplier theorems of AMANN were completely independent of the structure of the underlying Banach space X, WEIS observed that one could reduce the smoothness required of the multiplier in the most general setting by taking into account the Fourier-type of X. By doing so, he could relax the condition (2.14) to

$$(1+|\xi|)^{|\alpha|}\left\|D^\alpha m(\xi)\right\|_{\mathcal{L}(X)} \leq \kappa \qquad \text{for all } |\alpha| \leq \lfloor n/t \rfloor + 1,$$

when X has Fourier-type t. This reproduces AMANN's result for the trivial Fourier-type $t = 1$, but gives sharper results for "better" spaces. Thus, yet another connection was established between the structure of the Banach space X and the multiplier theory valid on that space.

Solution to the operator-valued multiplier problem on L^p, and further developments. An operator-valued version of MIHLIN's multiplier theorem had to wait until the turn of the millennium, although the seeds of the solution were already hidden in the work of BOURGAIN, or even, in a sense, in that of MARCINKIEWICZ. As PISIER's result showed that a complete analogue of MIHLIN's theorem (by means of replacing the absolute values in the condition (2.4) by operator-norms) was simply false, except in Hilbert spaces, the problem was now finding the right way of strengthening these asumptions, enough in order to make the result hold but not too much to exclude interesting applications. An interesting review of the state of the art only shortly before the solution was found is given by M. HIEBER [**41**].

To see where the problems occur with operator-valued multipliers, consider in the periodic situation a multiplier $(m_k)_{k\in\mathbb{Z}}$ for which $m_k = M_j$ whenever $2^{j-1} \leq |k| < 2^j$, $(M_j)_{j=0}^{\infty}$ being a bounded sequence on $\mathcal{L}(X)$. According to the dyadic decomposition (2.12), the L^p norm of the transformation of a function f by the multiplier m is then proportional to

$$\left(\mathbb{E}\left\|\sum \varepsilon_j M_j f_j(\cdot)\right\|_{L^p(\mathbb{T};X)}^p\right)^{1/p} = \left(\int_{\mathbb{T}} \mathbb{E}\left|\sum \varepsilon_j M_j f_j(t)\right|_X^p \mathrm{d}t\right)^{1/p}.$$

If the M_j were scalars, the multiplier transformation induced by $(m_k)_{k\in\mathbb{Z}}$ would be bounded by BOURGAIN's UMD-valued extension of MARCINKIEWICZ theorem. However, to show the boundedness of the present operator-valued multiplier transformation, one would have to pull out the operator-valued coefficients M_j from this summation, but there is, in general, no reason why this should be allowed.

E. BERKSON and T. A. GILLESPIE [**7**] observed the usefulness of having this property for particular collections of operators, and as it was not true in general, they included it as an *assumption* in a number of their results. More precisely, they formulated the notion of *R-property* of a collection $\mathcal{T} \subset \mathcal{L}(X)$, which is satisfied if the inequality

$$\left(\mathbb{E}\left|\sum_{j=1}^{N} \varepsilon_j T_j x_j\right|_X^p\right)^{1/p} \leq C \left(\mathbb{E}\left|\sum_{j=1}^{N} \varepsilon_j x_j\right|_X^p\right)^{1/p} \tag{2.15}$$

holds with a fixed $C < \infty$ and $p \in [1, \infty[$, for all $N \in \mathbb{Z}_+$ and all choices of $x_1, \ldots, x_N \in X$, $T_1, \ldots, T_N \in \mathcal{T}$. The R-property of a collection $\mathcal{T}$ is in fact independent of the exponent $p \in [1, \infty[$ (i.e., it either holds for all these p or for none of them), which follows from KAHANE's inequality (see e.g. [**67**]).

The R in the R-property of Berkson and Gillespie was used by these authors as a reference to M. Riesz; however, it can also be pronounced as either "randomized" or "Rademacher", following the (re)interpretations of other authors. Instead of the R-property, one usually speaks of *R-boundedness* in the recent literature; accordingly, the collection $\mathcal{T}$ with this property is said to be *R-bounded* and the *R-bound* of $\mathcal{T}$ is defined as

$$\mathcal{R}(\mathcal{T}) := \min\{C \geq 0 : \text{ (2.15) holds with } p = 1\}.$$

Although Berkson and Gillespie singled out this right notion of boundedness, they only used it as an auxiliary device in establishing scalar-valued multiplier theorems. A further study of the properties of R-boundedness and its relation to *unconditional Schauder decompositions* (of which the dyadic decomposition in (2.12) is an example) was carried out by Ph. Clément, B. de Pagter, F. A. Sukochev and H. Witvliet [**21**] (see also Witvliet's dissertation [**88**]), but it was Weis [**86, 87**] who first managed to fully exploit these ideas in proving the operator-valued extension of Mihlin's theorem: If X is a UMD-space and the multiplier function $m \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{L}(X))$ is such that the sets

$$\{m(\xi) : \xi \in \mathbb{R} \setminus \{0\}\} \text{ and } \{\xi m'(\xi) : \xi \in \mathbb{R} \setminus \{0\}\} \text{ are } R\text{-bounded,} \tag{2.16}$$

then m is a Fourier multiplier on $L^p(\mathbb{R}; X)$ for all $p \in \,]1, \infty[$.

Weis was able to give a partial converse to this result, showing that certain R-boundedness in the assumptions cannot be avoided, and Ph. Clément and J. Prüss [**22**] improved this converse argument, showing that, in fact, the R-boundedness of $\{m(\xi) : \xi \text{ a Lebesgue point of } m\}$ is necessary for m to be a Fourier multiplier on $L^p(\mathbb{R}^n; X)$. An abstraction of this necessary condition to more general groups (in place of $\mathbb{R}^n$) has been proved by S. Blunck [**9**].

While the results of Weis and Clément–Prüss did not exactly characterize the L^p multipliers on a UMD-space X, they clearly showed that R-boundedness is the right notion in this context. One could also say that the necessary and sufficient conditions were now as close as reasonably could be hoped for, recalling the lack of an exact characterization even in the scalar case.

First versions of the operator-valued multiplier theorem in n variables were proved by Ž. Štrkalj and Weis [**80**] and also by R. Haller, H. Horst and A. Noll [**40**]. Both of these works built on the ideas of Zimmermann and obtained analogous results stating the sufficiency of an R-version of Mihlin's condition on all UMD-spaces, and the sufficiency of an R-version of Lizorkin's condition on all UMD-spaces with so-called property (α) (see e.g. [**21**], Def. 3.11, or Chapter 2, Lemma 6.1, for a definition). Quite recently, M. Girardi and Weis [**36**] have proved operator-valued Hörmander-type theorems on UMD-Bôchner spaces, where the Fourier-type of the underlying spaces is taken into account to decrease the required smoothness. Similar results are obtained as an application of the boundedness results for singular integrals in Chapter 2, and Chapter 4 elaborates further on this theme.

W. Arendt and S. Bu [**2**] have given a simple proof of an operator-valued Marcinkiewicz-type theorem on $L^p(\mathbb{T};X)$ (which would also follow from the more general results in [**80**]). These authors also prove several interesting results, which shed some light on the nature of R-bounded operator collections and their relation to the structure of the underlying Banach space; e.g., they show that *every* bounded subset of $\mathcal{L}(X)$ is R-bounded if and only if X is isomorphic to a Hilbert space. With the help of this result and the necessity of R-boundedness of operator-valued multipliers, Arendt and Bu also give in [**3**] a proof of the earlier mentioned unpublished result of Pisier.

Of course, the solution of the multiplier question had profound implications on the problem of maximal regularity. In this field, a significant breakthrough had already been reached, independently of the above mentioned developments, by N. Kalton and G. Lancien [**49**]: They considered the restrictions on the structure of the Banach space X that would be imposed *provided* the maximal regularity property holds for *all* negative generators A on X, and came to the conclusion that, among all Banach spaces X with an unconditional basis, those isomorphic to a Hilbert space are the only ones for which this is true. In particular, this result applies to the L^q spaces, $q \in]1,\infty[$, of interest in the concrete applications, and shows that only on $X = L^2$ can one have maximal regularity for all negative generators. The proof exploited, in particular, a characterization of Hilbert spaces in terms of the complemented subspace property, dating back to the work of J. Lindenstrauss and L. Tzafriri [**58**] in 1971—one more instance were the geometry of Banach spaces entered the scene in a fundamental way.

While the result of Kalton and Lancien was a strong negative answer, similar in spirit to that of Pisier, one should bear in mind that it only implied the *existence* of some pathological generators $-A$, leaving open the problem of determining whether or not the maximal regularity property would hold for some particular A arising from a specific problem. The result of Weis, on the other hand, gave a *characterization* of those A for which the maximal regularity property does indeed hold: On a UMD-space X, the abstract Cauchy problem has maximal L^p-regularity for $p \in]1,\infty[$ *if and only if* $-A$ is the generator of a bounded analytic semigroup, such that the operator collection

$$\{A(\mathbf{i}2\pi\xi + A)^{-1} : \xi \in \mathbb{R} \setminus \{0\}\} \text{ is } R\text{-bounded;} \tag{2.17}$$

when this condition is satisfied, A is said to be R-sectorial, this being a strengthening of the usual notion of sectoriality of operators.

That one can indeed obtain a characterization may seem surprising, since for the multipliers there were only non-coinciding necessary and sufficient conditions; however, the better situation with the maximal regularity problem is due to the special form of the related multiplier m from (1.3). Since the maximal regularity for the Cauchy problem is equivalent to m inducing a bounded Fourier multiplier transformation, the necessity of R-boundedness proved by Clément and Prüss shows that (2.17) is indeed necessary for maximal regularity. (The original argument of Weis preceded the result of Clément and Prüss but was more complicated and less general.) But given that (2.17) holds, it already follows from (2.10) (and an easily established permanence of R-boundedness in operator products) that $\{\xi m'(\xi) : \xi \in \mathbb{R} \setminus \{0\}\}$, too, is R-bounded, and so Weis's conditions (2.16) are satisfied, showing that (2.17) is also sufficient.

After Weis' characterization of maximal L^p-regularity, the result of Kalton and Lancien can be reinterpreted as showing the existence of non-R-sectorial negative generators of analytic semigroups in all non-Hilbert UMD-spaces with an unconditional basis, whereas the problem of determining whether or not a particular operator A has the maximal regularity property is translated into the problem of investigating its R-sectoriality. Whether this has some use beyond theoretical interest, depends, of course, on the feasibility of checking the R-boundedness of particular collections of operators. It has turned out that this can indeed by done in many cases, and that the R-boundedness characterization in fact provides a very practical way of proving the maximal regularity of various concrete operators.

A comprehensive treatment of large classes of differential operators, using the R-boundedness techniques, is given by R. Denk, M. Hieber and J. Prüss [**25**]. Maximal regularity of analogues of the Cauchy problem (1.1) in the periodic situation has been considered by Arendt and Bu [**2**], and in the discrete time framework by S. Blunck [**9**] and by P. Portal [**68**]. P. C. Kunstmann and Weis [**52**] have proved perturbation theorems for maximal L^p-regularity via perturbation results for the equivalent R-boundedness conditions. A continuous time non-autonomous problem is treated with the R-boundedness methods by Ž. Štrkalj [**79**].

It is not difficult to find examples of R-bounded sets even in the more classical literature; in fact, the usual square-function estimates of harmonic analysis are actually statements of R-boundedness. Indeed, the R-boundedness of a collection $\mathfrak{T}$ of operators on $L^p(\mu)$, $p \in [1, \infty[$, is equivalent to the inequality

$$\left\| \Big(\sum |T_j f_j(\cdot)|^2 \Big)^{1/2} \right\|_{L^p(\mu)} \leq C \left\| \Big(\sum |f_j(\cdot)|^2 \Big)^{1/2} \right\|_{L^p(\mu)}$$

holding uniformly with a constant $C < \infty$ for all choices of $T_j \in \mathfrak{T}$ and $f_j \in L^p(\mu)$, and numerous estimates of this kind are proved, e.g., in the books of Stein [**76, 77**] and García-Cuerva and Rubio de Francia [**34**]. The fact that certain weighted estimates imply square-function estimates and thus R-boundedness has been exploited by A. Fröhlich [**32, 33**] in establishing maximal regularity results for the Stokes operator in weighted L^q spaces.

Several recent studies have revealed further abundance of R-bounded collections of operators. There is a remarkable "bootstrapping" property of R-boundedness which states, roughly speaking, that the operator-valued multiplier theorems, which contain R-boundedness in their assumptions, not only yield the boundedness of individual operators but in fact R-boundedness of collections of operators satisfying the assumptions uniformly. A. Venni [**84**] first observed this for scalar-valued multipliers acting on Bôchner spaces. Girardi and Weis [**37**] have extended the idea to the operator-valued setting and given a comprehensive treatment around this theme. The results of this kind again require the property (α) of the underlying Banach spaces.

The R-boundedness techniques have also been successfully exploited in other problems, such as the H^∞ functional calculus for Banach space operators (see Kalton and Weis [**50**]). The common factor in all these developments is the attempt to extend to the Banach space framework such results and ideas for which only Hilbert space theory was known so far. The remarkable success in this field in the recent years indicates that the notion of R-boundedness does indeed provide the right framework for such developments, many other of which are still likely to come. The present dissertation, too, is a part of and aims at furthering this "R-programme".

3. Comments on Chapters 1–4

Chapter 1. Chapter 1 is devoted to various aspects of translation-invariant operators on the Hardy spaces $H^p(\mathbb{R}^n;X)$, $p \in]0,1]$, of vector-valued functions. Both necessary and sufficient conditions are derived for Fourier multipliers acting on these spaces, and applications to the "maximal H^p-regularity" of the abstract Cauchy problem (1.1) are considered throughout the work.

The main results of the chapter include the following conditions on multipliers:

- If $m : \mathbb{R}^n \to \mathcal{L}(X,Y)$ is a Fourier multiplier from the Hardy space $H^1(\mathbb{R}^n;X)$ to $L^1(\mathbb{R}^n;X)$, then $\{m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\}$ is R-bounded.
- Conversely, if X and Y are UMD-spaces with Fourier-type $t \in]1,2]$, and $\{|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\}$ is R-bounded for all $|\alpha| \leq \lfloor n/t \rfloor + 1$, then m is a Fourier-multiplier from $H^1(\mathbb{R}^n;X)$ to $H^1(\mathbb{R}^n;Y)$.

The first statement is Theorem 4.2 of Chapter 1, whereas the second is actually a sufficient condition for checking the rather more technical, but also more general, assumptions of Theorem 5.13. This latter theorem also states that one can further ensure the boundedness of the multiplier transformation induced by m from $H^p(\mathbb{R}^n;X)$ to $H^p(\mathbb{R}^n;Y)$ for $p < 1$ by assuming more derivative conditions of the same form as above. In particular, assuming the R-boundedness of $\{|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\}$ for all $\alpha \in \mathbb{N}^n$, the function m will be a multiplier from $H^p(\mathbb{R}^n;X)$ to $H^p(\mathbb{R}^n;Y)$ for all $p \in]0,1]$, and, in fact, uniform boundedness would do in place of R-boundedness for the higher order derivatives.

As in the L^p setting, and for the same reason, it turns out that the generally non-coinciding necessary and sufficient conditions for multipliers give a characterization of boundedness when applied to the particular multiplier (1.3) related to the abstract Cauchy problem. The method also applies to some more general *fractional-order* equations, treated by Ph. Clément, G. Gripenberg and S.-O. Londen [**19**], and by Clément, Londen and G. Simonett [**20**], as well as to abstract PDE's.

Chapter 2. After the excursion to the Hardy spaces in Chapter 1, we return to the Bôchner space setting in Chapter 2, seeking for sufficient conditions for the boundedness on $L^p(\mathbb{R}^n; X)$ of $f \mapsto k * f$ in terms of the properties of the singular convolution kernel k. Nevertheless, it turns out that some of the ideas coming from the Hardy space set-up play a substantial rôle in the L^p framework, too; in fact, one of the key ingredients in the proof of the main theorem is a suitable *atomic decomposition* of a Schwartz function ϕ with zero integral. Other main ingredients include the systematic exploitation of the notion of R-boundedness and a deep lemma of Bourgain [**12**] concerning translations on $L^p(\mathbb{R}^n; X)$.

The main theorem (Theorem 4.1 of Ch. 2) is motivated by the classical result of Hörmander [**43**]. It shows that sufficient conditionss to guarantee the boundedness of the convolution transformation $f \mapsto k * f$ on $L^p(\mathbb{R}^n; X)$, when $p \in \,]1, \infty[$ and X is a UMD-space, are given in terms of an R-version of his integral condition (2.7), more precisely,

$$\text{(3.1)} \quad \int_{|x|>2|y|} \mathcal{R}(\{2^{-nj}(k(2^{-j}(x-y)) - k(2^{-j}x)) : j \in \mathbb{Z}\}) \log(2+|x|)\,\mathrm{d}x \\ \leq C \log(2+|y|)$$

for all $y \in \mathbb{R}^n \setminus \{0\}$, combined with the R-boundedness of the set $\{\hat{k}(\xi) : \xi \in \mathbb{R}^n\}$. Recall that $\mathcal{R}(\mathcal{T})$ designates the R-bound of the set $\mathcal{T}$.

While the R-boundedness reduces to uniform boundedness for scalar-valued kernels, the R-integral condition (3.1) remains stronger than the classical assumption (2.7) even in the scalar setting, not only because of the logarithmic weight but also because even the supremum $\sup_{j\in\mathbb{Z}} 2^{-nj} |k(2^{-j}(x-y)) - k(2^{-j}x)|$ inside the integral is in general larger than the single ($j = 0$)-term $|k(x-y) - k(x)|$ in the classical condition. However, the difference is not as substantial as it formally appears; when applied to regular singular integral operators, the main theorem of Chapter 2 yields conditions which are direct analogues of the classical theorems, only with boundedness replaced by R-boundedness. E.g., for an odd kernel $k \in C^1(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X))$, we find that the R-boundedness of $\{|x|^n k(x), |x|^{n+1} \nabla k(x) : x \in \mathbb{R}^n \setminus \{0\}\}$ is sufficient for the boundedness of $k*$ on $L^p(\mathbb{R}^n; X)$, $p \in \,]1, \infty[$.

A further illustration of the strength of the main theorem on singular integral operators is provided by applying it to deduce strong Fourier multiplier theorems.

Some results in this direction are proved in the last section of Chapter 2, and this theme is developed further in Chapter 4.

Chapter 3. Chapter 3 deals with operators that are formally similar to the ones considered in Chapter 2 but the function spaces on which the boundedness questions are considered now come from the Besov scale. The main theorem (Theorem 5.7 of Ch. 3) establishes the boundedness of a singular convolution operator $f \in B^{s,p}_q(\mathbb{R}^n; X) \mapsto k * f \in B^{s,p}_q(\mathbb{R}^n; X)$ for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$ under assumptions of the following type:

- $\hat{k} \in L^\infty(\mathbb{R}^n; \mathcal{L}(X, Y))$,
- $\int_{|t|>2|s|} \|k(t-s) - k(t)\|_{\mathcal{L}(X,Y)} \,\mathrm{d}t \le C$ for all $s \neq 0$, and
- $\int_{|t|>r} \|k(t)\|_{\mathcal{L}(X,Y)} \,\mathrm{d}t < \infty$ for some $r > 0$.

The conditions on the boundedness of the Fourier transform and the Hörmander-type estimate are already familiar from the L^p setting, and it is observed that the subtleties with R-boundedness and the logarithmic weight have disappeared from the conditions. This is in a close relation to the form of the dyadic decomposition which is valid on the two scales of spaces, and illustrates the simpler character of the Besov space setting as compared to the Bôchner spaces.

On the other hand, the third condition above, which imposes a rather heavy restriction on the size of the kernel at infinity, is related to the inhomogeneity of the Besov spaces (in the sense that the norms $\|f\|_{s,p;q}$ do not scale nicely with dilations $f \mapsto f(\delta\cdot)$ as do the L^p norms); in fact, it is shown in Chapter 3 that a condition of this type is necessary to ensure the boundedness of $f \mapsto k * f$ on all of the Besov spaces.

There are also some new phenomena to be encountered in connection with the Besov spaces $B^{s,p}_q$ when at least one of the exponents p or q is infinite. In the L^p setting, it is a classical result that $f \mapsto k * f$ is bounded on $L^\infty(\mathbb{R}^n)$ if and only if $k = \mu$ is a finite Borel measure on $\mathbb{R}^n$, so that all non-trivial singular integrals fail to be bounded on this space, and one is hence permitted, without loss of generality, to restrict the considerations to $p < \infty$. However, the situation is quite different for the Besov spaces $B^{s,p}_q$: Not only is it possible to have the boundedness of several interesting operators on the whole scale of these spaces where $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, but in fact some of the most important and concrete instances of Besov spaces are the Hölder (or Lipschitz) spaces $BUC^s = \Lambda^s = B^{s,\infty}_\infty$ (for $s \in \mathbb{R}_+ \setminus \mathbb{N}$), so that the restriction to $p, q < \infty$ would be a serious omission.

The problem with the infinite exponents is the fact that the Schwartz functions fail to be dense in $B^{s,p}_q$ as soon as p or q is ∞. As a result, proving an estimate $\|k * f\|_{s,p;q} \le C \|f\|_{s,p;q}$ for all Schwartz functions f does not automatically imply the existence of a bounded extension T of $k*$ to all of $B^{s,p}_q$. Thus the extension of the operator to the whole space requires a procedure different from the standard density argument; moreover, even if *an* extension is found, there is no guarantee of uniqueness.

It seems that this uniqueness problem of the extension has been set aside by many authors, but these problems are given special attention in Chapter 3. The idea of imposing additional conditions on the extended operator T so as to assure uniqueness was already considered by GIRARDI and WEIS [**35**] in connection with the multiplier problem. They showed that uniqueness is guaranteed by requiring the extension to satisfy an additional weak-to-weak type continuity assumption. We show in Chapter 3 that even somewhat weaker additional requirements are sufficient.

In fact, it is shown that the very natural requirement that the extended operator T have the properties

$$\psi * Tf = T(\psi * f), \qquad \text{and} \qquad (Tf)(\cdot - h) = T[f(\cdot - h)]$$

for all $f \in B_q^{s,p}(\mathbb{R}^n; X)$, all $\psi \in \mathcal{S}(\mathbb{R}^n)$, and all $h \in \mathbb{R}^n$ specifies it uniquely provided $p < \infty$ (but $q = \infty$ allowed); however, for $p = \infty$ we have to impose an additional "compact-to-weak" continuity requirement on T—this is weaker than the weak-to-weak continuity in [**35**]. Counterexamples are also given to show that the uniqueness fails, in general, unless such conditions are imposed.

Chapter 4. In many respects, Chapter 4 represents the culmination of the present work. Although it continues the development of the vector-valued theory, the results improve on several known theorems even in the scalar case.

The motivation of the investigation comes from the comparison of the two classical multiplier theorems of MIHLIN and HÖRMANDER, whose sufficient conditions for m to be a Fourier multiplier on $L^p(\mathbb{R}^n)$ are (2.4) and (2.8), respectively. Let us, for the sake of simplicity, consider $n = 3$, and compare the set of derivatives of m for which an estimate is required in the two sets of assumptions:

- by both: m, $\partial m/\partial x$, $\partial m/\partial y$, $\partial m/\partial z$, $\partial^2 m/\partial x\partial y$, $\partial^2 m/\partial y\partial z$, $\partial^2 m/\partial z\partial x$;
- by HÖRMANDER only: $\partial^2 m/\partial x^2$, $\partial^2 m/\partial y^2$, $\partial^2 m/\partial z^2$;
- by MIHLIN only: $\partial^3 m/\partial x\partial y\partial z$.

The intersection of the assumptions is rather large, but both theorems require some more estimates in addition to those common to both. It seems very natural a question to enquire whether some additional estimates really are necessary, or whether the pure intersection would be sufficient to conclude the boundedness of the multiplier transformation.

More generally, in n dimensions, MIHLIN requires estimates for the derivatives whose order $\alpha = (\alpha_1, \ldots, \alpha_n)$ satisfies the ∞-norm estimate $|\alpha|_\infty \leq 1$, whereas HÖRMANDER for those with the 1-norm restricted by $|\alpha|_1 \leq \lfloor n/2 \rfloor + 1$. In general, these describe partially overlapping sets of multi-indices, neither of which is included in the other. Again, one may wonder whether it would suffice to assume conditions only for those derivatives whose order satisfies both restrictions $|\alpha|_\infty \leq 1$ *and* $|\alpha|_1 \leq \lfloor n/2 \rfloor + 1$.

The answer, which is proved in Chapter 4 (Cor. 7.4), is *yes*.

In fact, I derive this result as a corollary of a rather more general theorem in which the smoothness required of the multiplier is measured with a continuous parameter; moreover, variants of the method apply not only to the Lebesgue–Bôchner spaces but to the Hardy and Besov spaces as well. (For the convenience of the reader who is interested in the main ideas behind the proof, rather than the most general form of the results, a simplified and self-contained proof of the above described “intersection theorem”, just in the scalar-valued case, is presented in Chapter 5.) The philosophy of the proof is to combine results for convolution operators, which were developed separately for Hardy, Bôchner and Besov spaces in Chapters 1–3, with sharp new Fourier embedding theorems, which are then used to check the appropriate conditions for the convolution kernel $k = \check{m}$ by means of the assumptions on the multiplier m.

As simple as it seems, the simultaneous improvement of MIHLIN's and HÖRMANDER's multiplier theorem illustrates the power of the approach. The strong results obtained even in the classical situation also provide one more motivating factor for the vector-valued analysis carried out in this work: It is often the case that the consideration of a problem from a somewhat generalized point of view may reveal aspects which were unnoticed in the particular special case of initial interest. In accordance with this principle, it is found here, in the end, that the elaborate machinery, which was developed to cope with the difficulties encountered in the operator-valued setting, is powerful enough not only to *extend* the theorems already known for the scalar case to the Banach space framework, but to actually *improve* even on the original scalar-valued results.

CHAPTER 1

Translation-invariant operators on Hardy spaces

We prove new operator-valued Fourier multiplier theorems on real-variable Hardy spaces of vector-valued functions. These are applied to the maximal regularity question of the abstract Cauchy problem (ACP) $\dot{u} + Au = f,\ u(0) = 0$ on $H^p(\bar{\mathbb{R}}_+; X)$, $p \in]0, 1]$, as well as other equations.

In particular, we extend the recent theorem of L. Weis, which says that the regularity of the ACP on $L^p(\bar{\mathbb{R}}_+; X)$ (where $p \in]1, \infty[$ and X is UMD) is equivalent to the R-boundedness of $A(\mathbf{i}\xi + A)^{-1}$, $\xi \in \mathbb{R}$, by showing that these conditions are further equivalent to the regularity of the ACP on H^1, and moreover they are sufficient for the regularity on H^p, $p \in]0, 1[$.

The results of this chapter have been submitted in the form of the paper [**45**].

1. Introduction

Since the establishment of the real-variable characterization of the Hardy spaces H^p, $p \in]0, 1]$, various operators (such as singular integrals, multipliers, maximal functions etc.) have been extensively studied in this setting. This new point of view has provided considerable insight into the nature of these operators, whose theory was classically concentrated on the L^p-setting, with $p \in]1, \infty[$ (cf. [**34, 77**]). The purpose of the present chapter is to carry out this programme for the maximal regularity problem of certain abstract differential equations and the related convolution and Fourier multiplier operators. This is motivated by the significant progress in the corresponding L^p-theory in recent years, after the realization, first by Weis [**86, 87**], of the meaning of R-boundedness in connection with these problems. See also [**2, 21, 22, 25, 36, 80**].

The extension of these recent results to the real-variable Hardy spaces is not only a theoretical challenge, but also a relevant task in view of some applications. In fact, such techniques are of interest in the analysis of nonlinear equations of elasticity, where local Hardy spaces (cf. e.g. [**77**], p. 134) seem to be the "right tool" (I. Lasiecka, personal communication; see also [**18**], Sect. 1.2). We should admit, though, that a gap remains between the abstract setting of the present chapter and the applied problems just mentioned; in fact, we consider Hardy space norms in the time-like variable t, whereas our space-like variables are hidden in the abstract space X, which in most cases is required to be UMD, a condition which immediately excludes non-reflexive function spaces. Nevertheless, the results given here appear to be the first steps towards bringing the R-boundedness techniques, succesfully applied in the L^p-setting, down to H^1 and below.

Let us be more precise about the particular problems we have in mind. As the simplest example, consider the abstract Cauchy problem

$$\dot{u}(t) + Au(t) = f(t) \ \text{ for } t \geq 0, \qquad u(0) = 0, \tag{1.1}$$

with A a closed linear operator with dense domain in the underlying Banach space X and $f \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}_+; X)$.

It is well-known that, if $-A$ is the generator of a strongly continuous semigroup $(T^t)_{t\geq 0}$, then an L^p-solution [$u \in W^{1,p}(\bar{\mathbb{R}}_+; X)$ with $u(t) \in \mathcal{D}(A)$ for a.e. $t \geq 0$, and which satisfies (1.1) a.e.], when it exists, is necessarily given by the *variation-of-constants formula*

$$u(t) = \int_0^t T^{t-s} f(s)\,\mathrm{d}s, \qquad \text{hence} \qquad Au(t) = A\int_0^t T^{t-s} f(s)\,\mathrm{d}s, \tag{1.2}$$

and the above formula for u can always be used to *define* what is called the *mild solution.*

Moreover, if (T^t) is bounded and analytic and f is appropriate, then

$$\hat{u}(\xi) = (\mathbf{i}2\pi\xi + A)^{-1}\hat{f}(\xi), \qquad \mathcal{F}[Au](\xi) = A(\mathbf{i}2\pi\xi + A)^{-1}\hat{f}(\xi), \tag{1.3}$$

where $\hat{\ } \equiv \mathcal{F}$ is the Fourier transform.

By *maximal regularity* of the ACP one means that, with any given data f in a certain function class, there exists a unique solution u (in an appropriate sense) such that both the terms on the left-hand side of (1.1) possess the same regularity (e.g., in the L^p-setting, are integrable to the same power on the positive half-line) as f. One also occasionally takes as a definition the somewhat stronger requirement that this regularity also hold for u itself, but, as it turns out, the latter condition is true if and only if the first one is and, in addition, A is boundedly invertible.

Some classical facts (due to different authors) concerning the maximal L^p-regularity of the ACP are contained in the following result.

Theorem 1.4 (de Simon, Sobolevskij 1964). *Let X be an arbitrary Banach space, and ACP have maximal $L^{\tilde{p}}$-regularity for some $\tilde{p} \in\,]1,\infty[$. Then $-A$ is the generator of a bounded analytic semigroup, and ACP has maximal L^p-regularity for all $p \in\,]1,\infty[$. If X is (isomorphic to) a Hilbert space, then $-A$ generating a bounded analytic semigroup is sufficient for maximal L^p-regularity for all $p \in\,]1,\infty[$.*

Proofs of the various statements and comments on the original works can be found in the review article [**27**] of Dore. As described in more detail there, whereas the equivalence of the various L^p-regularities was already observed by Sobolevskij [**75**], to see when this regularity property holds even for one $p \in\,]1,\infty[$

remained open for a much longer time (except in the Hilbert space case solved by de Simon [**74**]).

Several sufficient conditions for the maximal L^p-regularity of particular classes of operators were obtained over the years (again cf. [**27**]) but a fairly complete answer was given only recently: First, Kalton and Lancien [**49**] showed that, out of all Banach spaces with an unconditional basis, those isomorphic to a Hilbert space are the only ones were maximal regularity holds for *all* negative generators of analytic semigroups. Then, in the context of UMD-spaces, Weis [**87**] gave a *characterization* of those operators A for which the ACP does have maximal L^p-regularity. This characterization made use of the notion of *R-boundedness*, a concept first exploited by Bourgain [**12**] and systematically studied by Clément, de Pagter, Sukochev and Witvliet [**21**].

Theorem 1.5 (Weis 2000). *Let X be a UMD-space and $-A$ the generator of a bounded analytic semigroup. Then the following are equivalent:*

(W_1) *ACP has maximal L^p-regularity for all $p \in \,]1,\infty[$.*

(W_2) *The collection $\{A(\mathbf{i}2\pi\xi + A)^{-1} \mid \xi \in \mathbb{R} \setminus \{0\}\}$ is R-bounded.*

Let us now consider the question of whether, and to what extent, these results could be extended to H^p for $p \in \,]0,1]$.

First of all, we note that for the *differential equation* (1.1) to have a meaning as written, it is assumed that f is a *function*, yielding a value $f(t)$ for $t \geq 0$; however, for $p \in \,]0,1[$, an element of H^p is no longer a function in general, but a (tempered) distribution. (E.g., we have $\delta_a - \delta_b \in H^p(\mathbb{R})$ for $p \in]1/2,1[$, where δ_a is the Dirac mass at a.) Nevertheless, one can still consider the *operators* $f \mapsto Au$ defined by (1.2) or (1.3), initially on an appropriate dense subspace of H^p, and hope to establish their boundedness which then permits a unique continuous linear extension to all of H^p.

Taking the existence of such a continuous extension as the definition of maximal regularity in the general setting (for details, see Def. 3.10), one can establish the following extension of the classical equivalence of the various L^p-regularities:

> If the ACP has maximal $L^{\tilde{p}}$-regularity for some $\tilde{p} \in \,]1,\infty[$, then
> it also has maximal H^p-regularity for all $p \in \,]0,1]$.

In fact, this follows rather readily from general extension results due to Strömberg and Torchinsky [**81**]: Once a singular integral operator is bounded on some $L^{\tilde{p}}$, and its kernel (or the corresponding multiplier) satisfies certain conditions, the operator will also be bounded on H^p. Somewhat surprisingly, the scalar-valued results in [**81**] turn out to generalize to the setting of vector-valued functions and operator-valued kernels with essentially the same proofs, making only obvious, mostly notational modifications (Sect. 5).

Having obtained this sufficient condition for maximal H^p-regularity almost for free, one could also ask for a converse type implication, i.e., whether the knowledge of having maximal regularity on H^p could be used to deduce the corresponding property in the classical L^p-setting.

We are able to give a partial affirmative answer, and the key result in this direction is the following:

> If the ACP has (H^1, L^1) regularity, i.e., if $f \mapsto Au$ maps (a dense subset of) H^1 boundedly into L^1, then $-A$ generates a bounded analytic semigroup and the set $\{A(\mathbf{i}2\pi\xi + A)^{-1} | \ \xi \in \mathbb{R} \setminus \{0\}\}$ is R-bounded.

The R-boundedness assertion actually follows from the much more general and far-reaching Theorem 4.2, which asserts the R-boundedness of the essential range of any Fourier-multiplier mapping $H^1(\mathbb{R}^n; X)$ into $L^1(\mathbb{R}^n; Y)$. It extends the corresponding L^p-result due to CLÉMENT and PRÜSS [**22**] (which is restated as Theorem 4.1).

With the above mentioned sufficient criterion for R-boundedness and the implication $W_2 \Rightarrow W_1$ of Theorem 1.5, we obtain the fact that maximal H^1-regularity (or actually the formally weaker (H^1, L^1)-regularity) implies maximal L^p-regularity for all $p \in]1, \infty[$ in the UMD-setting.

Combining the results mentioned so far, we can augment WEIS' Theorem 1.5, so as to get the following more complete characterization of the maximal regularity of the ACP on a UMD-space:

THEOREM 1.6. *Let X be a UMD-space. Then the following are equivalent:*

(C_1) *ACP has maximal L^p-regularity for all $p \in]1, \infty[$.*
(C_2) *ACP has maximal H^1-regularity.*
(C_3) *ACP has (H^1, L^1)-regularity.*
(C_4) *$-A$ generates a bounded analytic semigroup and the collection*

$$\{A(\mathbf{i}2\pi\xi + A)^{-1} | \ \xi \in \mathbb{R} \setminus \{0\}\} \text{ is } R\text{-bounded.}$$

Moreover, any of these is sufficient to

(C_5) *ACP has maximal H^p-regularity for all $p \in]0, 1[$.*

REMARK 1.7. In fact, the implications $C_1 \Rightarrow C_2 \Rightarrow C_3 \Rightarrow C_4$ and $C_1 \Rightarrow C_5$ hold for any Banach space X.

We should note that the equivalence of C_2 and C_3 above is also a consequence of a more general result (Lemma 6.1) concerning Fourier multipliers acting on UMD-valued function spaces.

One could hope that the implication $C_3 \Rightarrow C_1$ above would turn out to be of use in some applications; indeed, to verify (H^1, L^1)-regularity, one would need to check that $f \mapsto Au$ maps *atoms* of H^1 uniformly into integrable functions, and this could in some cases be simpler than the direct verification of the other items in Theorem 1.6.

Having sketched the results we are going to prove for the Cauchy problem, we note that the methods applied are by no means restricted to this particular equation (although the existence of the simple and explicit variation-of-constants formula (1.2) can be used to simplify certain matters). We can also treat the more general *fractional order equations*

$$(1.8)\quad D^\alpha u(t) + Au(t) = f(t) \ \text{ for } t \geq 0, \qquad u(0) = 0, \qquad (\dot{u}(0) = 0 \text{ if } \alpha > 1),$$

where $\alpha \in]0, 2[$.

While the previous example was still a problem on the line, our methods work equally well in n dimensions, and as an example in this direction we are able to give a maximal regularity result similar to Theorem 1.6 for the abstract Laplace equation

$$(1.9)\qquad -\triangle u(t) + Au(t) = f(t) \ \text{ for } t \in \mathbb{R}^n.$$

The chapter is organized as follows: Sect. 2 is preliminary, collecting some general notation and facts to be used. Two lengthy proofs of lemmata concerning Hardy spaces are postponed to an appendix, Sect. 8. In Sect. 3 we discuss in detail the relation between different possible notions of maximal regularity in the setting of Hardy spaces, and we also prove the necessity of $-A$ generating an analytic semigroup (Theorem 3.1) for the (H^1, L^1)-regularity of (1.1). We go on with necessary conditions, from the point of view of general multipliers, in Sect. 4, where we prove the necessity of R-boundedness for (H^1, L^1)-multipliers (Theorem 4.2). Sufficient conditions for the boundedness of our operators are then taken up in Sect. 5. We return to the problem of maximal regularity in Sect. 6, where we complete the proof of Theorem 1.6, and also formulate and prove analogous results for the problems (1.8) and (1.9). Brief final remarks are given in Sect. 7.

2. General preliminaries

Let us fix some notation. The set of natural numbers is $\mathbb{N} := \{0, 1, 2, \ldots\}$ and that of positive integers is $\mathbb{Z}_+ := \{1, 2, \ldots\}$. Moreover, $\mathbb{R}_+ :=]0, \infty[$ and $\bar{\mathbb{R}}_+ := [0, \infty[$. For $\ell > 0$, we denote by $\lfloor \ell \rfloor$ the greatest integer at most ℓ, and by $\llbracket \ell \rrbracket$ the greatest integer strictly less than ℓ. Thus both functions give the integer part of a non-integer ℓ, but $\llbracket m \rrbracket = m - 1$, $\lfloor m \rfloor = m$ for $m \in \mathbb{Z}_+$.

X and Y are complex Banach spaces. The *Lebesgue–Bôchner spaces* of X-valued functions on Ω [usually $\mathbb{R}^n$ or $\bar{\mathbb{R}}_+$, always equipped with the Lebesgue measure] are denoted by $L^p(\Omega; X)$, and the *Hardy spaces* [whose definition is given later in this section] by $H^p(\Omega; X)$. If $X = \mathbb{C}$, we omit it from the notation and simply write $L^p(\mathbb{R}^n)$ etc.

Test function spaces. $\mathcal{S}(\mathbb{R}^n; X)$ denotes the *Schwartz class* of infinitely differentiable, rapidly decreasing X-valued functions. The X-valued tempered distributions are defined by $\mathcal{S}'(\mathbb{R}^n; X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n), X)$, where $\mathcal{L}(A, B)$ denotes the space of continuous linear operators between the topological vector spaces A and B.

Important test function classes include $\mathcal{D}(\mathbb{R}^n; X) := \mathcal{C}_c^\infty(\mathbb{R}^n; X) \subset \mathcal{S}(\mathbb{R}^n; X)$, where the subscript c indicates compact support, and

$$\hat{\mathcal{D}}_0(\mathbb{R}^n; X) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n; X) \,\middle|\, \hat{\psi} \in \mathcal{D}(\mathbb{R}^n; X),\ 0 \notin \operatorname{supp} \hat{\psi} \right\},$$

where $\hat{\psi}$ stands for the Fourier transform of ψ. It is well-known that all the test-function classes mentioned so far are dense in $L^p(\mathbb{R}^n; X)$ for $p \in]1, \infty[$; in fact, this is true even for the algebraic tensor products $X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n)$ etc.

Fourier transform and multipliers. The Fourier transform, of $f \in L^1(\mathbb{R}^n; X)$, is defined by

$$\hat{f}(\xi) \equiv \mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(t) e^{-\mathbf{i}2\pi\xi\cdot t} \, \mathrm{d}t.$$

It is an isomorphism on $\mathcal{S}(\mathbb{R}^n; X)$, as well as on $\mathcal{S}'(\mathbb{R}^n; X)$ [where it is defined by the duality $\left\langle \hat{f}, \psi \right\rangle := \left\langle f, \hat{\psi} \right\rangle$]. Moreover, the equality $\mathcal{F}^2 f(t) = f(-t)$ is always true in the sense of tempered distributions. The inverse Fourier transform is denoted by $\check{f} \equiv \mathcal{F}^{-1} f$.

Given $m \in L^1_{\text{loc}}(\mathbb{R}^n; \mathcal{L}(X, Y))$, we can consider the *Fourier multiplier operator* T, initially defined on $\hat{\mathcal{D}}_0(\mathbb{R}^n; X)$, say, by $Tf := \mathcal{F}^{-1}[m\hat{f}]$; or more explicitly,

$$Tf(t) := \int_{\mathbb{R}^n} m(\xi) \hat{f}(\xi) e^{\mathbf{i}2\pi\xi\cdot t} \, \mathrm{d}\xi. \tag{2.1}$$

It is an interesting question to determine whether, for a given m, the operator T has a bounded extension from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$, say. Several classical results are well-known in the scalar context; for the vector-valued situation, it has been known for some time that a reasonable theory is valid for the Banach spaces with the *UMD-property*. This means the *u*nconditionality of *m*artingale *d*ifference sequences in $L^p([0, 1]; X)$ for one (and then all) $p \in]1, \infty[$, or what is equivalent [according to results due to Burkholder and Bourgain], that the multiplier $m(\xi) := -\mathbf{i} \operatorname{sgn}(\xi)$ defines by means of (2.1) a bounded operator, the Hilbert transform, on $L^p(\mathbb{R}; X)$ for one (and then all) $p \in]1, \infty[$. See e.g. the review paper of Rubio de Francia [**71**] for more on UMD-spaces.

Fourier-type of Banach spaces. As another notion from the geometry of Banach spaces, we recall that a Banach space X is said to have *Fourier-type* p, if the Hausdorff–Young inequality

$$\|\hat{f}\|_{L^{p'}(\mathbb{R}; X)} \leq C \, \|f\|_{L^p(\mathbb{R}; X)}\,, \tag{2.2}$$

is true for every $f \in (L^1 \cap L^p)(\mathbb{R}; X)$ with some finite C. Obviously every Banach space satisfies this inequality with $p = 1$, and by interpolation the inequality holds for $q \in]1, p[$ if it holds for some $p > 1$. X is said to have a non-trivial Fourier-type, if it has a Fourier-type $p > 1$. Note that once (2.2) is true, the corresponding inequality with $\mathbb{R}$ replaced by $\mathbb{R}^n$ also holds due to the tensor nature of the Fourier transform.

The notion of Fourier-type is due to PEETRE [**65**]. He proved in [**65**], among other things, that every space $L^p(\Omega,\Sigma,\mu)$ (of scalar-valued functions) has Fourier-type $\min(p,p')$. KWAPIEŃ [**55**] has shown that B has Fourier-type 2 if and only if it is isomorphic to a Hilbert space.

Because of the significant rôle of the UMD-spaces in the theory of multipliers, it is useful to know that every UMD-space has a non-trivial Fourier-type. This is a consequence of the following results: (This argument was shown to me by S. GEISS.)

- A UMD-space does *not* contain uniformly $\ell^1(r) := (\mathbb{C}^r, |\cdot|_1)$ for $r\in\mathbb{Z}_+$.
- A Banach space X does not contain uniformly $\ell^1(r)$, $r\in\mathbb{Z}_+$, if and only if X has a non-trivial Rademacher-type.
- X has a non-trivial Rademacher-type if and only if it has a non-trivial Fourier-type.

The first assertion is easy to prove, since the non-reflexive sequence space ℓ^1 is not UMD (UMD-spaces being even super-reflexive, see [**71**, p. 205]), and so has infinite UMD-constants $M_p(\ell^1)=\infty$. By approximating ℓ^1-valued martingales by their projections to the r first coordinate directions, it follows readily that the UMD-constant of $\ell^1(r)$ is larger than any preassigned $M>0$ once r is large enough, i.e., $M_p(\ell^1(r))\to\infty$ as $r\to\infty$, which proves the assertion.

The second and in particular the third claim above are deeper, and we refer to [**67**], Theorems 4.4.7 and 5.6.30, and the references cited there, also for the definition of the Rademacher-type. These results are originally due to PISIER and BOURGAIN, respectively.

R-boundedness. This has become a prominent notion in connection with results for operator-valued Fourier multipliers and singular integrals. We denote by ε_j, $j=1,2,\ldots$, the Rademacher system of independent random variables on some probability space $(\Omega,\Sigma,\mathbb{P})$ which satisfy $\mathbb{P}(\varepsilon_j=1)=\mathbb{P}(\varepsilon_j=-1)=1/2$. $\mathbb{E}$ denotes the expectation related to the probability measure $\mathbb{P}$.

We recall [**22, 87**] that $\mathcal{T}\subset\mathcal{L}(X;Y)$ is called *R-bounded*, the R being short for Rademacher, randomized or Riesz, if for some $p\in\,]0,\infty[$ and $C<\infty$ and for all $N\in\mathbb{Z}_+$, $x_j\in X$, $T_j\in\mathcal{T}$ the inequality

$$\left(\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j T_j x_j\right|_Y^p\right)^{\frac{1}{p}} \le C\left(\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j x_j\right|_X^p\right)^{\frac{1}{p}}$$

holds. It follows from KAHANE's inequality that for each fixed $\mathcal{T}$, the condition in fact holds true either for all $p\in\,]0,\infty[$ (with C possibly depending on p) or for none. We shall be mostly concerned with the case $p=1$, and we define the R-bound of $\mathcal{T}$ as

$$\mathcal{R}$$

refer to the smallest C in this inequality as the R-bound of $\mathcal{T}$ and denote it by $\mathcal{R}(\mathcal{T})$.

One of the most standard tools related to R-boundedness is the *contraction principle* (of J.-P. Kahane) stating that

$$\left(\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j\lambda_j x_j\right|_X^p\right)^{\frac{1}{p}} \le \frac{\pi}{2}\left(\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|_X^p\right)^{\frac{1}{p}}$$

for $p \ge 1$, $N \in \mathbb{Z}_+$, $x_j \in X$ and $\lambda_j \in \mathbb{C}$ with $|\lambda_j| \le 1$.

In the literature, one usually finds this with the constant 2 in place of $\pi/2$. Even though the size of constants is quite immaterial for our purposes, we shall use the inequality in this sharper form, which is proved in [**67**, §3.5.4].

Atomic Hardy spaces. We recall the definition of the Hardy spaces of vector-valued functions and establish some of their properties that are relevant to us in the subsequent sections.

As is well-known in the scalar-valued setting, there exist various equivalent characterizations of the spaces H^p, $0 < p \le 1$. In the vector-valued situation, not all of these equivalences remain valid, and we must be more careful about the definition. Here we are concerned with the atomic Hardy spaces

$$H^p(\mathbb{R}^n;X) := \left\{\mathcal{S}'\text{-}\sum_{k=0}^{\infty}\lambda_k a_k \ : a_k \text{ an } H^p\text{-atom},\ \lambda_k \in \mathbb{C},\ \sum_{k=0}^{\infty}|\lambda_k|^p < \infty\right\}$$

(where $\mathcal{S}'\text{-}\sum$ indicates convergence of the series in the sense of tempered distributions), equipped with the quasi-norm

$$\|f\|_{H^p(\mathbb{R}^n;X)}^p := \inf\sum_{k=0}^{\infty}|\lambda_k|^p\,,$$

where the infimum is taken over all atomic decompositions of $f \in H^p$ as in the definition of the atomic Hardy space.

The definition of the atoms appearing above is the same as in the scalar-valued context: We say that $a \in L^q(\mathbb{R}^n;X)$ is a (p,q,N)-atom, where $0 < p \le 1 < q \le \infty$ and $N \in \mathbb{N}$, provided that

- a is supported in a ball $\bar{B}$,
- $\|a\|_{L^q} \le \left|\bar{B}\right|^{q^{-1}-p^{-1}}$, and
- $\int x^\alpha a(x)\,\mathrm{d}x = 0$ for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$.

The three requirements above are referred to as the *support* condition, the *size* condition and the *moment* condition, respectively.

We say that a is a (p,q)-atom if it is a (p,q,N)-atom for some $N \in \mathbb{N}$, and that a is an H^p-atom of L^q-type if it is a (p,q,N)-atom for some $N \ge n(p^{-1}-1)$. Finally, a is an H^p-atom, if it is an H^p-atom of some L^q-type, $q > 1$. In the definition of H^p above, we require that the a_k are (p,q)-atoms for some fixed $q > 1$. The spaces obtained with different values of q coincide and the norms are equivalent. In the sequel, we will freely use any of the equivalent norms of H^p

defined in terms of the different values of $q \in]1,\infty[$, whichever is most suited to a particular purpose.

Hardy spaces on a half-line. For the purposes of studying the Cauchy problem, a notion of Hardy spaces on $\bar{\mathbb{R}}_+$ is useful. Of course, one could simply *define* (and this could be done with any set in place of $\bar{\mathbb{R}}_+$)

$$H^p(\bar{\mathbb{R}}_+;X) := \left\{ \mathcal{S}'\text{-}\sum_{k=0}^{\infty} \lambda_k a_k : a_k \text{ an } H^p\text{-atom supported on } \bar{\mathbb{R}}_+, \lambda_k \in \mathbb{C},\ \sum_{k=0}^{\infty} |\lambda_k|^p < \infty \right\},$$

and take as the norm the same formal expression as in the case of the line, but restricted to those atomic decompositions living on the positive half-line only.

However, we would like to identify $H^p(\bar{\mathbb{R}}_+;X)$ with the subset of $H^p(\mathbb{R};X)$ consisting of those elements supported on the positive half-line, in the way familiar from the context of the L^p spaces. Contrary to the case of L^p where this identification, due to the fact that the size of a function is essentially the sum of the sizes of its local parts, is more or less obvious, it is not clear *a priori* that we can do the same in the H^p context; indeed, the fact that a distribution f is supported on $\bar{\mathbb{R}}_+$ and has an atomic decomposition does not imply that all (or in fact any) atoms of the particular decomposition should have their support contained in $\bar{\mathbb{R}}_+$. Nevertheless, things can be settled, as stated in the following:

LEMMA 2.3. *We have $H^p(\bar{\mathbb{R}}_+;X) \approx \{f \in H^p(\mathbb{R};X) : \operatorname{supp} f \in \bar{\mathbb{R}}_+\}$, in the sense of coincidence of the sets and equivalence of norms.*

More precisely, there exists $C = C_p < \infty$ such that for every $f \in H^p(\mathbb{R};X)$ supported on $\bar{\mathbb{R}}_+$, there exists an atomic decomposition $f = \mathcal{S}'\text{-}\sum_{k=1}^{\infty} \lambda_k a_k$ such that $\operatorname{supp} a_k \subset \bar{\mathbb{R}}_+$ and $\sum_{k=0}^{\infty} |\lambda_k|^p \leq C \|f\|^p_{H^p(\mathbb{R};X)}$, where the p-norm of $H^p(\mathbb{R};X)$ is defined using all atomic decompositions of f, possibly not supported on $\bar{\mathbb{R}}_+$.

The proof of this lemma, as well as that of the next one, is postponed to Sect. 8.

Dense subsets. It is always useful to have a convenient dense subspace to work with. The following lemma shows that the problem of finding dense subspaces of the Hardy spaces of vector-valued functions reduces to the correponding task in the scalar-valued context.

LEMMA 2.4. *Let Z be a dense subspace of X, and G a dense subspace of $H^p(\mathbb{R}^n)$ resp. $H^p(\bar{\mathbb{R}}_+)$. Then $Z \otimes G$ is a dense subspace of $H^p(\mathbb{R}^n;X)$ resp. $H^p(\bar{\mathbb{R}}_+;X)$. In particular,*

- *$X \otimes (\mathcal{D}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n))$ and $X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n)$ are dense in $H^p(\mathbb{R}^n;X)$,*
- *$X \otimes (\mathcal{C}_c^\infty(\mathbb{R}_+) \cap H^p(\bar{\mathbb{R}}_+))$ is dense in $H^p(\bar{\mathbb{R}}_+;X)$.*

Recall that

$$\mathcal{D}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n) \\ = \{f \in \mathcal{D}(\mathbb{R}^n) | \ f \text{ has the same vanishing moments as an atom of } H^p\},$$

and the same is true with $\mathcal{D}(\mathbb{R}^n)$ replaced by $\mathcal{S}(\mathbb{R}^n)$.

3. Notion of regularity of the ACP on Hardy spaces

In this section, we make slightly more precise the sense in which the regularity of the solutions of the Cauchy problem is equivalent to the boundedness of the (singular) integral and multiplier operators defined in (1.2) and (1.3). Although this is reasonably well-known in the L^p context, and the proofs in the H^p setting turn out to be quite standard, it seems appropriate to have a brief look at the very notion of regularity in this new setting, so as to underline the particular properties that are required of the function spaces [or more generally, spaces of distributions] in question for the results familiar from the L^p-context to make sense.

A necessary condition. First of all, one matter ought to be cleaned out of the way. Although it is possible to formulate a notion of regularity for the ACP with A any linear operator whatsoever, it is more convenient to work with a (negative) generator of a C_0-semigroup $(T^t)_{t\geq 0}$, or better still, a bounded analytic semigroup. According to the classical Theorem 1.4, in order to have maximal L^p-regularity for $p \in]1, \infty[$, it is necessary that $-A$ is a generator, so that there is no loss in generality in making this assumption when seeking for sufficient conditions in that setting.

The next theorem, which is the main result of this section, shows that the same property remains true for (H^1, L^1)-type regularity, as defined for general A in the statement of the theorem. Recall that $W^{1,p}(I; X)$, $I \subset \mathbb{R}$ being an interval, is the space of all $f \in L^p(I; X)$ whose distributional derivative f' is also in $L^p(I; X)$; then $W^{1,p}_{\text{loc}}(\bar{\mathbb{R}}_+; X)$ is the space of all f whose restriction to any finite interval $I = [0, b] \subset \bar{\mathbb{R}}_+$ is in $W^{1,p}(I; X)$.

THEOREM 3.1. *Let X be a complex Banach space and A a densely defined, closed, linear operator in X. Suppose that for every $f \in H^1(\bar{\mathbb{R}}_+; X)$ there exists a unique function $u \in W^{1,1}_{\text{loc}}(\bar{\mathbb{R}}_+; X)$ such that $u(t) \in \mathcal{D}(A)$ and $\dot{u}(t) + Au(t) = f(t)$ for a.e. $t \geq 0$, and moreover*

$$\|\dot{u}\|_{L^1(\bar{\mathbb{R}}_+;X)} + \|Au\|_{L^1(\bar{\mathbb{R}}_+;X)} \leq C\, \|f\|_{H^1(\bar{\mathbb{R}}_+;X)} .$$

Then $-A$ generates a bounded analytic semigroup.

If, in addition, we always have $\|u\|_{L^1(\bar{\mathbb{R}}_+;X)} \leq C\, \|f\|_{H^1(\bar{\mathbb{R}}_+;X)}$, then A is boundedly invertible.

The proof will essentially copy that in [**27**] for the L^p case. The only difference is that instead of the auxiliary functions $f_\lambda(t) = e^{\lambda t}\chi_{[0,1/\operatorname{Re}\lambda]}(t)$ used there, we will need [in order to ensure membership in H^1] the slightly more complicated expression

$$f_\lambda(t) := (A_\lambda e^{\lambda t} + B_\lambda)\chi_{[0,1/\operatorname{Re}\lambda]}(t) \qquad \text{for } \operatorname{Re}\lambda > 0, \tag{3.2}$$

where the constants A_λ and B_λ are chosen in an appropriate manner.

More precisely, we want to impose the conditions

$$0 \equiv \int_0^\infty f_\lambda(t)\,\mathrm{d}t = A\frac{e^{\lambda/\operatorname{Re}\lambda}-1}{\lambda} + B\frac{1}{\operatorname{Re}\lambda} = \frac{1}{\operatorname{Re}\lambda}\left(A\frac{e^{1+\mathbf{i}\theta}-1}{1+\mathbf{i}\theta} + B\right),$$

$$\theta := \frac{\operatorname{Im}\lambda}{\operatorname{Re}\lambda},$$

which is the requirement $f_\lambda \in H^1$ (since f_λ is bounded and compactly supported in any case), and

$$\frac{1}{\operatorname{Re}\lambda} \equiv \int_0^\infty e^{-\lambda t} f_\lambda(t)\,\mathrm{d}t = A\frac{1}{\operatorname{Re}\lambda} + B\frac{1-e^{-\lambda/\operatorname{Re}\lambda}}{\lambda} = \frac{1}{\operatorname{Re}\lambda}\left(A + B\frac{1-e^{-1-\mathbf{i}\theta}}{1+\mathbf{i}\theta}\right),$$

whose meaning will be clear later on.

[In [**27**], the vanishing of the first integral above is not needed, and the second condition is satisfied simply with the choice $A = 1,\ B = 0$.] The fact that we can choose A_λ and B_λ with the desired properties in a uniform manner follows from the following technical lemma:

LEMMA 3.3. *The pair of equations*

$$\begin{cases} A\frac{e^{1+\mathbf{i}\theta}-1}{1+\mathbf{i}\theta} + B & = 0 \\ A + B\frac{1-e^{-1-\mathbf{i}\theta}}{1+\mathbf{i}\theta} & = 1 \end{cases}$$

has for every $\theta \in \mathbb{R}$ a unique solution $(A(\theta), B(\theta))$, and $|A(\theta)| + |B(\theta)| \le C$ for some constant C independent of $\theta \in \mathbb{R}$.

PROOF. The matrix elements being bounded functions of θ, it follows from elementary linear algebra that a proof amounts to showing that the determinant

$$D(\theta) := \frac{e^{1+\mathbf{i}\theta}-1}{1+\mathbf{i}\theta}\cdot\frac{1-e^{-1-\mathbf{i}\theta}}{1+\mathbf{i}\theta} - 1 = \frac{e^{1+\mathbf{i}\theta} + e^{-1-\mathbf{i}\theta} - 3 + \theta^2 - 2\mathbf{i}\theta}{(1+\mathbf{i}\theta)^2}$$

satisfies $|D(\theta)| \ge c > 0$. Clearly $D(\theta) \to -1$ when $|\theta| \to \infty$, so that by continuity and compactness it suffices to show that $D(\theta)$ has no zeros. By considering separately the real and imaginary parts of the numerator of $D(\theta)$, we find that $D(\theta) = 0$ is equivalent to

$$\begin{cases} (e + e^{-1})\cos\theta + \theta^2 & = 3, \\ (e - e^{-1})\sin\theta - 2\theta & = 0. \end{cases}$$

One easily verifies, e.g., that the second equation has exactly the solutions $\theta = 0$ and $\theta = \pm\theta_0$, where $\theta_0 \approx 0.968$, and none of these is a solution of the first equation. $\square$

Now we are ready to prove the semigroup generation. Recall (cf. e.g. [**30**], Sect. II.4.a, where the terminology is slightly different though) that the condition that $-A$ generates a bounded analytic semigroup is equivalent to saying that A is sectorial of angle $\omega < \pi/2$. We recall the definition of sectoriality:

DEFINITION 3.4. We say that the linear operator A, with dense domain $\mathcal{D}(A)$ in X, is *sectorial* of angle $\omega \in]0,\pi[$ if

- the spectrum of A satisfies $\sigma(A) \subset \bar{\Sigma}_\omega$, where $\Sigma_\omega := \{\zeta \in \mathbb{C} \setminus \{0\} : |\arg(\zeta)| < \omega\}$, and
- for all $\theta \in]\omega,\pi[$ there exists a $C_\theta < \infty$ such that $\|\zeta(\zeta - A)^{-1}\|_{\mathcal{L}(X)} \leq C_\theta$ for all $\zeta \notin \bar{\Sigma}_\theta$.

For later use of this condition, note in particular that the above estimate holds with $\zeta \in \mathbf{i}\mathbb{R} \setminus \{0\}$ when $\omega < \pi/2$, and then also the similar estimate with $A(\zeta - A)^{-1}$ in place of $\zeta(\zeta - A)^{-1}$, since their difference is just the identity. Then finally to the proof:

PROOF OF THEOREM 3.1. Let $\operatorname{Re}\lambda > 0$ and let f_λ be defined by (3.2), with A_λ and B_λ chosen so that $|A_\lambda| + |B_\lambda| \leq C$ (independent of λ) and

$$\int_0^\infty f_\lambda(t)\,\mathrm{d}t = 0, \qquad \int_0^\infty e^{-\lambda t} f_\lambda(t)\,\mathrm{d}t = \frac{1}{\operatorname{Re}\lambda}. \tag{3.5}$$

[That this choice is possible is the content of Lemma 3.3 and the preceding remarks.] Since f_λ is bounded and compactly supported on the positive half-line, with vanishing integral, it is an element of $H^1(\bar{\mathbb{R}}_+)$, and more precisely, the norm is estimated by

$$\|f_\lambda\|_{H^1} \leq \|f_\lambda\|_{L^\infty} |\operatorname{supp} f_\lambda| \leq (|A_\lambda| \cdot e + |B_\lambda|)\,\frac{1}{\operatorname{Re}\lambda} \leq \frac{C}{\operatorname{Re}\lambda}.$$

Then for every $x \in X$, we have $f_\lambda(\cdot)x \in H^1(\bar{\mathbb{R}}_+; X)$, and by the assumptions of the theorem, to such a function corresponds a unique $u =: \mathcal{U}(f_\lambda x)$ with the properties listed in the assumptions.

Let us then define

$$R_\lambda x := \operatorname{Re}\lambda \int_0^\infty e^{-\lambda t}\mathcal{U}(f_\lambda x)(t)\,\mathrm{d}t = \frac{\operatorname{Re}\lambda}{\lambda}\int_0^\infty e^{-\lambda t}\dot{\mathcal{U}}(f_\lambda x)(t)\,\mathrm{d}t, \tag{3.6}$$

where the existence of the second integral is clear, since $\dot{u} \in L^1(\bar{\mathbb{R}}_+; X)$, and then $u \in L^\infty(\bar{\mathbb{R}}_+; X)$, so that the existence of the first integral also follows. The equality of the two follows from integration by parts.

With the help of $(\lambda + A)u = \lambda u + f - \dot{u}$ and (3.6), we find that

$$(\lambda + A)R_\lambda x = \operatorname{Re}\lambda \int_0^\infty e^{-\lambda t} f_\lambda(t)\,\mathrm{d}t\, x = x,$$

using for the last step the second equality in (3.5). Thus R_λ is a right inverse of $\lambda + A$, and the fact that it is also a left inverse follows, as in [**27**], from $A\mathcal{U}(f_\lambda x) = \mathcal{U}(f_\lambda A x)$ for $x \in \mathcal{D}(A)$; this equation is proved in Sect. 9.

Thus we have $\mathbb{C}_+ \subset \rho(-A)$ [the resolvent set of $-A$], and from (3.6) we obtain the estimate

$$\left|(\lambda + A)^{-1}x\right|_X \leq \frac{\operatorname{Re}\lambda}{|\lambda|}\left\|\dot{\mathcal{U}}(f_\lambda x)\right\|_{L^1(\bar{\mathbb{R}}_+;X)} \leq \frac{\operatorname{Re}\lambda}{|\lambda|} C\left\|f_\lambda x\right\|_{H^1(\bar{\mathbb{R}}_+;X)}$$
$$\leq \frac{\operatorname{Re}\lambda}{|\lambda|}\frac{\tilde{C}}{\operatorname{Re}\lambda}|x|_X = \frac{\tilde{C}}{|\lambda|}|x|_X .$$

Thus $\|\lambda(\lambda + A)^{-1}\|_{\mathcal{L}(X)} \leq C$ for $\lambda \in \mathbb{C}_+$, and by standard resolvent arguments, this inequality, which is uniform in the sector of angle $\pi/2$ continues to hold in a slightly larger sector, possibly adjusting the constant. Thus $-A$ is the generator of a bounded analytic semigroup.

Under the extra assumption on the regularity of u, we also have from the first form of $R_\lambda x$ in (3.6) that

$$\left|(\lambda + A)^{-1}x\right|_X \leq \operatorname{Re}\lambda\left\|\mathcal{U}(f_\lambda x)\right\|_{L^1(\bar{\mathbb{R}}_+;X)} \leq \operatorname{Re}\lambda \cdot C\left\|f_\lambda x\right\|_{H^1(\bar{\mathbb{R}}_+;X)}$$
$$\leq \operatorname{Re}\lambda\frac{\tilde{C}}{\operatorname{Re}\lambda}|x|_X = \tilde{C}\,|x|_X ,$$

and the bounded invertibility of A follows from the uniformity of this inequality as $\lambda \downarrow 0$. $\square$

Theorem 3.1 at our disposal, it will henceforth be assumed, in connection with the Cauchy problem, that $-A$ is the generator of a bounded analytic semigroup (T^t).

Different notions of regularity. As was mentioned in the Introduction, for our treatment of the ACP on Hardy spaces, we will deviate from the commonly used notion in the L^p-setting, where maximal regularity of the ACP is usually defined in terms of the regularity of the mild solution, simply because the notion of the mild solution assumes $f \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}_+;X)$, which in our case is not true in general. Thus we shall only examine the regularity of the solution when the data f is smooth enough (but requiring the quantitative estimates to be independent of this smoothness), and we relate this to the boundedness of the operators $f \mapsto Au$ in (1.2) and (1.3) on a dense subspace of the spaces of interest. Having examined the most general situation, we also show that this operator-based notion still agrees with the regularity defined in terms of the mild solution in the border-line case of H^1 and L^1.

Let us first recall the following fact. The result is more or less folklore, but a proof is nevertheless given for completeness.

Proposition 3.7. *Let $-A$ be the generator of a bounded, analytic semigroup on X. For $f \in C^1_c(\mathbb{R}_+;X)$, the classical solution u of the ACP satisfies, for a.e. $\xi \in \mathbb{R}$,*

$$\widehat{u}(\xi) = \mathbf{i}2\pi\xi(\mathbf{i}2\pi\xi + A)^{-1}\hat{f}(\xi), \qquad \widehat{Au}(\xi) = A(\mathbf{i}2\pi\xi + A)^{-1}\hat{f}(\xi) =: m(\xi)\hat{f}(\xi).$$

PROOF. Estimating crudely the variation-of-constants formula, we observe that

$$\begin{aligned} |u(t)|_X &\leq \int_0^t C\,|f(t-s)|_X \,\mathrm{d}s \leq C\,\|f\|_1\,, \\ |\dot{u}(t)|_X &\leq \int_0^t C\left|\dot{f}(t-s)\right|_X \mathrm{d}s \leq C\left\|\dot{f}\right\|_1 \end{aligned}$$

so that u and $\dot{u}$ are bounded functions.

Let $\lambda > 0$. We want to evaluate

$$\int_0^\infty e^{-(\lambda+i2\pi\xi)t}\dot{u}(t)\,\mathrm{d}t = (\lambda+i2\pi\xi)\int_0^\infty e^{-(\lambda+i2\pi\xi)t}u(t)\,\mathrm{d}t;$$

the equality follows from integration by parts, noting that $u(0)=0$.

To manipulate this equality further, we use the variation-of-constants formula to the result

$$\begin{aligned} &= (\lambda+i2\pi\xi)\int_0^\infty e^{-(\lambda+i2\pi\xi)t}\int_0^t e^{-A(t-s)}f(s)\,\mathrm{d}s\,\mathrm{d}t \\ &= (\lambda+i2\pi\xi)\int_0^\infty e^{-(\lambda+i2\pi\xi)s}\int_s^\infty e^{-(\lambda+i2\pi\xi)(t-s)}e^{-A(t-s)}f(s)\,\mathrm{d}t\,\mathrm{d}s \\ &= (\lambda+i2\pi\xi)\int_0^\infty e^{-(\lambda+i2\pi\xi)s}(\lambda+\mathbf{i}2\pi\xi+A)^{-1}f(s)\,\mathrm{d}s \\ &= (\lambda+i2\pi\xi)(\lambda+\mathbf{i}2\pi\xi+A)^{-1}\int_0^\infty e^{-(\lambda+i2\pi\xi)s}f(s)\,\mathrm{d}s. \end{aligned}$$

We now consider the limit $\lambda \downarrow 0$. Since $\dot{u}$ is a bounded function supported on $[0,\infty[$, we have $e^{-\lambda t}\dot{u}(t) \to \dot{u}(t)$ boundedly, and hence in the sense of tempered distributions. Thus $\int_0^\infty e^{-(\lambda+\mathbf{i}2\pi\xi)t}\dot{u}(t)\,\mathrm{d}t = \mathcal{F}[e^{-\lambda t}\dot{u}(t)](\xi) \to \widehat{\dot{u}}(\xi)$ in $\mathcal{S}'(\mathbb{R};X)$.

By the continuity of the resolvent, we have $(\lambda+\mathbf{i}2\pi\xi)(\lambda+\mathbf{i}2\pi\xi+A)^{-1} \to \mathbf{i}2\pi\xi(\mathbf{i}2\pi\xi+A)^{-1}$ whenever $\xi \neq 0$, the convergence being in the operator norm topology. Moreover, this convergence is uniformly bounded in ξ and λ. Furthermore, it is clear that $\int_0^\infty e^{-(\lambda+\mathbf{i}2\pi\xi)s}f(s)\,\mathrm{d}s \to \hat{f}(\xi)$, and the quantities involved are bounded in norm by $\|f\|_1$ uniformly in ξ and λ.

We conclude that

$$(\lambda+i2\pi\xi)(\lambda+\mathbf{i}2\pi\xi+A)^{-1}\int_0^\infty e^{-(\lambda+i2\pi\xi)s}f(s)\,\mathrm{d}s \underset{\lambda\downarrow 0}{\longrightarrow} i2\pi\xi(\mathbf{i}2\pi\xi+A)^{-1}\hat{f}(\xi)$$

boundedly and almost everywhere (in fact, everywhere except possibly at $\xi=0$). Thus the convergence also takes place in the distributional sense, and we obtain the first of the asserted equalities by the uniqueness of the distributional limit. The second equation follows from the first and the equality $\dot{u}+Au=f$. □

From this proposition, we immediately deduce the equivalence of three maximal regularity type notions on our spaces of interest: the *a priori* estimate for the classical solution of the ACP with test function data, and the boundedness of the operators defined by convolution with the operator-valued kernel

$$k(t) := AT^t \chi_{\mathbb{R}_+}(t) \tag{3.8}$$

and by transformation with the Fourier multiplier appearing in Prop. 3.7.

In the following result we treat several function spaces at a time, since the proofs are just the same. Thus we use the generic notation $\mathfrak{F}_0$ and $\mathfrak{F}_1$ for the spaces between which we consider the problem of regularity, i.e., $f \in \mathfrak{F}_0$ and (if there is regularity) $Au \in \mathfrak{F}_1$.

Proposition 3.9. *Let either* $(\mathfrak{F}_0, \mathfrak{F}_1) = (H^p, H^p)$ *with* $p \in]0, 1]$, *or* $(\mathfrak{F}_0, \mathfrak{F}_1) = (H^1, L^1)$. *Then the following conditions are equivalent:*

Dense class estimate: *For all* $f \in [\mathcal{D}(A) \otimes \mathcal{C}_c^\infty(\mathbb{R}_+)] \cap \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$ *and the corresponding classical solution* u *of the ACP, we have* $Au \in \mathfrak{F}_1(\bar{\mathbb{R}}_+; X)$ *with* $\|Au\|_{\mathfrak{F}_1} \leq K \|f\|_{\mathfrak{F}_0}$.

Integral condition: *The singular integral operator*

$$f \in \mathcal{D}(A) \otimes \mathfrak{D}(\mathbb{R}) \mapsto k * f$$

extends to a bounded linear mapping from $\mathfrak{F}_0(\mathbb{R}; X)$ *to* $\mathfrak{F}_1(\mathbb{R}; X)$, *of norm at most* K.

Multiplier condition: *The multiplier operator*

$$f \in \mathcal{D}(A) \otimes \mathfrak{D}(\mathbb{R}) \mapsto \mathcal{F}^{-1}(m\hat{f}) =: T_m f$$

extends to a bounded linear mapping from $\mathfrak{F}_0(\mathbb{R}; X)$ *to* $\mathfrak{F}_1(\mathbb{R}; X)$, *of norm at most* K.

The reader should not be confused by the typographical similarity of $\mathcal{D}$ (the domain of an operator) and $\mathfrak{D}$ (the set of test functions).

Proof. Note first that, for $f \in \mathcal{D}(A) \otimes \mathcal{C}_c^\infty(\mathbb{R}_+)$ and u the corresponding classical solution of the ACP, it follows from Prop. 3.7 and the variation-of-constants formula that $T_m f = k * f = Au$.

Assume the dense class estimate. If $f \in [\mathcal{D}(A) \otimes \mathcal{C}_c^\infty(\mathbb{R}_+)] \cap \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$, it follows from this estimate that $T_m f = k * f = Au \in \mathfrak{F}_1(\mathbb{R}; X)$ and $\|T_m f\|_{\mathfrak{F}_1} = \|k * f\|_{\mathfrak{F}_1} \leq K \|f\|_{\mathfrak{F}_0}$. Since T_m and $k*$ commute with translations and the norms of $\mathfrak{F}_\mu$, $\mu = 0, 1$, are translation-invariant, this inequality also holds for all $f \in [\mathcal{D}(A) \otimes \mathfrak{D}(\mathbb{R})] \cap \mathfrak{F}_0(\mathbb{R}; X)$, but this implies the multiplier and integral conditions by density.

Conversely, suppose that either the integral or the multiplier condition holds. For $f \in [\mathcal{D}(A) \otimes \mathcal{C}_c^\infty(\mathbb{R}_+)] \cap \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)]$ and u the corresponding classical solution of the ACP, we have $Au = k * f = T_m f \in \mathfrak{F}_1(\mathbb{R}; X)$ by the assumption, with $\|Au\|_{\mathfrak{F}_1} \leq K \|f\|_{\mathfrak{F}_0}$. Since u and hence Au is supported on $\bar{\mathbb{R}}_+$, we have in fact $Au \in \mathfrak{F}_1(\bar{\mathbb{R}}_+; X)$, and the dense class estimate holds. □

With the equivalence of the regularity notions, we adopt the following definition:

Definition 3.10. The ACP is said to have $(\mathfrak{F}_0, \mathfrak{F}_1)$-regularity if the three equivalent conditions in Prop. 3.9 are satisfied.

We should note that, since $\mathfrak{F}_0(\mathbb{R}; X) \hookrightarrow \mathfrak{F}_1(\mathbb{R}; X)$ in all the cases treated, the estimate $\|Au\|_{\mathfrak{F}_1} \leq K \|f\|_{\mathfrak{F}_0}$ already implies the similar estimate

$$\|\dot{u}\|_{\mathfrak{F}_1} = \|f - Au\|_{\mathfrak{F}_1} \leq (1 + K) \|f\|_{\mathfrak{F}_0},$$

where the first equality follows directly from the fact that u satisfies the ACP. This justifies our imposing regularity conditions only on Au.

Let us next verify that our notion of regularity implies the usual estimates for the mild solution in the borderline situation $p = 1$. What we need below is the embedding of our spaces in L^1_{loc}, a property that the spaces H^p fail to satisfy for $p < 1$.

Proposition 3.11. *Let $\mathfrak{F}_0 = H^1$ and $\mathfrak{F}_1 \in \{H^1, L^1\}$. Then the $(\mathfrak{F}_0, \mathfrak{F}_1)$-regularity of the ACP is equivalent to*

> **Regularity of the mild solution:** *For every $f \in \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$ and u the corresponding mild solution of the ACP, we have $u(t) \in \mathcal{D}(A)$ for a.e. $t \in \bar{\mathbb{R}}_+$, $Au \in \mathfrak{F}_1(\bar{\mathbb{R}}_+; X)$ with $\|Au\|_{\mathfrak{F}_1} \leq K \|f\|_{\mathfrak{F}_0}$. Moreover, we have $u \in W^{1,1}_{\mathrm{loc}}(\bar{\mathbb{R}}_+; X)$, $u(0) = 0$, $\dot{u} \in \mathfrak{F}_1(\bar{\mathbb{R}}_+; X)$, and the equation $\dot{u}(t) + Au(t) = f(t)$ holds for a.e. $t \in \bar{\mathbb{R}}_+$.*

Proof. The fact that the above mentioned regularity of the mild solution implies the dense class estimate in Prop. 3.9 is obvious, since a classical solution of the ACP is also a mild solution. Let us consider the converse.

Given $f \in \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$, let $f_n \in [\mathcal{D}(A) \otimes \mathcal{C}^\infty_c(\bar{\mathbb{R}}_+)] \cap \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$, $n \in \mathbb{N}$, be a sequence of functions converging to f in $\mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$. (The existence of such a sequence is guaranteed by Lemma 2.4.) By the dense class estimate we have $\|Au_n - Au_m\|_{\mathfrak{F}_1} \leq K \|f_n - f_m\|_{\mathfrak{F}_0} \to 0$; thus $(Au_n)_{n=1}^\infty \subset \mathfrak{F}_1(\mathbb{R}; X)$ is a Cauchy sequence, and by completeness we have $Au_n \to v$ for some $v \in \mathfrak{F}_1(\mathbb{R}; X)$. Since $\mathfrak{F}_1 \hookrightarrow L^1$ (L^1_{loc} would suffice), it follows, for a subsequence, that $Au_n(t) \to v(t)$ for a.e. $t \in \mathbb{R}$. We henceforth consider this subsequence.

We also have

$$u_n(t) = \int_0^t T^{t-s} f_n(s) \,\mathrm{d}s \to \int_0^t T^{t-s} f(s) \,\mathrm{d}s = u(t)$$

for all $t \in \mathbb{R}_+$, since (T^t) is bounded and $f_n \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}; X)$. From $u_n(t) \to u(t)$, $Au_n(t) \to v(t)$ (a.e. t), and the closedness of A we conclude that $u(t) \in \mathcal{D}(A)$ and $Au(t) = v(t)$ for a.e. t. Thus $\|Au\|_{\mathfrak{F}_1} = \|v\|_{\mathfrak{F}_1} = \lim \|Au_n\|_{\mathfrak{F}_1} \leq \lim K \|f_n\|_{\mathfrak{F}_0} = K \|f\|_{\mathfrak{F}_0}$.

The last statement in the regularity of the mild solution follows with similar reasoning from the closedness of $u \in W^{1,1}([0,t]; X) \mapsto \dot{u} \in L^1([0,t]; X)$ and the estimate

$$\|\dot{u}_n - \dot{u}_m\|_{\mathfrak{F}_1} \leq \|Au_n - Au_m\|_{\mathfrak{F}_1} + \|f_n - f_m\|_{\mathfrak{F}_1} \leq (K + 1) \|f_n - f_m\|_{\mathfrak{F}_0},$$

where the embedding $\mathfrak{F}_0 \hookrightarrow \mathfrak{F}_1$ was used. $\square$

4. R-boundedness is necessary for multipliers on H^1

In this section we show the necessity of R-boundedness for an operator-valued Fourier-multiplier from $H^1(\mathbb{R}^n; X)$ to $H^1(\mathbb{R}^n; Y)$, and in fact, even for a multiplier from $H^1(\mathbb{R}^n; X)$ to $L^1(\mathbb{R}^n; Y)$. According to our definition of the regularity of the ACP on these spaces, this result also yields the necessity of R-boundedness for the maximal H^1-regularity of the ACP.

We first recall the analogous L^p-result due to Clément and Prüss:

Theorem 4.1 ([**22**]). *Suppose $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}(X, Y))$ is such that the multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$ acts boundedly from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$ for some $p \in [1, \infty[$. Then $\{m(y) |\ y$ strong Lebesgue point of $m\}$ is R-bounded.*

In fact, the result is stated in [**22**] only for $p \in]1, \infty[$ and norm-topology Lebesgue points, but the proof works as such also for the slight generalization formulated above. Our purpose is to show that this assertion remains true even for multipliers from $H^1(\mathbb{R}^n; X)$ to $L^1(\mathbb{R}^n; Y)$, and this is a much larger class than the L^1-multipliers.

Theorem 4.2. *Suppose $m \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathcal{L}(X, Y))$ is such that the multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$ acts boundedly from $H^1(\mathbb{R}^n; X)$ to $L^1(\mathbb{R}^n; Y)$.*

Then m is strongly continuous away from the origin and moreover

$$\mathcal{R}\left(\{m(y) |\ y \neq 0\}\right) \leq C_n \left\|T_m\right\|_{\mathcal{L}(H^1(\mathbb{R}^n;X);L^1(\mathbb{R}^n;Y))},$$

where the constant C_n depends only on the dimension n. In particular, $m \in L^\infty(\mathbb{R}^n; \mathcal{L}(X, Y))$.

Proof of the necessity theorem. Before we prove Theorem 4.2, we need two lemmata. First of all, we require a tool for estimating the H^1-norms we will encounter. (Here, we are going to use the H^1 norm defined in terms of atoms of L^2-type.) Let B_r be the ball in $\mathbb{R}^n$ of radius r centered at the origin and $A_{r,R} := B_R \setminus B_r$ the annulus with inner and outer radii r and R, respectively.

Lemma 4.3. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n; X)$ with $\int \varphi(x)\,\mathrm{d}x = 0$. Then $\varphi \in H^1(\mathbb{R}^n; X)$, and the norm is estimated by*

$$\left\|\varphi\right\|_{H^1(\mathbb{R}^n;X)} \leq \sum_{k=1}^{\infty} |B_k|^{\frac{1}{2}} \left\|\varphi \mathbf{1}_{A_{k-1,k}}\right\|_{L^2(\mathbb{R}^n;X)} + (1 + 2^{n/2}) \sum_{k=1}^{\infty} \left\|\varphi \mathbf{1}_{B_k^c}\right\|_{L^1(\mathbb{R}^n;X)}$$

It is easy to see that the sum is indeed finite for a rapidly decreasing φ.

Proof. Let us denote

$$\varphi_k := \left(\varphi - \frac{1}{|B_k|} \int_{B_k} \varphi(y)\,\mathrm{d}y\right) \mathbf{1}_{B_k} = \left(\varphi + \frac{1}{|B_k|} \int_{B_k^c} \varphi(y)\,\mathrm{d}y\right) \mathbf{1}_{B_k},$$

where it is clear from the first form that $\int \varphi_k(x)\,\mathrm{d}x = 0$, and the latter equality follows from the assumption that the total integral of φ vanishes. Then

$$\begin{aligned}|\varphi(x)-\varphi_k(x)|_X &\leq |\varphi(x)|_X\,\mathbf{1}_{B_k^c}(x) + \frac{1}{|B_k|}\int_{B_k^c}|\varphi(y)|_X\,\mathrm{d}y\\ &\leq \max_{|y|\geq k}|\varphi(y)|_X + \frac{1}{|B_k|}\int_{B_k^c}|\varphi(y)|_X\,\mathrm{d}y \underset{k\to\infty}{\longrightarrow} 0;\end{aligned}$$

thus $\varphi_k \to \varphi$ uniformly as $k\to\infty$.

We then define $\phi_1 := \varphi_1$ and $\phi_k := \varphi_k - \varphi_{k-1}$ for $k>1$ so that $\sum_{k=1}^N \phi_k = \varphi_N \to \varphi$ uniformly as $N\to\infty$. Thus we have $\varphi = \sum_{k=1}^\infty \phi_k$, where $\operatorname{supp}\phi_k \subset B_k$ and $\int \phi_k(x)\,\mathrm{d}x = 0$. This is hence an atomic decomposition of φ, and we have

$$\|\varphi\|_{H^1(\mathbb{R}^n;X)} \leq \sum_{k=1}^\infty |B_k|^{\frac12}\,\|\phi_k\|_{L^2(\mathbb{R}^n;X)}.$$

Hence it remains to estimate the L^2-norm of

$$\phi_k = \varphi\mathbf{1}_{A_{k-1,k}} + \frac{\mathbf{1}_{B_k}}{|B_k|}\int_{B_k^c}\varphi(y)\,\mathrm{d}y - \frac{\mathbf{1}_{B_{k-1}}}{|B_{k-1}|}\int_{B_{k-1}^c}\varphi(y)\,\mathrm{d}y,$$

where the last term is interpreted as 0 for $k=1$, and this yields

$$\begin{aligned}&\|\phi_k\|_{L^2(\mathbb{R}^n;X)}\\ &\quad\leq \left\|\varphi\mathbf{1}_{A_{k-1,k}}\right\|_{L^2(\mathbb{R}^n;X)} + \frac{1}{|B_k|^{\frac12}}\left\|\varphi\mathbf{1}_{B_k^c}\right\|_{L^1(\mathbb{R}^n;X)} + \frac{1}{|B_{k-1}|^{\frac12}}\left\|\varphi\mathbf{1}_{B_{k-1}^c}\right\|_{L^1(\mathbb{R}^n;X)}.\end{aligned}$$

Multiplying by $|B_k|^{\frac12}$, observing that $|B_k|^{\frac12}/|B_{k-1}|^{\frac12} = (k/(k-1))^{n/2} \leq 2^{n/2}$ and summing over k we obtain the asserted estimate. □

The following simple result handles the easy part of the main theorem. It is not really crucial for the proof of the assertion concerning the R-boundedness of the multiplier m, since the strong continuity at $y\neq 0$ is only exploited via the fact that these points are strong Lebesgue points of m, and in any case we know that almost every point is a Lebesgue point. Nevertheless, we obtain a somewhat neater form of the theorem without the need for almost-every-qualifications.

LEMMA 4.4. *If $m\in L^1_{\mathrm{loc}}(\mathbb{R}^n;\mathcal{L}(X,Y))$ defines a bounded multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$, which maps $H^1(\mathbb{R}^n;X)$ boundedly into $L^1(\mathbb{R}^n;Y)$, then m is strongly continuous at every $y\neq 0$. In particular, every $y\neq 0$ is a strong Lebesgue point of m.*

PROOF. Let $y_0 \neq 0$. Then there exists a test function $\hat{\varphi}\in\mathcal{D}(\mathbb{R})$, which is supported away from the origin and equals unity in a neighbourhood of y_0. Then for $x\in X$ we have $\varphi(\cdot)x \in \mathcal{S}(\mathbb{R}^n;X)$ and $\int\varphi(y)x\,\mathrm{d}y = \hat{\varphi}(0)x = 0$. Hence $\varphi(\cdot)x\in H^1(\mathbb{R};X)$, and thus $T_m[\varphi(\cdot)x]\in L^1(\mathbb{R};Y)$. The Fourier transform of this latter function is $m(y)\hat{\varphi}(y)x$, and in a neighbourhood of y_0, this is just $m(y)x$.

But the Fourier transform of an L^1-function is continuous, thus $y \mapsto m(y)x$ is continuous in a neighbourhood of y_0, and this being true for every $x \in X$ the assertion is established. $\square$

Now we are ready to prove the necessity of R-boundedness:

Proof of Theorem 4.2. Let $N \in \mathbb{Z}_+$ and $x_1, \ldots, x_N \in X$, and let first

$$y_1, \ldots, y_N \in \{y = (y^1, \ldots, y^n) \in \mathbb{R}^n |\ y^n \geq 0,\ y \neq 0\},$$

i.e., the points are taken from the closed upper half-space, excluding the origin. Let us choose a (real-valued) test-function $\psi \in \mathcal{D}(\mathbb{R}^n)$ with support strictly contained in the lower half-space $\{y \in \mathbb{R}^n |\ y^n < 0\}$ and such that

$$\int_{\mathbb{R}^n} \psi^2(y)\,\mathrm{d}y = 1.$$

This function will be exploited in building an appropriate approximation of the identity; the reason for the support condition will become clear later. Since y_j is a Lebesgue point of $y \mapsto m(y)x_j$ by Lemma 4.4, we have

$$m(y_j)x_j = \lim_{k\to\infty} \int_{\mathbb{R}^n} m(y)x_j\psi^2(k(y_j - y))k^n\,\mathrm{d}y,$$

the convergence being in the norm of Y. Thus

$$\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j m(y_j)x_j\right|_Y = \lim_{k\to\infty} k^n\mathbb{E}\left|\int_{\mathbb{R}^n} \sum_{j=1}^N \varepsilon_j m(y)\psi(k(y_j - y))x_j\psi(k(y_j - y))\,\mathrm{d}y\right|_Y.$$

Note that since the Rademacher functions ε_j are simple random variables, the expectation $\mathbb{E}$ is nothing but a weighted finite sum, and thus it certainly commutes with limits. (Of course, for more general random variables we could have simply invoked Fatou's lemma to yield the above result with "$=\lim$" replaced by "$\leq \liminf$", and the rest of the proof would run in exactly the same way.)

We then write

$$\begin{aligned} m(y)\psi(k(y_j - y))x_j &= m(y)\mathcal{F}\mathcal{F}^{-1}[\psi(k(y_j - \cdot))x_j](y) \\ &= m(y)\mathcal{F}[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)/k^n = \mathcal{F}T_m[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)/k^n, \end{aligned}$$

and using the duality equality $\int \hat{g} f \,\mathrm{d}y = \int g \hat{f} \,\mathrm{d}y$ of the Fourier transform we arrive at

$$\begin{aligned}
&\mathbb{E}\Big|\sum_{j=1}^N \varepsilon_j m(y_j) x_j\Big|_Y \\
&\quad= \lim_{k\to\infty} k^{-n}\mathbb{E}\Big|\int_{\mathbb{R}^n}\sum_{j=1}^N \varepsilon_j T_m[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)e^{-\mathbf{i}2\pi y_j\cdot y}\hat{\psi}(-y/k)\,\mathrm{d}y\Big|_Y \\
&\qquad\le \liminf_{k\to\infty} k^{-n}\|\hat{\psi}\|_{L^\infty}\mathbb{E}\int_{\mathbb{R}^n}\Big|\sum_{j=1}^N \varepsilon_j e^{-\mathbf{i}2\pi y_j\cdot y} T_m[e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j](y)\Big|_Y\,\mathrm{d}y.
\end{aligned}$$

We now invoke the contraction principle to get rid of the exponential factors $e^{-\mathbf{i}2\pi y_j\cdot y}$ and then the assumed boundedness of the operator T_m to yield

$$\begin{aligned}
&\le \frac{\pi}{2}\|\hat{\psi}\|_{L^\infty}\,\|T_m\|_{\mathcal{L}(H^1,L^1)}\liminf_{k\to\infty} k^{-n}\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi y_j\cdot(\cdot)}\hat{\psi}(\cdot/k)x_j\Big\|_{H^1(\mathbb{R}^n;X)} \\
&= \frac{\pi}{2}\|\hat{\psi}\|_{L^\infty}\,\|T_m\|_{\mathcal{L}(H^1,L^1)}\liminf_{k\to\infty}\mathbb{E}\Big\|\sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}(\cdot)x_j\Big\|_{H^1(\mathbb{R}^n;X)},
\end{aligned}\tag{4.5}$$

where the last equality follows from the dilation property of the H^1-norm.

So far the proof has been completely parallel to that in [**22**] concerning the L^p situation, except for the choice of our auxiliary function ψ, but now we are faced with the H^1-norm, with which the contraction principle can no longer be applied. Instead, we invoke Lemma 4.3 for the evaluation of this norm. Let us first check that the assumptions of the lemma are satisfied by

$$\varphi(y) := \sum_{j=1}^N \varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot y}\hat{\psi}(y)x_j :$$

Certainly $\hat{\psi}\in\mathcal{S}(\mathbb{R}^n)$ since $\psi\in\mathcal{D}(\mathbb{R}^n)$, and since the exponential factors are $\mathcal{C}^\infty$ with bounded derivatives of all orders, the entire function φ belongs to $\mathcal{S}(\mathbb{R}^n;X)$. Moreover, recognizing the formula of the inverse Fourier transform, we have

$$\int_{\mathbb{R}^n} e^{\mathbf{i}2\pi k y_j\cdot y}\hat{\psi}(y)\,\mathrm{d}y = \psi(ky_j) = 0,$$

since $k > 0$ and y_j is in the upper half-space, whereas ψ is supported in the lower half-space.

Hence we get, for the H^1-norm appearing in (4.5), the estimate

$$\begin{aligned}\mathbb{E}\left\|\sum_{j=1}^{N}\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}(\cdot)x_j\right\|_{H^1(\mathbb{R}^n;X)} &\leq \sum_{\ell=1}^{\infty}|B_\ell|^{1/2}\,\mathbb{E}\left\|\sum_{j=1}^{N}\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}(\cdot)x_j\right\|_{L^2(\mathbb{R}^n;X)}\\ &\quad + (1+2^{n/2})\sum_{\ell=1}^{\infty}\mathbb{E}\left\|\sum_{j=1}^{N}\varepsilon_j e^{\mathbf{i}2\pi k y_j\cdot(\cdot)}\hat{\psi}\mathbf{1}_{B_\ell^c}(\cdot)x_j\right\|_{L^1(\mathbb{R}^n;X)}.\end{aligned}$$

Now we are back to L^p-norms, and the contraction principle applies again:

$$\begin{aligned}&\leq \frac{\pi}{2}\sum_{\ell=1}^{\infty}|B_\ell|^{1/2}\left(\mathbb{E}\left\|\sum\varepsilon_j\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}(\cdot)x_j\right\|^2_{L^2(\mathbb{R}^n;X)}\right)^{1/2}\\ &\qquad + (1+2^{n/2})\frac{\pi}{2}\sum_{\ell=1}^{\infty}\mathbb{E}\left\|\sum\varepsilon_j\hat{\psi}\mathbf{1}_{B_\ell^c}(\cdot)x_j\right\|_{L^1(\mathbb{R}^n;X)}\end{aligned}$$

$$\begin{aligned}&= \frac{\pi}{2}\sum_{\ell=1}^{\infty}|B_\ell|^{1/2}\left\|\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}\right\|_{L^2(\mathbb{R}^n)}\left(\mathbb{E}\left|\sum\varepsilon_j x_j\right|^2_X\right)^{1/2}\\ &\qquad + (1+2^{n/2})\frac{\pi}{2}\sum_{\ell=1}^{\infty}\left\|\hat{\psi}\mathbf{1}_{B_\ell^c}\right\|_{L^1(\mathbb{R}^n)}\mathbb{E}\left|\sum\varepsilon_j x_j\right|_X.\end{aligned}$$

Finally, combining (4.5) with the estimate above and applying KAHANE's inequality $\sqrt{\mathbb{E}\left|\sum\varepsilon_j x_j\right|^2_X} \leq \sqrt{2}\mathbb{E}\left|\sum\varepsilon_j x_j\right|_X$ (see [**67**, §4.1.10]), we get

$$\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j m(y_j)x_j\right|_Y \leq C_n\left\|T_m\right\|_{\mathcal{L}(H^1(\mathbb{R}^n;X);L^1(\mathbb{R}^n;Y))}\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j x_j\right|_X,$$

where the constant

$$\begin{aligned}C_n = \frac{\pi^2}{4}\|\hat{\psi}\|_{L^\infty(\mathbb{R}^n)}\Bigg(&\sqrt{2}\sum_{\ell=1}^{\infty}|B_\ell|^{1/2}\left\|\hat{\psi}\mathbf{1}_{A_{\ell-1,\ell}}\right\|_{L^2(\mathbb{R}^n)}\\ &+ (1+2^{n/2})\sum_{\ell=1}^{\infty}\left\|\hat{\psi}\mathbf{1}_{B_\ell^c}\right\|_{L^1(\mathbb{R}^n)}\Bigg) < \infty\end{aligned}$$

depends only on the dimension n and the choice of the auxiliary function ψ, thus fixing one ψ once and for all, only on the dimension n.

It is clear that we can repeat the same argument for points $y_1,\ldots,y_N$ in the lower half-space, exploiting another auxiliary function $\tilde{\psi} \in \mathcal{D}(\mathbb{R}^n)$ supported in the upper half-space (e.g., the reflection of ψ about the hyperplane $\{y \in \mathbb{R}^n |\ y^n = 0\}$). Thus we get the R-boundedness of $\{m(y)|\ y \neq 0\}$ with an R-bound of the asserted form. □

REMARK 4.6. Theorem 4.2 gives a nice extrapolation result in the case $X = \mathcal{H}_1$, $Y = \mathcal{H}_2$ are Hilbert spaces. Namely, if $T_m \in \mathcal{L}(H^1(\mathbb{R}^n; \mathcal{H}_1), L^1(\mathbb{R}^n; \mathcal{H}_2))$, the theorem tells us that m is essentially bounded, and then by Plancherel (for $p = 2$) and interpolation (for $p \in]1, 2[$) that $T_m \in \mathcal{L}(L^p(\mathbb{R}^n; \mathcal{H}_1), L^p(\mathbb{R}^n; \mathcal{H}_2))$ for $p \in]1, 2]$.

In the classical case $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{C}$, this would be immediate also without the above theorem; indeed, from duality we would obtain the boundedness of $T_m^* = T_{m(\cdot)^*}$ from $L^\infty(\mathbb{R}^n)$ to $\mathrm{BMO}(\mathbb{R}^n)$, and it is easy to see that $m(\cdot)$ and $m(\cdot)^*$ (the complex conjugate) are multipliers at the same time, and then the same conclusion (in fact for all $p \in]1, \infty[$) would follow from interpolation.

However, in the general Hilbert space situation, duality gives the boundedness of $T_m^* = T_{m(\cdot)^*}$ from $L^\infty(\mathbb{R}^n; \mathcal{H}_2)$ to $\mathrm{BMO}(\mathbb{R}^n; \mathcal{H}_1)$ and this is a statement concerning an operator different from T_m and acting "in the wrong direction", i.e., from $\mathcal{H}_2$-valued functions to $\mathcal{H}_1$-valued, so that there is no way to use the classical interpolation argument.

Another necessity proof for $n = 1$. For the one-dimensional domain, the result of Theorem 4.2, and actually a little more, can be derived with a simpler argument (avoiding Lemma 4.3 and the lengthy expressions following from its use at the end of the proof of Theorem 4.2 given above). The simplified proof we are going to give is, in fact, only a slight modification of the proof of [**22**] for the necessity of R-boundedness for L^p-multipliers. Note that the one-dimensional result is fully sufficient for application to the Cauchy problem, as well as to the fractional-order equations (1.8).

Instead of the atomic definition of the Hardy spaces used above, we here consider the Hardy space $\tilde{H}^1$ defined in terms of the conjugate operation or the Hilbert transform $\mathcal{H}$, which is the Fourier-multiplier operator with multiplier $-\mathbf{i}\,\mathrm{sgn}(\xi)$. We set

$$\tilde{H}^1(\mathbb{R}; X) := \{f \in L^1(\mathbb{R}; X) |\ \mathcal{H}f \in L^1(\mathbb{R}; X)\}$$

equipped with the graph norm

$$\|f\|_{\tilde{H}^1(\mathbb{R};X)} := \|f\|_{L^1(\mathbb{R};X)} + \|\mathcal{H}f\|_{L^1(\mathbb{R};X)}.$$

Our assumption in the following will be the boundedness of a multiplier operator T_m from $\tilde{H}^1(\mathbb{R}; X)$ to $L^1(\mathbb{R}; X)$, and we shall show the R-boundedness of $\{m(t)|\ t \neq 0\}$.

This result reproduces Theorem 4.2 in the case $n = 1$ and is a slight extension of it for non-UMD Banach spaces. Namely, in general we have $\tilde{H}^1(\mathbb{R}; X) \hookrightarrow H^1(\mathbb{R}; X)$, and if X is UMD, there is an equality of spaces with equivalence of norm. Of course, it is well-known that we have the equality $\tilde{H}^1(\mathbb{R}) = H^1(\mathbb{R})$ in the scalar-valued setting. The above mentioned results concerning the vector-valued Hardy spaces have been shown by O. Blasco [**8**] for the unit circle $\mathbb{T}$ in place of the real line $\mathbb{R}$, but the two results quoted are proved by methods which have direct analogues in the case of the line. Thus the assumption that T_m be bounded from the smaller space $\tilde{H}^1(\mathbb{R};X)$ to $L^1(\mathbb{R};Y)$ is clearly weaker than the boundedness from the possibly larger space $H^1(\mathbb{R};X)$. (Indeed, the Hilbert transform is always bounded from $\tilde{H}^1(\mathbb{R};X)$ to $L^1(\mathbb{R};X)$ by definition, but not in general from $H^1(\mathbb{R};X)$ to $L^1(\mathbb{R};X)$.)

What makes the one-dimensional proof for $\tilde{H}^1$ so simple, is the existence of a large class of functions for which the evaluation of the graph norm of the Hilbert transform is particularly easy: If the Fourier transform of f is supported only on the positive (resp. negative) half-axis, then $\mathcal{H}f$ is simply $-\mathbf{i}f$ (resp. $\mathbf{i}f$), and therefore $\|f\|_{\tilde{H}^1(\mathbb{R};X)} = 2\,\|f\|_{L^1(\mathbb{R};X)}$.

Now let us state and prove the result:

Proposition 4.7. *Suppose $m \in L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{L}(X;Y))$ is such that the multiplier operator $T_m f := \mathcal{F}^{-1}[m\hat{f}]$ acts boundedly from $\tilde{H}^1(\mathbb{R};X)$ to $L^1(\mathbb{R};Y)$.*

Then m is strongly continuous away from the origin, and the set $\{m(t)|\ t \neq 0\}$ is R-bounded in terms of an absolute constant times the operator norm of T_m.

Proof. The fact that m is strongly continuous outside $t = 0$ follows from Lemma 4.4 and the above mentioned embedding, or one can also give a direct proof parallel to Lemma 4.4. Indeed, if $t_0 \neq 0$ and $\hat{\psi} \in \mathcal{D}(\mathbb{R})$ is equal to unity in a neighbourhood of t_0 and supported on one half axis only, it is clear that $\psi(\cdot)x \in \tilde{H}^1(\mathbb{R};X)$ for all $x \in X$, and the rest of the proof is just like Lemma 4.4.

Let then $N \in \mathbb{Z}_+$, $t_1,\ldots,t_N > 0$ and $x_1,\ldots,x_N \in X$. We choose a real-valued test-function $\psi \in \mathcal{D}(\mathbb{R})$ supported on $]-\infty,0[$ and with the same integral condition as in the proof of Theorem 4.2. The proof runs in exactly the same fashion as there until we reach the estimate

$$\mathbb{E}\left|\sum_{j=1}^{N}\varepsilon_j m(t_j)x_j\right|_Y \leq \frac{\pi}{2}\left\|\hat{\psi}\right\|_{L^\infty}\|T_m\|\liminf_{k\to\infty}\frac{1}{k}\mathbb{E}\left\|\sum_{j=1}^{N}\varepsilon_j e^{\mathbf{i}2\pi t_j\cdot}\hat{\psi}(\cdot/k)x_j\right\|_{\tilde{H}^1(\mathbb{R};X)}.$$

We then observe that the Fourier transform of the function whose $\tilde{H}^1$-norm is to be evaluated is given by $\sum\varepsilon_j k\psi(k(t_j-\xi))x_j$, and for this to be non-zero, recalling the support condition imposed on ψ, we must have $t_j - \xi < 0$, i.e., $\xi > t_j > 0$.

Thus the support of the Fourier transform is contained on $]0,\infty[$, and so the $\tilde{H}^1$ norm is just twice the L^1 norm. Using this and the contraction principle, which is valid once we get back to L^1, we have

$$\leq 2\left(\frac{\pi}{2}\right)^2 \left\|\hat{\psi}\right\|_{L^\infty} \|T_m\|_{\mathcal{L}(\tilde{H}^1;L^1)} \liminf_{k\to\infty} \frac{1}{k}\mathbb{E}\left\|\sum_{j=1}^N \varepsilon_j \hat{\psi}(\cdot/k)x_j\right\|_{L^1(\mathbb{R};X)}$$

$$= \frac{\pi^2}{2}\left\|\hat{\psi}\right\|_{L^\infty}\left\|\hat{\psi}\right\|_{L^1}\|T_m\|_{\mathcal{L}(\tilde{H}^1;L^1)}\,\mathbb{E}\left|\sum_{j=1}^N \varepsilon_j x_j\right|_X,$$

and a parallel argument can be used to handle the negative half-axis. □

Of course, one should note in Proposition 4.7 that the origin has to be excluded. Indeed, for a non-zero $A \in \mathcal{L}(X,Y)$, the operator $A\mathcal{H}$ maps $\tilde{H}^1(\mathbb{R};X)$ boundedly into $L^1(\mathbb{R};Y)$, but the corresponding multiplier $-\mathbf{i}\,\mathrm{sgn}(t)A$ is certainly not even weakly continuous at $t=0$; the origin of the frequency domain has a genuinely special meaning in the spaces $\tilde{H}^1(\mathbb{R};X)$.

A sharpened necessary condition for L^p-multipliers, $p>1$**.** The idea of proof of Proposition 4.7 also applies to give a slightly sharpened form of the original result of [**22**] concerning the L^p-multipliers. To see how this comes out, consider the spaces

$$\tilde{H}^p(\mathbb{R};X) := \{f \in L^p(\mathbb{R};X)|\ \mathcal{H}f \in L^p(\mathbb{R};X)\}$$

with the graph norm, in analogy with the case $p=1$. Of course, for a UMD-space X, we have $\tilde{H}^p(\mathbb{R};X) = L^p(\mathbb{R};X)$ with equivalence of norms for $1<p<\infty$, and this condition actually characterizes UMD-spaces, but our intention is now to provide a piece of insight into the multiplier theory in non-UMD Banach spaces.

Now we observe the following: The proof of the result concerning the R-boundedness of $\{m(t)|\ t\neq 0$ a strong Lebesgue point of $m\}$ goes through with the assumption $T_m : \tilde{H}^1(\mathbb{R};X)\to L^1(\mathbb{R};Y)$ replaced by $T_m : \tilde{H}^p(\mathbb{R};X)\to L^p(\mathbb{R};Y)$. We just use the (equivalent) definition of R-boundedness in terms of the pth moment rather than the first, so that we can freely interchange the order of the L^p-norm with respect to the Lebesgue measure on the real line and the probability measure related to the Rademacher variables. Where we extracted the L^∞-norm of $\hat{\psi}$ from the integral, we now invoke HÖLDER's inequality to extract $\|\hat{\psi}\|_{L^{p'}}$, so that in place of the L^1 norm of the rest of the integrand we now have the L^p norm and we can apply the assumption. (This is also what was done in [**22**].) Due to the choice of the auxiliary function ψ, the evaluation of the $\tilde{H}^p$-norm also reduces to that of the L^p-norm, and we arrive at a similar conclusion as before but with $\|\hat{\psi}\|_{L^{p'}}\|\hat{\psi}\|_{L^p}$ instead of $\|\hat{\psi}\|_{L^\infty}\|\hat{\psi}\|_{L^1}$ in the constant. We formulate this result as a corollary, but it is a consequence of the proof rather than Proposition 4.7 itself.

COROLLARY 4.8. *If $m \in L^1_{\mathrm{loc}}(\mathbb{R};\mathcal{L}(X;Y))$ gives rise to a bounded multiplier operator $T_m = \mathcal{F}^{-1}m\mathcal{F} : \tilde{H}^p(\mathbb{R};X)\to L^p(\mathbb{R};Y)$ for some $p\in\,]1,\infty[$, then*

$$\mathcal{R}(\{m(t)|\ t\neq 0 \text{ a strong Lebesgue point of } m\}) \leq C\,\|T_m\|_{\mathcal{L}(\tilde{H}^p(\mathbb{R};X);L^p(\mathbb{R};Y))}.$$

Thus, even if we restrict the action of our multipliers to a function class on which non-trivial scalar-valued multipliers act boundedly (according to the condition of the boundedness of the Hilbert transform, which lies at the heart of $\tilde{H}^p(\mathbb{R}; X)$), this does not ease the problem of operator-valued multipliers in any essential way: they will not be bounded unless the multiplier function is R-bounded.

5. Multiplier theorems for Hardy spaces: sufficient conditions

Having examined necessary conditions for maximal regularity and multipliers in general, we now turn to the sufficient once. In this section, we collect the powerful machinery that will be used to deduce maximal regularity results on H^p from *a priori* regularity on L^p. This machinery consists of boundedness theorems for singular integral and multiplier operators, in the spirit of the classical Calderón–Zygmund theory. We also comment on conditions which guarantee the boundedness without *a priori* assumptions.

As indicated in the Introduction, this section contains straightforward generalizations of known results, mostly due to Strömberg and Torchinsky [**81**]. However, since the large extent to which these results carry over to vector-valued context appears to be unrecognised so far, it seems appropriate to allow them some space.

We present the results in a rather general form, which is somewhat excessive for the problems in maximal regularity which we have in mind. In particular, the sharp form of the conditions, with a minimum number of derivatives required, does not play a rôle in these applications where, as it turns out, infinitely many of the conditions are automatically satisfied. But the applicability of these results is of course not limited to maximal regularity.

Before passing to the general situation, we begin by recalling the classical (1962) result of Benedek, Calderón and Panzone [**5**], which already gives boundedness from H^1 to L^1, assuming boundedness on $L^{\tilde{p}}$ for some $\tilde{p} \in \,]1, \infty[$:

Theorem 5.1 ([**5**]; [**34**], Theorem V.3.4). *Consider an operator*

$$T \in \mathcal{L}(L^{\tilde{p}}(\mathbb{R}^n; X); L^{\tilde{p}}(\mathbb{R}^n; Y)),$$

given by $Tf(t) = \int_{\mathbb{R}^n} k(t-s) f(s)\,\mathrm{d}s$ *for* $f \in L_c^\infty(\mathbb{R}^n; X)$ *and* $t \notin \operatorname{supp} f$*, where* $k \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X, Y))$*. If k satisfies the Hörmander condition*

$$\int_{|t|>2|s|} |(k(t-s) - k(t))x|_Y \,\mathrm{d}t \le A\,|x|_X \tag{5.2}$$

then T has bounded extensions (*i*) *from* $H^1(\mathbb{R}^n; X)$ *to* $L^1(\mathbb{R}^n; X)$*,* (*ii*) *from* $L^{\tilde{p}} \cap L^\infty(\mathbb{R}^n; X)$ *(with L^∞-norm) to* $\mathrm{BMO}(\mathbb{R}^n; X)$*, and then by interpolation* (*iii*) *from* $L^p(\mathbb{R}^n; X)$ *to* $L^p(\mathbb{R}^n; Y)$ *for all* $p \in \,]1, \infty[$.

This result, which is stated essentially similarly in [**34**], does not yet appear in the given form in the classical paper [**5**]; however, as noted in [**34**], all essential ideas are already contained in [**5**].

The assertion (*iii*) of Theorem 5.1 was used by SOBOLEVSKIJ to derive the maximal regularity of the ACP for all $p \in \,]1,\infty[$ from the *a priori* assumption for one $\tilde{p} \in \,]1,\infty[$. Of course, exactly the same reasoning, which amounts only to checking the Hörmander condition (5.2) for $k(t) := AT^t\chi_{\mathbb{R}_+}(t)$, also gives regularity on the pairs (H^1, L^1) and (L^∞, BMO), using the other assertions of the theorem. The easy verification of (5.2) for this kernel is found explicitly in e.g. [**27**]. One can also find a direct proof of the (L^∞, BMO)-regularity of the ACP, assuming maximal $L^{\tilde{p}}$-regularity for some $\tilde{p} \in [1,\infty[$, in CANNARSA and VESPRI [**17**].

We then proceed to general H^p-H^p-results, $p \in \,]0,1]$.

H^p-boundedness with an *a priori* assumption on $L^{\tilde{p}}$. In order to get boundedness from $H^p(\mathbb{R}^n; X)$ to $H^p(\mathbb{R}^n; Y)$, somewhat stronger (and more technical) assumptions than (5.2) are required. Let us consider the following set of conditions:

DEFINITION 5.3. We say that a function k, with values in $\mathcal{L}(X, Y)$, belongs to the class $K(q, \ell; X, Y)$ [or just $K(q, \ell; X)$ if $Y = X$], where $1 \leq q < \infty$ and $\ell > 0$, provided that $k \in \mathcal{C}^{[\![\ell]\!]}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X, Y))$ and satisfies

$$\left(\frac{1}{r^n}\int_{r<|t|<2r} |D^\alpha k(t)x|_Y^q \,\mathrm{d}t\right)^{\frac{1}{q}} \leq A\, r^{-n-|\alpha|}\,|x|_X \tag{5.4}$$

for all $r > 0$, $x \in X$, and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq [\![\ell]\!]$, and moreover

$$\left(\frac{1}{r^n}\int_{r<|t|<2r} |(D^\alpha k(t) - D^\alpha k(t-s))x|_Y^q \,\mathrm{d}t\right)^{\frac{1}{q}} \leq \begin{cases} A\left(\frac{|s|}{r}\right)^{\ell-[\![\ell]\!]} r^{-n-[\![\ell]\!]}\,|x|_X & \ell \notin \mathbb{Z}_+ \\ A\frac{|s|}{r}\log\frac{r}{|s|}\cdot r^{-n-[\![\ell]\!]}\,|x|_X & \ell \in \mathbb{Z}_+ \end{cases}$$

for all $r > 0$, $z \in \mathbb{R}^n$ with $|s| < r/2$, $x \in X$, and $\alpha \in \mathbb{N}^n$ with $|\alpha| = [\![\ell]\!]$.

The corresponding conditions $K(\infty, \ell; X, Y)$ are defined by replacing the L^q-type integrals by essential suprema in the usual way.

REMARK 5.5. The estimate (5.4) is verified if $\|D^\alpha k(t)\|_{\mathcal{L}(X,Y)} \leq A\,|t|^{-n-|\alpha|}$.

These conditions are defined for scalar-valued functions k in STRÖMBERG and TORCHINSKY [**81**], p. 151. They use the notation $\widetilde{M}(q, \ell)$ for what we would call $K(q, \ell; \mathbb{C})$.

The significance of the conditions $K(q, \ell; X, Y)$ lies in the fact that they provide very satisfactory control over the action of the convolution $k * \cdot$ on atoms of Hardy spaces, and then, by definition, on general elements of $H^p(\mathbb{R}^n; X)$. The following result is proved in the scalar-valued context in [**81**] and "generalized" to the vector-valued case by a repetition of their argument.

THEOREM 5.6 ([**81**]). *Suppose that $k \in K(q, \ell; X, Y)$, where $q \in]1, \infty[$, and that the operator of convolution by k maps*

$$f \in L^q(\mathbb{R}^n; X) \mapsto k * f \in L^q(\mathbb{R}^n; Y) \qquad \textit{boundedly.}$$

Then also

$$f \in H^p(\mathbb{R}^n; X) \mapsto k * f \in H^p(\mathbb{R}^n; Y) \qquad \textit{boundedly for all } p \in \left] \frac{1}{1 + \ell/n}, 1 \right].$$

For the analysis of the abstract Cauchy problem, this result on convolution operators would be sufficient for our purposes, since we have the convolution-type variation-of-constants formula (1.2) at our disposal. However, this is not the case with the more general fractional-order equations (1.8) nor the Laplace equation (1.9) we have in mind, and therefore we also require extension results where the conditions are given in terms of the Fourier multiplier. Thus, we next define conditions similar to $K(q, \ell; X, Y)$ for the multipliers m on the Fourier transform side, and comment on the relations of the conditions satisfied by the multiplier and by the kernel.

DEFINITION 5.7 ([**54, 81**]). We say that a function $m \in L^\infty(\mathbb{R}^n; \mathcal{L}(X, Y))$ belongs to the class $M(q, \ell; X, Y)$ [or just $M(q, \ell; X)$ if $Y = X$] provided that $m \in \mathcal{C}^{\lfloor \ell \rfloor}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X, Y))$ and satisfies

$$\left(\frac{1}{r^n} \int_{r<|\xi|<2r} |D^\alpha m(\xi) x|_Y^q \, \mathrm{d}\xi \right)^{\frac{1}{q}} \le A \, r^{-|\alpha|} \, |x|_X \tag{5.8}$$

for all $r > 0$, $\alpha \in \mathbb{N}^n$ with $|\alpha| \le \lfloor \ell \rfloor$ and $x \in X$, and moreover, if $\ell \notin \mathbb{Z}_+$,

$$\left(\frac{1}{r^n} \int_{r<|\xi|<2r} |(D^\alpha m(\xi) - D^\alpha m(\xi - \zeta)) x|_Y^q \, \mathrm{d}\xi \right)^{\frac{1}{q}} \le A \left(\frac{|\zeta|}{r} \right)^{\ell - \lfloor \ell \rfloor} r^{-|\alpha|} \, |x|_X$$

for all $r > 0$, $\zeta \in \mathbb{R}^n$ with $|\zeta| < r/2$, $\alpha \in \mathbb{N}^n$ with $|\alpha| = \lfloor \ell \rfloor$ and $x \in X$.

REMARK 5.9. The estimate (5.8) holds if $\|D^\alpha m(\xi)\|_{\mathcal{L}(X,Y)} \le A \, |\xi|^{-|\alpha|}$.

This condition appears in KURTZ and WHEEDEN [**54**] for integer values of ℓ, and it was known to be related to the boundedness of multiplier operators even earlier. See [**54**] for some history and references. The definition of the multiplier condition for general ℓ [in the scalar-valued setting] is taken from STRÖMBERG and TORCHINSKY [**81**].

The usefulness of the conditions $M(q, \ell; X, Y)$ is related to the fact that $m \in M(q, \ell; X, Y)$ implies, in a certain sense, that $k \in K(\tilde{q}, \tilde{\ell}; X, Y)$ for certain $\tilde{q}$ and $\tilde{\ell}$, where k is the convolution kernel related to the multiplier m. Although we are not going to give the proof, which is again a repetition of the argument in the scalar case, it seems appropriate to outline the key lemma, since here the generalization of the scalar-valued argument requires an assumption concerning the geometry of the underlying Banach space Y.

The statement of the lemma involves a dyadic partition of unity defined as follows: Let $\eta \in \mathcal{D}(\mathbb{R}^n)$ be non-negative, equal to unity in $\bar{B}(0,1)$ and supported in $\bar{B}(0,2)$. Let $\phi(\xi) := \eta(\xi) - \eta(2\xi)$. Then $\phi(2^{-i}\cdot)$ is supported in the annulus $2^{i-1} \leq |x| \leq 2^{i+1}$, and $\eta(\xi) + \sum_{i=1}^{\infty} \phi(2^{-i}\xi) = 1$ for all $\xi \in \mathbb{R}^n$.

LEMMA 5.10 ([**81**]). *Suppose that the multiplier $m \in L^\infty(\mathbb{R}^n; \mathcal{L}(X,Y))$ satisfies $M(q,\ell;X,Y)$, where Y has Fourier-type $q \in [1,2]$. If we define*

$$m_0(\xi) := \eta(\xi)m(\xi), \quad m_i(\xi) := \phi(2^{-i}\xi)m(\xi), \text{ for } i \in \mathbb{Z}_+, \quad \text{and} \quad k_i := \check{m}_i,$$

then the kernels $k^N := \sum_{i=1}^{N} k_i$ satisfy the condition $K(q', \ell - n/q; X, Y)$ uniformly in N.

REMARK ON PROOF. The proof repeats the argument in the scalar context. The only point that does not directly generalize to the vector-valued situation is the use of the HAUSDORFF–YOUNG inequality, which is valid with a given exponent q if and only if (by definition) the underlying space has the correponding Fourier-type, but this is handled by the assumption. □

With Lemma 5.10, the following multiplier theorem is obtained as a corollary of Theorem 5.6.

THEOREM 5.11 ([**81**]). *Suppose $m \in L^\infty(\mathbb{R}^n; \mathcal{L}(X,Y))$, and that the corresponding multiplier operator T is bounded from $L^{\tilde{p}}(\mathbb{R}^n; X)$ to $L^{\tilde{p}}(\mathbb{R}^n; Y)$ for some $\tilde{p} \in]1,\infty[$. Suppose further that $m \in M(q,\ell;X,Y)$ for some q such that*

$$Y \text{ has Fourier-type } q, \qquad 1 \leq q \leq \tilde{p}' \qquad \text{and} \qquad \ell > n/q.$$

Then T extends boundedly to

$$f \in H^p(\mathbb{R}^n; X) \mapsto Tf \in H^p(\mathbb{R}^n; Y) \qquad \text{for all } \; p \in \left] \frac{1}{1/q' + \ell/n}, 1 \right]$$

REMARK 5.12. Observe that $1/q' + \ell/n > 1/q' + 1/q = 1$ under the assumptions, so that the asserted range of p is non-empty. Also note that only the Fourier-type of the image space Y is relevant, and moreover the theorem always contains the case $q = 1$, without any geometric conditions on the Banach spaces in question.

H^p-boundedness without a priori assumptions. The results quoted so far show that the problem of extending an operator to H^p, once its boundedness is known on some $L^{\tilde{p}}$-space *a priori*, has been solved to a large extent. It seems appropriate to conclude this section with a result which gives the boundedness without an *a priori* assumption. After the operator-valued extension due to WEIS [**87**] of the classical MIHLIN's multiplier theorem, several variants are now known which guarantee the boundedness of an operator-valued Fourier multiplier from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$ (e.g. in [**25, 36, 78**]). It is then clear that one only needs to impose on the multiplier m the union of the condition required by such a theorem and the conditions of Theorem 5.11. A particularly appealing result of this kind is obtained by combining Theorem 5.11 with results from Chapter 2, in particular Theorem 7.9 of that Chapter, where the L^p-boundedness is derived from a set of conditions which are in the same spirit as the conditions of Theorem 5.11. In fact, these (close to minimal) sufficient conditions for the L^p-boundedness already imply (the conditions of Theorem 5.11 for) the boundedness on H^p for p not too small, and always on H^1! (Cf. Remark 7.14 of Chapter 2.)

A Fourier multiplier theorem giving sufficient conditions for H^1-L^1-boundedness was also recently proved by GIRARDI and WEIS [**36**], Cor. 4.6, based on Theorem 5.1 of BENEDEK, CALDERÓN and PANZONE.

As has been known for some time, norm estimates are insufficient in the vector-valued situation, and one requires R-boundedness-type conditions. Thus, in the following Theorem 5.13, the membership in $M(q,\ell;X,Y)$ of m is not sufficient, but we require a similar condition for the *sequence-valued* multiplier $(m(2^j\cdot))_{-\infty}^{\infty}$, with the Rademacher classes $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$ [whose definition we recall after the statement of the theorem] in place of X and Y.

THEOREM 5.13. *Let X and Y be UMD-spaces with Fourier-type $q \in \,]1,2]$. Let $\ell > n/q$, and suppose*

$$
\begin{aligned}
&(m(2^j\cdot))_{-\infty}^{\infty} \in M(q,\ell;\mathrm{Rad}(X),\mathrm{Rad}(Y)) \text{ and}\\
&(m(2^j\cdot)')_{-\infty}^{\infty} \in M(q,\ell;\mathrm{Rad}(Y'),\mathrm{Rad}(X')).
\end{aligned}
\tag{5.14}
$$

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto \mathcal{F}^{-1}[m\hat f]$ extends boundedly to

$$
\begin{aligned}
&f \in L^p(\mathbb{R}^n;X) \mapsto \mathcal{F}^{-1}[m\hat f] \in L^p(\mathbb{R}^n;Y) \qquad \text{for all } p \in \,]1,\infty[\,, \qquad \text{and}\\
&f \in H^p(\mathbb{R}^n;X) \mapsto \mathcal{F}^{-1}[m\hat f] \in H^p(\mathbb{R}^n;Y) \qquad \text{for all } p \in \left]\frac{1}{1/q'+\ell/n}, 1\right].
\end{aligned}
$$

REMARK 5.15. The Rademacher class $\mathrm{Rad}(X)$ appearing in the statement of Theorem 5.13 is the closure in $L^p(\Omega;X)$ of the algebraic tensor product $X \otimes (\varepsilon_j)_{-\infty}^{\infty}$, where $(\varepsilon_j)_{-\infty}^{\infty}$ is the Rademacher system on the probability space Ω. The Rademacher classes are introduced and their properties presented in more detail in Chapter 2; at this point we only note that by KAHANE's inequality, any $p \in [1,\infty[$ yields the same definition. In particular, taking $p = q$, the version of the condition (5.8) for $(m(2^j\cdot))_{-\infty}^{\infty} \in M(q,\ell;\mathrm{Rad}(X),\mathrm{Rad}(Y))$ reads

$$
\mathbb{E}\int_{r<|\xi|<2r} \left|\sum_j \varepsilon_j 2^{j|\alpha|} D^\alpha m(2^j\xi) x_j\right|_Y^q \mathrm{d}\xi \le A^q r^{n-|\alpha|q}\,\mathbb{E}\left|\sum_j \varepsilon_j x_j\right|_X^q,
\tag{5.16}
$$

and the other conditions attain a similar form. By density, it suffices to restrict to finitely non-zero sequences $(x_j)_{-\infty}^{\infty} \in X^{\mathbb{Z}}$. Also note that the condition (5.16), as well as the corresponding dual condition, are satisfied if one assumes

$$
\{|\xi|^{|\alpha|}\, D^\alpha m(\xi) \mid \xi \neq 0\}
$$

to be R-bounded for the appropriate $\alpha \in \mathbb{N}^n$, a randomized Mihlin-type condition as first introduced by WEIS [**87**].

It should be emphasized that, once the conditions of Theorem 5.13 are satisfied, in order to get boundedness on H^p with smaller values of p than the bound given in Theorem 5.13, one does not need to impose more R-boundedness-type conditions (5.14), but the weaker conditions $m \in MS(q, \ell; X, Y)$ of Theorem 5.11 will do. Thus the assumptions of the following corollary are unnecessarily strong; nevertheless, they are satisfied by many multipliers encountered in the applications.

COROLLARY 5.17. *Let X, Y be UMD-spaces and $m \in \mathcal{C}^\infty(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X, Y))$ be a multiplier such that $\{|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \neq 0\}$ is R-bounded for every $\alpha \in \mathbb{N}^n$. Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto \mathcal{F}^{-1}[m\hat{f}]$ extends to a bounded mapping from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$ for all $p \in]1, \infty[$ and from $H^p(\mathbb{R}^n; X)$ to $H^p(\mathbb{R}^n; Y)$ for all $p \in]0, 1]$.*

6. Return to maximal regularity

We have now developed the necessary tools to prove the maximal regularity results indicated in the Introduction. As a last preparatory step on the general level, let us recall the following well-known (at least in the scalar-case) result, whose short proof is given for completeness.

LEMMA 6.1. *Let X and Y be Banach spaces, and $T_m : f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ be bounded from $\tilde{H}^1(\mathbb{R}; X)$ to $L^1(\mathbb{R}^n; Y)$. Then T_m is also bounded from $\tilde{H}^1(\mathbb{R}; X)$ to $\tilde{H}^1(\mathbb{R}; Y)$. In particular, if X and Y are UMD-spaces, this holds with $\tilde{H}^1(\mathbb{R}; Z) = H^1(\mathbb{R}; Z)$, $Z \in \{X, Y\}$.*

PROOF. With $C := \|T_m\|_{\mathcal{L}(\tilde{H}^1(\mathbb{R};X), L^1(\mathbb{R};Y))} < \infty$, we can estimate

$$\|T_m f\|_{L^1(\mathbb{R};Y)} \leq C \|f\|_{\tilde{H}^1(\mathbb{R};X)}$$

by definition, and

$$\|\mathcal{H} T_m f\|_{L^1(\mathbb{R};Y)} = \|T_m \mathcal{H} f\|_{L^1(\mathbb{R};Y)} \leq C \|\mathcal{H} f\|_{\tilde{H}^1(\mathbb{R};X)} = C \|f\|_{\tilde{H}^1(\mathbb{R};X)},$$

where the commutativity of the multiplier operators $\mathcal{H}$ and T_m is clear when investigated in terms of the Fourier transforms, and we used the fact that

$$\|\mathcal{H} f\|_{\tilde{H}^1(\mathbb{R};X)} = \|\mathcal{H} f\|_{L^1(\mathbb{R};X)} + \|-f\|_{L^1(\mathbb{R};X)} = \|f\|_{\tilde{H}^1(\mathbb{R};X)},$$

since $\mathcal{H}^2 = -1$, which is also clear from the Fourier transforms. □

We then concentrate on the problems we had in mind in the Introduction:

The abstract Cauchy problem $\dot{u} + Au = f$. Everything will be clear as soon as we verify that the conditions $K(q, \ell; X)$ and $M(q, \ell; X)$, required by the extension results, are verified by the convolution kernel and the multiplier, respectively, related to the ACP. In fact, it would suffice to consider just one of them, but we give both the short proofs for purposes of illustration. The other equations we consider below, namely (1.8) and (1.9), are treated only with multiplier methods, as we no longer have the variation-of-constants formula at our disposal.

We have the following lemma:

LEMMA 6.2. *Let $-A$ be the generator of a bounded analytic semigroup (T^t), and*

$$k(t) := AT^t\chi_{\mathbb{R}_+}(t), \qquad m(\xi) := A(\mathbf{i}2\pi\xi + A)^{-1}, \tag{6.3}$$

for $t \in \mathbb{R}$, $\xi \in \mathbb{R}\setminus\{0\}$. Then $k \in K(q,\ell;X)$ and $m \in M(q,\ell;X)$ for any $q \in [1,\infty]$, $\ell > 0$.

If, moreover, $\{m(\xi)|\ \xi \neq 0\}$ is R-bounded, then also $\{\xi^\nu D^\nu m(\xi)|\ \xi \neq 0\}$ is R-bounded for all $\nu \in \mathbb{N}$.

PROOF. According to Remarks 5.5 and 5.9, it suffices to verify that

$$\|D^\nu k(t)\|_{\mathcal{L}(X)} \le C_\nu\,|t|^{-1-\nu} \qquad \text{and} \qquad \|D^\nu m(\xi)\|_{\mathcal{L}(X)} \le C_\nu\,|\xi|^{-\nu} \qquad \forall\ \nu \in \mathbb{N},$$

i.e., we need the estimates

$$\begin{aligned} \left\|A^{1+\nu}T^t\right\|_{\mathcal{L}(X)} &\le C_\nu\,|t|^{-1-\nu} \qquad \text{and} \\ \nu!(2\pi)^\nu\left\|A(\mathbf{i}2\pi\xi + A)^{-1-\nu}\right\|_{\mathcal{L}(X,Y)} &\le C_\nu\,|\xi|^{-\nu}, \end{aligned}$$

but the first estimate is well-known and follows easily from $\|tAT^t\|_{\mathcal{L}(X)} \le C$ and the semigroup property, whereas for the second we only need recall that

$$\left\|(\lambda + A)^{-1}\right\|_{\mathcal{L}(X)} \le C\,|\lambda|^{-1} \qquad \forall\ \lambda \text{ with } |\arg(\lambda)| < \vartheta,$$

where $\vartheta > \pi/2$, in particular, for $\lambda = \mathbf{i}2\pi\xi$.

As for the last assertion, the R-boundedness of $\xi^\nu D^\nu m(\xi) = \nu!(2\pi\xi)^\nu A(\mathbf{i}2\pi\xi + A)^{-1-\nu}$ follows from the R-boundedness of $m(\xi)$ in exactly the same way as the norm boundedness of the derivatives followed from the norm boundedness of $m(\xi)$. □

Now the proof of Theorem 1.6 is a matter of collecting the pieces together.

PROOF OF THEOREM 1.6 AND REMARK 1.7. If the ACP has maximal L^p-regularity, $p \in\ ]1,\infty[$, then by the classical Theorem 1.4, $-A$ generates a bounded analytic semigroup, and by WEIS' Theorem 1.5, the collection $\{A(\mathbf{i}2\pi\xi{+}A)^{-1}|\ \xi \neq 0\}$ is R-bounded. Then by Lemma 6.2, the related convolution kernel and multiplier in (6.3) satisfy infinitely many of the conditions required to apply our extension results, and we obtain the boundedness of $f \mapsto k * f$ from Theorem 5.6, or equally well the boundedness of $f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ from either Theorem 5.11 or Corollary 5.17. Thus we have $C_1 \Rightarrow C_2, C_4, C_5$.

This did not really require UMD, since the operator extension theorems work for general Banach spaces, as soon as the boundedness on one $L^{\tilde{p}}(\mathbb{R};X)$ is known *a priori* (see Sect. 5); also the R-boundedness can be deduced from Theorem 4.1 of CLÉMENT and PRÜSS which holds for general X.

Clearly $C_2 \Rightarrow C_3$, but we also have $C_1 \Rightarrow C_3$ directly from the classical Theorem 5.1; thus the condition C_1 implies all the other conditions. (Still no UMD required!)

If the ACP has maximal (H^1, L^1)-regularity, then by Theorem 3.1, $-A$ generates a bounded analytic semigroup and by Theorem 4.2, $\{A(\mathbf{i}2\pi\xi + A)^{-1} | \xi \neq 0\}$ is R-bounded. Thus $C_3 \Rightarrow C_4$. Moreover, $C_4 \Rightarrow C_1$ by WEIS' Theorem 1.5, and here we need the UMD assumption.

Summarizing, we have $C_2 \Rightarrow C_3 \Rightarrow C_4 \Rightarrow C_1 \Rightarrow C_2, C_3, C_4, C_5$, and this is the theorem. □

REMARK 6.4. As the reader probably observed, the proof given above offers various alternative routes to check the conditions of our auxiliary results, either in terms of the multiplier or the convolution kernel. The fact that one can actually manage by investigating only the conditions for the kernel $k(t) = AT^t\chi_{\mathbb{R}_+}(t)$ from the variation-of-constants formula is worth emphasizing, since this means that the technical Lemma 5.10 of STRÖMBERG and TORCHINSKY, which is used to derive the multiplier theorem 5.11 from the convolution theorem 5.6, can be avoided, as long as only the Cauchy problem is concerned.

Moreover, if one is only interested in H^1 and not in H^p with $p < 1$, then the probably easiest argument runs as follows: $C_1 \Rightarrow C_3$ by the classical Theorem 5.1 of BENEDEK, CALDERÓN and PANZONE. $C_3 \Rightarrow C_2$ by Lemma 6.1, and the converse is trivial. The implications $C_3 \Rightarrow C_4 \Rightarrow C_1$ are proved as in the proof above. (This simplified version of the proof for this particular case was pointed out to me by L. WEIS.)

The remark only applies to the Cauchy problem, for which we have the variation-of-constants formula; the more general fractional-order equation (1.8), which we next treat, requires the multiplier approach, as does the Laplace equation (1.9).

The fractional-order equation $D^\alpha u + Au = f$, $\alpha \in \,]0, 2[$. We shall here give a treatment of the problem (1.8), somewhat analogous to that of the Cauchy problem (1.1), but with certain new features.

Let us first recall the relevant definition of the fractional derivative D^α appearing in our equation. (Cf. [**90**], §12.8 for the classical [scalar-valued] setting, or [**19, 20**] for the vector-valued context.)

DEFINITION 6.5. We say that $u \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}_+; X)$ has a fractional derivative of order $\alpha > 0$ provided $u = g_\alpha * f$ for some $f \in L^1_{\mathrm{loc}}(\bar{\mathbb{R}}_+; X)$, where

$$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}\chi_{\mathbb{R}_+}(t).$$

When such an f exists, it is, in fact, essentially unique, and we write $f =: D^\alpha u$.

REMARK 6.6. It is well-known that D^α is a closed operator on $L^p(\bar{\mathbb{R}}_+; X)$, $p \in [1, \infty[$. Indeed, suppose $u_j \to u$, $w_j := D^\alpha u_j \to w$ in $L^p(\bar{\mathbb{R}}_+; X)$. By definition, $u_j = g_\alpha * w_j$, and due to the local integrability of g_α, it follows that $u_j = g_\alpha * w_j \to g_\alpha * w$ in $L^p_{\mathrm{loc}}(\bar{\mathbb{R}}_+; X)$. But u_j also converges to u; hence $u = g_\alpha * w$, i.e., $D^\alpha u = w$.

In the L^p-setting, a result analogous to Weis' Theorem 1.5 was proved by Clément and Prüss:

Theorem 6.7 ([**22**], p. 85). *The initial value problem* (1.8) *admits a unique solution* $u \in L^p(\mathbb{R}_+; \mathcal{D}(A))$ *for every* $f \in L^p(\mathbb{R}_+; X)$ *provided that* $p \in]1, \infty[$ *and* A *is boundedly invertible and* R*-sectorial with* R*-angle* $\phi_A^R < \pi(1 - \alpha/2)$ [*see Def.* 6.10]. *The solution satisfies*

$$\|D^\alpha u\|_{L^p(\mathbb{R}_+;X)} + \|u\|_{L^p(\mathbb{R}_+;X)} + \|Au\|_{L^p(\mathbb{R}_+;X)} \le C \|f\|_{L^p(\mathbb{R}_+;X)}, \tag{6.8}$$

i.e., the problem (1.8) *has* (*strong*) *maximal* L^p*-regurity for all* $p \in]1, \infty[$.

As before, our aim will be the extension to H^p. In the lack of a candidate mild solution, our line of attack will be a little different from that adopted in treating the Cauchy problem [which is, of course, a special case of the present one, with $\alpha = 1$]. In particular, since the bounded invertibility of A was already assumed in Theorem 6.7, we will here treat (in the spirit of Theorem 6.7) the stronger notion of maximal regularity, in which the regularity of u and not only of Au and $D^\alpha u$ is required:

Definition 6.9. We say that (1.8) possesses strong regularity from $\mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$ to $\mathfrak{F}_1(\bar{\mathbb{R}}_+; X)$, where $\mathfrak{F}_0(\bar{\mathbb{R}}_+; X) \subset \mathfrak{F}_1(\bar{\mathbb{R}}_+; X) \subset L^1_{\mathrm{loc}}(\bar{\mathbb{R}}_+; X)$, if for every $f \in \mathfrak{F}_0(\bar{\mathbb{R}}_+; X)$, there exists a unique $u \in \mathfrak{F}_1(\bar{\mathbb{R}}_+; X)$ such that $u(t) \in \mathcal{D}(A)$ for a.e. $t \ge 0$, $D^\alpha u$ exists and the equation (1.8), and moreover the estimate

$$\|D^\alpha u\|_{\mathfrak{F}_1(\bar{\mathbb{R}}_+;X)} + \|u\|_{\mathfrak{F}_1(\bar{\mathbb{R}}_+;X)} + \|Au\|_{\mathfrak{F}_1(\bar{\mathbb{R}}_+;X)} \le C \|f\|_{\mathfrak{F}_0(\bar{\mathbb{R}}_+;X)}$$

holds with C independent of f. If $\mathfrak{F}_1 = \mathfrak{F}_0$, we speak of strong maximal regularity.

Before attacking the problem, let us clarify the notion that appeared in the assumptions of Theorem 6.7:

Definition 6.10. We say that a sectorial operator A is *R-sectorial* if its range is dense (which is trivially true if A is boundedly invertible) and moreover $\{A(t + A)^{-1} |\ t > 0\}$ is R-bounded, or equivalently (by a power series argument), if $\{A(z + A)^{-1} |\ |\arg(z)| \le \pi - \phi\}$ is R-bounded for some $\phi < \pi$. The *R-angle* ϕ_A^R of A is the infimum over all such $\phi \in]0, \pi[$ for which this condition holds.

The notion of R-sectoriality (R-boundedness on a sector) is connected with the R-boundedness estimates for multipliers (functions on the line) by means of the following lemma. It is a simple generalization of the corresponding result for $\alpha = 1$, which is proved in [**25**], Theorem 4.4.

Lemma 6.11. *Let* $\alpha \in]0, 2[$ *and let* A *be sectorial of angle* $\omega < \pi(1 - \alpha/2)$. *Then* A *is* R*-sectorial of some angle* $\theta < \pi(1 - \alpha/2)$ *if and only if* $\{A((\mathbf{i}\xi)^\alpha + A)^{-1} |\ \xi \in \mathbb{R} \setminus \{0\}\}$ *is* R*-bounded.*

PROOF. Observe that, for $z = re^{\pm\mathbf{i}\phi}$ (where $\phi \geq 0$), we have $z^\alpha = r^\alpha e^{\pm\mathbf{i}\alpha\phi} = -r^\alpha e^{\mp\mathbf{i}(\pi-\alpha\phi)} =: -\lambda$. Here λ, by the spectral assumption on A, belongs to the resolvent of A whenever $\pi - \alpha\phi > \omega =: \pi(1-\alpha/2) - 2\alpha\epsilon$, i.e. $\phi < \pi/2 + 2\epsilon$. For $\phi \leq \pi/2 + \epsilon$ we even know that $z \mapsto A(z^\alpha + A)^{-1} = -A(\lambda - A)^{-1}$ is a bounded function. Now $z \mapsto z^\alpha$ is analytic in $\mathbb{C} \setminus \bar{\mathbb{R}}_-$ and the resolvent operator $(\lambda - A)^{-1}$ admits a convergent power series expansion around every λ where it exists. Thus, for $x \in X$, $x' \in X'$, we know that $z \mapsto \langle x', A(z^\alpha + A)^{-1}x\rangle$ is bounded and analytic (in particular, harmonic) in the right half-plane. Thus it can be represented in terms of the boundary values by the classical Poisson formula

$$\left\langle x', A(z^\alpha + A)^{-1}x\right\rangle = \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{\sigma}{\sigma^2 + (\tau - \xi)^2}\left\langle x', A((\mathbf{i}\xi)^\alpha + A)^{-1}x\right\rangle \mathrm{d}\xi, \qquad z = \sigma + \mathbf{i}\tau \in \mathbb{C}_+.$$

Since this is true for all $x' \in X'$ and the corresponding Bôchner integrals exist, we also have the above representation without the x'. But the positive Poisson kernel has a unit integral, so that $A(z^\alpha + A)^{-1}x$ is then expressed as a (continuous) convex combination of $A((\mathbf{i}\xi)^\alpha + A)^{-1}x$, $\xi \in \mathbb{R}$, and the R-boundedness of $\{A(z^\alpha + A)^{-1} |\ z \in \mathbb{C}_+\}$ follows from that of $\{A((\mathbf{i}\xi)^\alpha + A)^{-1} |\ \xi \in \mathbb{R} \setminus \{0\}\}$ by the permanence of R-boundedness under convex hulls and strong operator closures. The fact that we even obtain R-boundedness in a slightly larger sector than the half-plane follows from the power series expansion of the resolvent on the imaginary axis.

Conversely, it is obvious that the R-boundedness of

$$\{A(\lambda + A)^{-1} : |\arg(\lambda)| \leq \alpha(\pi/2 + \epsilon)\} = \{A(z^\alpha + A)^{-1} : |\arg(z)| \leq \pi/2 + \epsilon\}$$

implies in particular that of $\{A((\mathbf{i}\xi)^\alpha + A)^{-1} |\ \xi \in \mathbb{R} \setminus \{0\}\}$. □

To apply the multiplier techniques, we want to transform our equation (1.8), and for this purpose we analyse the terms on the left-hand side. We first observe the following result [which we state for $\mathbb{R}^n$ instead of $\mathbb{R}$, since the proof is the same and the result will also be of use in the subsequent section]:

LEMMA 6.12. *Let $u \in L^p(\mathbb{R}^n; \mathcal{D}(A))$ [i.e., $u(t) \in \mathcal{D}(A)$ for a.e. $t \in \mathbb{R}^n$, and $u, Au \in L^p(\mathbb{R}^n; X)$], where $p \geq 1$ is a Fourier-type for X. Then $\hat{u}(\xi) \in \mathcal{D}(A)$ and $A\hat{u}(\xi) = \mathcal{F}[Au](\xi)$ for a.e. $\xi \in \mathbb{R}^n$.*

PROOF. Interpreting our functions as tempered distributions where appropriate, we have, for every $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}\langle \mathcal{F}[Au], \phi\rangle = \left\langle Au, \hat{\phi}\right\rangle = \int_{-\infty}^{\infty} Au(t)\hat{\phi}(t)\,\mathrm{d}t = A\int_{-\infty}^{\infty} u(t)\hat{\phi}(t)\,\mathrm{d}t \\ = A\left\langle u, \hat{\phi}\right\rangle = A\left\langle \hat{u}, \phi\right\rangle,\end{aligned} \tag{6.13}$$

where extracting A from the integral was legitimate due to the closedness of A and the integrability of $t \mapsto u(t)\hat{\phi}(t)$ and $t \mapsto Au(t)\hat{\phi}(t)$.

Taking in place of ϕ a sequence ϕ_n which provides an approximation of the Dirac mass at a point $\xi \in \mathbb{R}^n$, where ξ is a common Lebesgue point of both $\mathcal{F}[Au]$ and $\hat{u}$, we have, on the one hand, $\langle \hat{u}, \phi_n \rangle \to \hat{u}(\xi)$, and on the other, $A \langle \hat{u}, \phi_n \rangle = \langle \mathcal{F}[Au], \phi_n \rangle \to \mathcal{F}[Au](\xi)$. But then it follows from the closedness of A that $\hat{u}(\xi) \in \mathcal{D}(A)$ and $A\hat{u}(\xi) = \mathcal{F}[Au](\xi)$ for every such ξ, and that is, for a.e. $\xi \in \mathbb{R}^n$. □

It remains to compute the Fourier transform of $D^\alpha u$. The result is what one would expect from the well-known formula of the Fourier transform of a usual derivative of integral order. [In what follows, powers are always defined in terms of the principal branch of the logarithm.]

Lemma 6.14. *Suppose that $u \in L^p(\mathbb{R}_+; X)$ has a fractional derivative $D^\alpha u \in L^p(\mathbb{R}_+; X)$, where $p \geq 1$ is a Fourier-type for X. Then have*

$$\mathcal{F}[D^\alpha u](\xi) = (\mathbf{i}2\pi\xi)^\alpha \hat{u}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R}.$$

Proof. Denoting $v := D^\alpha u$, we have, by Def. 6.5, $u = g_\alpha * v$. We consider instead of g_α the modified kernel $g_\alpha^\mu(t) := g_\alpha(t)e^{-\mu t}$, where $\mu > 0$. Then $g_\alpha^\mu \in L^1(\mathbb{R}_+)$, and hence $g_\alpha^\mu * v \in L^p(\mathbb{R}_+; X)$ for $v \in L^p(\mathbb{R}_+; X)$, and its Fourier transform is a proper function in $L^{p'}(\mathbb{R}; X)$; in fact, it is given by

$$\mathcal{F}[g_\alpha^\mu * v](\xi) = \hat{g}_\alpha^\mu(\xi)\hat{v}(\xi). \tag{6.15}$$

The Fourier transform of g_α^μ can be computed explicitly, and it is given by

$$\int_0^\infty g_\alpha^\mu(t)e^{-\mathbf{i}2\pi\xi t}\,\mathrm{d}t = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-(\mu+\mathbf{i}2\pi\xi)t}\,\mathrm{d}t = \frac{1}{(\mu+\mathbf{i}2\pi\xi)^\alpha}, \tag{6.16}$$

which follows more or less directly from the definition of the Γ-function.

It is not difficult to see that $g_\alpha^\mu * v \to g_\alpha * v$ in $\mathcal{S}'(\mathbb{R}; X)$ as $\mu \downarrow 0$. Indeed, we observe that

$$\begin{aligned} |g_\alpha * v(t) - g_\alpha^\mu * v(t)|_X &\leq \int_0^t g_\alpha(s)\,|v(t-s)|_X\,(1-e^{-\mu s})\,\mathrm{d}s \\ &\leq g_\alpha * |v|_X\,(t)\cdot(1-e^{-\mu t}), \end{aligned}$$

and thus

$$\begin{aligned} \|g_\alpha * v - g_\alpha^\mu * v\|_{L^p(0,T;X)} &\leq \|g_\alpha\|_{L^1(0,T)}\,\|v\|_{L^p(\mathbb{R};X)}\,(1-e^{-\mu T}) \\ &\leq \frac{T^\alpha}{\Gamma(\alpha+1)}\,\|v\|_{L^p(\mathbb{R};X)}\,\mu T. \end{aligned}$$

Since this bound is only slowly increasing as a function of T, the convergence of $\int (g_\alpha^\mu * v)\phi$ to $\int (g_\alpha * v)\phi$, for a rapidly decreasing ϕ, follows immeadiately.

Using (6.16), we now write (6.15) in the form

$$(\mu+\mathbf{i}2\pi\xi)^m \mathcal{F}[g_\alpha^\mu * v](\xi) = (\mu+\mathbf{i}2\pi\xi)^{m-\alpha}\hat{v}(\xi), \tag{6.17}$$

where $\alpha \leq m \in \mathbb{Z}_+$. On the left-hand side, we have $g_\alpha^\mu * v \to g_\alpha * v$ in $\mathcal{S}'(\mathbb{R}; X)$ as $\mu \downarrow 0$, and due to the continuity of $\mathcal{F}$ on $\mathcal{S}'(\mathbb{R}; X)$, the same is true for the Fourier transforms. Moreover, multiplication by a bounded smooth function such as $(\mathbf{i}2\pi\xi)^k$ is also continuous on $\mathcal{S}'(\mathbb{R}^n;X)$, and it then follows easily [e.g., expanding the power of the binomial] that the left-hand side of (6.17) converges to $(\mathbf{i}2\pi\xi)^m\mathcal{F}[g_\alpha * v](\xi)$ in the sense of tempered distributions as $\mu \downarrow 0$.

On the right-hand side of (6.17), on the other hand, the pointwise convergence is obvious, and since the factor in front of $\hat{v}$ is only slowly increasing as a function of ξ, we again have convergence also in $\mathcal{S}'(\mathbb{R};X)$.

Hence, taking the limit in the sense of tempered distributions on both sides of (6.17), observing that both limits coincide with proper functions, and recalling that proper functions can only agree as distributions if they agree almost everywhere, we finally arrive at

$$(\mathbf{i}2\pi\xi)^m\mathcal{F}[g_\alpha * v](\xi) = (\mathbf{i}2\pi\xi)^{m-\alpha}\hat{v}(\xi) \qquad \text{for a.e. } \xi \in \mathbb{R},$$

which, after multiplication by $(\mathbf{i}2\pi\xi)^{\alpha-m}$, is the claim, since $g_\alpha * v = u$ and $v = D^\alpha u$. $\square$

From Lemmata 6.12 and 6.14 we see that the unique solution [guaranteed by Theorem 6.7] of (1.8) for $f \in L^q \cap H^p(\bar{\mathbb{R}}_+;X)$ [where $q > 1$ is a Fourier-type of X] satisfies

(6.18)
$$\hat{u}(\xi) = ((\mathbf{i}2\pi\xi)^\alpha + A)^{-1}\hat{f}(\xi) \qquad \text{and} \qquad \mathcal{F}[Au](\xi) = A((\mathbf{i}2\pi\xi)^\alpha + A)^{-1}\hat{f}(\xi).$$

Thus we are in a position to apply our multiplier theorems to deduce maximal regularity results similar to Theorem 6.7 also on $H^p(\bar{\mathbb{R}}_+;X)$, $p \in]0,1]$.

We need one last lemma dealing with the multipliers appearing in (6.18):

Lemma 6.19. *Let A be boundedly invertible and $\{A((\mathbf{i}2\pi\xi)+A)^{-1} | \ \xi \in \mathbb{R}\}$ be R-bounded. Then $m_\nu(\xi) := A^\nu((\mathbf{i}2\pi\xi)+A)^{-1}$, $\nu = 0,1$, satisfy the assumptions of Corollary 5.17.*

Proof. The R-boundedness of m_1 is assumed, and that of m_0 follows since multiplication by the bounded operator A^{-1} preserves R-boundedness.

We need to investigate the derivatives of the multipliers m_ν. For iterated derivatives of a composition of functions, one can show by induction that

$$D^k(f\circ g) = \sum_{j=1}^{k}(D^j f\circ g) \sum_{\substack{\sum j_\ell = j \\ \sum \ell j_\ell = k}} c_{(j_\ell)_{\ell=1}^k} \prod_{\ell=1}^{k}(D^\ell g)^{j_\ell},$$

where the $c_{(j_\ell)_{\ell=1}^k}$ are numerical constants depending only on the parameters indicated, and the second sum above runs over all sequences $(j_\ell)_{\ell=1}^k \in \mathbb{N}^k$ which satisfy the conditions indicated. Since the derivatives of the resolvent $(t+A)^{-1}$ have the same form as if A were just a number, and the resolvent thus just an ordinary rational function, we can also apply the above formula to $f(t) = A^\nu(t+A)^{-1}$,

$g(\xi) = (\mathbf{i}2\pi\xi)^\alpha$, to give

$$
\begin{aligned}
D^k m_\nu(\xi) &= \sum_{j=1}^{k} (-1)^j j! A^\nu((\mathbf{i}2\pi\xi)^\alpha + A)^{-1-j} \\
&\quad \times \sum_{(j_\ell)_{\ell=1}^k} c_{(j_\ell)_{\ell=1}^k} \prod_{\ell=1}^{k} \big(\alpha \cdots (\alpha - \ell + 1)(\mathbf{i}2\pi\xi)^{\alpha-\ell}(\mathbf{i}2\pi)^\ell\big)^{j_\ell} \\
&= \sum_{j=1}^{k} \sum_{(j_\ell)_{\ell=1}^k} C(j,k;(j_\ell)_{\ell=1}^k;\alpha) A^\nu((\mathbf{i}2\pi\xi)^\alpha + A)^{-1-j}(\mathbf{i}2\pi\xi)^{\alpha j-k}(\mathbf{i}2\pi)^k,
\end{aligned}
$$

where the conditions $\sum_{\ell=1}^k j_\ell = j$ and $\sum_{\ell=1}^k \ell j_\ell = k$ were used in the last step. Thus $\xi^k D^k m_\nu(\xi)$ is a linear combination of terms of the form

$$
A^\nu((\mathbf{i}2\pi\xi)^\alpha + A)^{-1} \cdot \big((\mathbf{i}2\pi\xi)^\alpha((\mathbf{i}2\pi\xi)^\alpha + A)^{-1}\big)^j = m_\nu(\xi)(1 - m_1(\xi))^j,
$$

and these are R-bounded by the R-boundedness of $m_\nu(\xi)$ for $\nu = 0, 1$. This being true for derivatives of any order, the assumptions of Corollary 5.17 are satified. ☐

We get the following result:

THEOREM 6.20. *Let $\alpha \in]0,2[$, let X be a UMD-space and A a boundedly invertible sectorial operator of angle $\omega < \pi(1-\alpha/2)$. Then the following are equivalent:*

(F_1) (1.8) *has strong maximal regularity on $L^p(\mathbb{R}^n;X)$ for $p \in]1,\infty[$.*
(F_2) (1.8) *has strong maximal regularity on $H^1(\mathbb{R}^n;X)$.*
(F_3) (1.8) *has strong regularity from $H^1(\mathbb{R}^n;X)$ to $L^1(\mathbb{R}^n;X)$.*
(F_4) $\{A((\mathbf{i}2\pi\xi)^\alpha + A)^{-1} \mid \xi \in \mathbb{R}\}$ *is R-bounded.*

Moreover, any of the above conditions implies

(F_5) *For all $p \in]0,1[$, for all (i) $f \in H^p \cap L^q(\bar{\mathbb{R}}_+;X)$ and all (ii) $f \in H^p \cap H^1(\bar{\mathbb{R}}_+;X)$, where $q > 1$ is a Fourier-type for X, the solution u of* (1.8) *satisfies*

$$
\|D^\alpha u\|_{H^p(\bar{\mathbb{R}}_+;X)} + \|u\|_{H^p(\bar{\mathbb{R}}_+;X)} + \|Au\|_{H^p(\bar{\mathbb{R}}_+;X)} \le C\,\|f\|_{H^p(\bar{\mathbb{R}}_+;X)}\,. \tag{6.21}
$$

PROOF. $F_1 \Rightarrow F_2, F_5(i)$. Let first $f \in L^q \cap H^p(\bar{\mathbb{R}}_+;X)$, where $p \in]0,1]$ and $q > 1$ is a Fourier-type for X. Let u be the solution of (1.8). Then we have the formulae (6.18) for $\hat{u}$ and $\mathcal{F}[Au]$. By Lemma 6.19, the corresponding multipliers give bounded operators on $H^p(\mathbb{R};X)$; thus u and Au and hence $D^\alpha u$ are also in $H^p(\bar{\mathbb{R}}_+;X)$, and the estimate (6.21) is valid. This shows that $F_1 \Rightarrow F_5(i)$.

Let then $p = 1$ and $f \in H^1(\bar{\mathbb{R}}_+;X)$ be arbitrary. We may choose $f_j \in H^1 \cap L^q(\bar{\mathbb{R}}_+;X)$ approaching f in $H^1(\bar{\mathbb{R}}_+;X)$. Let u_j be the corresponding solutions of (1.8) with f_j in place of f. Then, by the estimate established, $(u_j)_{j=1}^\infty$, $(Au_j)_{j=1}^\infty$ and $(D^\alpha u_j)_{j=1}^\infty$ are Cauchy sequences in $H^1(\bar{\mathbb{R}}_+;X)$, and then also in $L^1(\bar{\mathbb{R}}_+;X)$.

Denote the limits by u, v and w, respectively. By the closedness of A on X and D^α on L^1 we conclude that $Au = v$ and $D^\alpha u = w$ (meaning that the left-hand sides make sense and agree with the respective right-hand sides). But then u solves (1.8) and satisfies the asserted estimate. That the solution is unique follows by taking the Fourier transforms with the help of Lemmata 6.12 and 6.14, recalling that $H^1(\bar{\mathbb{R}}_+; X) \subset L^1(\bar{\mathbb{R}}_+; X)$.

$F_1, F_2 \Rightarrow F_5(ii)$. This is proved just like $F_1 \Rightarrow F_5(i)$, only considering $f \in H^1 \cap H^p(\bar{\mathbb{R}}_+; X)$.

$F_2 \Rightarrow F_3$ is obvious.

$F_3 \Rightarrow F_4$ follows from Theorem 4.2.

$F_4 \Rightarrow F_1$ follows from Lemma 6.11 and Theorem 6.7 of Clément and Prüss. □

Remark 6.22. As with the Cauchy problem, the UMD-condition was not needed for the implications $F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow F_4$ and $F_1, F_2 \Rightarrow F_5$.

The abstract Laplace equation $-\triangle u + Au = f$. We finally consider an abstract partial differential equation, which will give us a chance to exploit the n-dimensional versions of our multiplier theorems. An extensive treatment of partial differential operators in the vector-valued setting is found in Denk, Hieber and Prüss [**25**]; however, our problem is somewhat different from those treated there.

To deal with the present equation, let us first proceed in a formal manner to Fourier transform the equation to the form

$$4\pi^2 |\xi|^2 \hat{u}(\xi) + A\hat{u}(\xi) = \hat{f}(\xi).$$

If the appropriate inverse exists, this can be solved to give

$$u(t) = \int_{\mathbb{R}^n} (4\pi^2 |\xi|^2 + A)^{-1} \hat{f}(\xi) e^{\mathbf{i}2\pi\xi\cdot t} \,\mathrm{d}\xi. \tag{6.23}$$

Now we observe that if f is sufficiently nice, say $f \in \mathcal{S}(\mathbb{R}^n; X)$, and if $[0, \infty[$ is in the resolvent of $-A$, then both the Laplacian $\triangle$ and the operator A can be applied on the function defined by (6.23) under the integral, and we easily find that $-\triangle u(t) + Au(t) = f(t)$, using the Fourier inversion formula. Moreover, if $f \in \hat{\mathfrak{D}}_0(\mathbb{R}^n; X)$, it suffices to assume that the open interval $]0, \infty[$ is in the resolvent of $-A$, and the same remark applies.

Thus the existence of a solution is guaranteed at least for $f \in \hat{\mathfrak{D}}_0(\mathbb{R}^n; X)$, which is already dense in all $L^p(\mathbb{R}^n; X)$, $p \in]1, \infty[$, as well as in all $H^p(\mathbb{R}^n; X)$, $p \in]0, 1]$. If we require the solution to be sufficiently well-behaved, we also have uniqueness, since then our formal passing from the original equation (1.9) to (6.23) can be justified, i.e., the solution is necessarily given by (6.23). Thus we are able to discuss whether we have *dense class estimates* (cf. Prop. 3.10).

From (6.23) we have

$$Au(t) = \int_{\mathbb{R}^n} A(4\pi^2 |\xi|^2 + A)^{-1} \hat{f}(\xi) e^{\mathbf{i}2\pi\xi\cdot t} \,\mathrm{d}\xi,$$

and thus proving a dense class estimate for the regularity of Au amounts to showing that $m_1(\xi) := A(4\pi^2 |\xi|^2 + A)^{-1}$ is a multiplier between the appropriate spaces. The relevant multiplier for the regularity of u itself is, directly from (6.23), $m_0(\xi) := (4\pi^2 |\xi|^2 + A)^{-1}$. Once again these multipliers have the remarkable property of satisfying the infinity of the Mihlin-type conditions, as soon as they satisfy one.

LEMMA 6.24. *If* $\{m_1(\xi) |\ \xi \neq 0\}$ *resp.* $\{m_0(\xi), m_1(\xi) |\ \xi \neq 0\}$ *is* (R-)*bounded, then so is*

$$\{|\xi|^{|\alpha|} D^\alpha m_1(\xi) |\ \xi \neq 0\} \qquad \textit{resp.} \qquad \{|\xi|^{|\alpha|} D^\alpha m_0(\xi), |\xi|^{|\alpha|} D^\alpha m_1(\xi) |\ \xi \neq 0\}$$

for all $\alpha \in \mathbb{N}^n$.

PROOF. One can easily verify by induction that the αth partial derivative of a smooth radial function $g(x) = f(|x|^2)$ is given by

$$D^\alpha g(x) = \sum_{j \le |\alpha|} f^{(j)}(|x|^2) \sum_{\substack{L \subset \{1,\dots,|\alpha|\} \\ \#L = 2j - |\alpha|}} c(\alpha, L) \prod_{\ell \in L} x_{\alpha(\ell)}, \tag{6.25}$$

where $\alpha(\ell) := 1$ for $0 < \ell \le \alpha_1$, $\alpha(\ell) := 2$ for $\alpha_1 < \ell \le \alpha_1 + \alpha_2$ etc., and $c(\alpha, L) \in \mathbb{N}$ only depend on α and L.

Since the derivatives of the resolvent $(t + A)^{-1}$ have the same form as if A were just a number, we can also apply the above formula to $f(t) = A^\nu (t + A)^{-1}$, to get

$$D^\alpha A^\nu (|\xi|^2 + A)^{-1} = \sum_{j \le |\alpha|} (-1)^j j! A^\nu (|\xi|^2 + A)^{-1-j} \sum_{\#L = 2j - |\alpha|} c(\alpha, L) \prod_{\ell \in L} \xi_{\alpha(\ell)}.$$

To show that the set $\{|\xi|^{|\alpha|} D^\alpha m_\nu(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}\}$ is (R-)bounded, we need to consider (R-)bounds for

$$\begin{aligned} |\xi|^{|\alpha|} A^\nu (|\xi|^2 + A)^{-1-j} |\xi|^{\#L} \prod_{\ell \in L} \frac{\xi_{\alpha(\ell)}}{|\xi|} \\ = \left(A^\nu (|\xi|^2 + A)^{-1}\right) \left(|\xi|^2 (|\xi|^2 + A)^{-1}\right)^j \prod_{\ell \in L} \frac{\xi_{\alpha(\ell)}}{|\xi|}, \end{aligned}$$

where we used the equality $\#L + |\alpha| = 2j$ from (6.25) in the first step. But here the first factor is simply $m_\nu(\xi)$ while the second is $(1 - m_1(\xi))^j$ which were assumed (R-)bounded in the assumptions. The last factor is a scalar quantity of norm at most unity, so it is simply estimated by 1 in the norm-boundedness case, and dealt with by KAHANE's contraction principle in the R-boundedness case. $\square$

REMARK 6.26. Assuming m_1 (R-)bounded, m_0 will be (R-)bounded if and only if A is boundedly invertible.

Now we are going to formulate a result for the (strong) maximal regularity of (1.9). The definition of strong maximal regularity for the present problem is an obvious modification of the corresponding definition for the fractional-order equation (1.8), Def. 6.9. The statement of the theorem follows the same pattern as our previous maximal regularity results, Theorems 1.6 and 6.20. However, we do not here have an L^p-theorem as a starting point as we did above, so that we will have a chance to illustrate that the methods developed in this chapter can be used to show all the required implications. Nevertheless, since our main concern is $p \leq 1$, we are going to treat $p > 1$ only for those p which are Fourier-types for X.

THEOREM 6.27. *Let A be an invertible sectorial operator on a UMD-space X. Then the following are equivalent:*

(L_1) (1.9) *has strong maximal regularity on $L^p(\mathbb{R}^n; X)$ whenever $p > 1$ is a Fourier-type for X.*
(L_2) (1.9) *has strong maximal regularity on $H^1(\mathbb{R}^n; X)$.*
(L_3) (1.9) *has strong regularity from $H^1(\mathbb{R}^n; X)$ to $L^1(\mathbb{R}^n; X)$.*
(L_4) $\{A(t + A)^{-1} | \ t \geq 0\}$ *is R-bounded.*

Moreover, these imply

(L_5) *For all $p \in]0, 1[$, $q > 1$ a Fourier-type for X and $f \in H^p \cap L^q(\mathbb{R}^n; X)$ or $f \in H^p \cap H^1(\mathbb{R}^n; X)$, the unique solution u of* (1.9)*, together with $\triangle u$ and Au, also belong to this same space, and moreover*

$$\|\triangle u\|_{H^p(\mathbb{R}^n;X)} + \|u\|_{H^p(\mathbb{R}^n;X)} + \|Au\|_{H^p(\mathbb{R}^n;X)} \leq C \|f\|_{H^p(\mathbb{R}^n;X)}$$

PROOF. The proof can be modelled after that of Theorem 6.20. The only exception is the implication $L_4 \Rightarrow L_1$, which in the case of Theorem 6.20 was a direct application of Theorem 6.7 of CLÉMENT and PRÜSS. Now we argue as follows:

It was illustrated above that (1.9) has a solution u whenever $f \in \mathcal{S}(\mathbb{R}^n; X)$. From the assumptions of the theorem and Lemma 6.24 it follows that this u satisfies the estimate $\|\triangle u\|_{L^p(\mathbb{R}^n;X)} + \|u\|_{L^p(\mathbb{R}^n;X)} + \|Au\|_{L^p(\mathbb{R}^n;X)} \leq C \|u\|_{L^p(\mathbb{R}^n;X)}$. For a general $f \in L^p(\mathbb{R}^n; X)$, we consider a sequence $f_j \in \mathcal{S}(\mathbb{R}^n; X)$ which tends to f in $L^p(\mathbb{R}^n; X)$. Then the solution u of (1.9) is obtained as the limit of the Cauchy sequence $(u_j)_{j=1}^{\infty}$, just like in the proof of Theorem 6.20. This existence argument did not use the Fourier-type in any way. However, we show the uniqueness of the solution u by taking the Fourier transform, which shows that $\hat{u}(\xi) = (4\pi^2 |\xi|^2 + A)^{-1} \hat{f}(\xi)$ whenever $u, Au, \triangle u, f \in L^p(\mathbb{R}^n; X)$, and u is a solution of (1.9). The formal argument becomes precise when we have the Fourier-type p to assure that the L^p-functions above are boundedly mapped into $L^{p'}$-functions by the Fourier transform, and using Lemma 6.12 together with $\mathcal{F}[\triangle u](\xi) = -4\pi^2 |\xi|^2 \hat{u}(\xi)$. □

7. Final remarks

We mention two directions of simple extensions of the present theory:

REMARK 7.1. With the help of the vector-valued H^1-BMO-duality, one can also obtain boundedness results from $L^\infty(\mathbb{R}^n;X)$, or even from $\mathrm{BMO}(\mathbb{R}^n;X)$, to $\mathrm{BMO}(\mathbb{R}^n;Y)$. However, since the duality arguments involved are rather straightforward and quite standard, it seems appropriate to leave them as exercises for the reader. For more information concerning the H^1-BMO-duality in the vector-valued setting, we refer to BLASCO [**8**] (cf. also BOURGAIN [**12**]). An L^∞-BMO multiplier theorem (using duality as well) was also recently proved by GIRARDI and WEIS [**36**], Cor. 4.6.

REMARK 7.2. One can also consider the maximal regularity problem, as well as Fourier multipliers in general, in the periodic situation of the unit-circle $\mathbb{T}$ instead of $\mathbb{R}$. A good reference for the periodic L^p-case is ARENDT and BU [**2**]. In analogy with Prop. 4.7, one can show the necessity of R-boundedness of $\{m_k|\ k\in\mathbb{Z}\setminus\{0\}\}$ for $(m_k)_{k\in\mathbb{Z}}\subset\mathcal{L}(X,Y)^{\mathbb{Z}}$ to be a Fourier multiplier from $\tilde{H}^1(\mathbb{T};X)$ to $L^1(\mathbb{T};Y)$; in fact, it is obtained from the L^p-proof in [**2**] by a similar modification that yielded the proof of Prop. 4.7 from its L^p-version (Theorem 4.1 of CLÉMENT and PRÜSS).

8. Appendix: Proofs of two lemmata on Hardy spaces

PROOF OF LEMMA 2.3. Let $\mathcal{S}'\text{-}\sum_{k=0}^\infty\lambda_k a_k$ be any atomic decomposition of the distribution $f\in H^p(\mathbb{R};X)$, where $\operatorname{supp} f\subset\bar{\mathbb{R}}_+$, and let N be the required number of vanishing moments for these atoms. Let $a_k^+ := a_k\chi_{\bar{\mathbb{R}}_+}$, $a_k^- := a_k\chi_{\mathbb{R}_-}$.

Consider the generalized reflections

$$\tilde{a}_k^- := \sum_{j=0}^N b_j a_k^-(-c_j\cdot),\qquad b_j\in\mathbb{R},\ c_j>0.$$

We have

$$\int_0^\infty t^\alpha\tilde{a}_k^-(t)\,\mathrm{d}t=\sum_{j=0}^N b_j\int_0^\infty t^\alpha a_k^-(-c_jt)\,\mathrm{d}t=\sum_{j=0}^N b_j(-1)^\alpha c_j^{-(\alpha+1)}\int_{-\infty}^0 t^\alpha a_k^-(t)\,\mathrm{d}t.$$

We require the conditions $\sum_{j=0}^N b_j(-1)^\alpha c_j^{-(\alpha+1)}=1$, i.e., $\sum_{j=0}^N b_jc_j^{-(\alpha+1)}=(-1)^\alpha$, for $\alpha=0,1,\dots,N$, in which case $\tilde{a}_k^-$ has the same moments, up to the Nth, as a_k^-. For a fixed choice of the c_j, $j=0,1,\dots,N$, this is a linear system of $N+1$ equations and $N+1$ unkowns (the b_j's), and it can be solved for the b_j whenever the positive quantities c_j are all distinct [since then the matrix to be inverted is essentially that occuring in the uniquely solvable problem of polynomial fit].

When the reflection coefficients are chosen as above, we have

$$\int t^\alpha(a_k^+(t)+\tilde{a}_k^-(t))\,\mathrm{d}t=\int t^\alpha a_k^+(t)\,\mathrm{d}t+\int t^\alpha a_k^-(t)\,\mathrm{d}t=\int t^\alpha a_k(t)\,\mathrm{d}t=0$$
$$\text{for }\alpha=0,1,\dots,N.$$

This shows that $A_k := a_k^+ + \tilde{a}_k^-$, which is supported on $\bar{\mathbb{R}}_+$, has the same vanishing moments as a_k.

Concerning the atomic size condition of the A_k's, it is clear that A_k is supported on a ball $\bar{B}^k$, whose size is at most $c\left|\bar{B}_k\right|$, where $\bar{B}_k$ is the smallest ball containing $\operatorname{supp} a_k$ and c a constant depending on the choice of the reflection coefficients c_j. Moreover, we have

$$\begin{aligned}\|A_k\|_{L^q} &\leq \left\|a_k^+\right\|_{L^q} + \sum_{j=0}^{N} |b_j| \left\|a_k^-(-c_j\cdot)\right\|_{L^q} = \left\|a_k^+\right\|_{L^q} + \sum_{j=0}^{N} |b_j|\, c_j^{-1/q} \left\|a_k^-\right\|_{L^q} \\ &\leq \left(1 + \sum_{j=0}^{N} \frac{|b_j|}{c_j^{1/q}}\right) \|a_k\|_{L^q} =: C\, \|a_k\|_{L^q}\,.\end{aligned}$$

It follows that $\tilde{f} := \mathcal{S}'\text{-}\sum_{k=0}^{\infty} \lambda_k A_k$ defines an element of $H^p(\mathbb{R};X)$, whose p-norm satisfies $\|\tilde{f}\|_{H^p(\mathbb{R};X)} \leq \tilde{C}\sum_{k=0}^{\infty} |\lambda_k|^p$.

Since $\operatorname{supp} f \subset \bar{\mathbb{R}}_+$, we have (by definition) $\langle f,\psi\rangle = 0$ whenever $\psi \in \mathcal{S}(\mathbb{R})$ is supported on $\mathbb{R}_-$. This means, again by definition, that $\sum_{k=0}^{\infty} \lambda_k \int a_k^-(t)\psi(t)\,\mathrm{d}t = 0$ for such ψ, and by reflection, it follows that $\sum_{k=0}^{\infty} \lambda_k \int \tilde{a}_k^-(t)\psi(t)\,\mathrm{d}t = 0$ for $\psi \in \mathcal{S}(\mathbb{R})$ with support on $\mathbb{R}_+$. From the definition of $\tilde{f}$ it is clear that $\left\langle \tilde{f},\psi\right\rangle = 0$ for $\operatorname{supp}\psi \in \mathbb{R}_-$. Combining these facts we see that $\left\langle \tilde{f},\psi\right\rangle = \langle f,\psi\rangle$ whenever $\psi \in \mathcal{S}(\mathbb{R})$ is supported away from the origin, which means that $\operatorname{supp}(\tilde{f}-f) \subset \{0\}$. However, we also have $\tilde{f} - f \in H^p(\mathbb{R};X)$, and this cannot hold for a distribution supported only at the origin unless the distribution vanishes. This can be seen as follows: Denoting $g := \tilde{f} - f$, just note that, for $x' \in X'$, we have $x'g \in H^p(\mathbb{R})$ and obviously $\operatorname{supp} x'g \subset \{0\}$, where $\langle x'g,\phi\rangle := x'(\langle g,\phi\rangle)$. Thus we only need to know that a scalar-valued distribution $x'g \in H^p(\mathbb{R}^n)$ with one-point support must vanish (for which fact, see e.g. [**77**], Ch. III, §5.5(c)) in order to conclude that $x'(\langle g,\phi\rangle) = \langle x'g,\phi\rangle = 0$ for all $\phi \in \mathcal{S}(\mathbb{R})$ and $x' \in X'$; thus $\langle g,\phi\rangle = 0$ for all $\phi \in \mathcal{S}(\mathbb{R})$, and hence g vanishes as an element of $\mathcal{S}'(\mathbb{R};X)$.

We have shown that $f = \tilde{f}$ has a decomposition of the asserted form, and this completes the proof. □

Proof of Lemma 2.4. Let us first consider $\mathbb{R}^n$, and then indicate the appropriate modifications for $\bar{\mathbb{R}}_+$ in the end.

It follows from the atomic definition of the H^p norm that finite linear combinations of atoms are dense in $H^p(\mathbb{R}^n;X)$. The H^p norm of an atom can be controlled (from above) if its L^q norm can be controlled, preserving the appropriate moment and support conditions. Simple functions being dense in $L^q(\bar{B};X)$ for $q<\infty$, given a (p,q)-atom a supported on the ball $\bar{B}$, we can find a simple function $s = \sum x_k \chi_{E_k}$ with $E_k \subset \bar{B}$ measurable and $\|s-a\|_{L^q} < \epsilon$. Clearly, if we replace the x_k by $z_k \in Z$ taken sufficiently close to the respective x_k, we get a new simple function, still denoted by s, which approximates a as closely as desired in the L^q norm and belongs to $Z \otimes L^q(\bar{B})$.

For $g \in L^q(\bar{B}; X)$, let Pg denote the unique polynomial (with X-coefficients) of degree at most N and satisfying

$$\int_{\bar{B}} (g(t) - Pg(t)) t^\alpha \,\mathrm{d}t = 0 \qquad \text{for } |\alpha| \le N.$$

It is easy to see that P is a bounded operator on $L^q(\bar{B}; X)$, and moreover, it maps $Z \otimes L^q(\bar{B})$ to itself.

Now $s - Ps$ is an appropriate approximation of a: It is supported on the same ball $\bar{B}$, it has the appropriate number of vanishing moments by the very definition of P provided N is chosen large enough, and finally

$$\begin{aligned} \|(s - Ps) - a\|_{L^q(\bar{B};X)} &\le \|s - a\|_{L^q} + \|Ps - Pa\|_{L^q} \\ &\le (1 + \|P\|_{\mathcal{L}(L^q(\bar{B};X))}) \|s - a\|_{L^q(\bar{B};X)}, \end{aligned}$$

where the first estimate exploits the fact that $Pa = 0$, since a already has the appropriate vanishing moments. Since s can be chosen as close to a as desired, the same will be true of $b := s - Ps$.

Replacing each of the finite number of atoms a_i in the truncated atomic series of a given $f \in H^p(\mathbb{R}^n; X)$ by the corresponding b_i constructed as above, we can estimate f as closely as desired by a finite sum

$$\tilde{f} = \sum_{i=1}^{M} \lambda_i b_i = \sum_{i=1}^{M} \lambda_i \sum_{j=1}^{m_i} z_{i,j} f_{i,j}, \qquad z_{i,j} \in Z,\ f_{i,j} \in L^q_c(\mathbb{R}^n).$$

Since there are only finitely many of the $z_{i,j}$, all of them belong to some finite-dimensional subspace E of Z, for which we can find a basis $e_1, \ldots, e_m$. Expressing each $z_{i,j}$ as a linear combination of the e_k, the sum above gets the form $\tilde{f} = \sum_{k=1}^m e_k f_k$, where the f_k are compactly supported scalar L^q functions. Integrating this equality multiplied by t^α and using the linear independence of the e_k, we find that the f_k have (at least) the same vanishing moments as $\tilde{f}$. A compactly supported L^q function with appropriate moment conditions is clearly an atom, up to scaling, thus in particular an element of $H^p(\mathbb{R}^n)$.

Thus, so far an arbitrary $f \in H^p(\mathbb{R}^n; X)$ has been approximated with any desired precision by $\sum_{k=1}^m e_k f_k$, with $e_k \in Z$ and $f_k \in H^p(\mathbb{R}^n)$. It is clear that if the f_k are now replaced by suitable g_k in the dense subspace G of $H^p(\mathbb{R}^n)$, we can retain arbitrarily good approximation, and clearly $\sum_{k=1}^m e_k g_k \in Z \otimes G$, as desired.

To see the density of $\mathcal{D}(\mathbb{R}^n) \cap H^p(\mathbb{R}^n)$ in $H^p(\mathbb{R}^n)$, it suffices to convolute a truncated atomic series by a smooth, compactly supported approximation of the identity, observing that the amount in which this disturbes the supports of the atoms can be made as small as desired, that the vanishing moments are not disturbed by convolutions at all, and that approximation in L^q norm is reached by definition. The fact that $\hat{\mathfrak{D}}_0(\mathbb{R}^n)$ is a dense subspace of $H^p(\mathbb{R}^n)$ can be found in [**81**].

The case of the half-line $\bar{\mathbb{R}}_+$. This is essentially the same as that of the whole space, as we can use Lemma 2.3 to give us an atomic decomposition, also supported on $\bar{\mathbb{R}}_+$. Since the series converges to the given f in H^p, truncations of the series yield arbitrarily good approximations. Moreover, translation $g \mapsto g(\cdot - h)$ is strongly continuous on L^q, so that we can shift the finite number of atoms in the truncated series slightly to the right, preserving a good approximation and ensuring that the new truncated series is supported strictly right from the origin. The replacement of the atoms $a_k \in L^q(\bar{B}_k; X)$ by $b_k \in Z \otimes L^q(\bar{B}_k)$ can be done with the same algorithm as above, so as to yield an approximation $\sum_{k=1}^m e_k f_k$, with f_k now L^q functions compactly supported on $]0, \infty[$, and they can be further replaced by g_k in the desired dense class G of $H^p(\bar{\mathbb{R}}_+)$.

To see that $\mathcal{C}_c^\infty(\mathbb{R}_+) \cap H^p(\bar{\mathbb{R}}_+)$ is dense in $H^p(\bar{\mathbb{R}}_+)$, we can argue as in the case of $\mathbb{R}^n$ but now starting from the shifted truncated atomic series and noticing that if f, compactly supported in $]0, \infty[$, is convoluted with an approximation of the identity $\phi_\epsilon := \epsilon^{-1}\phi(\epsilon^{-1}\cdot)$ with compact support, then also $\phi_\epsilon * f$ is supported on $]0, \infty[$ as soon as ϵ is small enough.

Now all the assertions are verified. □

9. Appendix: Proof of $A\mathcal{U}(f_\lambda x) = \mathcal{U}(f_\lambda Ax)$

It was noted in the proof of Theorem 3.1 that the crucial step in proving that R_λ is a left inverse of $\lambda + A$ is to show that

$$\mathcal{U}(f_\lambda(\cdot)Ax) = A\mathcal{U}(f_\lambda(\cdot)x) \text{ whenever } x \in \mathcal{D}(A). \tag{9.1}$$

Indeed, once this is shown, it follows (for $x \in \mathcal{D}(A)$) that

$$\begin{aligned} R_\lambda(\lambda + A)x &= \operatorname{Re}\lambda \int_0^\infty e^{-\lambda t}\mathcal{U}(\lambda f_\lambda(\cdot)x + f_\lambda(\cdot)Ax)(t)\,\mathrm{d}t \\ &= \operatorname{Re}\lambda \int_0^\infty e^{-\lambda t}(\lambda + A)\mathcal{U}(f_\lambda(\cdot)x)\,\mathrm{d}t = (\lambda + A)R_\lambda x = x, \end{aligned}$$

since we already knew that R_λ is a right inverse.

We then turn to the proof of the equation (9.1). It is established in a sequence of three lemmata. The idea of the proof is the same as that of Lemma 2.3 in G. Dore [**28**]; however, that result is not directly applicable to the present purposes since the situation now only involves weak (H^1, L^1) regularity instead of *maximal* regularity.

Lemma 9.2. *Let* $\operatorname{Re}\mu > 0$, $f \in H^1(\bar{\mathbb{R}}_+; X)$ *and*

$$g(t) = (\mu + D)^{-1}f(t) = \int_0^t e^{-\mu s} f(t - s)\,\mathrm{d}s.$$

Then $g \in H^1(\bar{\mathbb{R}}_+;X)$ *and moreover* $\|g\|_{H^1(\bar{\mathbb{R}}_+;X)} \le C\,\|f\|_{H^1(\bar{\mathbb{R}}_+;X)}$ *with* $C < \infty$ *independent of* f.

PROOF. The convolution kernel $k(s) = e^{-\mu s}\chi_{\mathbb{R}_+}$ is integrable, so that the estimate $\|k * f\|_{L^p(\bar{\mathbb{R}}_+;X)} \le C\,\|f\|_{L^p(\bar{\mathbb{R}}_+;X)}$ holds for all $p \in [1,\infty]$. Moreover, the kernel satisfies the bounds $|D^\nu k(s)| = |\mu|^\nu e^{-\operatorname{Re}\mu s} \le C_\nu(\mu)\,|s|^{-1-\nu}$ (a very crude estimate!) for all $\nu \in \mathbb{N}$, so that Theorem 5.6 of Chapter 1 guarantees the boundedness of $k*$ acting on $H^1(\bar{\mathbb{R}}_+;X)$. (Since the convolution by $e^{-\mu s}\chi_{\mathbb{R}_+}$ is a translation-invariant operator, it makes no difference in norm of this operator whether we take it to act on function spaces on $\bar{\mathbb{R}}_+$ or on the whole line $\mathbb{R}$.) $\square$

LEMMA 9.3. *Let the assumptions of Theorem* 3.1 *be satisfied, and let* $\mathcal{U}$ *be the solution operator* (*as in the proof of that theorem*). *That is, we have* $\mathcal{U}f = u$ *if and only if*

- $f \in H^1(\bar{\mathbb{R}}_+;X)$,
- $u \in W^{1,1}_{\mathrm{loc}}(\bar{\mathbb{R}}_+;X)$, $u(t) \in \mathcal{D}(A)$ *for a.e.* $t \in \mathbb{R}_+$ *and* $\dot{u}, Au \in L^1(\bar{\mathbb{R}}_+;X)$.
- $\dot{u}(t) + Au(t) = f(t)$ *for a.e.* $t \in \mathbb{R}_+$ *and* $u(0) = 0$.

Then $(\mu + D)^{-1}\mathcal{U}f = \mathcal{U}(\mu + D)^{-1}f$ *for every* $\operatorname{Re}\mu > 0$ *and* $f \in H^1(\bar{\mathbb{R}}_+;X)$.

PROOF. Note first that, by Lemma 9.2, $(\mu + D)^{-1}f \in H^1(\bar{\mathbb{R}}_+;X)$, so that it makes sense to apply the solution operator $\mathcal{U}$ to this quantity.

Denote $u := \mathcal{U}f$, and $v(t) := (\mu+D)^{-1}u(t) = \int_0^t e^{-\mu s}u(t-s)\,\mathrm{d}s$. Then clearly $v(0) = 0$; moreover, since u is a solution of the ACP, we can evaluate

$$\dot{v}(t) = \int_0^t e^{-\mu s}\dot{u}(t-s)\,\mathrm{d}s \text{ (using } u(0)=0) \quad \text{and} \quad Av(t) = \int_0^t e^{-\mu s}Au(t-s)\,\mathrm{d}s$$

for a.e. $t \in \bar{\mathbb{R}}_+$. Since $\dot{u}, Au \in L^1(\bar{\mathbb{R}}_+;X)$, it follows from the above formulae that also $\dot{v}, Av \in L^1(\bar{\mathbb{R}}_+;X)$. Finally, we can add the two formulae above and use the fact that $u = \mathcal{U}f$ to get $\dot{v}(t) + Av(t) = (\mu+D)^{-1}f(t)$, and thus $v = \mathcal{U}(\mu+D)^{-1}f$. Since $v = (\mu + D)^{-1}\mathcal{U}f$, as well, the assertion is established. $\square$

LEMMA 9.4. *Let* $x \in \mathcal{D}(A)$ *and* $f \in H^1(\bar{\mathbb{R}}_+)$. *Then* $\mathcal{U}[f(\cdot)Ax] = A\mathcal{U}[f(\cdot)x]$.

PROOF. Manipulating the left side of the asserted equality, we get

$$\begin{aligned}
\mathcal{U}[f(\cdot)Ax] &= (\mu + D)(\mu + D)^{-1}\mathcal{U}[f(\cdot)Ax] \\
&= (\mu + D)\mathcal{U}(\mu + D)^{-1}f(\cdot)Ax \qquad \text{by Lemma 9.3} \\
&= (\mu + D)\mathcal{U}A(\mu + D)^{-1}f(\cdot)x \\
&= (\mu + D)\mathcal{U}((D + A) - (\mu + D) + \mu)(\mu + D)^{-1}f(\cdot)x.
\end{aligned}$$

Now note that $F(\cdot) := (D + A)(\mu + D)^{-1}f(\cdot)x = f(\cdot)x - \mu(\mu + D)^{-1}f(\cdot)x + (\mu + D)^{-1}f(\cdot)Ax \in H^1(\bar{\mathbb{R}}_+;X)$ (since all the summands are in this space). Now

$(\mu + D)^{-1} f(\cdot)x$ satisfies the ACP with data F by definition, and thus $\mathcal{U}F(\cdot) = (\mu + D)^{-1} f(\cdot)x$. Using this, we can continue our chain of equalities:

$$
\begin{aligned}
&= (\mu + D)(\mu + D)^{-1} f(\cdot)x - (\mu + D)\mathcal{U} f(\cdot)x + \mu(\mu + D)\mathcal{U}(\mu + D)^{-1} f(\cdot)x \\
&= f(\cdot)x - (\mu + D)\mathcal{U}[f(\cdot)x] + \mu\mathcal{U}[f(\cdot)x] \qquad \text{by Lemma 9.3} \\
&= f(\cdot)x - D\mathcal{U}[f(\cdot)x] \\
&= A\mathcal{U}[f(\cdot)x] \qquad \text{by the definition of a solution.}
\end{aligned}
$$

This completes the proof. $\square$

CHAPTER 2

Operator-valued singular integrals on UMD Bôchner spaces

We study operators $f \mapsto Kf$ of the form

$$(Kf)(t) = \int_{\mathbb{R}^n} k(t-s) f(s)\, \mathrm{d}s,$$

where f is a vector-valued function and k an operator-valued singular kernel. Sufficient conditions for boundedness on L^p-spaces of UMD-valued functions are given in terms of a Hörmander-type condition involving R-boundedness. The results cover large classes of kernels and also provide new proofs of some recent operator-valued Fourier multiplier theorems. Moreover, they give new information about families of singular integral operators.

This chapter is based on the joint paper [**47**] with L. Weis.

1. Introduction

Singular integrals have been the object of extensive study since the pioneering work of A. P. Calderón and A. Zygmund [**16**] in the 50's. Their results showed that large classes of singular integral operators are automatically bounded on the whole scale of the reflexive $L^p(\mathbb{R}^n)$ spaces (i.e., $p \in]1, \infty[$) as soon as they are bounded on $L^2(\mathbb{R}^n)$ and the kernels satisfy certain conditions which hold and can be verified for many operators appearing in applications. Moreover, the required L^2-boundedness is obtained for free (and therefore goes often almost without being mentioned) with the help of the Fourier transform and Plancherel's theorem.

The first results of Calderón and Zygmund concerning convolutions by homogeneous kernels $k(t) = \Omega(t^0)/|t|^n$, $t^0 := t/|t|$, have been generalized in several directions by the same authors and many others, and useful sufficient conditions for L^p-boundedness are now known both in terms of the kernel k (as in the original results) and in terms of the multiplier or the symbol $m = \hat{k}$ (Fourier transform in the sense of distributions). A classical theorem giving sufficient conditions in terms of the multiplier is due to S. G. Mihlin, and a variant was later proved by L. Hörmander as a corollary of his results on singular integrals [**43**]. In this connection Hörmander gave the world the condition bearing his name, today usually formulated as

$$\int_{|t|>2|s|} |k(t-s) - k(t)| \,\mathrm{d}t \leq A < \infty, \tag{1.1}$$

and being a sufficient condition on k to boundedly extend the operator $f \mapsto k * f$, bounded on $L^{\tilde{p}}(\mathbb{R}^n)$ for one $\tilde{p} \in]1,\infty[$, to the whole scale of the spaces $L^p(\mathbb{R}^n)$, $p \in]1,\infty[$.

The question of whether these results could be extended to the Lebesgue–Bôchner spaces $L^p(\mathbb{R}^n;X)$ of vector-valued functions was taken up by several authors already in the 60's. It was observed by A. Benedek, A. P. Calderón and R. Panzone [**5**] that the boundedness on $L^{\tilde{p}}(\mathbb{R}^n;X)$ for one $\tilde{p} \in]1,\infty[$ of a convolution operator, together with Hörmander's condition, implies the boundedness on $L^p(\mathbb{R}^n;X)$ for all $p \in]1,\infty[$ also in the general situation of vector-valued functions and an operator-valued kernel. However, to actually get the boundedness, without *a priori* assumptions, even for the single $\tilde{p}$ (something that was immediate for $\tilde{p} = 2$ in the scalar-valued, or more generally, a Hilbert space setting) turned out to be a significantly more difficult task.

By the 80's it was understood that the boundedness of vector-valued singular integrals, in particular, the prototype example given by the Hilbert transform, is intimately connected with the geometry of Banach spaces. Indeed, it was shown by D. L. Burkholder and J. Bourgain that the boundedness of the Hilbert transform on $L^p(\mathbb{T};X)$, $p \in]1,\infty[$, is equivalent to the so called UMD-property of the underlying Banach space X. Moreover, the boundedness of this one operator could already be used to show the boundedness of large classes of multipliers. In particular, the classical multiplier theorem of Mihlin was generalized (by F. Zimmermann [**89**] to the full generality on $\mathbb{R}^n$, based on the deep results of Bourgain in the one-dimensional case) to the setting of scalar-valued multipliers acting on UMD-valued functions.

However, the general situation of operator-valued kernels or multipliers, which is of interest in the theory of evolution equations, remained open until the turn of the millennium. As the naïve generalization of the classical Mihlin condition by means of replacing absolute values by norms was found, by G. Pisier (unpublished), to imply the desired boundedness only in the Hilbert space setting, a new idea was required to build a condition strong enough to get the desired conclusion but reasonable enough to cover a wide range of relevant applications. This idea turned out to be the notion of R-boundedness, already implicit in the work of Bourgain and later Zimmermann and explicitly formulated by Ph. Clément, B. de Pagter, F. A. Sukochev and H. Witvliet [**21**] and by L. Weis [**87**] who first generalized the Mihlin theorem to allow for operator-valued multipliers but requiring R-boundedness instead of norm boundedness in reformulating Mihlin's conditions. Clément and J. Prüss [**22**] showed that the R-boundedness of the range of the multiplier is also necessary.

The realization of R-boundedness as the right notion for operator-valued multiplier theorems has spurred significant activity in the field, leading to several generalizations and improvements of the first results in this direction, as well as to applications in differential equations (see [**25**], [**53**] for a survey). In the present chapter, we make use of these ideas to attack the operator-valued versions of the problems originally treated by CALDERÓN and ZYGMUND, i.e., to search for conditions on the operator-valued singular kernel k to yield a bounded operator $f \mapsto k * f$ from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$.

In the scalar-valued context it follows from PLANCHEREL's theorem that $k*$ is bounded on $L^2(\mathbb{R}^n)$ if and only if $\hat{k}$ is bounded, and in the general situation we know from CLÉMENT and J. PRÜSS [**22**] that the range of $\hat{k}$ must even be R-bounded. Thus it is natural to impose the condition

$$\mathcal{R}(\{\hat{k}(\xi) |\ \xi \in \mathbb{R}^n\}) \leq A < \infty, \tag{1.2}$$

where $\mathcal{R}(\mathcal{T})$ denotes the R-bound (cf. Def. 3.2) of the set $\mathcal{T}$.

In the context of multiplier theorems, appropriate additional conditions are obtained by imposing Mihlin-type bounds, but replaced by R-bounds, for the derivatives of $\hat{k}$ (see [**2, 36, 40, 80, 87**]). However, we now search for conditions directly on the convolution kernel k, and it will be shown (Theorem 4.1) that sufficient conditions are obtained by incorporating the notion of R-boundedness into the classical Hörmander conditions so as to require that

$$\int_{|t|>2|s|} \mathcal{R}(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t)) |\ j \in \mathbb{Z}\}) \log(2+|t|)\,\mathrm{d}t \leq A \log(2+|s|), \tag{1.3}$$

Besides the R-bound, the new feature compared to the classical situation is the additional logarithmic factor, which arises from the use of a deep result of BOURGAIN concerning UMD-spaces. Nevertheless, this condition is still satisfied by large classes of singular kernels (cf. Theorem 5.12), and it also gives new information about collective properties (the R-boundedness) of families of singular integral operators (Theorem 6.4).

Besides being of interest in their own right, the results for the convolution operators can also be used to derive some recent operator-valued multiplier theorems (e.g. from [**36**], cf. Theorem 7.9). This is not surprising in view of the historical fact that HÖRMANDER used his results on singular integrals to derive a variant of the theorem of MIHLIN on Fourier multipliers. As a general remark, which will be given more quantitative content in the body of the chapter, it seems that the understanding of the multipliers and convolution integrals greatly benefits from the interaction of the two different points of view. As an additional illustration of its usefulness, we give an alternative proof of the characterization (from [**87**]) of maximal regularity in terms of R-boundedness (Example 5.15).

2. A framework for vector-valued singular integrals

In this section we set up a convenient framework for vector-valued singular integrals of the form

$$Kf(t) = \int_{\mathbb{R}^n} k(t-s)f(s)\,\mathrm{d}s, \qquad t \in \mathbb{R}^n, \tag{2.1}$$

which will allow us to use the basic tools of harmonic analysis.

In the scalar case it is customary to assume that k is a tempered distribution which agrees on $\mathbb{R}^n \setminus \{0\}$ with a locally integrable function. For $\phi \in \mathcal{S}(\mathbb{R}^n)$, one interprets (2.1) as

$$K\phi(t) := (k * \phi)(t) = \langle k, \phi(t-\cdot)\rangle, \qquad t \in \mathbb{R}^n.$$

If one can prove an L^p-estimate $\|K\phi\|_{L^p} \le C\|\phi\|_{L^p}$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$, then by the density of $\mathcal{S}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, the operator K can be extended to a bounded operator on $L^p(\mathbb{R}^n)$, and we can think of this operator as formally given by (2.1).

In this chapter, we are typically interested in the case where k is an operator-valued function, say $t \in \mathbb{R}^n \setminus \{0\} \mapsto k(t) \in \mathcal{L}(X,Y)$, and f is in $L^p(\mathbb{R}^n;X)$. To give a meaning to (2.1), we therefore assume that k is an operator-valued distribution in

$$\mathcal{S}'(\mathbb{R}^n;\mathcal{L}(X;Y)) := \mathcal{L}(\mathcal{S}(\mathbb{R}^n);\mathcal{L}(X;Y)). \tag{2.2}$$

But to avoid annoying technicalities about the convolutions of vector-valued distributions, we choose a special class of test-functions, namely $X \otimes \mathcal{S}(\mathbb{R}^n)$: for $x \in X$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ we define a linear functional $x \otimes \phi$ on $\mathcal{S}'(\mathbb{R}^n;\mathcal{L}(X,Y))$ by

$$[x \otimes \phi](k) := \langle k, \phi\rangle\, x,$$

and extend this definition by linearity from $X \times \mathcal{S}(\mathbb{R}^n)$ to the algebraic tensor product $X \otimes \mathcal{S}(\mathbb{R}^n)$. In particular, for $f = x \otimes \phi$, we can now interpret (2.1) as $\langle k, \phi(t-\cdot)\rangle\, x$, which we may also write as $k * \phi(t)x$ or $k * x\phi(t)$ or even $k(\cdot)x * \phi(t)$, whichever seems convenient in a particular context. Recall that the convolution $k * \phi(t) := \langle k, \phi(t-\cdot)\rangle$ of a tempered distribution k with a Schwartz function $\phi \in \mathcal{S}(\mathbb{R}^n)$ is an infinitely differentiable function with polynomially bounded derivatives of all orders; the vector-valued situation does not bring any complications at this point, and one can simply repeat the standard proofs from the scalar-valued theory.

Note that $X \otimes \mathcal{S}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n;X)$ for $1 < p < \infty$, so that the class $X \otimes \mathcal{S}(\mathbb{R}^n)$ is sufficient to prove the boundedness of the operator K in (2.1) from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n;Y)$. For that matter, it will be enough to consider the even smaller class $X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n)$, where

$$\hat{\mathcal{D}}_0(\mathbb{R}^n) := \left\{ \psi \in \mathcal{S}(\mathbb{R}^n) \,\middle|\, \hat{\psi} \in \mathcal{D}(\mathbb{R}^n),\ 0 \notin \operatorname{supp}\hat{\psi} \right\}.$$

This leads us to the following basic assumption for the kernel of a singular integral operator as in (2.1):

ASSUMPTION 2.3. For every $x \in X$, the distribution $k(\cdot)x \in \mathcal{S}'(\mathbb{R}^n; Y)$ (defined by $\langle k(\cdot)x, \phi\rangle := \langle k, \phi\rangle\, x$) agrees away from the origin with a locally integrable Y-valued function, which we denote by the same symbol $k(\cdot)x$. That is, we have

$$\langle k(\cdot)x, \phi\rangle = \int_{\mathbb{R}^n} k(t)x\,\phi(t)\,\mathrm{d}t \qquad \text{for } \phi \in \mathcal{S}(\mathbb{R}^n),\ 0 \notin \operatorname{supp}\phi.$$

REMARK 2.4. (i) By the definition of the convolution, and linearity, this gives

$$k * f(t) = \int_{\mathbb{R}^n} k(s)f(t-s)\,\mathrm{d}s \qquad \text{for } f \in X \otimes \mathcal{S}(\mathbb{R}^n),\ t \notin \operatorname{supp} f.$$

It is easy to see that, for $t \notin \operatorname{supp} f$, the representation $\sum x_j \otimes \phi_j$ of $f \in X \otimes \mathcal{S}(\mathbb{R}^n)$ can be chosen in such a way that $t \notin \operatorname{supp}\phi_j$ for any j.

(ii) It might seem a little technical to assume the integrability of the point-evaluations $k(\cdot)x$ only, rather than $k(\cdot)$ itself as an operator-valued function. However, this is essential to include many of the natural examples of operator-valued kernels. For instance, the function $t \in \mathbb{R} \mapsto \tau_t \in \mathcal{L}(L^p(\mathbb{R}))$, where $\tau_t f(x) := f(x-t)$ and $p \in [1, \infty[$, is strongly continuous and hence strongly integrable, but it is not Bôchner integrable as an operator-valued function. To see this, recall ([**26**]) that the differentiation theorem of LEBESGUE is also true for the Bôchner integral.

However, we have

$$\left(\frac{1}{2h}\int_0^{2h} \tau_t\chi_{[0,h]}(x)\,\mathrm{d}t\right)(x) = \frac{1}{2h}\int_0^{2h} \chi_{[t,h+t]}(x)\,\mathrm{d}t = \frac{1}{2h}\int_{0\vee(x-h)}^{2h\wedge x} \mathrm{d}t \le \frac{1}{2},$$

and thus

$$\left\|\chi_{[0,h]} - \frac{1}{2h}\int_0^{2h} \tau_t\chi_{[0,h]}\,\mathrm{d}t\right\|_{L^p(\mathbb{R})}^p \ge \int_0^h \left(\frac{1}{2}\right)^p \mathrm{d}x = 2^{-p}h = 2^{-p}\left\|\chi_{[0,h]}\right\|_{L^p(\mathbb{R})}^p.$$

It follows that

$$\left\|\tau_x - \frac{1}{2h}\int_x^{x+2h} \tau_t\,\mathrm{d}t\right\|_{\mathcal{L}(L^p(\mathbb{R}))} = \left\|I - \frac{1}{2h}\int_0^{2h} \tau_t\,\mathrm{d}t\right\|_{\mathcal{L}(L^p(\mathbb{R}))} \ge \frac{1}{2},$$

and hence $\frac{1}{h}\int_x^{x+h} \tau_t\,\mathrm{d}t \not\to \tau_x$ as $h \to 0$ for any $x \in \mathbb{R}$, whereas, if $x \mapsto \tau_x$ were Bôchner integrable as an operator-valued function, the convergence should take place for a.e. $x \in \mathbb{R}$.

Let us look at some examples of singular integral operators satisfying our assumptions.

EXAMPLE 2.5. Repeating the argument for the scalar-valued situation, e.g. pp. 193–4 of [**34**], one can show that the following prominent class of operators provides singular integrals in the sense of the above definition: Let $t \mapsto k(t) \in \mathcal{L}(X;Y)$ be strongly locally integrable on $\mathbb{R}^n \setminus \{0\}$ and satisfy the conditions

$$\int_{r<|t|<2r} |k(t)x|_Y \,\mathrm{d}t \leq A_1 |x|_X \qquad \text{for all } r>0,\ x \in X, \tag{2.6}$$

$$\left| \int_{r<|t|<R} k(t)x \,\mathrm{d}t \right|_Y \leq A_2 |x|_X \qquad \text{for all } R>r>0,\ x \in X, \qquad \text{and} \tag{2.7}$$

$$\lim_{r\downarrow 0} \int_{r<|t|<1} k(t)x \,\mathrm{d}t \qquad \text{exists as a norm limit in } Y \text{ for all } x \in X. \tag{2.8}$$

Then the operator p.v.-k defined on $\phi \in \mathcal{S}(\mathbb{R}^n)$ by

$$\langle \text{p.v.-}k, \phi \rangle \, x := \lim_{\epsilon\downarrow 0} \int_{|t|>\epsilon} k(t)x\,\phi(t) \,\mathrm{d}t \tag{2.9}$$

gives a well-defined tempered distribution p.v.-$k \in \mathcal{S}'(\mathbb{R}^n; \mathcal{L}(X;Y))$; actually

$$|\langle \text{p.v.-}k, \phi \rangle \, x|_Y \leq \Big(2A_1(\|\nabla\phi(t)\|_{L^\infty(\mathrm{d}t)} + \||t|\,\phi(t)\|_{L^\infty(\mathrm{d}t)}) + A_2\,|\phi(0)| \Big) \, |x|_X \,. \tag{2.10}$$

It is obvious from the definition that this distribution satisfies Assumption 2.3.

While the previous example showed that certain results simply carry over to the operator-valued situation with essentially no modifications, the purpose of the next one is to illustrate the new phenomena not present in the scalar-valued context.

Example 2.11. We show that the integrability conditions for $t \mapsto k(t)x$ of the previous example do not imply anything similar for $t \mapsto k(t)f(t)$, where $f \in L^\infty(\mathbb{R}^n; X)$, even compactly supported away from the origin. This fact motivates the procedure adopted above first to define our operators on the rather restricted algebraic tensor products, where they make sense without any further assumptions. It is then a different matter to search for conditions guaranteeing the boundedness of these operators; it seems wise to do the hard work with the theorems and not the definitions.

Consider $X := \ell^p(\mathbb{Z})$, $1 \leq p < \infty$, which we identify with $L^p(\mathbb{R}, \sigma([0,1) + \mathbb{Z}), \mathrm{d}s)$ in the obvious way, and let $Y := \mathbb{K}$, the field of scalars. Note in particular that the example includes $\ell^2(\mathbb{Z})$, the prototype of all separable Hilbert spaces, so that there is certainly nothing pathological in the geometry of the Banach spaces in question.

For $t > 0$, $\log_2 t \notin \mathbb{Z} + 1/2$, we set $\alpha(t) := \tan(\pi \log_2(t))$; this map restricted to any of the intervals $(2^{j-1/2}, 2^{j+1/2})$ with $j \in \mathbb{Z}$ is an increasing bijection onto $(-\infty, \infty)$. Let further $g \in \ell^{p'}(\mathbb{Z}) \setminus \ell^1(\mathbb{Z})$. We can then define the operators $k(t) : X \to Y$ by

$$k(t)x := \operatorname{sgn}(t) \cdot x(\alpha(|t|)) \cdot g(\alpha(|t|)) \cdot \alpha'(|t|)$$

for $t \neq 0$, $\log_2(|t|) \notin \mathbb{Z} + 1/2$, and $k(t)x := 0$, say, for the countably many values of t just mentioned. Clearly these operators are linear, and moreover $\|k(t)\|_{X \to Y} = |g(\alpha(|t|))| \, \alpha'(|t|)$ (or 0 for the countably many special cases).

The kernel $k(\cdot)$ is manifestly odd, so that it satisfies (2.7) and (2.8) rather trivially, and moreover

$$
\begin{aligned}
\int_r^{2r} |k(t)x|_Y \, \mathrm{d}t &= \left(\int_r^{2^{j+1/2}} + \int_{2^{j+1/2}}^{2r} \right) |x(\alpha(t))| \cdot |g(\alpha(t))| \, \alpha'(t) \, \mathrm{d}t \\
&= \left(\int_{\alpha(r)}^{\infty} + \int_{-\infty}^{\alpha(2r)} \right) |x(s)| \cdot |g(s)| \, \mathrm{d}s = \int_{-\infty}^{\infty} |x(s)| \cdot |g(s)| \, \mathrm{d}s \\
&\leq \|x\|_{L^p} \|g\|_{L^{p'}} = c \, |x|_X \,,
\end{aligned}
$$

where j is the unique integer such that $\log_2 r \leq j + 1/2 < \log_2(2r) = \log_2 r + 1$, and we have taken into account that $\alpha(r) = \alpha(2r)$ by the π-periodicity of the tangent.

Now we define our function $f \in L^\infty(\mathbb{R}^n; X)$. Let $\eta \in \mathcal{D}(\mathbb{R})$ be $= 1$ in $[-1, 1]$, have range $[0, 1]$ and vanish outside $[-2, 2]$, and define

$$
f(t)(s) := f(t, s) := \begin{cases} \eta(\alpha(|t|) - \lfloor s \rfloor) & \text{if } t \neq 0, \ \log_2(|t|) \notin \mathbb{Z} + 1/2, \\ 0 & \text{else.} \end{cases}
$$

This f is actually not only bounded, but it is $\mathcal{C}^\infty$ in the regions $(2^{j-1/2}, 2^{j+1/2})$, $j \in \mathbb{Z}$.

Since our integrability conditions concern compact subsets of $\mathbb{R} \setminus \{0\}$, we can take f to be compactly supported away from 0 by simply making a cut-off outside our domain of integration. We then have

$$
\begin{aligned}
\int_{1/\sqrt{2}}^{\sqrt{2}} |k(t) f(t)|_Y \, \mathrm{d}t &= \int_{1/\sqrt{2}}^{\sqrt{2}} |f(t, \alpha(t))| \cdot |g(\alpha(t))| \, \alpha'(t) \, \mathrm{d}t \\
&= \int_{1/\sqrt{2}}^{\sqrt{2}} \eta(\alpha(t) - \lfloor \alpha(t) \rfloor) \, |g(\alpha(t))| \, \alpha'(t) \, \mathrm{d}t \\
&= \int_{-\infty}^{\infty} \eta(s - \lfloor s \rfloor) \, |g(s)| \, \mathrm{d}s = \int_{-\infty}^{\infty} |g(s)| \, \mathrm{d}s = \infty,
\end{aligned}
$$

since $g \notin L^1(\mathbb{R}, \mathrm{d}s)$, and this shows quite explicitly that $k(\cdot) f(\cdot)$ is not integrable.

Note that the failure of integrability in the last computation in no way depended on the singularity of $k(\cdot)$ at the origin. In fact, we could have defined k as above only in the annulus $1/\sqrt{2} < |t| < \sqrt{2}$, say, and set $k(t) := 0$ elsewhere. Then we would have even global integrability $\int_{-\infty}^{\infty} |k(t)x|_Y \, \mathrm{d}t \leq c \, |x|_X$ for every fixed $x \in X$, and yet a blow-up of even the local integrals for a function $f \in L^\infty(\mathbb{R}; X)$ in place of x, as above.

3. Some estimates for random series

In this section we review some techniques related to vector-valued random series that have proved to be fundamental for the vector-valued extension of classical results of harmonic analysis.

Denote by ε_j, $j \in \mathbb{Z}$, the Rademacher system of independent random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$ verifying $\mathbb{P}(\varepsilon_j = 1) = \mathbb{P}(\varepsilon_j = -1) = 1/2$. Let $\mathbb{E} := \int (\cdot)\, \mathrm{d}\mathbb{P}$ be the corresponding expectation.

For a Banach space X, let $\mathrm{Rad}(X)$ be the closure in $L^2(\Omega; X)$ of the algebraic tensor product $X \otimes \mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}$ equipped with the norm of $L^2(\Omega; X)$. By Kahane's inequality, any $p \in [1, \infty)$ in place of 2 gives the same space (as a set) with an equivalent norm.

If X is *B-convex*, then various useful properties of $\mathrm{Rad}(X)$ follow readily from the boundedness of the Rademacher projection

$$(Rf)(\omega) := \sum_{-\infty}^{\infty} \mathbb{E}[\varepsilon_j f]\varepsilon_j(\omega); \tag{3.1}$$

in fact, the property that the operator R above is well-defined and bounded on $L^2(\Omega; X)$ can be taken as the definition of the B-convexity of X. See [**67**], §4.14. The theorem of Pisier contained there also shows that X is B-convex if and only if it does not contain uniformly the spaces $\ell^1(r)$, $r \in \mathbb{Z}_+$. Then the argument given in Chapter 1, p. 35, shows that every UMD space is B-convex. Thus we do not get any new geometric restrictions, since the places where we exploit serious analytic (as opposed to algebraic) properties of $\mathrm{Rad}(X)$ are such that the UMD-condition is required anyway. Note that the boundedness of R implies the uniform boundedness of its partial sum projections by the Banach–Steinhaus theorem.

Denoting by $\mathcal{R}(R)$ the range of R, it is obvious that $X \otimes \mathrm{span}(\varepsilon_j)_{-\infty}^{\infty} \subset \mathcal{R}(R)$. On the other hand, the fact that the partial sums of the series in (3.1) (which are in $X \otimes \mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}$ by definition) converge to Rf for every $f \in L^2(\Omega; X)$ shows that $\mathcal{R}(R) \subset \overline{X \otimes \mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}}$. Finally, since R as a bounded projection has a closed range, we conclude that $\mathrm{Rad}(X) = \mathcal{R}(R : L^2(\Omega; X) \to L^2(\Omega; X))$ whenever X is a B-convex space. This allows us to identify $f = Rf \in \mathrm{Rad}(X)$ with the sequence appearing in (3.1),

$$f = Rf \approx (\mathbb{E}[\varepsilon_j f])_{-\infty}^{\infty} \in X^{\mathbb{Z}}.$$

The density of finitely non-zero sequences in $\mathrm{Rad}(X)$ follows from the very definition of $\mathrm{Rad}(X)$ as the closure of $X \otimes \mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}$.

Let us make a useful observation concerning the dual of $\mathrm{Rad}(X)$. Since the unit ball of $L^2(\Omega; X')$ is norming for $L^2(\Omega; X) \supset \mathrm{Rad}(X)$, we have, for $f = Rf \in$

$\mathrm{Rad}(X)$,

$$\begin{aligned}\|f\|_{\mathrm{Rad}(X)} = \sup_{\|g\|_{L^2(\Omega;X')}\le 1} \langle g, Rf\rangle_{\langle L^2(\Omega;X'),L^2(\Omega;X)\rangle} \\ = \sup \langle Rg, f\rangle \le \sup_{\substack{h\in \mathrm{Rad}(X') \\ \|h\|_{L^2(\Omega;X)}\le C}} \langle h, f\rangle \le C\,\|f\|_{\mathrm{Rad}(X)}\end{aligned}$$

where the easily verified self-adjointness of R was used, and C is the operator norm of R on $L^2(\Omega; X)$, thus also the norm of its adjoint. This shows that the unit ball of $\mathrm{Rad}(X')$ is equivalently norming for $\mathrm{Rad}(X)$.

As a consequence of Fubini's theorem and the equivalence of the definitions of $\mathrm{Rad}(X)$ in terms of different exponents we also have

$$L^p(\Gamma; \mathrm{Rad}(X)) \approx \mathrm{Rad}(L^p(\Gamma; X))$$

whenever Γ is a σ-finite measure space. (We really need this only for $\Gamma = \mathbb{R}^n$.)

The Rademacher classes $\mathrm{Rad}(X)$, $\mathrm{Rad}(Y)$ provide a straightforward but occasionally useful reformulation of the concept of *R-boundedness*, whose definition we recall:

Definition 3.2. A collection $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called R-bounded if, for some $C < \infty$, the inequality

$$\left(\mathbb{E}\left|\sum_{j=-N}^{N} \varepsilon_j T_j x_j\right|_Y^p\right)^{\frac{1}{p}} \le C\left(\mathbb{E}\left|\sum_{j=-N}^{N} \varepsilon_j x_j\right|_X^p\right)^{\frac{1}{p}} \tag{3.3}$$

holds for all $N \in \mathbb{N}$ and all $x_j \in X$, $T_j \in \mathcal{T}$ and some [equivalently, all] $p \in [1, \infty[$. The smallest constant C [when $p = 1$, say] is called the *R-bound* of $\mathcal{T}$ and denoted by $\mathcal{R}(\mathcal{T})$.

With the understanding that $x_j = 0$ for $|j| > N$,we can write (3.3) as

$$\left\|(T_j x_j)_{-\infty}^{\infty}\right\|_{\mathrm{Rad}(Y)} \le C\left\|(x_j)_{-\infty}^{\infty}\right\|_{\mathrm{Rad}(X)},$$

and by the density of finitely non-zero sequences $(x_j)_{-\infty}^{\infty} \in \mathrm{Rad}(X)$, the condition is simply that of boundedness of the diagonal operators $(T_j)_{-\infty}^{\infty}$ from $\mathrm{Rad}(X)$ to $\mathrm{Rad}(Y)$.

The following permanence property of R-boundedness will be useful.

Lemma 3.4. *Let X be a B-convex space and $\mathcal{T} \subset \mathcal{L}(X; Y)$ be R-bounded. Then $\mathcal{T}' := \{T' | \ T \in \mathcal{T}\} \subset \mathcal{L}(Y'; X')$ is also R-bounded, and more precisely $\mathcal{R}(\mathcal{T}') \le C\mathcal{R}(\mathcal{T})$, where C is a geometric constant.*

Proof. For $g \in L^2(\Omega; X)$, we have

$$\begin{aligned}
\mathbb{E}\left\langle \sum_{-N}^{N} \varepsilon_j T_j' y_j', g \right\rangle &= \mathbb{E} \sum_{-N}^{N} \left\langle \varepsilon_j T_j' y_j', \varepsilon_j \mathbb{E}[\varepsilon_j g] \right\rangle = \mathbb{E}\left\langle \sum_{-N}^{N} \varepsilon_j y_j', \sum_{-N}^{N} \varepsilon_i T_i \mathbb{E}[\varepsilon_i g] \right\rangle \\
&\leq \left(\mathbb{E} \left| \sum_{-N}^{N} \varepsilon_j y_j' \right|_{X'}^2 \right)^{1/2} \mathcal{R}(\mathcal{T}) \left(\mathbb{E} \left| \sum_{-N}^{N} \varepsilon_j \mathbb{E}[\varepsilon_j g] \right|_{X}^2 \right)^{1/2} \\
&\leq \left(\mathbb{E} \left| \sum_{-N}^{N} \varepsilon_i y_i' \right|_{X'}^2 \right)^{1/2} \mathcal{R}(\mathcal{T}) C \left\| g \right\|_{L^2(\Omega;X)},
\end{aligned}$$

recalling the uniform boundedness of the partial sum projections of the Rademacher projection R. Taking supremum over $g \in L^2(\Omega; X)$ of unit norm, we find that $\mathcal{R}(\mathcal{T}') \leq C\mathcal{R}(\mathcal{T})$, where C is the same constant as above. □

We also recall (e.g. from [**87**]) that the family $\tilde{\mathcal{T}}$ of canonical extensions $(\tilde{T} f)(t) := T[f(t)]$ of $T \in \mathcal{T} \subset \mathcal{L}(X; Y)$ to $L^p(\Gamma; X) \to L^p(\Gamma; Y)$ is R-bounded whenever $\mathcal{T}$ is, with the same R-bound and without any geometric assumptions. (This is easy to see.)

An R-bounded collection is always uniformly bounded, but the converse is not true in general. Perhaps the simplest example of a uniformly bounded, non-R-bounded family of operators is the group of translations acting on $L^p(\mathbb{R}^n)$, $p \neq 2$. However, there is a remarkable result due to Bourgain [**12**] providing a partial substitute of this R-boundedness of translations under appropriate restrictions on the support of the Fourier transforms of the functions involved. This result plays an important rôle in Bourgain's paper [**12**], as well as in the present work. The difficult part of the proof, the case $n = 1$ for the unit-circle $\mathbb{T}$ in place of $\mathbb{R}^n$, is given in [**12**], Lemma 10. The transference to $\mathbb{R}^n$ uses standard methods and is detailed in [**36**], Lemma 3.5.

Lemma 3.5 ([**12, 36**]). *Let X be a UMD-space and $(f_j)_{-\infty}^{\infty} \subset L^p(\mathbb{R}^n; X)$ a finitely non-zero sequence such that* $\operatorname{supp} \hat{f}_j \subset \bar{B}(0, 2^j)$*. Let $(h_j)_{-\infty}^{\infty} \subset \mathbb{R}^n$ be a sequence, lying on the same line through the origin and such that $|h_j| < K2^{-j}$ for some constant K. Then*

$$\mathbb{E} \left\| \sum \varepsilon_j f_j(\cdot - h_j) \right\|_{L^p(\mathbb{R}^n;X)} \leq C \log(2 + K) \mathbb{E} \left\| \sum \varepsilon_j f_j \right\|_{L^p(\mathbb{R}^n;X)}.$$

Remark 3.6. Although we do not need it, we mention that one can get away from the assumption that the h_j lie on the same line, with the cost of getting $\log^n(2 + K)$ in place of $\log(2 + K)$. While the case $n = 1$ is obviously handled already, the case of $n > 1$ dimensions can be reached by induction on n.

To ensure the support condition of the Fourier transforms for the application of Bourgain's lemma, we will exploit (a smooth version of) a Littlewood–Paley-type dyadic decomposition. Let $\eta \in \mathcal{D}(\mathbb{R}^n)$ have range $[0,1]$, equal 1 for $|\xi| < 1/4$ and vanish for $|\xi| > 1/2$. Let then $\hat{\varphi}_0(\xi) := \eta(\xi) - \eta(2\xi)$, and $\hat{\varphi}_j(\xi) := \hat{\varphi}(2^{-j}\xi)$. Then $\sum_{-\infty}^{\infty} \hat{\varphi}_j(\xi) = 1$ for $\xi \neq 0$ and $\hat{\varphi}_j$ is supported in the annulus $2^{j-3} \leq |\xi| \leq 2^{j-1}$. Moreover, $\hat{\Phi}_j := \hat{\varphi}_{j-1} + \hat{\varphi}_j + \hat{\varphi}_{j+1}$ is equal to unity on the support of $\hat{\varphi}_j$, and is supported in the annulus $2^{j-4} \leq |\xi| \leq 2^j$. Our indices are slightly shifted from the usual choice, the sole purpose of which being to ensure the condition $\operatorname{supp} \hat{\Phi}_j \subset [-2^j, 2^j]^n$ so as to avoid playing with indices when applying Lemma 3.5.

The next lemma allows us to estimate deterministic L^p-norms with randomized ones, i.e., to incorporate the Rademacher functions ε_j into our equations. Slight variants of this lemma and the next one appear in several papers, cf. e.g. Girardi and Weis [**36**], Cor. 3.3.

Lemma 3.7. *Let X be a UMD-space, $1 < p < \infty$, and*

$$(g_j)_{-\infty}^{\infty} \subset (\mathcal{S}' \cap L^{1,\mathrm{loc}})(\mathbb{R}^n; X)$$

be a finitely non-zero sequence. Assume further that $\hat{g}_j$ is supported in the annulus $|\xi| \in 2^j[a,b]$ for some $0 < a < b$. Then

$$\Big\| \sum g_j \Big\|_{L^p(\mathbb{R}^n;X)} \leq C\mathbb{E} \Big\| \sum \varepsilon_j g_j \Big\|_{L^p(\mathbb{R}^n;X)}, \tag{3.8}$$

where the constant depends only on a and b (and the geometry of X).

Proof. Let us first observe that we can assume that $g_j \in L^p(\mathbb{R}^n; X)$ for all j, since otherwise the right-hand side is ∞. Indeed, let $E_m := \{t \in \mathbb{R}^n |\ |t| \leq m,\ |g_j(t)|_X \leq m \text{ for all } j\}$. Then

$$\begin{aligned}\|g_i \mathbf{1}_{E_m}\|_{L^p(\mathbb{R}^n;X)} \\ \leq \frac{1}{2} \Big\| \Big(g_i + \sum_{j\neq i} g_j \Big) \mathbf{1}_{E_m} \Big\|_{L^p(\mathbb{R}^n;X)} + \frac{1}{2} \Big\| \Big(g_i - \sum_{j\neq i} g_j \Big) \mathbf{1}_{E_m} \Big\|_{L^p(\mathbb{R}^n;X)}.\end{aligned}$$

As $m \to \infty$, the left-hand side becomes the $L^p(\mathbb{R}^n; X)$-norm of g_i, whereas on the right-hand side we have two terms appearing on the right-hand side of (3.8). Should we have $\|g_i\|_{L^p(\mathbb{R}^n;X)} = \infty$, the right-hand side of (3.8) would also be ∞, and there is nothing to prove.

Let us hence assume that $g_j \in L^p(\mathbb{R}^n; X)$ for all j. We choose $N \in \mathbb{N}$ large enough so that $2^N > b/a$. Then, by the triangle inequality,

$$\Big\| \sum_{j=-\infty}^{\infty} g_j \Big\|_{L^p(\mathbb{R}^n;X)} \leq \sum_{k=0}^{N-1} \Big\| \sum_{j\equiv k (\mathrm{mod}\ N)} g_j \Big\|_{L^p(\mathbb{R}^n;X)}. \tag{3.9}$$

The motivation for this rearrangement is the fact that the supports of $\hat{g}_j$ for $j \equiv k (\mathrm{mod}\ N)$ are disjoint for any fixed k.

Choose $\phi \in \mathcal{D}(\mathbb{R}^n)$ with range $[0,1]$, equal to unity in $[a,b]$ and with support in $[2^{-N}b, 2^N a]$. Then $\hat{g}_j = \phi(2^{-j}\cdot)\hat{g}_j$ and $\phi(2^{-i}\cdot)\hat{g}_j = 0$ for $i \neq j$. Thus, for $(\epsilon_j)_{-\infty}^{\infty} \in \{-1,1\}^{\mathbb{Z}}$,

$$\sum_{j\equiv k} \epsilon_j \hat{g}_j = \sum_{j\equiv k} \epsilon_j \phi(2^{-j}\cdot)\hat{g}_j = \left(\sum_{i\equiv k} \epsilon_i \phi(2^{-i}\cdot)\right) \sum_{j\equiv k} \hat{g}_j =: m \sum_{j\equiv k} \hat{g}_j,$$

and the Fourier multiplier m satisfies infinitely many of the Mihlin-type conditions

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq \sum_{j\equiv k} |\xi|^{|\alpha|} 2^{-j|\alpha|} \left|(D^\alpha \phi)(2^{-j}\xi)\right| \leq 2 \sup_{\xi} |\xi|^{|\alpha|} |D^\alpha \phi(\xi)| < \infty,$$

where the factor 2 follows from the fact that at most two of the functions $\phi(2^{-j}\cdot)$, $j \equiv k$, are supported at any given point.

The UMD-space version of Mihlin's multiplier theorem (which is due to Zimmermann [**89**]) implies that

$$\left\|\sum_{j\equiv k} \epsilon_j g_j\right\|_{L^p(\mathbb{R}^n;X)} \leq K \left\|\sum_{j\equiv k} g_j\right\|_{L^p(\mathbb{R}^n;X)},$$

and the inequality is readily seen to be two-sided by taking $\epsilon_j g_j$ in place of g_j. Then, taking $\epsilon_j := \varepsilon_j(\omega)$ and integrating over $\omega \in \Omega$, we have

$$\left\|\sum_{j\equiv k} g_j\right\|_{L^p(\mathbb{R}^n;X)} \leq K\mathbb{E} \left\|\sum_{j\equiv k} \varepsilon_j g_j\right\|_{L^p(\mathbb{R}^n;X)} \leq K\mathbb{E} \left\|\sum_{j=-\infty}^{\infty} \varepsilon_j g_j\right\|_{L^p(\mathbb{R}^n;X)},$$

where the last inequality follows from Kahane's contraction principle.

Combining this with (3.9), we have the assertion with $C = NK$. □

We also need to be able to get rid of the randomization, and for this we have the following:

Lemma 3.10. *For $f \in L^p(\mathbb{R}^n;X)$ we have*

$$\mathbb{E} \left\|\sum \varepsilon_j \Phi_j * f\right\|_{L^p(\mathbb{R}^n;X)} \leq C \|f\|_{L^p(\mathbb{R}^n;X)}.$$

Proof. Since $\mathcal{F} \sum \varepsilon_j \Phi_j * f = \sum \varepsilon_j \hat{\Phi}_j \hat{f} =: m\hat{f}$, and

$$|\xi|^{|\alpha|} |D^\alpha m(\xi)| \leq \sum |\xi|^{|\alpha|} 2^{-j|\alpha|} \left|(D^\alpha \hat{\Phi}_0)(2^{-j}\xi)\right| \leq 3 \sup_{\xi} |\xi|^{|\alpha|} \left|D^\alpha \hat{\Phi}_0(\xi)\right| < \infty,$$

even a stonger result with the expectation replaced by the supremum norm over the random variables ε_j is an immediate consequence of the Mihlin–Zimmermann theorem. □

4. A Hörmander-type condition for singular integrals

The classical result for scalar-valued singular integral operators (see HÖRMANDER [**43**]) states that the formal convolution (2.1), interpreted as explained in Sect. 2, defines a bounded operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$, if $\hat{k}$ is bounded and k satisfies the Hörmander condition (1.1). Our main result in this section is the following version of this theorem for operator-valued kernel functions:

THEOREM 4.1. *Let X, Y be UMD-spaces. Assume that $k \in \mathcal{S}'(\mathbb{R}^n; \mathcal{L}(X;Y))$ has a Fourier-transform $\hat{k} \in L^\infty_{\mathrm{str}}(\mathbb{R}^n; \mathcal{L}(X,Y))$ (strongly measurable, essentially bounded) and satisfies Assumption 2.3, as well as the following conditions:*

$$\mathcal{R}\left(\{\hat{k}(\xi) \mid \xi \in \mathbb{R}^n\}\right) \leq A_0, \tag{4.2}$$

and

$$\int_{|t|>2|s|} \mathcal{R}\left(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t)) \mid j \in \mathbb{Z}\}\right) w(t)\,\mathrm{d}t \leq A_1 w(s), \tag{4.3}$$

where $w(t) = \log(2 + |t|)$.

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto k * f$ extends to a bounded linear operator

$$f \in L^p(\mathbb{R}^n; X) \mapsto k * f \in L^p(\mathbb{R}^n; Y)$$

with norm at most $C(A_0 + A_1)$, where C is a geometric constant.

REMARK 4.4. (*i*) For $X = Y$ and $n = 1$, the Hilbert transform $\mathcal{H} = (\text{p.v.-}1/\pi t)*$, with $k(t) = 1/\pi t$, $\hat{k}(\xi) = -\mathbf{i}\,\mathrm{sgn}(\xi)$, is easily seen to verify the conditions of the theorem – note in particular that the R-boundedness reduces to uniform boundedness for a scalar kernel – which shows that the UMD-assumption is necessary in this case. On the other hand, if Z is a UMD-space, $A \in \mathcal{L}(X, Z)$ and $B \in \mathcal{L}(Z, Y)$, then the singular integral operator of the special form $B\mathcal{H}A : L^p(\mathbb{R}; X) \to L^p(\mathbb{R}; Y)$ satisfies the conclusion of the theorem for arbitrary Banach spaces X and Y.

(*ii*) The operator $f \mapsto k * f$ can also be interpreted as a Fourier multiplier transformation $\hat{f} \mapsto \hat{k}\hat{f}$, with operator-valued multiplier $\hat{k} \in L^\infty(\mathbb{R}^n; \mathcal{L}(X;Y))$. Thus a result of CLÉMENT and PRÜSS [**22**] shows that the R-boundedness condition (4.2) of the operators $\hat{k}(\xi)$ is necessary.

(*iii*) The R-boundedness assumption in our version of the Hörmander condition enables us to use the Littlewood–Paley decomposition, whereas the logarithmic factor is forced on us by Lemma 3.5. Note that while the usual Hörmander condition (1.1) is sufficient (also in the vector-valued context) to obtain the boundedness on the whole scale $p \in]1, \infty[$ as soon as the boundedness is known for one $L^{\tilde{p}}$, we *do not* assume any *a priori* boundedness.

(*iv*) For the verification of the weighted Hörmander condition (4.3) in concrete situations, it is useful to note the estimate

$$\int_r^\infty t^{-(1+\delta)}\log(2+t)\,\mathrm{d}t \le C(\delta)r^{-\delta}\log(2+r) \qquad \text{for all } r,\delta>0, \tag{4.5}$$

whose verification is elementary calculus.

Theorem 4.1 will be a special case of Theorem 4.21 below. As a preparation for the proof, we first give a somewhat technical condition for the boundedness of singular integral operators. It is a version of Proposition 3.7 in Girardi and Weis [**36**].

Proposition 4.6. *Let X, Y be UMD-spaces and $k \in \mathcal{S}'(\mathbb{R}^n;\mathcal{L}(X;Y))$ satisfy Assumption 2.3. Define, for every $t\in\mathbb{R}^n$, an operator from* $\mathrm{Rad}(X)$ *to* $\mathrm{Rad}(Y)$ *by*

$$K(t) := ((\varphi_0 * 2^{-nj}k(2^{-j}\cdot))(t))_{j=-\infty}^{\infty}, \tag{4.7}$$

and assume that the Banach adjoints $K(t)' : \mathrm{Rad}(Y') \to \mathrm{Rad}(X')$, canonically extended to $L^{p'}(\mathbb{R}^n;\mathrm{Rad}(Y')) \to L^{p'}(\mathbb{R}^n;\mathrm{Rad}(X'))$, satisfy the condition

$$\int_{\mathbb{R}^n} \|K(t)'g\|_{L^{p'}(\mathbb{R}^n;\mathrm{Rad}(X'))}\,w(t)\,\mathrm{d}t \le A\,\|g\|_{L^{p'}(\mathbb{R}^n;\mathrm{Rad}(Y'))}, \tag{4.8}$$

with $w(t) := \log(2+|t|)$, for every $g\in\mathrm{Rad}(L^{p'}(\mathbb{R}^n;X))$.

Then $f \in X\otimes\hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto k*f$ extends to a bounded linear operator from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n;Y)$, of norm at most CA, where C is a geometric constant.

Proof. We have $\mathcal{F}[k*f] = \hat{k}\hat{f} = \sum_{-\infty}^{\infty}\hat{\varphi}_j\hat{k}\hat{f}$, where the sum contains only finitely many non-zero terms for $f\in\hat{\mathcal{D}}_0$. Moreover, we have $\hat{\varphi}_j\hat{k}\hat{f} = \hat{\varphi}_j\hat{k}\,\hat{\Phi}_j\hat{f} = \mathcal{F}[(\varphi_j*k)*(\Phi_j*f)]$. Denoting $f_j := \Phi_j * f$, we have the decomposition $k*f = \sum_{-\infty}^{\infty}(\varphi_j*k)*f_j$.

As a last preparatory manipulation, we write

$$\begin{aligned}(\varphi_j*k)*f_j(t) &= \int_{\mathbb{R}^n}(\varphi_j*k)(2^{-j}s)f_j(t-2^{-j}s)2^{-jn}\,\mathrm{d}s \\ &= \int_{\mathbb{R}^n}2^{-jn}(\varphi_0*k(2^{-j}\cdot))(s)f_j(t-2^{-j}s)\,\mathrm{d}s,\end{aligned}$$

where a simple change of variable was performed, recalling that $\hat{\varphi}_j = \hat{\varphi}(2^{-j}\cdot)$, whence $\varphi_j = 2^{jn}\varphi(2^j\cdot)$. A functional notation is used to denote the dilation of the distribution k for simplicity, but this is defined by the duality $\langle k(\delta\cdot),\phi\rangle := \langle k,\delta^{-n}\phi(\delta^{-1}\cdot)\rangle$.

We now invoke the UMD-property by means of the Littlewood–Paley decomposition (more precisely, Lemma 3.7), which allows us to write

$$\begin{aligned}\Big\|\sum_{-\infty}^{\infty}(\varphi_j * k) * f_j\Big\|_{L^p(\mathbb{R}^n;X)} &\leq C\mathbb{E}\Big\|\sum_{-\infty}^{\infty}\varepsilon_j(\varphi_j * k) * f_j\Big\|_{L^p(\mathbb{R}^n;Y)} \\ &= C\mathbb{E}\Big\|\sum_{-\infty}^{\infty}\varepsilon_j\int_{\mathbb{R}^n}2^{-jn}[\varphi_0 * k(2^{-j}\cdot)](s)f_j(\cdot - 2^{-j}s)\,\mathrm{d}s\Big\|_{L^p(\mathbb{R}^n;Y)} \\ &= C\Big\|\int_{\mathbb{R}^n}K(s)\left(f_j(\cdot - 2^{-j}s)\right)_{-\infty}^{\infty}\,\mathrm{d}s\Big\|_{\mathrm{Rad}(L^p(\mathbb{R}^n;Y))},\end{aligned}$$

where we recalled the definition of our auxiliary sequence-valued kernel K from equation (4.7).

To estimate the norm on the right of the previous inequality, we pick an arbitrary $g \in \mathrm{Rad}(L^{p'}(\mathbb{R}^n;Y'))$. We have

$$\begin{aligned}(4.9)\quad &\Big\langle g, \int_{\mathbb{R}^n}K(s)\left(f_j(\cdot - 2^{-j}s)\right)_{-\infty}^{\infty}\,\mathrm{d}s\Big\rangle_{\left\langle L^{p'}(\mathbb{R}^n;\mathrm{Rad}(Y')),L^p(\mathbb{R}^n;\mathrm{Rad}(Y))\right\rangle} \\ &= \int_{\mathbb{R}^n}\mathrm{d}s\,\Big\langle K(s)'g, \left(f_j(\cdot - 2^{-j}s)\right)_{-\infty}^{\infty}\Big\rangle_{\left\langle L^{p'}(\mathbb{R}^n;\mathrm{Rad}(X')),L^p(\mathbb{R}^n;\mathrm{Rad}(X))\right\rangle} \\ &\leq \int_{\mathbb{R}^n}\mathrm{d}s\,\|K(s)'g\|_{L^{p'}(\mathbb{R}^n;\mathrm{Rad}(X'))}\Big\|\left(f_j(\cdot - 2^{-j}s)\right)_{-\infty}^{\infty}\Big\|_{\mathrm{Rad}(L^p(\mathbb{R}^n;X))}.\end{aligned}$$

The second factor can be estimated with the help of Bourgain's lemma to the result

$$\begin{aligned}\mathbb{E}\Big\|\sum_{-\infty}^{\infty}\varepsilon_j f_j(\cdot - 2^{-j}s)\Big\|_{L^p(\mathbb{R}^n;X)} &\leq C\log(2+|s|)\Big\|\sum_{-\infty}^{\infty}\varepsilon_j f_j\Big\|_{L^p(\mathbb{R}^n;X)} \\ &\leq \tilde{C}\log(2+|s|)\,\|f\|_{L^p(\mathbb{R}^n;X)},\end{aligned}$$

the last step being again a consequence of the Littlewood–Paley decomposition for UMD-valued functions (more precisely, Lemma 3.10).

It remains to estimate the integral over s in (4.9) by means of the assumption, invoke the assumption (4.8), and consider the supremum over all appropriate $g \in \mathrm{Rad}(L^{p'}(\mathbb{R}^n;X))$ of norm at most unity, to conclude that

$$\|k * f\|_{L^p(\mathbb{R}^n;X)} \leq CA\,\|f\|_{L^p(\mathbb{R}^n;X)}$$

for all f in the dense subspace considered. Thus the proposition is proved. □

In the previous proposition, the boundedness of a singular integral operator acting on the space $L^p(\mathbb{R}^n;X)$ was related to a boundedness condition of another operator acting on the Rademacher class $\mathrm{Rad}(X)$ and related spaces. The new kernel $K(t)$ in (4.7) has some special structure, in particular, the convolution with a nice test function φ_0. To be able to exploit this particular structure, so as to find a sufficient condition more explicitly in terms of the original kernel k, we need the following decomposition lemma, which is a variant of Lemma 4.3 in Chapter 1.

LEMMA 4.10. *Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ have a vanishing integral. Then there exists a decomposition $\varphi = \sum_{m=0}^{\infty} \psi_m$ with the following properties:*

$$\psi_m \in \mathcal{D}(\mathbb{R}^n), \qquad \operatorname{supp} \psi_m \subset \bar{B}(0, 2^m), \qquad \int \psi_m(y)\,\mathrm{d}y = 0,$$

and for every pair of multi-indices $\alpha, \beta \in \mathbb{N}^n$ and every $M > 0$ the sequence of Schwartz norms

$$\|\psi_m\|_{\alpha,\beta} := \left\| x^\beta D^\alpha \psi_m(x) \right\|_\infty ,$$

as well as $\left\|\hat{\psi}_m\right\|_{\alpha,\beta}$, is $\mathcal{O}(2^{-mM})$ as $m \to \infty$.

In particular, for every $p \in [1, \infty]$ and every $M > 0$, the sequence of Lebesgue norms $\|\psi_m\|_{L^p}$, as well as $\left\|\hat{\psi}_m\right\|_{L^p}$, is $\mathcal{O}(2^{-mM})$ as $m \to \infty$.

PROOF. Fix $\eta \in \mathcal{D}(\mathbb{R}^n)$, with range $[0, 1]$, equal to 1 for $|x| \leq 1/2$ and vanishing for $|x| \geq 1$. We set, for $r > 0$,

$$\varphi_r(x) := \eta(x/r) \left(\phi(x) + \frac{1}{r^n \int \eta(y)\,\mathrm{d}y} \int \varphi(y)(1 - \eta(y/r))\,\mathrm{d}y \right).$$

Then φ_r has a vanishing integral since φ does, and φ_r is supported in $\bar{B}(0, r)$ by the support condition imposed on η. Moreover,

$$\begin{aligned} |\varphi_r(x) - \varphi(x)| &\leq |\eta(x/r) - 1| \cdot |\varphi(x)| + \frac{\eta(x/r)}{r^n \int \eta(y)\,\mathrm{d}y} \int |\varphi(y)| \,(1 - \eta(y/r))\,\mathrm{d}y \\ &\leq \max_{|y| \geq r} |\phi(y)| + c r^{-n} \int_{|y| \geq r} |\varphi(y)| \,\mathrm{d}y, \end{aligned}$$

which tends to zero as $r \to \infty$; thus $\varphi_r \to \varphi$ uniformly.

We next set $\psi_0 := \varphi_1$ and $\psi_m := \varphi_{2^m} - \varphi_{2^{m-1}}$ for $m > 0$; whence $\sum_{m=0}^{\infty} \psi_m = \lim_{m \to \infty} \varphi_{2^m} = \varphi$, uniformly. Explicitly, for $m > 0$, we have

$$\begin{aligned} \psi_m(x) := {} & \varphi(x) \left(\eta(2^{-m}x) - \eta(2^{-(m-1)}x) \right) \\ & + \frac{\eta(2^{-m}x)}{2^{nm} \int \eta(y)\,\mathrm{d}y} \int \varphi(y)(1 - \eta(2^{-m}y))\,\mathrm{d}y \\ & - \frac{\eta(2^{-(m-1)}x)}{2^{n(m-1)} \int \eta(y)\,\mathrm{d}y} \int \varphi(y)(1 - \eta(2^{-(m-1)}y))\,\mathrm{d}y. \end{aligned} \tag{4.11}$$

It remains to estimate the order of the size of the Schwartz norms of the terms appearing here.

Let us first have a look at the last two terms in (4.11). We have, by a simple change of variable, $\|\eta(2^{-m}\cdot)\|_{\alpha,\beta} = 2^{m(|\beta| - |\alpha|)} \|\eta\|_{\alpha,\beta}$, which looks a little bad for

$|\beta| > |\alpha|$. However, the thing that settles the matters is the constant factor, whose size is estimated by

$$\int |\varphi(y)|\,(1-\eta(2^{-m}y))\,\mathrm{d}y \le \int_{|y|>2^m} |\varphi(y)|\ \mathrm{d}y$$
$$\le C(M,n,\varphi)\int_{|y|>2^m} |y|^{-M-n}\ \mathrm{d}y = \tilde{C}2^{-Mm}.$$

We then turn to estimate the first term in (4.11) and denote for simplicity $\phi(x) := \eta(x)-\eta(2x)$, so that this term is $\varphi(x)\phi(2^{-m}x)$. Note that $\phi \in \mathcal{D}(\mathbb{R}^n)$ is supported away from the origin. By LEIBNIZ' rule we have

$$x^\beta D_x^\alpha(\varphi(x)\phi(2^{-m}x)) = \sum_{\theta\le\alpha}\binom{\alpha}{\theta} x^\beta D^{\alpha-\theta}\varphi(x)2^{-m|\theta|}D^\theta\phi(2^{-m}x). \tag{4.12}$$

Let us make a Taylor expansion of $D^\theta\phi(2^{-m}x)$ at the origin; since $D^\vartheta\phi(0)=0$ for all $\vartheta\in\mathbb{N}^n$, all we get is the error term:

$$D^\theta\phi(2^{-m}x) = \sum_{|\vartheta|=M}\frac{2^{-mM}x^\vartheta}{\vartheta!}\int_0^1 D^{\theta+\vartheta}\phi(u2^{-m}x)M(1-u)^{M-1}\,\mathrm{d}u.$$

Now a typical term in the sum in (4.12) is estimated as

$$\sum_{|\vartheta|=M} 2^{-mM}\frac{1}{\vartheta!}\left|x^{\beta+\vartheta}D^{\alpha-\theta}\varphi(x)\right|\int_0^1 \left|D^{\theta+\vartheta}\phi(u2^{-m}x)\right| M(1-u)^{M-1}\,\mathrm{d}u$$
$$\le 2^{-mM}\sum_{|\vartheta|=M}\frac{1}{\vartheta!}\left\|\varphi\right\|_{\alpha-\theta,\beta+\vartheta}\left\|\phi\right\|_{\theta+\vartheta,0}.$$

Summing over the finite number of bounds of this type, we have established the desired rate of convergence of the sequence $\|\psi_m\|_{\alpha,\beta}$ as $m\to\infty$.

The assertion concerning the Schwartz norms of the Fourier transforms $\hat{\psi}_m$ follows from the continuity of the Fourier transform on $\mathcal{S}(\mathbb{R}^n)$. The assertion concerning the Lebesgue norms follows by estimating $\|\psi_m\|_{L^p}$ by a finite number of Schwartz norms. $\square$

REMARK 4.13. (*i*) The result is equally valid for vector-valued functions, with the same proof, but we only need it here for scalar-valued ones.

(*ii*) The decomposition established is an *atomic decomposition* of φ, and shows the well-known fact that $\varphi\in\mathcal{S}(\mathbb{R}^n)$ belongs to the Hardy space $H^1(\mathbb{R}^n)$ provided it has a vanishing integral. However, more than this we are interested in the particular type of the decomposition, with the rapid rate of convergence.

LEMMA 4.14. *Consider a mapping* $t\in\mathbb{R}^n\mapsto\mathcal{L}(X;Y)$ *having the form*

$$K(t) := k*\varphi(t)$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int \varphi(y)\,\mathrm{d}y = 0$*, and* $k \in \mathcal{S}'(\mathbb{R}^n; \mathcal{L}(X;Y))$ *is an operator-valued tempered distribution satisfying Assumption 2.3, and moreover* $\hat{k}$ *agrees with a function in* $L^\infty(\mathbb{R}^n; \mathcal{L}(X;Y))$*. In addition, suppose that*

(4.15)
$$\left\|\hat{k}(\xi)\right\|_{X\to Y} \leq A_0 \qquad \textit{for a.e. } \xi \in \mathbb{R}^n,$$

(4.16)
$$\int_{|t|>2|s|} |(k(t-s)-k(t))x|_Y\, w_0(t)\,\mathrm{d}t \leq A_1 w_1(s)\,|x|_X \qquad \forall\ s \in \mathbb{R}^n \setminus \{0\},\ x \in X_1,$$

where w_0 *and* w_1 *are positive, measurable and polynomially bounded functions, and* $X_1 \subset X$*.*

Then, for every $x \in X_1$*,*

$$\int |K(t)x|_Y\, w_0(t)\,\mathrm{d}t \leq (A_0 C(\varphi, w_0) + A_1 C(\varphi, w_1))\,|x|_X\,,$$

where the C*'s are finite quantities depending only on the objects indicated.*

Proof. We apply Lemma 4.10 to write $\varphi = \sum_{m=0}^{\infty} \psi_m$, where the ψ_m have the properties stated in that lemma. Then we devide $K(t)$ into the pieces

$$K_m(t) := k * \psi_m(t),$$

and investigate each of them separately.

Recall that ψ_m is supported in the ball $\bar{B}_m := \bar{B}(0, 2^m)$. We first investigate the integral of $|K_m(t)x|_Y$, with $x \in X_1$, well away from this ball, i.e., outside the larger ball $\bar{B}_{m+1}$:

$$\int_{\bar{B}^c_{m+1}} |K_m(t)x|_Y\, w_0(t)\,\mathrm{d}t = \int_{\bar{B}^c_{m+1}} \mathrm{d}t\, w_0(t)\ \left|\int_{\bar{B}_m} k(t-s)x\,\psi_m(s)\,\mathrm{d}s\right|_Y.$$

Since the integral of ψ_m vanishes, we can continue with

$$\begin{aligned}
&= \int_{\bar{B}^c_{m+1}} \mathrm{d}t\, w_0(t)\ \left|\int_{\bar{B}_m} (k(t-s)-k(t))x\,\psi_m(s)\,\mathrm{d}s\right|_Y \\
&\leq \int_{\bar{B}_m} \mathrm{d}s\,|\psi_m(s)| \int_{|t|>2|s|} |(k(t-s)-k(t))x|_Y\ w_0(t)\,\mathrm{d}t \\
&\leq \int_{\bar{B}_m} \mathrm{d}s\,|\psi_m(s)|\,A_1 w_1(s)\,|x|_X \leq A_1 \|\psi_m\|_{L^\infty}\,\nu_1(\bar{B}_m)\,|x|_X\,,
\end{aligned}$$

where we have denoted $\mathrm{d}\nu_1(t) := w_1(t)\,\mathrm{d}t$.

Inside the ball $\bar{B}_{m+1}$ we argue as follows, with the obvious definition of ν_0:

$$\begin{aligned}\int_{\bar{B}_{m+1}} |K_m(t)x|_Y\, w_0(t)\,\mathrm{d}t &\leq \nu_0(\bar{B}_{m+1})\, \|K_m(\cdot)x\|_{L^\infty(\mathbb{R}^n;Y)} \\ &\leq \nu_0(\bar{B}_{m+1}) \left\|\hat{K}_m(\cdot)x\right\|_{L^1(\mathbb{R}^n;Y)} = \nu_0(\bar{B}_{m+1}) \int_{\mathbb{R}^n} \left|\hat{k}(\xi)x\,\hat{\psi}_m(\xi)\right|_Y \mathrm{d}\xi \\ &\leq A_0\nu_0(\bar{B}_{m+1}) \left\|\hat{\psi}_m\right\|_{L^1} |x|_X\,.\end{aligned}$$

Summing over the estimates, we have

$$\int_{\mathbb{R}^n} |K(t)x|_Y\, w_0(t)\,\mathrm{d}t \leq \sum_{m=0}^{\infty} \left(A_1\nu_1(\bar{B}_m)\,\|\psi_m\|_{L^\infty} + A_0\nu_0(\bar{B}_{m+1}) \left\|\hat{\psi}_m\right\|_{L^1}\right) |x|_X\,.$$

The convergence of the series follows from the properties of the decomposition $\varphi = \sum_{m=0}^{\infty}\psi_m$. Indeed, the at most polynomial growth of w_i guarantees that $\nu_i(\bar{B}(0,r)) \leq Cr^N$ as $r \to \infty$; hence $\nu_0(\bar{B}_{m+1}), \nu_1(\bar{B}_m) \leq C2^{mN}$, but we have $\|\psi_m\|_{L^\infty}, \left\|\hat{\psi}_m\right\|_{L^1} \leq C_M 2^{-mM}$ for any $M > 0$, so it suffices to take $M > N$. This completes the proof. $\square$

Remark 4.17. (*i*) The result of the lemma is rather general, since no conditions on the Banach space geometry are required.

(*ii*) Although our application of the lemma is to the boundedness of operators acting on the usual Bôchner spaces with respect to the plain Lebesgue measure, where a specific choice of the weight w is relevant, the above result itself has some taste of a more general weighted norm inequality. The assumption that the weights w_i be polynomially bounded is exploited via the growth condition of the size of the balls $\bar{B}(0,r)$ in terms of the measures $\mathrm{d}\nu_i(t) := w_i(t)\,\mathrm{d}t$. Such a growth estimate would also follow from the *doubling condition* $\nu(\bar{B}(x,2r)) \leq C\nu(\bar{B}(x,r))$, which is the usual regularity assumption when dealing with more general measure spaces.

The following corollary simply specializes Lemma 4.14 to the spaces $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$ in place of X and Y. The subset $X_1 \subset X$ that appeared in Lemma 4.14 will now be the set $X \otimes \mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}$ of finitely non-zero elements of $\mathrm{Rad}(X)$.

Corollary 4.18. *Consider a mapping* $t \in \mathbb{R}^n \mapsto \mathcal{L}(\mathrm{Rad}(X);\mathrm{Rad}(Y))$ *having the form*

$$K(t) := (k_j * \varphi(t))_{-\infty}^{\infty}$$

where $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *with* $\int \varphi(y)\,\mathrm{d}y = 0$, *and* $k_j \in \mathcal{S}'(\mathbb{R}^n;\mathcal{L}(X;Y))$ *are operator-valued tempered distributions satisfying Assumption 2.3, and moreover* $\hat{k}_j$ *agrees* *with a function in* $L^\infty(\mathbb{R}^n;\mathcal{L}(X;Y))$ *for every* $j\in\mathbb{Z}$. *In addition, suppose that*

$$\left\|(\hat{k}_j(\xi))_{-\infty}^{\infty}\right\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)}\leq A_0\qquad \text{for a.e. } \xi\in\mathbb{R}^n, \tag{4.19}$$

$$\int_{|t|>2|s|}\left\|((k_j(t-s)-k_j(t))x_j)_{-\infty}^{\infty}\right\|_{\mathrm{Rad}(Y)} w_o(t)\,\mathrm{d}t\leq A_1 w_1(s)\,\|x\|_{\mathrm{Rad}(X)} \tag{4.20}$$

for all $s\in\mathbb{R}^n\setminus\{0\}$ *and* $x\in X\otimes\operatorname{span}(\varepsilon_j)_{\infty}^{\infty}$, *where* w_0 *and* w_1 *are positive, measurable and polynomially bounded.*

Then

$$\int\|K(t)x\|_{\mathrm{Rad}(Y)}\,w_0(t)\,\mathrm{d}t\leq (A_0C(\varphi,w_0)+A_1C(\varphi,w_1))\,\|x\|_{\mathrm{Rad}(X)}$$

for all $x\in X\otimes\operatorname{span}(\varepsilon_j)_{-\infty}^{\infty}$.

Combining Corollary 4.18 with Proposition 4.6, we have the following result, which contains Theorem 4.1 as a special case, as shown below:

THEOREM 4.21. *Let* X, Y *be UMD-spaces,* $k\in\mathcal{S}'(\mathbb{R}^n;\mathcal{L}(X;Y))$ *satisfy Assumption 2.3,* $\hat{k}\in L^\infty_{\mathrm{str}}(\mathbb{R}^n;\mathcal{L}(X;Y))$, *and* $p\in\,]1,\infty[$. *Let the following conditions be satisfied:*

$$\left\|(\hat{k}(2^j\xi))_{j=-\infty}^{\infty}\right\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)}\leq A_0\qquad \text{for a.e. } \xi\in\mathbb{R}^n, \tag{4.22}$$

and

$$\begin{aligned}\int_{|t|>2|s|}\left\|(2^{-nj}(k(2^{-j}(t-s))'-k(2^{-j}t)')g_j)_{j=-\infty}^{\infty}\right\|_{\mathrm{Rad}(L^{p'}(\mathbb{R}^n;X'))} w(t)\,\mathrm{d}t&\\ \leq A_1 w(s)\,\|g\|_{\mathrm{Rad}(L^{p'}(\mathbb{R}^n;Y'))}&\end{aligned} \tag{4.23}$$

for all $s\in\mathbb{R}^n\setminus\{0\}$ *and* $x\in X\otimes\operatorname{span}(\varepsilon_j)_{-\infty}^{\infty}$, *where* $w(t):=\log(2+|t|)$.

Then $f\in X\otimes\hat{\mathfrak{D}}_0(\mathbb{R}^n)\mapsto k*f$ *extends to a bounded linear operator*

$$f\in L^p(\mathbb{R}^n;X)\mapsto k*f\in L^p(\mathbb{R}^n;Y)$$

with norm at most $C(A_0+A_1)$, *where* C *is a geometric constant.*

PROOF. By the permanence properties of R-bounds (see Lemma 3.4 and the paragraph after it), we also have analogue of conditions (4.22) valid for the extensions of the adjoint operators $\hat{k}(\xi)'$ to $L^{p'}(\mathbb{R}^n;Y')\to L^{p'}(\mathbb{R}^n;X')$. Let us then define $k_j(t):=2^{-nj}k(2^{-j}t)$, whence (4.23) is the same as (4.20) with $L^{p'}(\mathbb{R}^n;Y')$ in place of X and $L^{p'}(\mathbb{R}^n;X')$ in place of Y. Moreover, $\hat{k}_j(\xi)=\hat{k}(2^j\xi)$, so that (4.22) implies the analogue of (4.19) with the same substitutions. Thus Corollary 4.18 shows that the kernel $K(t)$ defined in (4.7) satisfies the assumption (4.8) of Proposition 4.6, and hence that proposition implies the assertion of the theorem. □

Now we can also give

Proof of Theorem 4.1. Clearly the assumption (4.2) implies (4.22). As for the condition (4.3), we can again use the permanence properties of R-bounds to obtain the same condition first for $k(\cdot)' : Y' \to X'$ in place of $k(\cdot)$, and finally for the extension $k(\cdot)' : L^{p'}(\mathbb{R}^n; Y') \to L^{p'}(\mathbb{R}^n; X')$. Thus (4.3) implies the operator norm version of the strong condition (4.23), and hence Theorem 4.1 follows as a special case of Theorem 4.21. □

Remark 4.24. As we saw in the proof of Theorem 4.21, the Hörmander condition (4.3) with operator norms could be used to deduce the more technical condition (4.23). While it is nice to have the sufficiency of the strong condition (4.23) for its own sake, the sufficiency of strong estimates becomes essential when applying Theorem 4.21 to prove multiplier theorems. Namely, as soon as our Banach space X has a Fourier-type $q > 1$ (and a UMD-space always has), the Hausdorff–Young inequality allows us to pass from estimates for $\|\hat{f}\|_{L^q(\mathbb{R}^n;X)}$ to those for $\|f\|_{L^{q'}(\mathbb{R}^n;X)}$, i.e., we are able to transform strong estimates for the q-norm in the frequency domain to strong estimates for the q'-norm in the spatial domain. However, the operator spaces $\mathcal{L}(X)$ only have trivial Fourier-type, and thus the transference of norm conditions does not work.

5. Application to special classes of singular integrals

For the application of Theorem 4.21 to classical operator-valued kernels (see Theorem 5.12, Cor. 5.14), we first provide criteria for checking the condition (4.22) without the need to know the Fourier transform $\hat{k}$ of the distribution of interest. This is done in the following lemma, the core of whose proof is simply a repetition of the classical argument. Nevertheless, we need to consider several technical points to reduce the considerations to this classical situation.

Lemma 5.1. *Consider a principal value distribution* p.v.-$k \in \mathcal{S}'(\mathbb{R}^n; \mathcal{L}(X;Y))$ *as in* (2.6)–(2.9)*, whose related sequence valued kernel* $K(t) := (2^{-nj}k(2^{-j}t))_{-\infty}^{\infty}$ *verifies the analogues of the properties* (2.6)–(2.7)*. More precisely, assume that, for every finitely non-zero* $x = (x_j)_{-\infty}^{\infty} \in \mathrm{Rad}(X)$ *we have*

$$\int_{r<|t|<2r} \|K(t)x\|_{\mathrm{Rad}(Y)}\,\mathrm{d}t \le A\,\|x\|_{\mathrm{Rad}(X)} \qquad \text{for all } r > 0, \tag{5.2}$$

$$\left\|\int_{r<|t|<R} K(t)x\,\mathrm{d}t\right\|_{\mathrm{Rad}(Y)} \le A\,\|x\|_{\mathrm{Rad}(X)} \qquad \text{for all } R > r > 0, \tag{5.3}$$

and moreover

$$\int_{|t|>2|s|} \|(K(t-s) - K(t))x\|_{\mathrm{Rad}(Y)}\,\mathrm{d}t \le A\,\|x\|_{\mathrm{Rad}(X)}\,. \tag{5.4}$$

Then, given that Y has the Radon–Nikodým property, the Fourier transform $\hat{k}$ (taken in the sense of distributions) is identified with an essentially bounded *strongly measurable function, and actually*

$$\hat{K}(\xi) := (\hat{k}(2^j\xi))_{-\infty}^{\infty} \qquad \textit{satisfies} \qquad \left\|\hat{K}(\xi)\right\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)} \le cA \tag{5.5}$$

for a.e. $\xi \in \mathbb{R}$, *where* c *is a numerical constant.*

Remark 5.6. (*i*) Of course, the assumption of the conditions (2.6)–(2.7), which are related to the existence of the principal value integral (2.9), is superfluous, since they follow from the stronger estimates (5.2)–(5.3). On the other hand, the analogue of (2.8),

$$\lim_{r\downarrow 0}\int_{r<|t|<1} K(t)x\,\mathrm{d}t \qquad \text{exists as in } \mathrm{Rad}(Y) \text{ for finitely non-zero } x\in\mathrm{Rad}(X), \tag{5.7}$$

already follows from (2.8), since we have the existence of the finite number of non-zero limits

$$\lim_{r\downarrow 0}\int_{r<|t|<1} 2^{-nj}k(2^{-j}t)x_j\,\mathrm{d}t = \lim_{r\downarrow 0}\int_{2^{-j}r<|t|<2^{-j}} k(t)x_j\,\mathrm{d}t,$$

and we just add them up.

(*ii*) The assumption (5.4) obviously follows if we have (4.23). The conditions (5.3) and (5.7) are trivial if k is odd, or slightly more generally, if its strong integral vanishes over almost every origin-centered sphere rS^{n-1}.

(*iii*) In the situation where we use the lemma, Y is already required to be UMD, hence reflexive, and thus has the RNP automatically. (See e.g. [**26**] for more on the RNP.)

Proof of Lemma 5.1. The same classical argument, which could be repeated to show that the conditions (2.6)–(2.8) imply that (2.9) gives a well-defined tempered distribution p.v.-$k : X \times \mathcal{S}(\mathbb{R}^n) \to Y$ verifying the estimate (2.10), can equally well be used to give from (5.2), (5.3) and (5.7) the analogous estimates with X and Y replaced by $\mathrm{Rad}(X)$ and $\mathrm{Rad}(Y)$. Thus we have

$$\text{p.v.-}K \in \mathcal{S}'(\mathbb{R}^n; \mathcal{L}(\mathrm{Rad}(X); \mathrm{Rad}(Y))).$$

We then make a cut-off to define

$$K^{\epsilon,R}(t) := K(t)\chi_{\epsilon<|t|<R} \qquad \text{for } R>\epsilon>0.$$

We claim that

$$\left\langle K^{\epsilon,R},\phi\right\rangle x \underset{\epsilon\downarrow 0,R\uparrow\infty}{\longrightarrow} \langle K,\phi\rangle x \qquad \text{in } \mathrm{Rad}(Y)\ \forall\,\phi\in\mathcal{S}(\mathbb{R}^n),\ x\in X\otimes\mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}. \tag{5.8}$$

Indeed, for a finitely non-zero $x \in \mathrm{Rad}(X)$,

$$\begin{aligned}\left\langle K^{\epsilon,R}, \phi\right\rangle x = \sum \varepsilon_j \int_{\epsilon<|t|<R} 2^{-nj} k(2^{-j}t) x_j \, \phi(t) \, \mathrm{d}t \\ = \sum \varepsilon_j \int_{2^{-j}\epsilon<|s|<2^{-j}R} k(s) x_j \, \phi(2^j s) \, \mathrm{d}s \\ \to \sum \varepsilon_j \, \text{p.v.-} \int_{\mathbb{R}^n} k(s) x_j \, \phi(2^j s) \, \mathrm{d}s = \sum \varepsilon_j \, \text{p.v.-} \int_{\mathbb{R}^n} 2^{-nj} k(2^{-j}t) x_j \, \phi(t) \, \mathrm{d}t \\ = \langle K, \phi \rangle \, x,\end{aligned}$$

since we can separately take each of the finite number of limits whose existence we know, and add them up.

With these technicalities out of the way, we are effectively in the classical situation, and the proof of [**34**], pp. 206–7, can merely be reproduced to estimate the integrals defining the Fourier transform of $K^{\epsilon,R}(\cdot)x$ with $x \in X \otimes \mathrm{span}(\varepsilon_j)_{j \in J}$ and $J \subset \mathbb{Z}$ a finite subset, to the result

$$\left\| \hat{K}^{\epsilon,R}(\cdot)x \right\|_{L^\infty(\mathbb{R}^n;\mathrm{Rad}(Y))} \leq cA \left\| x \right\|_{\mathrm{Rad}(X)}.$$

By the obvious estimate, this implies for $\phi \in \mathcal{S}(\mathbb{R}^n)$

$$\left\| \left\langle K^{\epsilon,R}, \hat{\phi} \right\rangle x \right\|_{\mathrm{Rad}(Y)} = \left\| \left\langle \hat{K}^{\epsilon,R}, \phi \right\rangle x \right\|_{\mathrm{Rad}(Y)} \leq cA \left\| x \right\|_{\mathrm{Rad}(X)} \left\| \phi \right\|_{L^1(\mathbb{R}^n)},$$

and it follows from (5.8) that the same inequality holds with $K^{\epsilon,R}$ replaced by K. But this means that $\phi \in \mathcal{S}(\mathbb{R}^n) \mapsto \langle K, \phi \rangle \, x \in Y \otimes \mathrm{span}(\varepsilon_j)_{j \in J}$ extends to a bounded linear operator, of norm at most $cA \left\| x \right\|_{\mathrm{Rad}(X)}$,

$$\phi \in L^1(\mathbb{R}^n) \mapsto \left\langle \hat{K}, \phi \right\rangle x \in Y \otimes \mathrm{span}(\varepsilon_j)_{j \in J},$$

where the closed subspace $Y \otimes \mathrm{span}(\varepsilon_j)_{j \in J} \approx Y^J$ of $\mathrm{Rad}(Y)$ is equipped with the norm of $\mathrm{Rad}(Y)$.

Now we can invoke the RNP of Y, or actually of Y^J which follows, by means of the equivalent condition of validity of the vector-valued Riesz Representation Theorem; see [**26**], Theorem III.1.5. (It is easy to see that the finiteness of the measure space, assumed in the theorem cited, can be replaced by σ-finiteness.) This gives an essentially unique $g[x](\cdot) = \sum \varepsilon_j g_j[x](\cdot) \in L^\infty(\mathbb{R}^n; Y \otimes \mathrm{span}(\varepsilon_j)_{j \in J})$ such that

$$\left\langle \hat{K}, \phi \right\rangle x = \int_{\mathbb{R}^n} g[x](\xi) \phi(\xi) \, \mathrm{d}\xi \tag{5.9}$$

and

$$\left\| g[x](\cdot) \right\|_{L^\infty(\mathbb{R}^n;\mathrm{Rad}(Y))} = \left\| \phi \mapsto \left\langle \hat{K}, \phi \right\rangle x \right\|_{L^1(\mathbb{R}^n) \to \mathrm{Rad}(Y)} \leq cA \left\| x \right\|_{\mathrm{Rad}(X)}. \tag{5.10}$$

It follows easily that the jth component $g_j[x]$ of $g[x]$ depends only on the jth component x_j of x, and the mappings $x_j \in X \mapsto g_j[x_j \varepsilon_j](\xi) =: G_j(\xi) x_j \in Y$ are linear, and by (5.10) they are bounded uniformly in $\xi \in \mathbb{R}^n$, disregarding a set of measure zero. If we consider the particular case with $x = \varepsilon_j x_j$, then (5.9) yields

$$\left\langle \hat{k}(2^j \cdot), \phi \right\rangle x_j = \int_{\mathbb{R}^n} G_j(\xi) x_j \, \phi(\xi) \, \mathrm{d}\xi$$

Thus $\hat{k}(2^j \cdot) = G_j$, and in particular $\hat{k} = G_0$, with the equality in the sense of distributions. This gives the asserted identification, and (5.10) gives the asserted estimate (5.5), now that we know that $g[x](\xi) = \hat{K}(\xi)x$. $\square$

With Theorem 4.21 and Lemma 5.1 at our disposal, it becomes a routine task to obtain operator-valued generalizations of classical results on the boundedness of singular integrals, with the receipt "replace any boundedness assumption by R-boundedness". In this spirit, we have the following:

Lemma 5.11. *Suppose that for $k \in L^{1,\mathrm{loc}}_{\mathrm{str}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$ and some $\delta > 0$, the collection*

$$\mathcal{T} := \{ |t|^{n+\delta} \, |s|^{-\delta} \, (k(t-s) - k(t)) : |t| > 2\,|s| > 0 \}$$

is R-bounded. Then the condition (4.23) *holds with a constant* $c(n,\delta)\mathcal{R}(\mathcal{T})$.

Proof. We have

$$\begin{aligned}
&\mathcal{R}(\{2^{-nj}(k(2^{-j}(t-s)) - k(2^{-j}t))\}_{j=-\infty}^{\infty}) \\
&\qquad = \mathcal{R}(\{(2^{-j}\,|t|)^{n+\delta}(2^{-j}\,|s|)^{-\delta}(k(2^{-j}(t-s)) - k(2^{-j}t))\}_{-\infty}^{\infty}) \, |t|^{-(n+\delta)} \, |s|^{\delta} \\
&\qquad\qquad \leq \mathcal{R}(\mathcal{T}) \, |t|^{-(n+\delta)} \, |s|^{\delta},
\end{aligned}$$

and with this estimate, (4.23) follows by integrating in the polar coordinates and using (4.5). $\square$

Theorem 5.12. *Let X and Y be UMD-spaces and suppose*

$$k \in L^{1,\mathrm{loc}}_{\mathrm{str}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$$

is an odd kernel satisfying

$$\mathcal{R}(\{ |t|^n \, k(t), \ |t|^{n+\delta} \, |s|^{-\delta} \, (k(t-s) - k(t)) : |t| > 2\,|s| > 0 \}) =: A < \infty.$$

Then k gives rise to a tempered distribution p.v.-k *in the sense of* (2.9), *and $f \in X \otimes \mathcal{S}(\mathbb{R}^n) \mapsto$* p.v.-$k * f$ *extends to a bounded mapping from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n;Y)$, for all $p \in \,]1,\infty[$, of norm at most CA with C geometric.*

Proof. By Lemma 5.11, k satisfies the condition (4.23) of Theorem 4.21. Since k is odd, to verify the assumptions of Lemma 5.1 so as to get the condition (4.22), only (5.2) needs checking, but this follows immediately from the assumed R-boundedness of $\{|t|^n \, k(t) : t \neq 0\}$. $\square$

REMARK 5.13. (*i*) With $X = Y = \mathbb{C}$, Theorem 5.12 is classical. Observe that, despite its general geometric setting and the complications on the way here, our theorem is strong enough to recover the classical result, since R-boundedness then reduces to uniform boundedness.

(*ii*) A generalization to the vector-valued situation, with $Y = X$ a UMD-space, but with a scalar-valued kernel, is first due to BOURGAIN [**12**], who considers the periodic domain $\mathbb{T}$.

As in the classical case, Theorem 5.12 has the following immediate corollary which is sufficient for many concrete examples of kernels.

COROLLARY 5.14. *Let X and Y be UMD-spaces, $k \in \mathcal{C}^1(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$ be odd and*

$$\mathcal{R}(\{|t|^n \, k(t), \ |t|^{n+1} \, \nabla k(t) : t \neq 0\}) =: A < \infty.$$

Then k satisfies the conclusion of Theorem 5.12

While Theorem 5.12 was, for simplicity, formulated for an odd kernel k, in which case the conditions (5.3) and (2.8) (or (5.7)) of Lemma 5.1 were trivially satisfied, the general theory we have developed is powerful enough to handle more general situations. We next give an illustration of a situation where the kernel is manifestly not odd.

EXAMPLE 5.15 (R-bounded semigroups). The question of maximal regularity for the abstract Cauchy problem (cf. [**27, 87**], or Chapter 1)

$$\dot{u}(t) = Au(t) + f(t) \quad \text{for } t \geq 0, \qquad u(0) = 0,$$

with A the generator of a bounded analytic semigroup and f a given function, leads one to consider the mild solution given by the variation-of-constants formula

$$Au(t) = \int_0^t AT^{t-s} f(s) \, \mathrm{d}s. \tag{5.16}$$

This is obviously a (singular) convolution integral with the kernel

$$k(t) := \begin{cases} AT^t & t > 0, \\ 0 & t \leq 0, \end{cases}$$

When (T^t) is bounded and analytic, $\mathcal{R}(T^t) \subset \mathcal{D}(A)$ for $t > 0$, and the AT^t are bounded operators whose norm behaves like $1/t$; thus tAT^t are uniformly bounded operators for $t > 0$. If we assume a little more, i.e., R-boundedness of T^t and tAT^t instead of uniform boundedness, then $|t| \, k(t)$ is obviously R-bounded, and in this special case, this already implies that $|t|^2 \, k'(t) = t^2 A^2 T^t = 4(t/2)^2 A^2 (T^{t/2})^2 = 4(t/2 \, AT^{t/2})^2$ is also R-bounded. Hence the assumptions of Cor. 5.14 are verified, except for the oddness of k, which is clearly false. In view of the proof of Theorem 5.12 we nevertheless know that k then satisfies the Hörmander condition (4.23) of Theorem 4.21, as well as (5.2) and (5.4) of Lemma 5.1. Thus only the conditions (5.3) and (2.8) need verification.

As for (5.3), we have

$$\int_r^R K(t)x\,\mathrm{d}t = \sum \varepsilon_j \int_r^R 2^{-j} A T^{2^{-j}t} x_j\,\mathrm{d}t = \sum \varepsilon_j \int_{2^{-j}r}^{2^{-j}R} AT^t x_j\,\mathrm{d}t$$
$$= \sum \varepsilon_j \int_{2^{-j}r}^{2^{-j}R} \frac{\mathrm{d}T^t x_j}{\mathrm{d}t}\,\mathrm{d}t = \sum \varepsilon_j (T^{2^{-j}R} - T^{2^{-j}r})x_j,$$

and (5.3) follows from the R-boundedness of T^t, $t > 0$.

Concerning (2.8), we have [and here the R-boundedness conditions play no rôle]

$$\int_r^1 k(t)x\,\mathrm{d}t = \int_r^1 AT^t x\,\mathrm{d}t = \int_r^1 \frac{\mathrm{d}T^t x}{\mathrm{d}t}\,\mathrm{d}t = T^1x - T^r x \underset{r\downarrow 0}{\longrightarrow} T^1 x - x,$$

which shows the existence of the limit required in (2.8).

Thus we have shown that, on a UMD-space, the mapping $f \mapsto Au$ defined by (5.16) maps $L^p(\bar{\mathbb{R}}_+; X)$ to $L^p(\bar{\mathbb{R}}_+; X)$ [and hence the Cauchy problem has maximal L^p-regularity] whenever A is the generator of the analytic semigroup (T^t) for which the sets $\{T^t \mid t > 0\}$ and $\{tAT^t \mid t > 0\}$ are R-bounded. Thus our results on singular integrals provide a direct approach to the recent maximal regularity results, allowing one to work with the variation-of-constants formula (5.16) instead of Fourier multipliers.

6. R-boundedness of families of singular integral operators

An interesting general phenomenon in the world of vector-valued inequalities is that they almost immediately self-improve to give related statements for large families of kernels; cf. e.g. [**34**], p. 493. In the more specific context of R-boundedness, this was observed by Girardi and Weis [**37**], and following these ideas, we next show how Theorem 5.12 can in fact be used to derive not only boundedness of certain singular integrals but in fact R-boundedness of families of singular integral operators which satisfy the assumptions of the theorem in such a way that the ranges of the kernels belong to the *same* R-bounded set $\mathcal{T}$.

The precise formulation of the result vaguely described above requires Pisier's notion of the *property* (α) from the geometry of Banach spaces. We exploit this notion via the following lemma which is essentially in [**21**], Lemma 3.13. The "traditional" definition of the property (α) can also be found in this same article (Def. 3.11), but actually, for $Y = X$, one could (equivalently) take the assertion of the lemma as the definition of X having the property (α). While the property (α) is independent of the UMD-condition, it is also satisfied by the most common reflexive spaces appearing in analysis; cf. [**21, 37**].

Lemma 6.1. *Let X and Y be Banach spaces with property (α). Then*

$$\mathbb{E}\,\mathbb{E}' \left| \sum_{i,j=-N}^{N} \varepsilon_i \varepsilon_j' T_{ij} x_{ij} \right|_Y \le \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})\mathbb{E}\,\mathbb{E}' \left| \sum_{i,j=-N}^{N} \varepsilon_i \varepsilon_j' x_{ij} \right|_X$$

whenever $N \in \mathbb{N}$, $x_{ij} \in X$, $T_{ij} \in \mathcal{T} \subset \mathcal{L}(X;Y)$, *and the* (ε_i) *and* (ε'_j) *are two independent systems of Rademacher functions, the related expectation operators of which are denoted by* $\mathbb{E}$ *and* $\mathbb{E}'$, *respectively. Here* $\alpha(X), \alpha(Y) < \infty$ *are geometric constants.*

Corollary 6.2. *Let* $\mathcal{T} \subset \mathcal{L}(X;Y)$ *be* R*-bounded, where the Banach spaces* X *and* Y *have the property* (α). *Then*

$$\tilde{\mathcal{T}} := \{(T_j)_{-\infty}^{\infty} \text{ finitely non-zero}, \ T_j \in \mathcal{T}\} \subset \mathcal{L}(\operatorname{Rad}(X); \operatorname{Rad}(Y)) \tag{6.3}$$

is R*-bounded, and in fact* $\mathcal{R}(\tilde{\mathcal{T}}) \leq \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})$.

Proof. For $N \in \mathbb{N}$, $\tilde{T}_i = (T_{ij})_{j=-\infty}^{\infty} \in \tilde{\mathcal{T}}$ and $\tilde{x}_i = (x_{ij})_{j=-\infty}^{\infty} \in X \otimes \operatorname{span}(\varepsilon_j)_{-\infty}^{\infty}$ we have

$$\begin{aligned}
\mathbb{E}\left\|\sum_{i=-N}^{N} \varepsilon_i \tilde{T}_i \tilde{x}_i\right\|_{\operatorname{Rad}(Y)} &= \mathbb{E}\,\mathbb{E}'\left|\sum_{i,j} \varepsilon_i \varepsilon'_j T_{ij} x_{ij}\right|_Y \\
&\leq \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})\mathbb{E}\,\mathbb{E}'\left|\sum_{i,j}\varepsilon_i\varepsilon'_j x_{ij}\right|_X = \alpha(X)\alpha(Y)\mathcal{R}(\mathcal{T})\mathbb{E}\left\|\sum_{i=-N}^{N}\varepsilon_i\tilde{x}_i\right\|_{\operatorname{Rad}(X)},
\end{aligned}$$

as a direct consequence of Lemma 6.1. □

Now we are ready to state and prove the theorem.

Theorem 6.4. *Let* X *and* Y *be UMD-spaces with property* (α), *and* $k_\lambda \in L^{1,\mathrm{loc}}_{\mathrm{str}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$, *where* $\lambda \in \Lambda$ *(any index set), be odd kernels which satisfy*

$$\{|t|^n k_\lambda(t),\ |t|^{n+\delta}|s|^{-\delta}(k_\lambda(t-s) - k_\lambda(t)) \mid\ |t| > 2|s| > 0\} \subset \mathcal{T},$$

where $\mathcal{T} \subset \mathcal{L}(X;Y)$ *is* R*-bounded. Then the family*

$$\{k_\lambda * : L^p(\mathbb{R}^n;X) \to L^p(\mathbb{R}^n;Y) \mid \lambda \in \Lambda\}$$

is R*-bounded for all* $p \in]1,\infty[$.

Proof. Let $k_j := k_{\lambda_j}$ for some $\lambda_j \in \Lambda$ when $|j| \leq N$, and $k_j := 0$ otherwise. Consider the sequence-valued kernel $K(t) := (k_j(t))_{j=-\infty}^{\infty}$. With $\tilde{\mathcal{T}}$ defined as in (6.3), it is clear that

$$\{|t|^n K(t),\ |t|^{n+\delta}|s|^{-\delta}(K(t-s) - K(t))\} \subset \tilde{\mathcal{T}}.$$

But then by Corollary 6.2 and Theorem 5.12, the operator $f \in \operatorname{Rad}(X) \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto K * f$ extends to a bounded linear operator from $L^p(\mathbb{R}^n; \operatorname{Rad}(X)) \approx \operatorname{Rad}(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; \operatorname{Rad}(Y)) \approx \operatorname{Rad}(L^p(\mathbb{R}^n;Y))$, of norm at most $C\mathcal{R}(\mathcal{T})$. But this boundedness means, by definition, that

$$\mathbb{E}\left\|\sum_{j=-N}^{N}\varepsilon_j k_{\lambda_j} * f_j\right\|_{L^p(\mathbb{R}^n;Y)} \leq C\mathcal{R}(\mathcal{T})\mathbb{E}\left\|\sum_{j=-N}^{N}\varepsilon_j f_j\right\|_{L^p(\mathbb{R}^n;X)}$$

for all $f_j \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n)$. In the above argument, the $N \in \mathbb{N}$ and the $\lambda_j \in \Lambda$ were fixed but arbitrary, and hence the result obtained is exactly the asserted R-boundedness of the collection $\{k_\lambda * \mid \lambda \in \Lambda\}$. □

Let us note some immediate consequences of this theorem:

REMARK 6.5. (*i*) The conclusion of the theorem follows in particular if

$$\{|t|^n\, k_\lambda(t),\ |t|^{n+1}\, \nabla k_\lambda(t)|\ t \neq 0\} \subset \mathcal{T}.$$

(*ii*) If X and Y are Hilbert spaces, the R-boundedness assumptions reduce to the norm-boundedness of $\mathcal{T}$.

(*iii*) If all the kernels k_λ are scalar-valued (but the spaces X and Y are any UMD-spaces with property (α)), then again the R-boundedness means just norm-boundedness of $\mathcal{T}$.

If $X = Y = \mathbb{C}$, then it is well-known that R-boundedness of operators on $L^p(\mathbb{R}^n)$ is equivalent to square-function estimates (see [**25**], Prop. 3.3, or [**53**], Sect. 2, for details). Therefore Theorem 6.4 implies classical results of the following kind (cf. [**34**], pp. 494–5):

COROLLARY 6.6. *For all $\lambda \in \Lambda$ (any index set), let $k_\lambda \in L^{1,\mathrm{loc}}(\mathbb{R}^n)$ be odd kernels satisfying*

$$|k_\lambda(t)| \le \frac{A}{|t|^n}, \qquad |k_\lambda(t-s) - k_\lambda(t)| \le A\frac{|s|^\delta}{|t|^{n+\delta}}$$

for all $|t| > 2\,|s| > 0$. Then the family

$$\{k_\lambda * : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)|\ \lambda \in \Lambda\}$$

is R-bounded for all $p \in\,]1,\infty[$; equivalently, we have the square-function inequality

$$\left\| \Big(\sum \left|k_{\lambda_j} * f_j(\cdot)\right|^2 \Big)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \le CA \left\| \Big(\sum |f_j(\cdot)|^2 \Big)^{1/2} \right\|_{L^p(\mathbb{R}^n)} \tag{6.7}$$

for all $f_j \in L^p(\mathbb{R}^n; X)$ and $\lambda_j \in \Lambda$.

7. Application to Fourier multipliers

We can also use Theorem 4.21 to obtain sufficient conditions for the L^p-boundedness of an operator $f \mapsto k * f$ entirely in terms of the symbol $\hat{k} =: m$. We present a Hörmander-type multiplier theorem in a rather general form, with a continuous smoothness parameter ℓ. The Hölder continuity assumptions (7.12) and (7.13) of the highest derivatives, which can be used to relax by one the number of classical derivatives required, is introduced in the classical context by STRÖMBERG and TORCHINSKY [**81**]. An operator-valued multiplier theorem with the slightly stronger assumptions (7.3) and (7.4) for all $|\alpha| \le \lfloor n/q \rfloor + 1$ is proved by GIRARDI and WEIS [**36**] as a consequence of a general multiplier theorem assuming Besov norm estimates for the multiplier function. Instead of using this result, we follow here an alternative approach which is closer to the classical proof of these theorems in the scalar setting, as found e.g. in [**34**], and which sheds some light on the interplay of multiplier theorems and singular integrals.

We first formulate a somewhat technical result, nevertheless containing the essential flavour of the actual theorem which is then readily derived from this intermediate result.

PROPOSITION 7.1. *Let X and Y be UMD-spaces and Y have Fourier type $q \in]1,2]$. Let $\ell > n/q$,*

$$m \in \mathcal{C}^{\lfloor \ell \rfloor}_{\mathrm{str}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X;Y)) \qquad \text{and} \qquad M(\xi) := (m(2^j\xi))_{j=-\infty}^{\infty}. \tag{7.2}$$

Suppose further that

$$\|M(\xi)\|_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)} \leq A \qquad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{7.3}$$

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)' g\|^q_{\mathrm{Rad}(L^q(\mathbb{R}^n;X'))}\,\mathrm{d}\xi\right)^{1/q} \leq A r^{-|\alpha|}\,\|g\|_{\mathrm{Rad}(L^q(\mathbb{R}^n;Y'))} \tag{7.4}$$

for all $|\alpha| \leq \lfloor \ell \rfloor$, and finally

$$\begin{aligned}\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|(D^\alpha M(\xi-\zeta)' - D^\alpha M(\xi)')g\|^q_{\mathrm{Rad}(L^q(\mathbb{R}^n;X'))}\,\mathrm{d}\xi\right)^{1/q}& \\ \leq A r^{-\ell}\,|\zeta|^{\ell-\lfloor \ell\rfloor}\,\|g\|_{\mathrm{Rad}(L^q(\mathbb{R}^n;Y'))} \qquad \text{for } |\alpha| = \lfloor \ell \rfloor,\ |\zeta| \leq r/2,&\end{aligned} \tag{7.5}$$

where (7.4)–(7.5) are assumed for all finitely non-zero $g \in \mathrm{Rad}(L^q(\mathbb{R}^n;Y'))$ and all $r \in]0,\infty[$.

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto \mathcal{F}^{-1}[m\hat{f}]$ extends to a bounded linear mapping

$$f \in L^p(\mathbb{R}^n;X) \mapsto \mathcal{F}^{-1}[m\hat{f}] \in L^p(\mathbb{R}^n;Y)$$

for all $p \in [q',\infty[$, with norm at most C_pA, where C_p is a geometric constant.

REMARK 7.6. (*i*) From the "periodicity" of the sequence-valued multiplier M [in the sense that the sequence $M(2^i\xi) = (m(2^{i+j}\xi))_{j=-\infty}^{\infty}$ is just the sequence $M(\xi) = (m(2^j\xi))_{j=-\infty}^{\infty}$ with indexing shifted by i steps], it follows easily that the conditions (7.4) and (7.5) for a general $r \in]0,\infty[$ are already implied by the corresponding conditions for (say) $r = 1$.

(*ii*) Using, as in the proof of Theorem 4.21, the permanence properties of R-bounds, it is immediate that the conditions (7.4) and (7.5) are verified if instead of (7.4) we assume

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)\|^q_{\mathrm{Rad}(X)\to\mathrm{Rad}(Y)}\,\mathrm{d}\xi\right)^{1/q} \leq A r^{-|\alpha|} \qquad \text{for } |\alpha| \leq \lfloor \ell \rfloor,$$

and instead of (7.5) a similar modification obtained in the obvious way.

(*iii*) Recall that UMD-spaces automatically have some Forier-type $q \in]1,2]$.

It is also possible to verify (7.4) and (7.5) by strong integral conditions instead of operator norm conditions, yet avoiding considerations of the extended operators acting on $L^q(\mathbb{R}^n;Y')$. Indeed, assume

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)'y'\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi\right)^{1/q} \le A\,r^{-|\alpha|}\,\|y'\|_{\mathrm{Rad}(Y')}\,. \tag{7.7}$$

Then

$$\begin{aligned}
&\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|D^\alpha M(\xi)'g(\cdot)\|^q_{\mathrm{Rad}(L^q(\mathbb{R}^n;X'))}\,\mathrm{d}\xi\right)^{1/q} \\
&\qquad \le C\left(\frac{1}{r^n}\int_{r<|\xi|<2r}\left(\int_{\mathbb{R}^n}\|D^\alpha M(\xi)'g(t)\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}t\right)\mathrm{d}\xi\right)^{1/q} \\
&\qquad = C\left(\int_{\mathbb{R}^n}\left(\frac{1}{r^n}\int_{r<|\xi|<2r}\|D^\alpha M(\xi)'g(t)\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi\right)\mathrm{d}t\right)^{1/q} \\
&\qquad \le C\left(\int_{\mathbb{R}^n}(A\,r^{-|\alpha|}\,\|g(t)\|_{\mathrm{Rad}(Y')})^q\,\mathrm{d}t\right)^{1/q} \le \tilde{C}A\,r^{-|\alpha|}\,\|g\|_{\mathrm{Rad}(L^{p'}(\mathbb{R}^n;Y'))}\,,
\end{aligned}$$

where we used the isomorphism of $\mathrm{Rad}(L^{p'}(\mathbb{R}^n;Z))$ and $L^{p'}(\mathbb{R}^n;\mathrm{Rad}(Z))$ in the first and last steps, Fubini's theorem in the second, and the assumption (7.7) in the third. What we have proved is that (7.7) (for all $|\alpha|\le\lfloor\ell\rfloor$) implies (7.4), and exactly the same reasoning yields out of

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} \|(D^\alpha M(\xi-\zeta)'-D^\alpha M(\xi)')y'\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi\right)^{1/q} \le Ar^{-\ell}\,|\zeta|^{\ell-\lfloor\ell\rfloor}\,\|y'\|_{\mathrm{Rad}(Y')} \tag{7.8}$$

(for appropriate α and ζ) the condition (7.5).

These remarks lead us to the following refinement of Corollaries 4.9 and 4.10 in Girardi and Weis [**36**], where one takes $\ell=\lfloor n/q\rfloor+1$ so that the difference estimates below are replaced by having some more derivatives, and moreover the pair of strong conditions as in (7.10), (7.11) is replaced by a single norm condition.

Theorem 7.9. *Let X and Y be UMD-spaces with Fourier-type $q\in\,]1,2]$. Let $\ell>n/q$, and assume* (7.2), (7.3), *and moreover the conditions [for all $x\in X\otimes\mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}$, $y'\in Y'\otimes\mathrm{span}(\varepsilon_j)_{-\infty}^{\infty}$]*

$$\int_{1<|\xi|<2}\|D^\alpha M(\xi)x\|^q_{\mathrm{Rad}(Y)}\,\mathrm{d}\xi \le A^q\,\|x\|^q_{\mathrm{Rad}(X)} \qquad \textit{for } |\alpha|\le\lfloor\ell\rfloor \tag{7.10}$$

$$\int_{1<|\xi|<2}\|D^\alpha M(\xi)'y'\|^q_{\mathrm{Rad}(X')}\,\mathrm{d}\xi \le A^q\,\|y'\|^q_{\mathrm{Rad}(Y')} \qquad \text{"} \qquad \text{"} \tag{7.11}$$

$$(7.12)\quad \int_{1<|\xi|<2} \|(D^\alpha M(\xi-\zeta) - D^\alpha M(\xi))x\|^q_{\mathrm{Rad}(Y)}\, \mathrm{d}\xi \le A^q \,\|x\|^q_{\mathrm{Rad}(X)}$$

$$\textit{for } |\alpha| = \lfloor \ell \rfloor,\ |\zeta| < \frac{1}{2}$$

$$(7.13)\quad \int_{1<|\xi|<2} \|(D^\alpha M(\xi-\zeta)' - D^\alpha M(\xi)')y'\|^q_{\mathrm{Rad}(X')}\, \mathrm{d}\xi \le A^q \,\|y'\|^q_{\mathrm{Rad}(Y')}$$

” ”

Then $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n) \mapsto \mathcal{F}^{-1}[m\hat{f}]$ *extends to a bounded linear mapping*

$$f \in L^p(\mathbb{R}^n; X) \mapsto \mathcal{F}^{-1}[m\hat{f}] \in L^p(\mathbb{R}^n; Y) \qquad \textit{for all } p \in]1,\infty[$$

with norm at most C_pA*, where* C_p *is a geometric constant.*

Proof. By the computations before the statement of the theorem, (7.11) implies (7.4) and (7.13) implies (7.5). Using Remark 7.6(i), Proposition 7.1 yields the assertion for $p \in [q', \infty[$. On the other hand, the conditions (7.10) and (7.12) are the analogues, respectively, of (7.11) and (7.13) for the dual multiplier $\xi \mapsto m(\xi)' \in \mathcal{L}(Y', X')$. Moreover, the condition (7.3) already implies its analogue for $m(\cdot)'$ by the permanence properties of R-bounds. Thus we also obtain the boundedness of

$$g \in L^p(\mathbb{R}^n; Y') \mapsto \mathcal{F}^{-1}[m(\cdot)'\hat{f}] \in L^p(\mathbb{R}^n; X') \qquad \text{for } p \in [q', \infty[.$$

By a well-known duality argument, the boundedness of the operator corresponding to the multiplier $m(\cdot)'$ from $L^{p'}(\mathbb{R}^n; Y')$ to $L^{p'}(\mathbb{R}^n; X')$ is equivalent to the boundedness of the operator with multiplier m from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$. Thus we also obtain the assertion of the theorem for $p \in]1, q]$. If $q = 2$, we have already covered all $p \in]1, \infty[$, and otherwise the boundedness for the remaining exponents $p \in]q, q'[$ is obtained by interpolation. □

Remark 7.14. Combining Theorem 7.9 with results from Chapter 1 shows that the same assumptions already imply the boundedness also from the Hardy spaces $H^p(\mathbb{R}^n; X)$ to $H^p(\mathbb{R}^n; Y)$ for all $p \in](1/q' + \ell/n)^{-1}, 1]$, in particular, from $H^1(\mathbb{R}^n; X)$ to $H^1(\mathbb{R}^n; Y)$ since $\ell > n/q \implies \ell/n + 1/q' > 1/q + 1/q' = 1$. It is shown in Chapter 1 that a multiplier operator satisfying (7.10) and (7.12) [somewhat weaker conditions without randomization will do], and which is bounded from $L^{\tilde{p}}(\mathbb{R}^n; X)$ to $L^{\tilde{p}}(\mathbb{R}^n; Y)$ for some $\tilde{p} \in]1, \infty[$, extends boundedly to the scale of the Hardy spaces mentioned. See Theorem 5.13 of Chapter 1; also [**36**], Cor. 4.6.

As a very particular case of Theorem 7.9, we state the following corollary which was already proved in [**36**].

COROLLARY 7.15. *Let X, Y be UMD-spaces and Y have Fourier type $q > 1$. If $m \in \mathcal{C}^{\lfloor n/q \rfloor+1}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X;Y))$ satisfies*

$$\mathcal{R}(\{|\xi|^{|\alpha|} D^\alpha m(\xi) | \ \xi \in \mathbb{R}^n \setminus \{0\}) \le A \qquad \text{for all } |\alpha| \le \lfloor n/q \rfloor + 1,$$

then m is a Fourier multiplier from $L^p(\mathbb{R}^n; X)$ to $L^p(\mathbb{R}^n; Y)$ with norm at most $C_p A$.

We then return to prove our Proposition 7.1 [which was already used to prove Theorem 7.9]. The proof becomes a simple modification of the reasoning in the scalar-valued context (cf. [**43**] or [**34**], §II.6), as soon as one realizes the right way to make these modifications. Let us elaborate a little on this.

In the scalar-valued case, the R-boundedness-type assumptions (7.3)–(7.5) are unnecessary, and one simply assumes the same conditions with m in place of M. The idea of the proof is to smoothly cut the multiplier m into pieces, say m_j, which are well-behaved enough so that they correspond to Fourier transforms of integrable functions k_j. It remains to investigate how the multiplier conditions (7.3)–(7.5) transform to the properties of the kernels k_j, so that results on singular integral operators (classical analogues of Theorem 4.21) can be applied.

In the present situation, the assumptions involve the sequence-valued multiplier M, and also in the case of singular integral, the sequence-valued kernel K. Yet the actual operators of interest are defined in terms of the multiplier m and the kernel k. To make sense of our passing from the Fourier domain to the non-transformed domain, some truncations are to be first performed, as in the scalar-case. However, it is not at all the same whether we first truncate m and then form the corresponding sequence-valued multiplier, or if we first form the sequence M, and perform a cut-off (in the variable ξ) on this sequence. In fact, we shall need to apply both types of truncations mentioned, in the appropriate order.

In the following lemma, the new features compared to the classical situation are the Fourier-type condition required to use the HAUSDORFF–YOUNG inequality (which is, of course, a mere additional statement), and the weight function $\log(2+t)$ arising from BOURGAIN's lemma (which is also easily dealt with).

LEMMA 7.16. *Let X have Fourier type $q \ge 1$ and let $\ell > n/q$. Let $k \in (L^{1,\mathrm{loc}} \cap \mathcal{S}')(\mathbb{R}^n; X)$ and let its Fourier transform be $\mathcal{C}^{\lfloor \ell \rfloor}$ and satisfy*

$$\left(\int_{\mathbb{R}^n} \left| D^\alpha \hat{k}(\xi - \zeta) - D^\alpha \hat{k}(\xi) \right|_X^q \mathrm{d}\xi \right)^{1/q} \le A\, |\zeta|^{\ell - \lfloor \ell \rfloor} \qquad \text{for } |\alpha| = \lfloor \ell \rfloor,\ |\zeta| \le \delta.$$

Then, with $w(t) := w(|t|) := \log(2 + |t|)$, we have

$$\int_{|t|>r} |k(t)|_X\, w(t)\, \mathrm{d}t \le CA\, r^{n/q - \ell} w(r) \qquad \text{for } r \ge \frac{1}{4\delta}.$$

PROOF. Observe that $\sum_{i=1}^n |\sin(\pi t_i)| = 0$ if and only if $t \in \mathbb{Z}^n$. Thus, for $0 < a \le |t| \le b < 1$, we have, by compactness, $\sum_{i=1}^n |\sin(\pi t_i)| \ge c(a,b) > 0$.

Thus, when the variable t is appropriately restricted, we can majorize unity by the sum of sines, and we use this idea to estimate

$$\begin{aligned}\int_{2^j r<|t|\le 2^{j+1}r} |k(t)|_X\, w(t)\,\mathrm{d}t &\le C\sum_{i=1}^{n}\int_{2^j r<|t|\le 2^{j+1}r} \left|k(t)\sin(\pi t_i/2^{j+2}r)\right|_X w(t)\,\mathrm{d}t\\ &\le C\sum_{i=1}^{n}\sum_{|\alpha|=\lfloor \ell\rfloor}\left(\int_{2^j r<|t|\le 2^{j+1}r}\left|t^\alpha k(t)(e^{2\pi t\cdot e_i/2^{j+2}r}-1)\right|_X^{q'}\mathrm{d}t\right)^{1/q'}\\ &\qquad\times\left(\int_{2^j r}^{2^{j+1}r} w^q(\rho)\rho^{-\ell q+n-1}\,\mathrm{d}\rho\right)^{1/q}.\end{aligned}$$

Using the assumptions and the Hausdorff–Young inequality, the first factor is estimated by

$$\left(\int_{\mathbb{R}^n}\left|D^\alpha\hat{k}(\xi-e_i/2^{j+2}r)-D^\alpha\hat{k}(\xi)\right|_X^q\mathrm{d}\xi\right)^{1/q}\le A\,2^{-j-2}r^{-1},$$

provided $2^{-j-2}r^{-1}\le\delta$, which holds for $j\in\mathbb{N}$, since $r\ge 1/4\delta$, and the second factor is easily seen to be bounded by $c(1+j)w(r)2^{j(n/q-\ell)}r^{n/q-\ell}$.

Summing over $j\in\mathbb{N}$ we get the desired conclusion, since the series $\sum_{j=0}^{\infty}(1+j)2^{j(n/q-\ell)}$ converges to a finite quanitity for $n/q-\ell<0$. $\square$

In the next two lemmata, we present the two kinds of cut-offs we perform on the multiplier. The proofs involve straightforward computations, and we merely mention the new features compared to the classical situation. It is convenient to adopt the abbreviations

$$U:=L^q(\mathbb{R}^n;X'),\qquad V:=L^q(\mathbb{R}^n;Y'),\tag{7.17}$$

since for the proof these are just two Banach spaces, whose "internal structure" is of no interest to us. Note, however, that the spaces U and V (as well as $\mathrm{Rad}(U)$ and $\mathrm{Rad}(V)$ have Fourier-type $q\in\,]1,2]$ whenever X and Y have.

Lemma 7.18. *For m as in Proposition 7.1 and $\phi\in\mathcal{S}(\mathbb{R}^n)$, the new multipliers $m(\cdot)\phi(\delta\cdot)$, $\delta>0$, satisfy the assumptions of Proposition 7.1 uniformly in δ. More precisely, the inequalities (7.3)–(7.5) hold with $M(\xi)=(m(2^{-j}\xi))_{j=-\infty}^{\infty}$ replaced by $(m(2^{-j}\xi)\phi(\delta 2^{-j}\xi))_{j=-\infty}^{\infty}$ with a constant $C(\phi)A$ in place of A.*

Sketch of proof. The proof uses straighforward estimates. The only new feature related to the sequence-valuedness of the kernel is the use of Kahane's contraction principle: Leibniz' rule yields terms of the form

$$(D_\xi^\theta[m(2^j\xi)](\delta 2^j)^{|\alpha|-|\theta|}D^{\alpha-\theta}\phi(\delta 2^j\xi)x_j)_{-\infty}^{\infty},\tag{7.19}$$

and since the scalar quantities $(\delta 2^j\,|\xi|)^{|\alpha|-|\theta|}D^{\alpha-\theta}\phi(\delta 2^{-j}\xi)$ are bounded by a constant $C(\phi)$, the contraction principle gives a bound of the form

$$C(\phi)\,|\xi|^{|\theta|-|\alpha|}\left\|D_\xi^\theta(m(2^j\xi)'g_j)_{-\infty}^{\infty}\right\|_{\mathrm{Rad}(U)}$$

for the Rademacher norm of the quantity in (7.19). Using estimates of this type, the proof is a routine computation along entirely classical lines. □

LEMMA 7.20. *For M as in Proposition 7.1, and $\sum_{\mu=-\infty}^{\infty} \hat\varphi_0(2^{-\mu}\xi) = 1$ the partition of unity used in the radial Littlewood–Paley decomposition, denote $M_\mu(\xi) := M(\xi)\hat\varphi_0(2^{-\mu}\xi)$. Then we have the inequalities*

(7.21)
$$\left(\int_{\mathbb{R}^n} \|D^\alpha M_\mu(\xi)'g\|^q_{\mathrm{Rad}(U)}\,\mathrm{d}\xi\right)^{1/q} \le CA\,2^{\mu(n/q-|\alpha|)}\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| \le \lfloor \ell \rfloor,$$

and

$$\left(\int_{\mathbb{R}^n} \|(D^\alpha M_\mu(\xi-\zeta)' - D^\alpha M_\mu(\xi)')g\|^q_{\mathrm{Rad}(U)}\,\mathrm{d}\xi\right)^{1/q} \le CA\,2^{\mu(n/q-\ell)}\,|\zeta|^{\ell-\lfloor\ell\rfloor}\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| = \lfloor \ell \rfloor \tag{7.22}$$

as well as

$$\left(\int_{\mathbb{R}^n} \left\|D^\alpha_\xi[M_\mu(\xi)'(e^{\mathbf{i}2\pi s\cdot\xi}-1)]g\right\|^q_{\mathrm{Rad}(U)}\,\mathrm{d}\xi\right)^{1/q} \le CA\,2^{\mu(n/q-|\alpha|+1)}\,|s|\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| \le \lfloor \ell \rfloor,\ |s| \le 2^{-\mu}, \tag{7.23}$$

and finally

$$\begin{aligned}\Bigg(\int_{\mathbb{R}^n} \Big\|&(D^\alpha[M_\mu(\cdot)'(e^{\mathbf{i}2\pi s\cdot(\cdot)}-1)](\xi-\zeta) \\ &- D^\alpha[M_\mu(\cdot)'(e^{\mathbf{i}2\pi s\cdot(\cdot)}-1)](\xi))g\Big\|^q_{\mathrm{Rad}(U)}\,\mathrm{d}\xi\Bigg)^{1/q} \\ &\le CA\,2^{\mu(n/q-\ell+1)}\,|s|\cdot|\zeta|^{\ell-\lfloor\ell\rfloor}\,\|g\|_{\mathrm{Rad}(V)} \quad \text{for } |\alpha| = \lfloor \ell \rfloor,\ |s| \le 2^{-\mu},\end{aligned} \tag{7.24}$$

where C is a numerical constant, and the inequalities hold for all finitely non-zero $g \in \mathrm{Rad}(V) := \mathrm{Rad}(L^{p'}(\mathbb{R}^n;Y'))$.

NOTE ON PROOF. The proof is straightforward and entirely classical. The fact that M and M_μ are sequence-valued plays no rôle here. A direct computation only gives (7.22) and (7.24) for $|\zeta| \le c2^\mu$ [with c a numerical constant] but for $|\zeta| > c2^\mu$ one can obtain the corresponding estimates by the triangle inequality from (7.21) or (7.23), respectively. □

As the final preparatory step towards proving Proposition 7.1, we note the following reduction:

LEMMA 7.25. *Without loss of generality, the multiplier m is compactly supported in $\mathbb{R}^n \setminus \{0\}$. Thus, without loss of generality, m is strongly integrable and $k := \check m$, taken in the strong sense, is a strongly measurable, essentially bounded function.*

Proof. To see this, let $\eta \in \mathcal{D}(\mathbb{R}^n)$, as before, have range $[0,1]$, be 1 for $|\xi| \leq 1/2$ and 0 for $|\xi| \geq 1$. Then $\eta(\cdot/R) - \eta(\cdot/\epsilon)$ will have the same range, be 1 for $\epsilon \leq |\xi| \leq R/2$ and 0 for $|\xi| < \epsilon/2$ or $|\xi| > R$. Thus, $m_\epsilon^R(\xi) := m(\xi)(\eta(\xi/R) - \eta(\xi/\epsilon))$ is compactly supported in $\mathbb{R}^n \setminus \{0\}$, and for any $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n)$, we have $m\hat{f} = m_\epsilon^R \hat{f}$ as soon as ϵ is small and R large enough. Moreover, by Lemma 7.18, the multipliers m_ϵ^R satisfy the same conditions as those assumed for m, with a constant CA in place of A. Thus, provided we can prove the assertion of Proposition 7.1 with the additional support condition on m, then for a general m and $f \in X \otimes \hat{\mathcal{D}}_0(\mathbb{R}^n)$, we have

$$\left\|\mathcal{F}^{-1}[m\hat{\phi}]\right\|_{L^p(\mathbb{R}^n;Y)} = \lim_{\epsilon\downarrow 0,\, R\uparrow\infty} \left\|\mathcal{F}^{-1}[m_\epsilon^R\hat{\phi}]\right\|_{L^p(\mathbb{R}^n;Y)} \leq CA\,\|\phi\|_{L^p(\mathbb{R}^n;X)},$$

and hence also the general form of the assertion follows.

That m is strongly integrable is clear, since it is strongly measurable [being even strongly continuous by (7.2)], essentially bounded [by (7.3)] and compactly supported. □

Now we are ready to prove the multiplier theorem, and with Lemma 7.25 at our disposal, it is reduced to showing that $k := \check{m}$ satisfies the appropriate conditions required for an integral kernel to give a bounded operator.

Proof of Proposition 7.1. We need to show that $k := \check{m}$ satisfies the Hörmander condition (4.23) of Theorem 4.21. Denote $K(t) := (2^{-nj}k(2^{-j}t))_{j=-\infty}^{\infty}$; i.e., $K := \check{M}$, and moreover $K_\mu := \check{M}_\mu$, where the M_μ are the pieces of M from the radial Littlewood–Paley decomposition, as in Lemma 7.20.

We derive two different estimates for

$$\int_{|t|>2|s|} \|(K_\mu(t-s)' - K_\mu(t)')g\|_{\mathrm{Rad}(U)}\, w(t)\,\mathrm{d}t, \tag{7.26}$$

which are useful for different ranges of s and μ:

As a *first case*, we can make the crude estimate by

$$2\int_{|t|>|s|} \|K_\mu(t)'g\|_{\mathrm{Rad}(U)}\, w(t)\,\mathrm{d}t.$$

The Fourier transform of $K_\mu(t)'g$ is $M_\mu(\xi)'g$, which satisfies (7.22), and so, applying Lemma 7.16, we get the bound

$$CA\,2^{\mu(n/q-\ell)}\,|s|^{n/q-\ell}\,w(s)\,\|g\|_{\mathrm{Rad}(V)}\,. \tag{7.27}$$

As a *second case*, we observe that the Fourier transform of $t \mapsto K_\mu(t-s)'g - K_\mu(t)'g$ is $M_\mu(\xi)'(e^{\mathrm{i}2\pi s\cdot\xi} - 1)g$, which satisfies (7.24); whence Lemma 7.16 gives the bound

$$CA\,2^{\mu(n/q-\ell+1)}\,|s|\cdot|s|^{n/q-\ell}\,w(s)\,\|g\|_{\mathrm{Rad}(V)}\,. \tag{7.28}$$

Using one of the two estimates (7.27) or (7.28) for (7.26) when appropriate, we have

$$\sum_{\mu=-\infty}^{\infty} \int_{|t|>2|s|} \|(K_\mu(t-s)' - K_\mu(t)')g\|_{\mathrm{Rad}(U)}\, w(t)\, \mathrm{d}t$$

$$\leq CA\, w(s)\, \|g\|_{\mathrm{Rad}(V)} \left(\sum_{\mu\colon 2^\mu|s|\geq 1} (2^\mu\,|s|)^{n/q-\ell} + \sum_{\mu\colon 2^\mu|s|<1} (2^\mu\,|s|)^{n/q-\ell+1} \right).$$

We recall that $n/q - \ell > 0$ by the assumption in Proposition 7.1. On the other hand, since the assumptions of Proposition 7.1 are the stronger the larger ℓ we have, we may assume that $\ell < n/q+1$, i.e., $n/q-\ell+1>0$. When this is the case, the two geometric series above are bounded by finite quanities depending only on n, q and ℓ.

The estimate established shows that the sequence-valued kernels

$$K^\nu := \sum_{\mu=-\nu}^{\nu} K_\mu$$

satisfy uniformly the weighted Hörmander condition

$$\int_{|t|>2|s|} \|(K^\nu(t-s)' - K^\nu(t)')g\|_{\mathrm{Rad}(U)}\, w(t)\, \mathrm{d}t \leq CA\, w(s)\, \|g\|_{\mathrm{Rad}(V)}\,. \tag{7.29}$$

The Fourier transform of $K^\nu(t)'g$ is

$$\sum_{\mu=-\nu}^{\nu} M_\mu(\xi)'g = \sum_{\mu=-\nu}^{\nu} M(\xi)'\hat{\varphi}_0(2^{-\mu}\xi)g = \left(m(2^j\xi)' \sum_{\mu=-\nu}^{\nu} \hat{\varphi}_0(2^{-\mu}\xi) g_j \right)_{j=-\infty}^{\infty}.$$

We recall that m is compactly supported away from 0; hence also $\xi \mapsto m(2^j\xi)'$ has the same property. Thus, for any finitely non-zero $g = (g_j)_{-\infty}^{\infty} \in \mathrm{Rad}(V) := \mathrm{Rad}(L^{p'}(\mathbb{R}^n; Y'))$, we observe that $\sum_{\mu=-\nu}^{\nu} \hat{\varphi}_0(2^{-\mu}\xi) = 1$ for ξ on the union of the supports of $m(2^j\xi)'g_j$, as soon as ν is large enough. Whence for all large enough ν (depending on g), $K^\nu(\cdot)'g = K(\cdot)'g$, and we find that the weighted Hörmander condition (4.23) (with $p'=q$), which we need in order to apply Theorem 4.21, is already contained in the uniform estimate (7.29). Thus the assertion for $p=q'$ follows from Theorem 4.21.

To show the assertion for $p \in\,]q', \infty[$, we invoke the classical theory of singular integrals. We take in the estimate (4.23), which we already proved, $g = g_0\varepsilon_0$, where $g_0(\cdot) = \psi(\cdot)y'$ for some non-zero $\psi \in L^q$, and some $y' \in Y'$. In this case, (4.23) reduces to

$$\int_{|t|>2|s|} |(k(t-s)' - k(t)')y'|_{X'}\, \mathrm{d}t \leq CA\, |y'|_{Y'}\,;$$

we also dropped the weight w, as we clearly can, since the weighted condition is stronger than the unweighted one. But this is just the vector-valued generalization of the usual Hörmander condition for the kernel $k(\cdot)'$. Moreover, it is well-known (from a duality argument) that the operator $k(\cdot)*$ belongs to

$$\mathcal{L}(L^{q'}(\mathbb{R}^n;X), L^{q'}(\mathbb{R}^n;Y))$$

if and only if $k(\cdot)'*$ belongs to $\mathcal{L}(L^q(\mathbb{R}^n;Y'), L^q(\mathbb{R}^n;X'))$ and the operator-norms agree.

We conclude, by duality, that $k(\cdot)'*$ is bounded from $L^q(\mathbb{R}^n;Y')$ to $L^q(\mathbb{R}^n;X')$, and then from the fact that $k(\cdot)'$ satisfies Hörmander's condition that it is bounded from $L^{p'}(\mathbb{R}^n;Y')$ to $L^{p'}(\mathbb{R}^n;X')$ for $p' \in]1,q]$, with a constant C_pA. Finally, again by duality, we have that $k(\cdot)*$ is bounded from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n;Y)$ for $p \in [q',\infty[$, and this completes the proof. $\square$

CHAPTER 3

Singular integrals on Besov spaces

The boundedness of singular convolution operators $f \mapsto k * f$ is studied on Besov spaces of vector-valued functions, the kernel k taking values in $\mathcal{L}(X, Y)$. The main result is a Hörmander-type theorem giving sufficient conditions for the boundedness of such an operator on these spaces.

The chapter is based on the joint manuscript [**48**] written with L. Weis.

1. Introduction

The scale of Besov spaces has the remarkable property that these function spaces retain their good character in the vector-valued setting, even when the underlying Banach space lacks all "good" properties such as reflexivity, separability, etc. Not surprisingly thus, before the right line of attack to the multiplier theorems on the vector-valued L^p spaces was found, the setting of the Besov spaces $B_q^{s,p}$ was found to be more fertile, as explained in Chapter 0 and briefly recalled:

Whereas in the X-valued L^p spaces the analogues of the classical multiplier theorems require special geometry of the underlying Banach space X, it was observed (independently) by H. Amann [**1**] and L. Weis [**85**] (see also [**35**]) in the second half of the 90's that the situation was quite different for the Besov spaces. In fact, even operator-valued multiplier theorems were obtained on $B_q^{s,p}(X)$ (the Besov space of X-valued functions) with no geometric restrictions on the underlying Banach space X. Moreover, norm boundedness conditions on the derivatives of the multiplier function (imitating the classical ones due to Mihlin and Hörmander, and some generalizations) were found to be sufficient to give the boundedness of the associated operator on $B_q^{s,p}(X)$. In a sharp contrast to this, recent studies [**21, 22, 87**] of operator-valued $L^p(X)$-multipliers have revealed the necessity of a strengthened notion of uniform boundedness, namely *R-boundedness*, in this connection.

Now that the situation is better understood on both scales of spaces, the results on $L^p(X)$ and $B_q^{s,p}(X)$ are seen to complement each other: Although it is perhaps desirable to work with the more concrete and familiar Bôchner spaces when this is possible, it is not always possible, and one is therefore forced to use substitute results when X is non-reflexive, or more generally, non-UMD. On the other hand, the results on the Besov spaces remain invariant, to a large extent at least, under the geometry of the underlying Banach space X. Continuity results on more classical function spaces can then be derived using sharp embedding theorems by which spaces such as $L^p(X)$ and BUC are related to the Besov scale. Moreover, the Besov spaces include as subscales several "semi-classical" function spaces such as BUC^s $(= B_\infty^{s,\infty})$ and $W^{s,p}$ $(= B_p^{s,p}$, $p \in [1,\infty[)$ for non-integral values of $s > 0$. The reader is referred to [**1**] for details on these points.

The philosophy of the present chapter is to adopt the convolution-integral point of view to the translation-invariant operators on $B_q^{s,p}(X)$, i.e., instead of thinking $f \mapsto \mathcal{F}^{-1}[m\hat{f}]$ (the Fourier multiplier point of view), we write this directly as $f \mapsto k * f$, where $k = \mathcal{F}^{-1}m$, and the goal is to find sufficient conditions for the boundedness on $B_q^{s,p}(X)$ in terms of the (singular) convolution kernel k.

This approach has several advantages: First of all, operators appear in applications which are naturally given in the convolution form, so that it is desirable to be able to determine the boundedness from the structure of the convolution kernel, without the need to first transform everything to the frequency domain. Second, such an approach helps to decouple the boundedness conditions in the theorems from certain properties of the underlying Banach space X. In fact, when the conditions are expressed in terms of the multiplier m, the minimal order of smoothness required for the boundedness of the associated operator depends on the *Fourier-type* of the underlying Banach space X (see [**35**]). On the other hand, the Fourier-type does not enter the present results in any way; yet these results are strong enough to be used to rederive many of the multiplier theorems in [**35**]. The Fourier-type only enters the scene when we want to show that the conditions assumed on the multiplier actually imply the kind of conditions we need on the corresponding kernel, so as to apply the convolution results. (We are not going to consider this point any further here, but dedicate this chapter [except for the last section] to the convolution point-of-view. Multiplier theorems on Besov spaces are then derived, using the results of this chapter, in Chapter 4.)

The chapter is organized as follows: Sect. 2 collects preliminary results and notation, including the definition of the vector-valued Besov spaces and the operators to be studied. In Sect. 3, we formulate the problems we address and we study the convolution operators $k*$ on $B_q^{s,p}(X)$ in rather general terms. The main result of this section, Theorem 3.15, gives a characterization of the convolutors (see Def. 3.11) on $B_q^{s,p}(X)$ in terms of convolutors on $L^p(X)$. In this way, the original problem of boundedness is reduced to a sequence of subproblems on $L^p(X)$ (related to the "dyadic pieces" of the kernel k obtained from a Littlewood–Paley decomposition). Sect. 4 collects some results for the treatment of the above-mentioned L^p-subproblems, and the results so far combine to give Theorem 4.10, where the sufficient conditions for k to be a $B_q^{s,p}$-convolutor are expressed more explicitly, without reference to L^p-convolutors. This result is used in Sect. 5 to derive our Hörmander-type Theorem 5.7, where the sufficient conditions are expressed in a style as classical as possible. Although this is no longer an exact characterization, partial converse results are proved along the way which show that the assumptions cannot be essentially weakened in general.

An application to evolutionary integral equations is considered in Sect. 6, showing that the conditions of Theorem 5.7 are also satisfied by operators naturally arising from that field. In Sect. 7, some counterexamples are given to further demonstrate the necessity of some of the conditions imposed. Finally, Sect. 8 aims at clarifying the difference between the theories of translation-invariant operators on vector-valued Besov and Bôchner spaces—this comparison seems to be most easily carried out in the multiplier set-up which is hence adopted in this last section of the present chapter.

2. Preliminaries

Spaces of functions and distributions. We are mostly concerned with functions (or distributions) defined on all of $\mathbb{R}^n$, where n is arbitrary but fixed throughout the discussion. Hence the domain $\mathbb{R}^n$ will not be indicated explicitly, and we write, e.g., $L^p(X)$ for the space of Bôchner measurable X-valued functions on $\mathbb{R}^n$, with $\|f\|_p := \big(\int |f(t)|_X^p \,\mathrm{d}t\big)^{1/p} < \infty$. Here and below an integral always refers to integration over the whole space $\mathbb{R}^n$, unless another domain is specified explicitly.

$\mathcal{S}(X)$ is the Schwartz space of smooth, X-valued, rapidly decreasing functions, and $\mathcal{S} := \mathcal{S}(\mathbb{C})$. $\mathcal{S}(X)$ is endowed with its usual topology generated by the countable collection of seminorms $\|\psi\|_{\alpha,\beta} := \big\|t \mapsto t^\beta D^\alpha \psi(t)\big\|_\infty$, $\alpha, \beta \in \mathbb{N}^n$. $\mathcal{D}(X)$ consists of the compactly supported elements of $\mathcal{S}(X)$. The space of X-valued tempered distributions is $\mathcal{S}'(X) := \mathcal{L}(\mathcal{S}, X)$.

As with $\mathcal{S}$, we write more generally $\mathfrak{F} := \mathfrak{F}(\mathbb{C})$ for the scalar-valued version of any function (or distribution) space $\mathfrak{F} \in \{\mathcal{D}, L^p, \mathcal{S}', \ldots\}$. In some rare occasions where the vector-valuedness of a function space is immaterial, we may depart from this convention and simply write $\mathfrak{F}$ even for the vector-valued function-space, but this is always indicated explicitly.

Rather than $\mathcal{S}(X)$, our most important test-function class will be the smaller algebraic tensor product $X \otimes \mathcal{S}$, a reason for which will appear below. We note that this is dense in $\mathcal{S}(X)$ w.r.t. its usual topology. A sketch of the proof is as follows: First, it is well known that $\mathcal{D}(X)$ is dense in $\mathcal{S}(X)$; thus it suffices to approximate a compactly supported ψ by functions in $X \otimes \mathcal{S}$. We take a (fine enough) finite partition of unity $(\varphi_j)_{j=1}^m$ of the support of ψ. Let ψ_j be the Nth degree Taylor expansion of ψ at t_j (a point chosen from the support of φ_j), where N is chosen large enough. Then $\psi_j(\cdot)\varphi_j(\cdot) \in X \otimes \mathcal{S}$, and $\sum_{j=1}^m \varphi_j \psi_j$ can be chosen as close to ψ as desired, the closedness being measured in terms of any preassigned finite collection of the seminorms $\|\cdot\|_{\alpha,\beta}$.

Fourier transform and convolutions. The Fourier transform is defined by

$$\hat{f}(\xi) \equiv \mathcal{F}f(\xi) := \int f(t)e^{-\mathbf{i}2\pi t\cdot\xi}\,\mathrm{d}t$$

for $f \in L^1(X)$. It is an isomorphism on $\mathcal{S}(X)$, and on $\mathcal{S}'(X)$ where it is defined by duality: $\langle \mathcal{F}f, \psi\rangle := \langle f, \mathcal{F}\psi\rangle$. The inverse Fourier transform is denoted $\check{f} \equiv \mathcal{F}^{-1}f$. We recall the identity $\mathcal{F}^2 f = \widetilde{f}$, where $\widetilde{f}(t) = f(-t)$.

The convolution of a tempered distribution $k \in \mathcal{S}'(X)$ and a Schwartz function $\psi \in \mathcal{S}$ is defined pointwise by $k * \psi(t) := \langle k, \psi(t - \cdot)\rangle$. It can be shown, and the vector-valued situation brings no complications at this point, that $k * \psi$ is a smooth, slowly increasing function. It can be identified with a tempered distribution, and satisfies $\langle k * \psi, \varphi\rangle = \left\langle k, \widetilde{\psi} * \varphi\right\rangle$

Besov spaces. The Besov spaces $B^{s,p}_q(X)$ can be defined in various ways. For the Fourier analytic definition which we use, we require the following Littlewood–Paley-type decomposition: Let $(\varphi_j)_{j=0}^\infty$ be a resolution of the identity, defined (in terms of the corresponding Fourier transforms) as follows: Let $\hat{\varphi}_0 \in \mathcal{D}$ be radial, equal to unity in $\bar{B}(0,1)$, and supported in $\bar{B}(0,2)$. (The definition of the Besov spaces is [up to equivalence of norms] independent of this choice; in fact, one could allow much more general resolutions of the identity than considered here.) Denote $\hat{\phi} := \hat{\varphi}_0 - \hat{\varphi}_0(2\cdot)$ and $\hat{\varphi}_j := \hat{\phi}(2^{-j}\cdot)$ for $j = 1, 2, \ldots$ We can then decompose $\hat{f} = \sum_{j=0}^\infty \hat{f}\hat{\varphi}_j$, i.e., $f = \sum_{j=0}^\infty f * \varphi_j$, where the series converges in $\mathcal{S}(X)$ for $f \in \mathcal{S}(X)$ and in $\mathcal{S}'(X)$ for $f \in \mathcal{S}'(X)$. Then, for $s \in \mathbb{R}$, $p, q \in [1, \infty]$ the space $B^{s,p}_q(X)$ consists of those $f \in \mathcal{S}'(X)$ for which

$$\|f\|_{s,p;q} := \left\| \left(2^{js}\,\|f * \varphi_j\|_p\right)_{j=0}^\infty \right\|_{\ell^q}$$

is finite.

We have $\mathcal{S}(X) \hookrightarrow B^{s,p}_q(X) \hookrightarrow \mathcal{S}'(X)$, where $\hookrightarrow$ denotes continuous embedding, and $B^{s,p}_q(X)$ are Banach spaces for all values of the indices as above.

It is convenient to define $\chi_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ (where $\varphi_{-1} := 0$), so that $\hat{\chi}_j = 1$ on the support of $\hat{\varphi}_j$.

The operators of interest. We study convolution transformations $f \mapsto k * f$, where $k \in \mathcal{S}'(\mathcal{L}(X,Y))$. These are initially defined on the algebraic tensor product $X \otimes \mathcal{S}$ as follows: For $\psi \in \mathcal{S}$ and $k \in \mathcal{S}'(\mathcal{L}(X,Y))$, the convolution $k * \psi$ is defined as above; for every $t \in \mathbb{R}^n$, we have a well-defined pointwise value $k * \psi(t) \in \mathcal{L}(X,Y)$. Then also $[k * \psi(t)]x$ is well-defined for $x \in X$. Thus $(k * f)(t) := [k * \psi(t)]x$ for $f = x \otimes \psi$, and this definition extends to $f \in X \otimes \mathcal{S}$ by linearity. The transformation $f \mapsto k * f$ maps $X \otimes \mathcal{S}$ into the subset of $\mathcal{S}'(Y)$ consisting of smooth, slowly increasing functions.

We note in passing that there is an elaborate method for defining the action of $k*$ on the whole of $\mathcal{S}(X)$ instead of only $X \otimes \mathcal{S}$. An interested reader should consult the paper of Amann [**1**] for this. However, the more modest approach adopted here suffices for our purposes. In fact, it is shown in [**1**] that $\mathcal{S}(X)$ is dense in $B^{s,p}_q(X)$ iff both $p < \infty$ and $q < \infty$, and the same argument can be used to show the density of the smaller class $X \otimes \mathcal{S}$ in exactly the same range of Besov spaces. Actually, since $X \otimes \mathcal{S}$ is dense in $\mathcal{S}(X)$ w.r.t. the usual topology of $\mathcal{S}(X)$, which is stronger than the topology of $B^{s,p}_q(X)$, the $B^{s,p}_q(X)$-closures of $X \otimes \mathcal{S}$ and $\mathcal{S}(X)$ always coincide. Thus, to have an *a priori* estimate $\|k * f\|_{s,p;q} \leq C \|f\|_{s,p;q}$ for all $f \in X \otimes \mathcal{S}$ is just as good as the corresponding estimate for all $f \in \mathcal{S}(X)$: When $p, q < \infty$, either one allows us to conclude the existence of a unique extension $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ s.t. $T|_{X\otimes\mathcal{S}} = k*$. When p or q is infinite, we only get an extension to a closed subspace of $B^{s,p}_q(X)$, and this subspace is the same irrespective of whether we started with $X \otimes \mathcal{S}$ or $\mathcal{S}(X)$. To have an extension to the whole space, we require an extra argument in either case.

3. General theory

In this section we investigate general conditions for the boundedness of convolution operators from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$. The task is essentially two-fold: For $k \in \mathcal{S}'(\mathcal{L}(X,Y))$, our operator $k*$ is initially defined on the subspace $X \otimes \mathcal{S}$ of $B^{s,p}_q(X)$. Thus the first problem is

Problem 3.1. When do we have $k * f \in B^{s,p}_q(Y)$ and $\|k * f\|_{s,p;q} \leq C \|f\|_{s,p;q}$ for all $f \in X \otimes \mathcal{S}$, with $C < \infty$ independent of f?

Of course, this is the only problem if $X \otimes \mathcal{S}$ is dense in $B^{s,p}_q(X)$, since a unique operator $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ with the property $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$ is then determined by k, as soon as k satisfies the condition searched in Problem 3.1. However, we know that the density holds iff $p, q < \infty$. Thus, in general, we are faced with another problem:

Problem 3.2. When and how can we extend $k*$ to $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ s.t. $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$?

Moreover, it is natural to ask

Problem 3.3. Is the extension T unique? If not, is it possible to choose it in some canonical manner so as to have uniqueness by requiring some additional property? Is the extended operator translation-invariant, like the original operator $T|_{X\otimes\mathcal{S}} = k*$ was?

By translation-invariance, we will mean not only the property $T(f(\cdot - h)) = (Tf)(\cdot - h)$ for $h \in \mathbb{R}^n$, but also $T(\psi * f) = \psi * Tf$ for all $\psi \in \mathcal{S}$. Formally, the latter property is a consequence of the former, but making this precise requires suitably continuity, and it is easier to study the validity of this condition directly. Moreover, it appears that the property $T(\psi * f) = \psi * Tf$ is actually the more useful of the two in applications. Both these properties are easily seen to be satisfied by the operator $k*$ acting on $X \otimes \mathcal{S}$.

We first consider Problem 3.1. To facilitate notation related to this problem, we denote

$$\mathcal{L}^\circ(\mathfrak{F}(X),\mathfrak{F}(Y)) := \{T : X\otimes\mathcal{S}\to\mathfrak{F}(Y)|\ \|Tf\|_{\mathfrak{F}} \le C\,\|f\|_{\mathfrak{F}}\ \forall f\in X\otimes\mathcal{S}\},$$

where $\mathfrak{F}$ means either L^p or $B^{s,p}_q$, and $\|T\|_{\mathcal{L}^\circ(\mathfrak{F}(X),\mathfrak{F}(Y))}$ is the smallest possible C, as usual. Moreover, since the vector-valuedness plays no rôle in the proof of the next result, we make even further simplification, and only write $B^{s,p}_q$ instead of $B^{s,p}_q(X)$, and $\mathcal{L}^\circ(B^{s,p}_q)$ instead of $\mathcal{L}^\circ(B^{s,p}_q(X);B^{s,p}_q(Y))$ in the proof.

Since the membership and the norm of a distribution f in the spaces $B^{s,p}_q$ is determined solely in terms of the L^p norm of its dyadic pieces, it is not surprising that the boundedness of a convolution operator $k*$ on $B^{s,p}_q$ depends only on the boundedness on L^p of the convolution operators induced by the dyadic pieces of k. More precisely, the following proposition holds:

Proposition 3.4. *For arbitrary Banach spaces X and Y, there is an equivalence of norms*

$$\begin{aligned}\|k*\|_{\mathcal{L}^\circ(B^{s,p}_q(X),B^{s,p}_q(Y))} &\approx \sup_j \|(k*\varphi_j)*\|_{\mathcal{L}^\circ(L^p(X),L^p(Y))}\\ &\approx \sup_j \|(k*\chi_j)*\|_{\mathcal{L}^\circ(L^p(X),L^p(Y))},\end{aligned}$$

and the constants of equivalence depend only on s and q.

Proof. The latter comparability is elementary, since

$$\|(k*\chi_j)*\|_{\mathcal{L}^\circ(L^p)} \le \sum_{i=j-1}^{j+1} \|(k*\varphi_i)*\|_{\mathcal{L}^\circ(L^p)},$$

and as to the other direction, we have $\|(k*\varphi_j)*\|_{\mathcal{L}^\circ(L^p)} = \|\varphi_j*(k*\chi_j)*\|_{\mathcal{L}^\circ(L^p)} \le \|\varphi_j\|_1\,\|(k*\chi_j)*\|_{\mathcal{L}^\circ(L^p)}$, and $\|\varphi_j\|_1\le C$.

In view of the fact that $\hat\chi_j = 1$ on $\operatorname{supp}\hat\varphi_j$, we have

$$\|(k*f)*\varphi_j\|_p = \|(k*\chi_j)*(f*\varphi_j)\|_p \le \sup_i \|(k*\chi_i)*\|_{\mathcal{L}^\circ(L^p)}\,\|f*\varphi_j\|_p,$$

and thus

$$\begin{aligned}\|k*f\|_{B^{s,p}_q} &= \left\|\left(2^{js}\,\|k*f*\varphi_j\|_p\right)_j\right\|_{\ell^q}\\ &\le \sup_i \|(k*\chi_i)*\|_{\mathcal{L}^\circ(L^p)} \left\|\left(2^{js}\,\|f*\varphi_j\|_p\right)_j\right\|_{\ell^q} = \sup_i \|(k*\chi_i)*\|_{\mathcal{L}^\circ(L^p)}\,\|f\|_{B^{s,p}_q},\end{aligned}$$

which shows that $\|k*\|_{\mathcal{L}^\circ(B^{s,p}_q)} \le \sup_i \|(k*\chi_i)*\|_{\mathcal{L}^\circ(L^p)}$.

For the converse inequality, note first that all is clear if $\|(k*\varphi_j)*\|_{\mathcal{L}^\circ(L^p)} = 0$ for all j. Otherwise, we fix an index j_0 with $\|(k*\varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)} > 0$, consider an arbitrary positive $M < \|(k*\varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)}$, and let $g\in\mathcal{S}\setminus\{0\}$ satisfy

$$\|(k*\varphi_{j_0})*g\|_p \ge M\,\|g\|_p. \tag{3.5}$$

(Note that such choices can be made whether $\|(k*\varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)}$ is finite or infinite.) Take $f := g*\chi_{j_0} \in \mathcal{S}$. Then $\hat{f} = \hat{g}$ on $\operatorname{supp}\hat{\varphi}_{j_0}$, and hence $(k*\varphi_{j_0})*g = (k*\varphi_{j_0})*f$. This equality in combination with (3.5) shows that f is non-zero.

Moreover, we have

$$\|\varphi_j * f\|_p \leq \|\varphi_j * \chi_{j_0}\|_1 \|g\|_p \leq C\|g\|_p ,$$

and in fact $\varphi_j * f = 0$ for $|j-j_0|>1$, again by considering the supports of the Fourier transforms. These facts show that

$$\|f\|_{B_q^{s,p}} = \left\| \left(2^{(j_0+i)s} \|f*\varphi_{j_0+i}\|_p \right)_{i=-1}^{1} \right\|_{\ell^q} \leq 2^{j_0 s} C(s,q)\|g\|_p .$$

Finally, we have

$$\begin{aligned}\|k*f\|_{B_q^{s,p}} = \left\| \left(2^{js} \|(k*f)*\varphi_j\|_p \right)_{j=0}^{\infty} \right\|_{\ell^q} &\geq 2^{j_0 s} \|(k*\varphi_{j_0})*f\|_p \\ &= 2^{j_0 s}\|(k*\varphi_{j_0})*g\|_p \geq 2^{j_0 s} M\|g\|_p \geq C^{-1}(s,q)M\|f\|_{B_q^{s,p}} .\end{aligned}$$

Since this holds for arbitrary j_0 and any $M < \|(k*\varphi_{j_0})*\|_{\mathcal{L}^\circ(L^p)}$, with some non-zero $f \in \mathcal{S}$, we conclude that $\|k*\|_{\mathcal{L}^\circ(B_q^{s,p})} \geq C^{-1}(s,q)\sup_j \|(k*\varphi_j)*\|_{\mathcal{L}^\circ(L^p)}$. □

The proposition shows that the question of boundedness of the convolution operator $k*$ in the $B_q^{s,p}$-norm reduces to the problem of L^p-boundedness of the convolution operators $(k*\varphi_j)*$, which will be studied in detail in the subsequent section. For a while, we turn to Problems 3.2 and 3.3. As mentioned above, these only require consideration if either p or q is infinite. The rest of this section will be concerned with developing a theory applicable to these cases. Thus, a reader mainly interested in the case $p,q<\infty$ might wish to move immediately to the beginning of the next section. For those who stay, we are next going to give a preliminary result for the solution of Problem 3.3 when $p=\infty$; it has also some use in understanding Problem 3.2, which is the reason for taking up this consideration at this early stage.

Lemma 3.6. *Let T be a linear and $\sigma(L^p(X), L^{p'}(X'))$-to-$\sigma(\mathcal{S}'(Y), Y'\otimes\mathcal{S})$-continuous operator from $L^p(X)$ to $\mathcal{S}'(Y)$, such that $T|_{X\otimes\mathcal{S}} = k*$, for some $k \in \mathcal{S}'(\mathcal{L}(X,Y))$. Then $\psi*Tg = T(\psi*g)$ and $(Tg)(\cdot-h) = T[g(\cdot-h)]$ for all $\psi\in\mathcal{S}$, $h\in\mathbb{R}^n$.*

Proof. Suppose $g \in X\otimes\mathcal{S}$. Then $\mathcal{F}[\psi*Tg] = \mathcal{F}[\psi*(k*g)] = \hat{\psi}\hat{k}\hat{g}$, and $\mathcal{F}[T(\psi*g)] = \mathcal{F}[k*(\psi*g)] = \hat{k}\hat{\psi}\hat{g}$, so everything is clear.

For arbitrary $g\in L^p(X)$, we consider a sequence $g_n \in X\otimes\mathcal{S}$ which converges to g in $\sigma(L^p(X), L^{p'}(X'))$. Observe that $X\otimes\mathcal{S}$ is $\sigma(L^p(X), L^{p'}(X'))$-dense in $L^p(X)$; for $p\in[1,\infty[$ it is even norm-dense, as is well known, and for $p=\infty$ the verification of this assertion is an exercise in vector-valued integration.

Now $Tg_n \to Tg$ in $\sigma(\mathcal{S}'(Y), Y'\otimes\mathcal{S})$, i.e., $y'(\langle Tg_n,\phi\rangle) \to y'(\langle Tg,\phi\rangle)$ for all $y'\in Y'$, $\phi\in\mathcal{S}$. With $\tilde{\psi}*\phi$ in place of ϕ this gives $y'(\langle\psi*Tg_n,\phi\rangle) \to y'(\langle\psi*Tg,\phi\rangle)$.

From $g_n \to g$ in $\sigma(L^p(X), L^{p'}(X'))$ we easily have $\psi * g_n \to \psi * g$ in the same topology. By assumption then, $y'(\langle T(\psi * g_n), \phi\rangle) \to y'(\langle T(\psi * g), \phi\rangle)$ for all y' and ϕ as above.

Since the assertion was shown for the g_n and the limit is unique, we conclude that $y'(\langle \psi * Tg, \phi\rangle) = y'(\langle T(\psi * g), \phi\rangle)$ for all $y' \in Y'$, $g \in L^p(X)$, $\psi, \phi \in \mathcal{S}$, and this implies $\psi * Tg = T(\psi * g)$ as tempered distributions, thus a.e. (since both sides are locally integrable functions), and this is the assertion for convolutions. The proof for the translations is similar. $\square$

Remark 3.7. For $p \in [1, \infty[$, the same conclusion follows from the continuity assumption $T \in \mathcal{L}(L^p(X), L^p(Y))$; indeed, for these p, the class $X \otimes \mathcal{S}$ is norm-dense in $L^p(X)$, and we could have simply argued that $T(\psi * g) = \lim T(\psi * g_n) = \lim \psi * Tg_n = \psi * Tg$, and similarly for translations.

However, the same is not true for $p = \infty$ (counterexamples can be constructed with the help of Banach limits; see Sect. 7), and this is the reason for establishing the result for weak-to-weak-type continuity.

We are now in a position to present the extension procedure to obtain from the original convolution operator $k* \in \mathcal{L}^\circ(B_q^{s,p}(X), B_q^{s,p}(Y))$ an operator $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$. The idea of the method comes from Amann [**1**], and Girardi and Weis [**35**]. The previous Lemma 3.6 will play a rôle in establishing that the equivalence of the two slightly differing extensions used by these authors do agree under mild weak-to-weak-continuity assumptions. We note that we always obtain *an* extension, as soon as $k* \in \mathcal{L}^\circ(B_q^{s,p}(X), B_q^{s,p}(Y))$; however, under the additional assumptions, as illustrated in the subsequent results, we are more justified to call it *the* extension.

Proposition 3.8. *Let* $T_j|_{X\otimes\mathcal{S}} = (k * \chi_j)*$. *Suppose that* $\|T_j\|_{\mathcal{L}(L^p(X), L^p(Y))} \leq \kappa < \infty$ *for all* $j \in \mathbb{N}$. *Then, for every* $f \in B_q^{s,p}(X)$, *the formal series*

$$Tf := \sum_{j=0}^{\infty} \chi_j * T_j(\varphi_j * f)$$

converges in $B_q^{s,p}(Y)$ *if* $q < \infty$ *and always in* $\mathcal{S}'(Y)$ *to an element of* $B_q^{s,p}(Y)$ *of norm at most* $C\kappa$. *We have* $T|_{X\otimes\mathcal{S}} = k*$.

If, moreover, either $p < \infty$, *or* $p = \infty$ *and each* T_j *is* $\sigma(L^\infty(X), L^1(X'))$*-to-*$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$*-continuous, the above series agrees, term by term, with*

$$\tilde{T}f := \sum_{j=0}^{\infty} T_j(\varphi_j * f),$$

and hence the same assertions hold for $\tilde{T}f$.

Proof. That $Tf = k * f$ for $f \in X \otimes \mathcal{S}$ is clear from $\mathfrak{F}[\chi_j * T_j(\varphi_j * f)] = \hat{\chi}_j(\hat{k}\hat{\chi}_j)(\hat{\varphi}_j\hat{f}) = \hat{\varphi}_j(\hat{k}\hat{f}) = \mathfrak{F}[\varphi_j * (k * f)]$.

For $p = \infty$ and under the $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$-continuity assumption, we have $\chi_j * T_j(\varphi_j * f) = T_j((\chi_j * \varphi_j) * f) = T_j(\varphi_j * f)$ by Lemma 3.6. When $p < \infty$, this is clear from Remark 3.7. Thus it remains to establish the assertions for Tf.

Convergence in $B_q^{s,p}(Y)$, $q < \infty$. We have

$$\varphi_i * \sum_{j=M}^{N} \chi_j * T_j(\varphi_j * f) = \sum_{j=M}^{N} (\varphi_i * \chi_j) * T_j(\varphi_j * f),$$

and $\varphi_i * \chi_j = 0$ for $|i - j| > 1$. Thus, denoting by $T_M^N f$ the truncated series of Tf above, we have

$$\left\|\varphi_i * T_M^N f\right\|_p \le \sum_{j=i-2\vee M}^{i+2\wedge N} \|\varphi_i * \chi_j\|_1 \|T_j(\varphi_j * f)\|_p \le C\kappa \sum_{j=i-2\vee M}^{i+2\wedge N} \|\varphi_j * f\|_p,$$

and then, for $q < \infty$,

$$\sum_{i=0}^{\infty} 2^{isq} \left\|\varphi_i * T_M^N f\right\|_p^q \le C\kappa^q \sum_{j=M}^{N} 2^{jsq} \|\varphi_j * f\|_p^q \to 0 \qquad \text{as } M, N \to \infty.$$

Thus $T_0^N f$ is a Cauchy sequence in $B_q^{s,p}(Y)$. Once we know that the formal series has a meaning, we can set in the above equations $M = 0$, $N = \infty$, and we deduce that $\|Tf\|_{s,p;q} \le C\kappa \|f\|_{s,p;q}$.

Convergence in $\mathcal{S}'(Y)$. For $\psi \in \mathcal{S}$ we have

$$\begin{aligned}
&\sum_{j=0}^{\infty} |\langle \chi_j * T_j(\varphi_j * f), \psi\rangle|_Y \\
&\quad = \sum_{j=0}^{\infty} |\langle T_j(\varphi_j * f), \chi_j * \psi\rangle|_Y \le \sum_{j=0}^{\infty} \|T_j(\varphi_j * f)\|_p \|\chi_j * \psi\|_{p'} \\
&\qquad \le \sum_{j=0}^{\infty} \kappa 2^{js} \|\varphi_j * f\|_p 2^{-js} \|\chi_j * \psi\|_{p'} \le C\kappa \|f\|_{s,p;q} \|\psi\|_{-s,p';q'},
\end{aligned}$$

which is finite, since $\psi \in \mathcal{S} \subset B_{q'}^{-s,p'}$, and this gives the convergence. Then we can evaluate $\varphi_i * Tf$ just as above, and we get that $Tf \in B_q^{s,p}(Y)$, with a norm estimate of the same form as before. □

Remark 3.9. Amann [**1**] uses [somewhat implicitly, with an intermediate notion of sequence-spaces denoted $B_q^s(L^p(X))$] the series Tf, whereas Girardi and Weis [**35**] use $\tilde{T}f$. The operators of the latter authors are always even $\sigma(L^p(X), L^{p'}(X'))$-to-$\sigma(L^p(Y), L^{p'}(Y'))$-continuous, so that the definitions agree. The convergence of the series Tf in $\mathcal{S}'(Y)$ was shown in [**35**] under this stronger continuity assumption.

Proposition 3.10. *Under (all) the assumptions of Prop. 3.8, the operator T also have the following properties:*

Translation-invariance: $\varphi * Tf = T(\varphi * f)$ *and* $(Tf)(\cdot - h) = T(f(\cdot - h))$ *for all* $\varphi \in \mathcal{S}$, $h \in \mathbb{R}^n$, *and*

Compact-to-weak continuity: *If* $p = \infty$, *whenever* $\operatorname{supp} \hat{f}_m, \operatorname{supp} \hat{f} \subset K$, *a compact set, and* $f_m \to f$ *in* $\sigma(L^\infty(X), L^1(X'))$, *then* $Tf_m \to Tf$ *in* $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$.

Moreover, the T in Prop. 3.8 is the only operator in $\mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ *which satisfies* $T|_{X \otimes \mathcal{S}} = k*$ *the above properties.*

Proof. Translation-invariance follows from the corresponding properties of the operators T_j (Lemma 3.6 and Remark 3.7), and of $\chi_j *$ and $\varphi_j *$, from the $\mathcal{S}'(Y)$-convergence of the series defining Tf, and from the continuity of $f \mapsto \varphi * f$ and $f \mapsto f(\cdot - h)$ on $\mathcal{S}'(Y)$.

For f_m and f as in the continuity assertion, we have $\sum_{i=0}^M \varphi_i \equiv 1$ on K for some large enough M. It is then clear from the definition of the Besov norm that the norms $\|f\|_p$ and $\|f\|_{s,p;q}$ are equivalent for all $f \in \mathcal{S}'(X)$ with $\operatorname{supp} \hat{f} \subset K$. In particular, $f_m, f \in B_q^{s,\infty}(X)$, so that Tf_m, Tf make sense. Moreover,

$$Tf_m = \sum_{j=0}^{M} \chi_j * T_j(\varphi_j * f_m) \to \sum_{j=0}^{M} \chi_j * T_j(\varphi_j * f) = Tf,$$

where the convergence is in $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$. Indeed, $\varphi_j * f_m \to \varphi_j * f$ in $\sigma(L^\infty(X), L^1(X'))$ when $f_m \to f$ in this topology, and T_j is $\sigma(L^\infty(X), L^1(X))$-to-$\sigma(\mathcal{S}'(Y), Y \otimes \mathcal{S})$-continuous by assumption.

Uniqueness of T. To establish the last assertion, let $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ now be any operator which extends $k*$ and satisfies the translation-invariance and compact-to-weak continuity assertions of Prop. 3.10. For any $f \in B_q^{s,p}(X)$, we have $\varphi_j * Tf = T(\varphi_j * f)$, where $\varphi_j * f \in L^p(X)$. By density, we can find a sequence $g_m \in X \otimes \mathcal{S}$ s.t. $g_m \to \varphi_j * f$ in $L^p(X)$ if $p < \infty$ and in $\sigma(L^\infty(X), L^1(X'))$ if $p = \infty$. Then also $\chi_j * g_m \to \chi_j * (\varphi_j * f) = \varphi_j * f$ in the same topology, and it is clear that the Fourier transforms of $\chi_j * g_m$ and of $\varphi_j * f$ are supported on a fixed compact set K. Thus, when $p = \infty$, the compact-to-weak continuity guarantees that

$$T(\varphi_j * f) = \sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})\text{-} \lim_{m \to \infty} T(\chi_j * g_m) = \lim_{m \to \infty} k * (\chi_j * g_m).$$

When $p < \infty$, we have $\chi_j * g_m \to \varphi_j * f$ in $L^p(X)$, and then also in $B_q^{s,p}(X)$, due to the support condition on the Fourier transforms. Since $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$, this guarantees that $\varphi_j * Tf = T(\varphi_j * f) = \lim T(\chi_j * g_m) = \lim k * (\chi_j * g_m)$, the limit now taken in the norm-topology of $B_q^{s,p}(X)$.

Thus, in either case, $\varphi_j * Tf = T(\varphi_j * f)$ is uniquely determined by k. Since $Tf = \mathcal{S}'(Y)\text{-} \sum_{j=0}^{\infty} \varphi_j * Tf$, the same is true of Tf. □

Prop. 3.10 at hand, the following definition seems justified:

Definition 3.11. Let $k \in \mathcal{S}(\mathcal{L}(X,Y))$. We say that k is a *convolutor* from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ if there exists a $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ with the following properties:

- $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$, and
- T is translation-invariant and compact-to-weak continuous in the sense of Prop. 3.10.

T is called the *operator associated with* k.

Remark 3.12. It follows from Prop. 3.10 that the operator associated with k is unique (which would not be the case without imposing the second condition in Def. 3.11; see Sect. 7). For $p = \infty$ or $q = \infty$, this would not be the case if we only required $T \in \mathcal{L}(B^{s,p}_q(X), B^{s,p}_q(Y))$ s.t. $T|_{X\otimes\mathcal{S}} = k*$.

Our definition of a convolutor could be contrasted with that of a *Fourier multiplier* from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ used in [**35**]. The uniqueness question of the associated operator is there settled by requiring that T be $\sigma(B^{s,p}_q(X), B^{-s,p'}_{q'}(X'))$-to-$\sigma(B^{s,p}_q(Y), B^{-s,p'}_{q'}(Y'))$ continuous. This is a stronger requirement than that in Def. 3.11. Indeed, the translation-invariance of such an operator can be derived by continuity from the fact that it holds for the restriction of T to $X \otimes \mathcal{S}$, a dense subspace of $B^{s,p}_q(X)$ w.r.t. the topology $\sigma(B^{s,p}_q(X), B^{-s,p'}_{q'}(X'))$. The compact-to-weak continuity of T also follows; in fact, if $f_m \to f$ in $\sigma(L^\infty(X), L^1(X'))$ and $\operatorname{supp} \hat{f}_m, f \subset K$, then, since $f_m = \sum_{i=0}^{M} \varphi_i * f_m$ for some fixed finite M, it follows easily that also $f_m \to f$ in $\sigma(B^{s,\infty}_q(X), B^{-s,1}_{q'}(X'))$ (see [**35**] for the definition of the duality pairing in this context), and then $Tf_m \to Tf$ in $\sigma(B^{s,\infty}_q(Y), B^{-s,1}_{q'}(Y'))$ and hence in $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$. In particular, if m is a Fourier multiplier from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ in the sense of the definition in [**35**], then $\check{m}$ is a convolutor in the sense of Def. 3.11, and the associated operators agree.

Because of the intimate connection of convolution operators on $B^{s,p}_q(X)$ and those on $L^p(X)$, which was already demonstrated in Prop. 3.4 and will be described in even more detail below, we also give a parallel definition on $L^p(X)$:

Definition 3.13. Let $k \in \mathcal{S}(\mathcal{L}(X,Y))$. We say that k is a *convolutor* from $L^p(X)$ to $L^p(Y)$ if there exists a $T \in \mathcal{L}(L^p(X), L^p(Y))$ with the following properties:

- $Tf = k * f$ for all $f \in X \otimes \mathcal{S}$, and
- if $p = \infty$, T is $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$-continuous.

Again, T is called the *operator associated with* k.

Remark 3.14. Again, the operator T associated with k is unique. This follows from the density of $X \otimes \mathcal{S}$ in the norm-topology of $L^p(X)$ when $p < \infty$ and in the $\sigma(L^\infty(X), L^1(X'))$-topology when $p = \infty$. Lemma 3.6 and Remark 3.7 show that T is translation-invariant. The compact-to-weak continuity required in the Besov space setting holds now rather trivially.

The following theorem completes the general description of convolutors from $B_q^{s,p}(X)$ to $B_q^{s,p}(Y)$.

THEOREM 3.15. *Let $k \in \mathcal{S}'(\mathcal{L}(X,Y))$. Then k is a convolutor from $B_q^{s,p}(X)$ to $B_q^{s,p}(Y)$ if and only if*

- *$\chi_i * k$ is a convolutor from $L^p(X)$ to $L^p(Y)$ for all $i \in \mathbb{N}$, and*
- *$\sup_{i\in\mathbb{N}} \|T_i\|_{p\to p} < \infty$, where T_i is the operator associated with $\chi_i * k$.*

When this is the case, we have

$$Tf = \sum_{i=0}^{\infty} T_i(\varphi_i * f)$$

for all $f \in B_q^{s,p}(X)$, with convergence in $B_q^{s,p}(X)$ if $q < \infty$ and always in $\mathcal{S}'(Y)$.

PROOF. The implication "$\Leftarrow$" is the content of Prop.'s 3.8 and 3.10. Let us establish "$\Rightarrow$".

Suppose k is a convolutor, and let $T \in \mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))$ be the associated operator. Define $T_j f := T(\chi_j * f)$ for $f \in L^p(X)$ (then $\chi_j * f \in B_q^{s,p}(X)$, so this makes sense). Now we need to observe certain properties of the operators T_j:

- For $f \in X \otimes \mathcal{S}$, also $\chi_j * f \in X \otimes \mathcal{S}$, and $T_j f = T(\chi_j * f) = k * (\chi_j * f) = (\chi_j * k) * f$; thus $T_j|_{X\otimes\mathcal{S}} = (\chi_j * k)*$.
- Denoting $\kappa := \|T\|_{\mathcal{L}(B_q^{s,p}(X), B_q^{s,p}(Y))}$, we have the norm estimate

$$\begin{aligned}\|T_j f\|_p &\leq 4 \cdot 2^{-j} \sum_{i=-2}^{2} 2^{j+i} \|\varphi_{j+i} * T(\chi_j * f)\|_p \\ &\leq 4 \cdot 2^{-j} \left\| \left(2^{is} \|\varphi_i * T(\chi_j * f)\|_p \right)_{i=0}^{\infty} \right\|_{\ell^q} = 4 \cdot 2^{-j} \|T(\chi_j * f)\|_{s,p;q} \\ &\leq 4 \cdot 2^{-j} \kappa \|\chi_j * f\|_{s,p;q} \leq C(s,q) 2^{-j} \kappa 2^j \|f\|_p ;\end{aligned}$$

  hence $\sup_{j\in\mathbb{N}} \|T_j\|_{p\to p} \leq C(s,q)\kappa < \infty$.
- Suppose $p = \infty$, and $f_m \to f$ in $\sigma(L^\infty(X), L^1(X'))$. Then $\chi_j * f_m \to \chi_j * f$ in the same topology, and obviously the supports of the Fourier transforms of these functions are contained in a fixed compact set K. Thus $T(\chi_j * f_m) \to T(\chi_j * f)$ in $\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$, i.e., $T_j f_m \to T_j f$ in the same topology.

Thus every $\chi_j * k$ is a convolutor from $L^p(X)$ to $L^p(Y)$, and the T_j's defined above are the associated operators.

If we now *define* $\tilde{T} f := \sum_{j=0}^{\infty} T_j(\varphi_j * f)$ for all $f \in B_q^{s,p}(X)$, then Prop.'s 3.8 and 3.10 show that $\tilde{T}$ is the operator associated with k, and by uniqueness we have $\tilde{T} f = Tf$. □

4. Auxiliary results for L^p-convolutors

The previous section culminated in Theorem 3.15, which completely reduced the problem of boundedness of convolution operators on $B_q^{s,p}(X)$ to a related problem in $L^p(X)$. It is worth observing that this related problem is *not* the question of boundedness of general convolution operators on $L^p(X)$; rather, it deals with the convolution kernels $k * \chi_j$ having a very special structure: they are $\mathcal{C}^\infty$ and moreover have Fourier transforms supported on dyadic annuli. Nevertheless, it is convenient first to collect some general criteria for the L^p-boundedness of convolution operators with an operator-valued kernel. These results are mostly taken from [**38**] (where also more general versions are contained), and hence some of the proofs are omitted. A proof of the following proposition, which is somewhat different from the original one of Girardi and Weis, is nevertheless given.

Proposition 4.1 ([**38**]). *Suppose*

$$\int_{\mathbb{R}^n} |k(t)x|_Y \,\mathrm{d}t \le \kappa_1 |x|_X \,, \qquad \int_{\mathbb{R}^n} |k(t)'y'|_{X'} \,\mathrm{d}t \le \kappa_\infty |y'|_{Y'} \,. \tag{4.2}$$

Then $k*$, initially defined on simple functions, extends to a bounded mapping from $L^p(X)$ to $L^p(Y)$ for all $p \in [1,\infty[$, of norm at most $\kappa_1^{1/p}\kappa_\infty^{1/p'}$.

Proof. It is immediate that the action of $k*$ on $f = \sum_{j=1}^N x_k \chi_{E_k}$ given by

$$k * f(t) = \int_{\mathbb{R}^n} k(t-s)f(s)\,\mathrm{d}s = \sum_{j=1}^N \int_{E_j} k(t-s)x_k \,\mathrm{d}s$$

gives a well-defined operator mapping simple X-valued functions into stongly measurable Y-valued functions. Moreover, we have the estimates

$$\|k * f\|_{L^1} \le \int \mathrm{d}s \int |k(t-s)f(s)|_Y \,\mathrm{d}t \le \int \kappa_1 |f(s)|_Y \,\mathrm{d}s = \kappa_1 \|f\|_{L^1}$$

and

$$\left|\left\langle y', \int k(t-s)f(s)\,\mathrm{d}s\right\rangle\right| \le \int |k(t-s)'y'|_{X'} \,\mathrm{d}s \,\|f\|_{L^\infty} \le \kappa_\infty |y'|_{Y'} \|f\|_{L^\infty} \,;$$

hence $\|k * f\|_{L^\infty} \le \kappa_\infty \|f\|_{L^\infty}$.

Of course, the L^1-estimate is already the assertion for $p = 1$ by the density of simple functions, and the general assertion looks very much like something we should be able obtain by the convexity theorem from the two extremes. However, in the upper extreme we only have an estimate for simple functions, which are not dense in L^∞ and so do not give us an extension of $k*$ to the whole space. However, we can make a Marcinkiewicz-type argument as follows:

Consider $a, b \in]0,1[$ with $a + b = 1$, whose values will be chosen later on. For a simple function f and a measurable set E, obviously $f\chi_E$ is also simple. Then, for $\lambda > 0$, let $f^\lambda := f\chi_{\{|f(\cdot)|_X \le b\lambda\kappa_\infty^{-1}\}}$ and $f_\lambda := f - f^\lambda$, so that f_λ and f^λ are simple functions. Now

$$\begin{aligned}\|k * f\|_p^p &= \int_0^\infty p\lambda^{p-1} |\{|k * f(\cdot)|_Y > \lambda\}| \,\mathrm{d}\lambda \\ &\le \int_0^\infty p\lambda^{p-1} |\{|k * f_\lambda(\cdot)|_Y > a\lambda\}| \,\mathrm{d}\lambda + \int_0^\infty p\lambda^{p-1} \left|\{\left|k * f^\lambda(\cdot)\right|_Y > b\lambda\right|\} \,\mathrm{d}\lambda\end{aligned}$$

Since $\left\|k * f^\lambda\right\|_{L^\infty} \le \kappa_\infty \left\|f^\lambda\right\|_{L^\infty} \le b\lambda$ by the choice of f^λ, the last term vanishes.

The first term on the right is bounded by

$$\begin{aligned}\int_0^\infty p\lambda^{p-1}\frac{\kappa_1}{a\lambda}\|f_\lambda\|_{L^1} \,\mathrm{d}\lambda &= \frac{p\kappa_1}{a}\int_0^\infty \mathrm{d}\lambda\, \lambda^{p-2} \int_{\{|f(\cdot)|_X > b\lambda\kappa_\infty^{-1}\}} |f(x)|_X \,\mathrm{d}x \\ = \frac{p\kappa_1}{a}\int_{\mathbb{R}^n} \mathrm{d}x\, |f(x)|_X \int_0^{\kappa_\infty |f(x)|_X / b} \lambda^{p-2}\,\mathrm{d}\lambda &= \frac{p\kappa_1}{a}\int_{\mathbb{R}^n} |f(x)|_X \frac{|f(x)|_X^{p-1}\kappa_\infty^{p-1}}{(p-1)b^{p-1}} \\ &= \frac{p\kappa_1\kappa_\infty^{p-1}}{(p-1)ab^{p-1}}\|f\|_p^p = \frac{p^{p+1}}{(p-1)^p}\kappa_1\kappa_\infty^{p-1}\|f\|_p^p,\end{aligned}$$

where the last equality follows from the choice $a = 1/p$, $b = 1/p'$.

Thus, taking pth roots, we have shown that

$$\|k * f\|_p \le p' p^{1/p}\kappa_1^{1/p}\kappa_\infty^{1/p'}\|f\|_p$$

for all $p \in]1, \infty[$ and all simple functions f. By density we conclude that $k*$ extends to a bounded operator on all L^p with p in this range, of norm at most $p' p^{1/p}\kappa_1^{1/p}\kappa_\infty^{1/p'}$. This is like the assertion, but the bound for the norm is worse than we claimed. However, now that we have the operator $k*$ defined and bounded on the whole space L^q, where we fix (momentarily) some $q \in]1, \infty[$, we can apply the convexity theorem between L^q and L^1. This gives, for $p \in]1, q[$, a new bound for the operator norm of $k*$ on L^p, namely

$$\|k*\|_{\mathcal{L}(L^p)} \le \|k*\|_{\mathcal{L}(L^1)}^{\frac{1/p-1/q}{1-1/q}} \|k*\|_{\mathcal{L}(L^q)}^{\frac{1-1/p}{1-1/q}} \le \left((q')^{q'} q^{1/(q-1)}\right)^{1/p'} \kappa_1^{1/p}\kappa_\infty^{1/p'}.$$

We then consider p fixed, and let $q \to \infty$. Then $q' \to 1$, and so $(q')^{q'} \to 1$; moreover, $\log(q^{1/(q-1)}) = \log(q)/(q-1) \to 0$, so that $q^{1/(q-1)} \to 1$, and we have shown that $\|k*\|_{\mathcal{L}(L^p)} \le \kappa_1^{1/p}\kappa_\infty^{1/p'}$, as we claimed. □

Let us make a few remarks concerning the necessity of the assumptions above. For a general $k \in \mathcal{S}'(\mathcal{L}(X,Y))$ with $k(\cdot)x \in L^{1,\mathrm{loc}}(Y)$ for all $x \in X$, it is easy to see that the first condition in (4.2) is also necessary for $k*$ to give a bounded operator from $L^1(X)$ to $L^1(Y)$. Indeed, fix a non-negative $\psi \in \mathcal{S}$ with $\hat{\psi}(0) = \int \psi = 1$, and denote $\psi_\epsilon := \epsilon^{-n}\psi(\epsilon^{-1}\cdot)$. Then $\|\psi_\epsilon\|_1 = \|\psi\|_1 = 1$ for all $\epsilon > 0$, and hence $\|k * \psi_\epsilon(\cdot)x\|_1 \le \kappa |x|_X$ where $\kappa := \|k*\|_{1\to 1}$.

It is known that $k * \psi_\epsilon \to k$ as $\epsilon \to 0$ in the sense of distributions, thus in particular $\langle k * \psi_\epsilon(\cdot)x, \Phi\rangle \to \langle k(\cdot)x, \Phi\rangle$ for all $x \in X$ and $\Phi \in Y' \otimes \mathcal{S}$. By the norm estimate above, we conclude that

$$\sup_{\|\Phi\|_\infty \leq 1} |\langle k(\cdot)x, \Phi\rangle| \leq \kappa \, |x|_X \,,$$

and this gives the estimate $\|k(\cdot)x\|_1 \leq \kappa \, |x|_X$ for the L^1-norm of the locally integrable function $k(\cdot)x$.

Note that we cannot make such a conclusion unless we presuppose the local integrability of $k(\cdot)x$. In fact, even in the scalar-valued context we know that $k*$ will be bounded on L^1 iff $k = \mu$ is a finite Borel measure. In the vector-valued situation the previous considerations show that, if $k*$ is bounded from $L^1(X)$ to $L^1(Y)$ with norm κ as above, then $\Phi \in Y' \otimes \mathcal{S} \mapsto \langle k(\cdot)x, \Phi\rangle$ extends to a bounded functional on $C_0(Y')$ (since $Y' \otimes \mathcal{S}$ is dense in this space), of norm at most $\kappa \, |x|_X$, i.e., $k(\cdot)x \in C_0(Y')'$. For $Y = \mathbb{C}$, we get the "only if" part of the classical result by the duality $(C_0)' = M^1$, where M^1 is the space of finite Borel measures on $\mathbb{R}^n$.

In order to ensure the existence of an extension from $L^\infty(X)$ to $L^\infty(Y)$, further assumptions are required. For this, we introduce the following notion:

Definition 4.3. We say that an operator-valued function $k(\cdot) : \mathbb{R}^n \to \mathcal{L}(X, Y)$ is **uniformly strongly integrable**, for short $k \in L^1_u(\mathcal{L}(X, Y))$, if it is strongly integrable and the following property holds: For all measurable sets $E_m, E \subset \mathbb{R}^n$,

$$\sup_{|x|_X \leq 1} \int_{E_m} |k(t)x|_Y \, \mathrm{d}t \underset{m\to\infty}{\longrightarrow} 0 \qquad \text{whenever } E_m \downarrow E, \ |E| = 0.$$

Remark 4.4. It is easy to see that the condition of uniform strong integrability of k can be separated (equivalently) into the following two parts concerning sets of finite and infinite measure:

$$\sup_{|x|_X \leq 1} \int_{E_m} |k(t)x|_Y \, \mathrm{d}t \underset{m\to\infty}{\longrightarrow} 0 \qquad \text{whenever } E_m \downarrow E \text{ and } |E_m| \to 0$$

and

$$\sup_{|x|_X \leq 1} \int_{|t|>r} |k(t)x|_Y \, \mathrm{d}t \underset{r\to\infty}{\longrightarrow} 0. \tag{4.5}$$

It is clear that norm integrability $k \in L^1(\mathcal{L}(X, Y))$ implies uniform strong integrability $k \in L^1_u(\mathcal{L}(X, Y))$; the point of introducing this notion is exactly to avoid the rather strong notion of norm integrability.

Now we state the result:

Proposition 4.6 ([**38**]). *Assume the second condition in* (4.2)*, and define the integrals (which exist for all variables as below)*

$$\langle Kf(t), y'\rangle := \int_{\mathbb{R}^n} \langle y', k(t-s)f(s)\rangle \, \mathrm{d}s \tag{4.7}$$

for all $t \in \mathbb{R}^n$ and all $f \in L^\infty(X)$. Then $Kf(t) \in Y''$ and in fact $\|Kf(t)\|_{Y''} \leq \kappa_\infty \|f\|_\infty$.

If moreover $k(\cdot)' \in L^1_u(\mathcal{L}(Y', X'))$, then $Kf(t) \in Y$ for all $t \in \mathbb{R}^n$ and $t \mapsto Kf(t)$ is strongly measurable; thus, by the norm estimate, $Kf \in L^\infty(Y)$ and $\|Kf\|_\infty \leq \kappa_\infty \|f\|_\infty$. Moreover,

$$\tilde{K}g(t) := \int k(s-t)' g(s)\,\mathrm{d}s,$$

initially defined on $Y' \otimes [L^1 \cap L^\infty]$, extends to a bounded operator $\tilde{K}$ from $L^1(Y')$ to $L^1(X')$), and $\tilde{K} = K'|_{L^1(Y')}$, where K' is the adjoint of K.

Remark 4.8. It is clear that $Kf = k * f$ for $f \in X \otimes [L^1 \cap L^\infty]$.

It is also shown in [**38**] that the assertion of Prop. 4.6 remains valid even without $k(\cdot)' \in L^1_u(\mathcal{L}(Y', X'))$ provided that the Banach space Y *does not contain* c_0.

Corollary 4.9. *Suppose k satisfies* (4.2), *and $k(\cdot)' \in L^1_u(\mathcal{L}(Y', X'))$. Then k is a convolutor from $L^p(X)$ to $L^p(Y)$ for all $p \in [1, \infty]$, and the associated operator $K_p \in \mathcal{L}(L^p(X), L^p(Y))$ has norm at most $\kappa_1^{1/p} \kappa_\infty^{1/p'}$.*

Conversely, suppose $k \in \mathcal{S}'(\mathcal{L}(X, Y))$ coincides with a strongly locally integrable function, and that $k(\cdot)'$ is also strongly locally integrable. If k is a convolutor from $L^1(X)$ to $L^1(Y)$ [resp. from $L^\infty(X)$ to $L^\infty(Y)$], then it satisfies the first [resp. second] condition in (4.2).

Proof. Concerning the first assertion, everything else is contained in Prop.'s 4.1 and 4.6, except for the $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(\mathcal{S}'(Y), Y' \otimes \mathcal{S})$-continuity of $K = K_\infty$. However, even more follows easily from Prop. 4.6: Suppose $f_m \to f$ in $\sigma(L^\infty(X), L^1(X'))$, and let $g \in L^1(Y')$. Then

$$\langle g, Kf_m \rangle = \langle K'g, f_m \rangle \to \langle K'g, f \rangle = \langle g, Kf \rangle\,,$$

where the convergence follows from the assumption and the fact that $K'g = \tilde{K}g \in L^1(X')$ by Prop. 4.6. Thus K_∞ is even $\sigma(L^\infty(X), L^1(X'))$-to-$\sigma(L^\infty(Y), L^1(Y'))$ continuous.

The necessary conditions. Suppose now that $\|k*\|_{1\to 1} = \kappa_1 < \infty$. We fix a non-negative $\psi \in \mathcal{S}$ with $\hat{\psi}(0) = \int \psi = 1$, and denote $\psi_\epsilon := \epsilon^{-n}\psi(\epsilon^{-1}\cdot)$. Then $\|\psi_\epsilon\|_1 = \|\psi\|_1 = 1$ for all $\epsilon > 0$, and hence $\|k * \psi_\epsilon(\cdot)x\|_1 \leq \kappa_1 |x|_X$.

It is known that $k * \psi_\epsilon \to k$ as $\epsilon \to 0$ in the sense of distributions, thus in particular $\langle k * \psi_\epsilon(\cdot)x, \Phi \rangle \to \langle k(\cdot)x, \Phi \rangle$ for all $x \in X$ and $\Phi \in Y' \otimes \mathcal{S}$. By the norm estimate above, we conclude that

$$\sup_{\Phi \in Y' \otimes \mathcal{S},\, \|\Phi\|_\infty \leq 1} |\langle k(\cdot)x, \Phi \rangle| \leq \kappa_1 |x|_X\,,$$

and this gives the estimate $\|k(\cdot)x\|_1 \leq \kappa |x|_X$ for the L^1-norm of the locally integrable function $k(\cdot)x$.

Finally, if $\|k * f\|_\infty \le \kappa_\infty \|f\|_\infty$ for all $f \in X \otimes \mathcal{S}$, then

$$\begin{aligned}\|k(\cdot)'y'\|_{L^1(X')} = \|k(t-\cdot)'y'\|_{L^1(X')} &= \sup_{f\in X\otimes\mathcal{D}, \|f\|_\infty\le 1} |\langle k(t-\cdot)'y', f\rangle| \\ = \sup_f \left|\left\langle y', \int k(t-s)f(s)\,\mathrm{d}s\right\rangle\right| &= \sup_f |\langle y', (k*f)(t)\rangle| \\ &\le \sup_f \|k*f\|_\infty |y'|_Y \le \kappa_\infty |y'|_Y .\end{aligned}$$

Now all the assertions have been verified. □

The following result is now an immediate consequence of the previous ones. Recall that the φ_j denote the resolution of unity in the frequency domain which was used in the definition of the Besov spaces.

THEOREM 4.10. *Let $k \in \mathcal{S}'(\mathcal{L}(X,Y))$, and suppose that we have the estimates*

$$\|\varphi_j * k(\cdot)x\|_1 \le \kappa |x|_X , \qquad \|\varphi_j * k(\cdot)'y'\|_1 \le \kappa |y'|_{Y'} , \tag{4.11}$$

and moreover that $\varphi_j * k(\cdot)'$ is uniformly strongly integrable. Then k is a convolutor from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p, q \in [1,\infty]$.

Conversely, the estimates (4.11) *are also necessary.*

PROOF. Cor. 4.9 shows that every $\varphi_j * k$ (and then every $\chi_j * k = \sum_{i=-1}^{i} \varphi_{j+i} * k$) is a convolutor from $L^p(X)$ to $L^p(Y)$, and the associated operators are uniformly bounded. Then Theorem 3.15 shows that k is a convolutor from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$. The converse statement is obtained from the converse assertions of these same results. □

REMARK 4.12. The uniform strong integrability can be dropped if Y does not contain c_0, or else if only the exponents $p < \infty$ are considered.

5. A Hörmander-type condition for singular integrals

We are now approaching our main goal of giving sufficient criteria for $B^{s,p}_q$-convolutors in terms of conditions with the flavour of L. Hörmander's classical theorem. In particular, we want to express our conditions more explicitly in terms of the kernel k itself, rather than using the auxiliary kernels $\varphi_j * k$ or $\chi_j * k$ appearing in Theorems 3.15 and 4.10.

In the context of the reflexive L^p spaces of scalar-valued functions, it is well-known (cf. e.g. [**34**]) that a sufficient condition for the boundedness of $k*$, where $k \in \mathcal{S}'$ coincides with a locally integrable function outside the origin, is obtained by requiring $\hat{k} \in L^\infty$ and, in addition, the Hörmander condition (see Def. 5.1 below). As we will see, in the context of the Besov spaces, it is necessary to strengthen these assumptions by imposing a stronger integrability condition in a neighbourhood of the infinity. This arises from the inhomogeneity of the Besov spaces, more precisely, the requirement that we should have $k * \varphi_0 \in L^1$, where $\hat{\varphi}_0(0) = \int \varphi_0 \ne 0$.

We first formulate several conditions that will play a rôle in our Hörmander-type convolution theorem.

DEFINITION 5.1. Let $k \in L^{1,\mathrm{loc}}_{\mathrm{str}}(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(X,Y)) \cap \mathcal{S}'(\mathcal{L}(X,Y))$. We define the following conditions, which k may or may not satisfy:

Hörmander's condition: This holds if, for some $b > 1$, $\kappa < \infty$,

$$\int_{|t|>b|s|} |(k(t-s)-k(t))x|_Y \,\mathrm{d}t \le \kappa\, |x|_X\,, \qquad \text{for all } x \in X,\ s \in \mathbb{R}^n \setminus \{0\},$$

and we write for short (following Hörmander's original notation [**43**]) $k \in K^1(X,Y)$ in this case.

Principal value condition: We say that k satisfies the strong (resp. weak resp. weak*) principal value condition, and write $k \in PV(X,Y)$ (resp. $k \in w\text{-}PV(X,Y)$ resp. $k \in w^*\text{-}PV(X,Y)$) provided

$$\int_{r<|t|<2r} |k(t)x|_Y \,\mathrm{d}t \le \kappa\, |x|_X \qquad \text{for all } r > 0,$$

$$\left| \int_{r<|t|<R} k(t)x \,\mathrm{d}t \right|_Y \le \kappa\, |x|_X \qquad \text{for all } R > r > 0,$$

and moreover the limit

$$\lim_{r\downarrow 0} \int_{r<|t|<1} k(t)x \,\mathrm{d}t \tag{5.2}$$

exists in the norm (resp. weak resp. weak*) topology of Y for every $x \in X$. (It is assumed that Y is a dual space when dealing with the condition $w^*\text{-}PV(X,Y)$.)

Strong integrability at infinity: By this we mean that

$$\int_{|t|>r} |k(t)x|_Y \,\mathrm{d}t \le \kappa\, |x|_X$$

for some $r \in \,]0,\infty[$.

Strong vanishing at infinity: This is said to hold provided the condition of strong integrability holds for $r > r_0$, and moreover the smallest allowable κ tends to zero as $r \to \infty$. In other words, this is the second half of uniform strong integrability, i.e. (4.5).

REMARK 5.3. If k satisfies the strong (resp. weak resp. weak*) principal value condition, then the limit

$$\begin{aligned} \lim_{\epsilon\downarrow 0} \int_{|t|>\epsilon} k(t)x\,\phi(t)\,\mathrm{d}t &= \lim_{\epsilon\downarrow 0} \int_{\epsilon<|t|\le 1} k(t)x(\phi(t)-\phi(0))\,\mathrm{d}t \\ &\quad + \lim_{\epsilon\downarrow 0} \int_{\epsilon<|t|\le 1} k(t)x\,\mathrm{d}t\,\phi(0) + \int_{|t|>1} k(t)x\,\phi(t)\,\mathrm{d}t \end{aligned}$$

exists in the norm (resp. weak resp. weak*) topology of Y for every $x \in X$ and $\phi \in \mathcal{S}$. (This explains the name.) The above mentioned limit defines the action of the tempered distribution p.v.-k, or just k, on the Schwartz function ϕ, and we also have the estimate

$$|\langle k, \phi\rangle\, x|_Y \leq 2\kappa(\|\nabla\phi\|_\infty + |\phi(0)| + \||s|\,\phi(s)\|_\infty)\, |x|_X\,.$$

All this follows from the assumed principal value condition by a direct adaptation of the scalar-valued calculations in [**34**], pp. 193–4.

Remark 5.4. If the limit (5.2) exists in $\sigma(Y, Y')$, then the limit

$$\lim_{r\downarrow 0} \int_{r<|t|<1} k(t)'y'\,\mathrm{d}t$$

exists in $\sigma(X', X)$ for every $y' \in Y'$. In this way, the weak* principal value condition arises naturally in connection with the adjoint kernel $k(\cdot)'$.

Of course, whenever the sets are well-defined, $PV(X,Y) \subset w\text{-}PV(X,Y) \subset w^*\text{-}PV(X,Y)$.

Assuming conditions like those in the previous definition, we now wish to derive good estimates for the dyadic pieces $\varphi_i * k$. This is naturally divided into two cases: the inhomogeneous term $i = 0$, and the homogeneous terms $i = 1, 2, \ldots$ For $i > 0$, we can exploit the fact that φ_i then has a vanishing integral. But we need a uniform estimate for all such i, since otherwise the conditions of Theorem 4.10 will not be fulfilled.

In the following, we examine $k * \phi$, where k satisfies some of the conditions above, and $\phi \in \mathcal{S}$ is assumed to have a vanishing integral, so that it serves as a prototype of the functions φ_i, $i > 0$.

Lemma 5.5. *Suppose that* $k \in K^1(X,Y)$ (*with constant* κ) *and either satisfies any one of the principal value conditions* (*const.* κ), *or* $\hat{k} \in L^\infty(\mathcal{L}(X,Y))$ *with* $\|\hat{k}\|_\infty \leq \kappa$. *Let* $\phi \in \mathcal{S}$ *with* $\int \phi = 0$. *Then* $\|k * \phi(\cdot)x\|_1 \leq C(\phi)\kappa\, |x|_X$ *for all* $x \in X$.

If, moreover, k *vanishes strongly at* ∞, *then* $k * \phi$ *is uniformly strongly integrable.*

Proof. Assuming one of the principal-value conditions, from Rem. 5.3 we have the estimate

$$\begin{aligned}|k * \phi(t)x|_Y &= |\langle k(\cdot)x, \phi(t-\cdot)\rangle|_Y \\ &\leq 2\kappa(\|\nabla\phi(t-\cdot)\|_\infty + |\phi(t)| + \||t-\cdot|\,\phi(t-\cdot)\|_\infty \\ &\qquad + |t|\,\|\phi(t-\cdot)\|_\infty)\,|x|_X \leq C(\phi)\kappa(1+|t|)\,|x|_X\,.\end{aligned}$$

On the other hand, the Fourier condition gives

$$|(k * \phi)(t)x|_Y = \left|\mathcal{F}^{-1}[\hat{k}(\cdot)x\,\hat{\phi}](t)\right|_Y \leq \left\|\hat{k}(\cdot)x\,\hat{\phi}\right\|_1 \leq \kappa\,|x|_X\,\|\hat{\phi}\|_1.$$

Thus in either case we can say that $|k * \phi(t)x|_Y \leq \kappa C(\phi)(1+|t|)\,|x|_X$. From this it is already clear that $k * \phi$ satisfies the first half of the condition of uniform strong integrability, cf. Remark 4.4.

Estimation of the L^1-norm. We invoke the Decomposition Lemma 4.10 of Chapter 2: For $\phi \in \mathcal{S}$ with $\int \phi = 0$, there exists a decomposition $\phi = \sum_{m=0}^{\infty} \psi_m$ s.t. $\operatorname{supp} \psi_m \subset \bar{B}(0, 2^m) =: \bar{B}_m$, $\int \psi_m = 0$, and finally for any fixed $\alpha, \beta \in \mathbb{N}^n$ and $M > 0$, the sequence of Schwartz norms $\|\psi_m\|_{\alpha,\beta}$ is $\mathcal{O}(2^{-mM})$. The same is true for $\|\hat{\psi}_m\|_{\alpha,\beta}$ as well as for $\|\psi_m\|_p$, $\|\hat{\psi}_m\|_p$ for all $p \in [1, \infty]$.

Outside $b\bar{B}_m$, we estimate $k * \psi_m$ by the Hörmander condition:

$$\begin{aligned}\int_{b\bar{B}_m^c} |k * \psi_m(t)x|_Y \,\mathrm{d}t &= \int_{b\bar{B}_m^c} \left| \int_{\bar{B}_m} (k(t-s)x - k(t)x)\psi_m(s)\,\mathrm{d}s \right|_Y \mathrm{d}t \\ &\leq \int_{\bar{B}_m} \mathrm{d}s\, |\psi_m(s)| \int_{|t|>b|s|} |(k(t-s) - k(t))x|_Y \,\mathrm{d}t \leq \|\psi_m\|_1 \,\kappa\, |x|_X .\end{aligned}$$

Inside $b\bar{B}_m$ we invoke the estimate $|\psi_m * k(t)x|_Y \leq \kappa C(\psi_m)(1 + |t|)$, which gives

$$\int_{b\bar{B}_m} |\psi_m * k(t)x|_Y \,\mathrm{d}t \leq \kappa C(\psi_m) c_n b^{n+1} 2^{m(n+1)},$$

after integrating $1 + |t|$ in polar coordinates and recalling that $\bar{B}_m$ has radius 2^m. The two estimates combine to give $\|\psi_m * k(\cdot)x\|_1 \leq \kappa C_{n,b}(\psi_m) 2^{m(n+1)} |x|_X$.

Recalling the estimates in which the size of ψ_m entered in the constant $C_{n,b}(\psi_m)$, as well as the properties of the sequence $(\psi_m)_{m=0}^{\infty}$ from the decomposition lemma, it follows that $C_{n,b}(\psi_m)$ is $\mathcal{O}(2^{-mM})$ for any preassigned $M > 0$ as $m \to \infty$. It suffices to take $M > n + 1$ to conclude that $\sum_{m=0}^{\infty} \|\psi_m * k(\cdot)x\|_1$ converges, and thus we obtain $\phi * k(\cdot)x \in L^1(Y)$ with a norm estimate of the desired form.

Uniform integrability at ∞. Concerning the strong uniform integrability of $k * \phi$, only the estimate at infinity (cf. Remark 4.4) remains to be established. We estimate

$$\begin{aligned}&\int_{|t|>r} \sum_{m=0}^{\infty} |k * \psi_m(t)x|_Y \,\mathrm{d}t \\ &\qquad \leq \sum_{m:2^{m+1} \leq r} \int_{|t|>r} |k * \psi_m(t)x|_Y \,\mathrm{d}t + \sum_{m:2^{m+1}>r} \|k * \psi_m(\cdot)x\|_1 ,\end{aligned}$$

In the sum with large m's, $\|k * \psi_m(\cdot)x\|_1$ is $\mathcal{O}(2^{-mM})$ uniformly in $|x|_X \leq 1$, and this shows that the entire sum is $\mathcal{O}(r^{-M})$, $M > 0$.

The sum with small m's will be dealt with as follows, in analogy with the estimate of the L^1 norm above:

$$\int_{|t|>r} |k * \psi_m(t)x|_Y \,\mathrm{d}t \leq \int_{\bar{B}_m} \mathrm{d}s\, |\psi_m(s)| \int_{|t|>r/2} |k(t)x|_Y \,\mathrm{d}t.$$

The t-integral, which is independent of m, has the desired property by the assumption of k vanishing uniformly at the infinity, and we can sum over m, since the $\|\psi_m\|_1$ is $\mathcal{O}(2^{-mM})$. This completes the proof. □

The previous Lemma 5.5 is essentially all we need to handle the homogeneous terms $k * \varphi_i$ with $i > 0$. However, it clearly fails to apply directly to the inhomogeneity $k * \varphi_0$, since $\int \varphi_0 = \hat{\varphi}_0(0) = 1 \neq 0$. The following result shows that, in order to get a similar estimate for this term, it is necessary and sufficient to add the condition of strong integrability at ∞.

LEMMA 5.6. *The following conditions are equivalent for any $k \in \mathcal{S}'(\mathcal{L}(X,Y))$:*

- $\|k * \varphi(\cdot)x\|_1 \leq C\,|x|_X$ *for some $\varphi \in \mathcal{S}$ with $\int \varphi \neq 0$.*
- *In a neighbourhood of the origin, $\hat{k}$ coincides with some $\hat{f}$ s.t. $\|f(\cdot)x\|_1 \leq \tilde{C}\,|x|_X$.*

If $k(\cdot) \in K^1(X,Y)$, and moreover k satisfies one of the principal-value conditions or $\hat{k} \in L^\infty(\mathcal{L}(X,Y))$, then these are further equivalent to either of the following:

- $\|k * \varphi(\cdot)x\|_1 \leq C(\varphi)\,|x|_X$ *for all $\varphi \in \mathcal{S}$.*
- *k is strongly integrable at infinity.*

PROOF. Let us first establish the equivalence of the two properties valid for general k. Let φ be as in the first condition. Since $\hat{\varphi}(0) = \int \varphi \neq 0$ and $\hat{\varphi}$ is continuous, there are $\epsilon, r > 0$ such that $|\hat{\varphi}(\xi)| > \epsilon$ for $|\xi| < 2r$. Let $\eta \in \mathcal{D}$ have support in $\bar{B}(0,2r)$ and equal to unity in $\bar{B}(0,r)$. Then $\hat{\psi} := \eta \cdot \hat{\varphi}^{-1} \in \mathcal{D}$, and $\psi \in \mathcal{S} \subset L^1$. Then, since $k * \varphi(\cdot)x \in L^1(Y)$, we also have $(k * \varphi)(\cdot)x * \psi \in L^1(Y)$ and $\|k * \varphi * \psi(\cdot)x\|_1 \leq \|\psi\|_1 \|k * \varphi(\cdot)x\|_1 \leq C\,\|\psi\|_1\,|x|_X$. But the Fourier transform of $k * \varphi * \psi$ is $\hat{k}\hat{\varphi}\hat{\psi}$, and in $\bar{B}(0,r)$, this agrees with $\hat{k}$.

Conversely, if $\hat{k} = \hat{f}$ in $\bar{B}(0,r)$ [in the sense that $\left\langle \hat{k} - \hat{f}, \psi \right\rangle = 0$ for $\psi \in \mathcal{S}$ supported in $\bar{B}(0,r)$], where $\|f(\cdot)x\|_1 \leq C\,|x|_X$, let $\hat{\varphi} \in \mathcal{D}$ be 1 at the origin and supported in $\bar{B}(0,r)$. Then $\hat{k}\hat{\varphi} = \hat{f}\hat{\varphi}$, i.e., $k * \varphi = f * \varphi$, so $\|k * \varphi(\cdot)x\|_1 \leq \|\varphi\|_1 \|f(\cdot)x\|_1 \leq C\,\|\varphi\|_1\,|x|_X$, and $\int \varphi = \hat{\varphi}(0) = 1 \neq 0$.

To show that, with the additional conditions on k, the estimate

$$\|k * \varphi_0(\cdot)x\|_1 \leq C_0\,|x|_X$$

for some $\varphi_0 \in \mathcal{S}$ with non-vanishing integral implies the same property for $k * \varphi$ and any $\varphi \in \mathcal{S}$, it suffices to observe that any $\varphi \in \mathcal{S}$ is [uniquely] decomposed as $\varphi = \lambda\varphi_0 + \psi$, where $\lambda \in \mathbb{C}$ and $\int \psi = 0$. Then $\|k * \lambda\varphi_0(\cdot)x\|_1 \leq C\,|\lambda|\,|x|_X$ by assumption, and the fact that $\|k * \psi(\cdot)x\|_1 \leq C(\psi)\,|x|_X$, whenever k has the properties assumed and $\psi \in \mathcal{S}$ a vanishing integral, was shown in Lemma 5.5.

Next, let us assume $\|k * \varphi(\cdot)x\|_1 \leq C(\varphi)\,|x|_X$ for all $\varphi \in \mathcal{S}$ and show that k is strongly integrable in a neighbourhood of the infinity. To this end, fix a non-negative $\varphi \in \mathcal{D}$, supported in $\bar{B} := \bar{B}(0, r)$ and with $\int \varphi = 1$. We then have

$$C(\varphi) |x|_X \geq \|k * \varphi(\cdot)x\|_1 \geq \int_{(b\bar{B})^c} |k * \varphi(t)x|_Y \, \mathrm{d}t$$
$$= \int_{(b\bar{B})^c} \left| \int_{\bar{B}} k(t-s)\varphi(s)x \, \mathrm{d}s \right|_Y \mathrm{d}t,$$

where $b\bar{B} := \bar{B}(0, br)$. On the other hand, we have

$$\int_{(b\bar{B})^c} \int_{\bar{B}} |(k(t-s) - k(t))\varphi(s)x|_Y \, \mathrm{d}s \, \mathrm{d}t$$
$$\leq \int_{\bar{B}} \mathrm{d}s \int_{|t|>b|s|} |(k(t-s) - k(t))\varphi(s)x|_Y \, \mathrm{d}t \leq \kappa \int_{\bar{B}} |\varphi(s)| \, \mathrm{d}s \, |x|_X = \kappa \, |x|_X \,,$$

by Hörmander's condition. Estimating by the triangle inequality, we then obtain

$$\int_{(2\bar{B})^c} |k(t)x|_Y \, \mathrm{d}t = \int_{(2\bar{B})^c} \left| \int_{\bar{B}} k(t)\varphi(s) \, \mathrm{d}s \right|_Y \mathrm{d}t \leq (C(\varphi) + \kappa) \, |x|_X$$

but this means exactly the integrability of k in a neighbourhood of the infinity.

Finally, we show that the estimate $\int_{(b-1)\bar{B}^c} |k(t)x|_Y \, \mathrm{d}t \leq \kappa \, |x|_X$ implies the inequality $\|k * \varphi(\cdot)x\|_1 \leq C \, |x|_X$ for all $\varphi \in \mathcal{D}$, supported in $\bar{B} := \bar{B}(0, r)$. Indeed, we have

$$\int_{b\bar{B}} |k * \varphi(t)x|_Y \, \mathrm{d}t \leq \int_{b\bar{B}} \kappa C(\phi)(1 + |t|) \, |x|_X \, \mathrm{d}t = C(\phi, b\bar{B}) \, |x|_X \,,$$

where the estimate was shown in the first part of the proof of Lemma 5.5. Moreover,

$$\int_{(b\bar{B})^c} \left| \int_{\bar{B}} k(t-s)\varphi(s)x \, \mathrm{d}s \right|_Y \mathrm{d}t \leq \int_{\bar{B}} \mathrm{d}s \int_{(b\bar{B})^c} |k(t-s)\varphi(s)x|_Y \, \mathrm{d}t$$
$$\leq \int_{\bar{B}} |\varphi(s)| \, \mathrm{d}s \int_{(b-1)\bar{B}^c} |k(t)x|_Y \, \mathrm{d}t \leq \|\varphi\|_1 \, \kappa \, |x|_X \,.$$

This completes the proof. □

THEOREM 5.7. *Let $k \in \mathcal{S}'(\mathcal{L}(X; Y))$ satisfy the following conditions:*

- *$k(\cdot) \in K^1(X, Y)$ and $k(\cdot)' \in K^1(Y', X')$,*
- *$\hat{k} \in L^\infty(\mathcal{L}(X, Y))$, or both $k(\cdot)$ and $k(\cdot)'$ satisfy a principal value condition,*
- *$k(\cdot)$ and $k(\cdot)'$ are strongly integrable at infinity.*

Then $k*$ is a convolutor from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p \in [1, \infty[$ and $q \in [1, \infty]$.

The assertion remains true for $p = \infty$ under either of the following additional assumptions:

- *$k(\cdot)'$ vanishes strongly at infinity, or*

• *Y does not contain c_0.*

Proof. The plan is to verify the conditions in Theorem 4.10 for the $\varphi_i * k$, $i = 0, 1, 2, \ldots$.

Case $i > 0$. First of all, we observe that if k satisfies Hörmander's conditions resp. the principal value condition resp. $\|\hat{k}\|_\infty \leq \kappa$, then then the same holds for $2^{-in}k(2^{-i}\cdot)$ with the same constant κ. Moreover, $k * \varphi_i = k * 2^{in}\phi(2^i\cdot) = 2^{in}(2^{-in}k(2^{-i}\cdot) * \phi)(2^i\cdot)$, and then by the dilation-invariance of the L^1-norm, we have

$$\|k * \varphi_i(\cdot)x\|_1 = \left\|2^{-in}k(2^{-i}\cdot) * \phi(\cdot)x\right\|_1 \leq \kappa C(\phi)\,|x|_X$$

by the assumptions, Lemma 5.5 and the above-mentioned invariance of the conditions on k under dilation. Now this estimate is uniform in $i = 1, 2, \ldots$ The same argument with $k(\cdot)'y'$ in place of $k(\cdot)x$ clearly yields $\|k(\cdot)' * \varphi_i(\cdot)y'\|_1 \leq \kappa C(\phi)\,|x|_X$.

Under the assumption that $k(\cdot)'$ vanishes strongly at ∞, Lemma 5.5 shows that $\varphi_i * k(\cdot)'$ is uniformly strongly integrable.

Case $i = 0$. According to Lemma 5.6, we have $\|\varphi_0 * k(\cdot)x\|_1 \leq C(\varphi_0)\,|x|_X$ and $\|\varphi_0 * k(\cdot)'y'\|_1 \leq C(\varphi_0)\,|y'|_{Y'}$.

As for the uniform strong integrability of $\varphi_0 * k(\cdot)$ (under the additional assumption of strong vanishing of $k(\cdot)$ at ∞), we write $\varphi_0 = \varphi + \psi$, where $\varphi \in \mathcal{D}$ is supported in $\bar{B}(0,\epsilon)$ and $\int \psi = 0$. Then $\psi * k(\cdot)' \in L^1_u(\mathcal{L}(Y', X'))$ by Lemma 5.5, and moreover

$$\begin{aligned}\int_{|t|>r} |\varphi * k'(t)y'|_{X'}\,\mathrm{d}t &\leq \int_{|t|>r} \mathrm{d}t \int_{|s|\leq\epsilon} |\varphi(s)k(t-s)'y'|_{X'}\,\mathrm{d}s \\ &\leq \|\varphi\|_1 \int_{|t|>r-\epsilon} |k(t)'y'|_{X'}\,\mathrm{d}t.\end{aligned}$$

Thus also $\varphi * k(\cdot)'$ satisfies the second half of the uniform strong integrability, and the first half (cf. Rem. 4.4) is proved just like in the first part of the proof of Lemma 5.5. (This part of the proof did not require the vanishing integral of the test function, as is easily seen.)

Now all the conditions required for Theorem 4.10 (and Remark 4.12) have been verified. □

6. Application to evolutionary integral equations

We will here apply our results to kernels arising from solution formulae for certain evolutionary integral equations considered in [**69**]. It is there shown (see § 7.4 of [**69**]) that a related maximal regularity problem leads one to investigate the boundedness on $B^{s,p}_q([0,t_0];X)$ of the operator $f \mapsto u$ given by

$$u(t) = f(t) + \int_0^t \dot{S}_0(t-\tau)f(\tau)\,\mathrm{d}\tau, \tag{6.1}$$

where the *resolvent* or *solution operator* $S_0 \in \mathcal{C}^1(]0,\infty[;\mathcal{L}(X))$ is strongly continuous at the origin and satisfies the estimates

$$\|S_0(t)\|_{\mathcal{L}(X)} + \left\|t\dot{S}_0(t)\right\|_{\mathcal{L}(X)} \le \kappa, \qquad 0 < t < t_0 \tag{6.2}$$

and

$$\left\|\dot{S}_0(t) - \dot{S}_0(t-s)\right\|_{\mathcal{L}(X)} \le \kappa \frac{s}{t(t-s)}\left(1 + \log\frac{t}{s}\right), \qquad 0 < s < t < t_0. \tag{6.3}$$

It is clear from (6.1) that the values of $u(t)$ for $t \in [0, t_0]$ remain unchanged if we truncate the kernel at t_0, so that we are lead to consider the convolution operator with the kernel $k(t) := \dot{S}_0(t)\chi_{]0,t_0[}(t)$. Let us check the conditions of Theorem 5.7 for this kernel.

Hörmander's condition. For a kernel supported on the positive half-line, it is easily seen that it suffices to consider the case $s > 0$. If $2s \ge t_0 + s$, the condition is trivial; if $t_0 \le 2s < t_0 + s$, i.e., $t_0/2 \le s < t_0$, then

$$\begin{aligned}\int_{t>2s} \|k(t) - k(t-s)\|_{\mathcal{L}(X)}\,\mathrm{d}t &= \int_{2s}^{t_0+s} \left\|\dot{S}_0(t-s)\right\|_{\mathcal{L}(X)}\,\mathrm{d}t \\ &\le \kappa \int_{2s}^{t_0+s} \frac{\mathrm{d}t}{t-s} = \kappa \log\frac{t_0}{s} \le \kappa\log 2.\end{aligned}$$

Finally, let $0 < 2s < t_0$. For $2s < t < t_0$, (6.3) gives

$$\|k(t) - k(t-s)\|_{\mathcal{L}(X)} \le 2\kappa s t^{-2}(1 + \log(t/s)),$$

and for $t_0 \le t < t_0 + s$ we have

$$\|k(t) - k(t-s)\|_{\mathcal{L}(X)} = \left\|\dot{S}_0(t-s)\right\|_{\mathcal{L}(X)} \le \kappa(t-s)^{-1}$$

by (6.2). Hence

$$\begin{aligned}\int_{t>2s} \|k(t) - k(t-s)\|_{\mathcal{L}(X)}\,\mathrm{d}t &\le 2\kappa\int_{2s}^{t_0} \frac{s}{t^2}(1 + \log(t/s))\,\mathrm{d}t + \int_{t_0}^{t_0+s} \frac{\kappa}{t-s}\,\mathrm{d}t \\ &\le 2\kappa\int_2^\infty \frac{1}{u^2}(1 + \log u)\,\mathrm{d}u + \kappa\log\frac{t_0}{t_0 - s} \le (1 + 2\log 2)\kappa,\end{aligned}$$

since $t_0/(t_0 - s) < 2$. The norm version of the condition established implies in particular the corresponding strong estimates as well as their dual versions.

Principal value condition. Using the assumption (6.2) only, we have

$$\begin{aligned}&\int_r^{2r} \|k(t)\|_{\mathcal{L}(X)}\,\mathrm{d}t \le \int_r^{2r} \frac{\kappa}{t}\,\mathrm{d}t = \kappa\log 2, \\ &\left\|\int_r^R k(t)\,\mathrm{d}t\right\|_{\mathcal{L}(X)} = \left\|\int_r^{R\wedge t_0} \dot{S}_0(t)\,\mathrm{d}t\right\|_{\mathcal{L}(X)} = \|S(R\wedge t_0) - S(r)\|_{\mathcal{L}(X)} \le 2\kappa.\end{aligned}$$

These norm estimates imply the first two conditions in the definition of the principal value conditions. Finally, from the strong continuity we have

$$\int_{\epsilon}^{1} k(t)x\,\mathrm{d}t = \int_{\epsilon}^{1\wedge t_0} \dot{S}_0(t)x\,\mathrm{d}t = S_0(1\wedge t_0)x - S_0(\epsilon)x \underset{\epsilon\downarrow 0}{\longrightarrow} S_0(1\wedge t_0)x - S_0(0)x,$$

which shows that $k \in PV(X)$. Then in particular $k \in w\text{-}PV(X)$, and thus $k(\cdot)' \in w^{*}\text{-}PV(X')$ (cf. Rem. 5.4).

Conditions at infinity. These are trivially satisfied, since k vanishes outside a compact set.

Conclusion. Having verified all the conditions of Theorem 5.7, we conclude that the solution map $f \mapsto u$ defined in (6.1) is indeed bounded on $B_q^{s,p}([0,t_0];X)$. This we knew, of course, from [**69**] already; but the ease with which the conditions of our Theorem 5.7 were verified for this operator illustrates the applicability of this general theorem in concrete situations.

7. Counterexamples to uniqueness of extensions

We present some counterexamples to demonstrate the non-uniqueness of the extended operators studied in Sect. 3 unless we impose some additional conditions as we did there. All the examples are based on Banach limits.

The first one concerns operators on $L^\infty(\mathbb{R}^n)$; this is not only instructive as an example of the non-uniqeness phenomenon in a very concrete space, but it will also be exploited in constructing the counterexamples in the Besov space setting.

Example 7.1 (Non-trivial extensions of zero to $\mathcal{L}(L^\infty)$). It suffices to establish the examples for the scalar case, since operators between $L^\infty(X)$ and $L^\infty(Y)$ are then obtained by mapping $f \in L^\infty(X) \mapsto y \otimes L(\langle x', f(\cdot)\rangle) \in L^\infty(Y)$, where $x' \in X'$, $y \in Y$ and $L \in \mathcal{L}(L^\infty)$.

The purpose is to show that there exist non-trivial operators $L \in \mathcal{L}(L^\infty)$ which annihilate $\mathcal{S}$. We even want to show that there are both translation-invariant and translation-variant operators of this kind.
Construction. Consider for every $f \in L^\infty$ the sequence

$$(f_j)_{j=0}^{\infty} := \left(\frac{1}{\sigma_n R_j^n}\int_{B(0,R_j)} f(t)\,\mathrm{d}t\right)_{j=0}^{\infty} \in \ell^\infty,$$

where $R_j \to \infty$ and σ_n is the volume of the unit ball of $\mathbb{R}^n$. Let $\lambda(f) := \Lambda((f_j)_{j=0}^\infty)$, where Λ is a *Banach limit.* For the present purposes, it suffices to take for Λ any Hahn–Banach extension to ℓ^∞ of the linear functional $\tilde{\Lambda} : c \subset \ell^\infty \to \mathbb{K}$, $(a_j)_{j=0}^\infty \mapsto \lim_{j\to\infty} a_j$, where c is the closed subspace of ℓ^∞ of all convergent sequences. Then $|\lambda(f)| \le \sup_j |f_j| \le \|f\|_\infty$. Clearly $\lambda(f) = 0$ for $f \in \mathcal{S}$, or in fact for $f \in L^p \cap L^\infty$ for any $p < \infty$ and also for $f \in C_0$. On the other hand, if $f \equiv c$ is a constant, or more generally has the limit c at the infinity, then $\lambda(f) = c$.

Evaluation of $\lambda(\psi * f)$. Observe first that

$$\left|\int_{B(0,R)} f(t)\,\mathrm{d}t - \int_{B(s,R)} f(t)\,\mathrm{d}t\right| \le |B(0,R)\Delta B(s,R)|\,\|f\|_\infty\,, \tag{7.2}$$

where Δ denotes the symmetric difference of two sets. Its measure can be estimated by observing that $B(0,R)\cup B(s,R)\subset B(s/2,R+|s|/2)$ and, if $|s|/2<R$, $B(0,R)\cap B(s,R)\supset B(s/2,R-|s|/2)$. These show that $|B(0,R)\Delta B(s,R)|\le \sigma_n((R+|s|/2)^n-(R-|s|/2)^n)\le \sigma_n n(R+|s|/2)^{n-1}|s|\le \sigma_n n2^{n-1}R^{n-1}|s|$. Thus the difference of the two integrals in (7.2) is at most $\sigma_n n 2^{n-1}\|f\|_\infty |s| R^{n-1}$ for $R>|s|/2$. Clearly it is always at most $2\sigma_n R^n\|f\|_\infty$.

We are now ready to see how the functional λ behaves w.r.t. convolutions with $\psi\in\mathcal{S}$. Let us denote $a:=\int\psi$. Then

$$\begin{aligned}\frac{1}{\sigma_n R^n}&\left(\int_{B(0,R)} a\,f(t)\,\mathrm{d}t-\int_{B(0,R)} f*\psi(t)\,\mathrm{d}t\right)\\&=\frac{1}{\sigma_n R^n}\int_{B(0,R)}\int_{\mathbb{R}^n}(f(t)-f(t-s))\,\psi(s)\,\mathrm{d}s\,\mathrm{d}t\\&=\int_{\mathbb{R}^n}\psi(s)\frac{1}{\sigma_n R^n}\left(\int_{B(0,R)}-\int_{B(-s,R)}\right)f(t)\,\mathrm{d}t\,\mathrm{d}s.\end{aligned}\tag{7.3}$$

Using the estimates for the t-integral obtained above, we find that the absolute value of the whole quantity above is estimated from above by

$$\int_{|s|<2R}|\psi(s)|\cdot|s|\,\mathrm{d}s\frac{c\,\|f\|_\infty}{R}+\int_{|s|\ge 2R}|\psi(s)|\,\mathrm{d}s\cdot 2\,\|f\|_\infty \underset{R\to\infty}{\longrightarrow} 0.$$

Thus $\lambda(a\,f-f*\psi)=0$, which means that $\lambda(f*\psi)=a\lambda(f)=(\int\psi)\lambda(f)$ for $\psi\in\mathcal{S}$.

The proof that $\lambda(f(\cdot-h))=\lambda(f)$ for $h\in\mathbb{R}^n$ is similar and essentially contained above.

The operators. It remains to pick some $g\in L^\infty$ and set $Lf:=\lambda(f)g$. Let us see what properties the operator L has, depending on the choice of g. We have $L(f(\cdot-h))=\lambda(f(\cdot-h))g=\lambda(f)g$, whereas $(Lf)(\cdot-h)=\lambda(f)g(\cdot-h)$. Since there exist L^∞-functions which are not annihilated by λ (e.g., the constants), we see that L is translation-invariant if and only if g is a constant.

Concerning convolutions, we have $L(\psi*f)=\lambda(\psi*f)g=\lambda(f)(\int\psi)g$, whereas $\psi*Lf=\lambda(f)(\psi*g)$. Again, if $g\equiv c$ is a constant, it is easy to see that $\psi*g\equiv(\int\psi)c\equiv(\int\psi)g$, and so L is translation-invariant. Conversely, if L is translation-invariant and hence the previous equality holds, we have in particular $\psi*g=g$, i.e. $\hat\psi\hat g=\hat g$ whenever $\hat\psi(0)=\int\psi=1$. This being the only restriction on the $\hat\psi\in\mathcal{S}$, we see that $\hat g$ cannot have support except possibly at the origin, and so $\hat g=\sum c_\alpha D^\alpha\delta_0$ (a finite sum). Then g is a polynomial, and the requirement that $g\in L^\infty$ forces it to be a constant. Thus L is also translation-invariant if and only if the g is chosen to be a constant.

Summarizing, we have seen that, if only norm-continuity conditions are required, the zero operator on $\mathcal{S}$ (which is manifestly translation-invariant) possesses various non-trivial extensions, both translation-variant and translation-invariant, to $\mathcal{L}(L^\infty)$.

Example 7.4 (Non-trivial extensions of zero to $\mathcal{L}(B_q^{s,\infty})$). This is essentially contained in Example 7.1 already, but let us see how to obtain the example in the Besov space setting.

We first observe that, for every s and q, the constant function 1 is in $B_q^{s,\infty}$ and $\|1\|_{s,\infty;q} = 1$. Indeed, $1 * \varphi_j = \int \varphi_j(s)\,\mathrm{d}s = \hat{\varphi}_j(0) = \delta_{0,j}$, and thus $\|1\|_{s,\infty;q} = \left\|(2^{js}\delta_{0,j})_{j=0}^\infty\right\|_{\ell^q} = 1$.

Now let $\lambda \in (L^\infty)'$ be the functional from Example 7.1, $g \in B_q^{s,\infty}$ some fixed element, and define, for $f \in B_q^{s,\infty}$, the operator $Lf := \lambda(f * \varphi_0)g$. Clearly this is linear and continuous. If $f \in \mathcal{S}$, then also $f * \varphi_0 \in \mathcal{S}$, and $\lambda(f * \varphi_0) = 0$. However, for a constant function $f \equiv c \in B_q^{s,\infty}$, we have $c * \varphi_0 \equiv c\hat{\varphi}_0(0) = c$; thus $\lambda(c * \varphi_0) = \lambda(c) = c$ and $Lc = c\,g$. Now the rest of the example works exactly as in Example 7.1; in particular, we obtain both translation-variant and translation-invariant non-trivial extensions of zero, depending on the choice of g.

Example 7.5 (Non-trivial extensions of zero to $B_\infty^{s,p}$). The example is based on the same ideas as the previous ones. Note that $B_\infty^{s,p}$ is the space of all $f \in \mathcal{S}'$ such that $(2^{js}\|\varphi_j * f\|_p)_{j=0}^\infty \in \ell^\infty$. Let $(g_j)_{j=0}^\infty \in \ell^\infty(L^{p'})$ be a sequence to be fixed later, and consider the linear functional $\lambda(f) := \Lambda((2^{js}\langle g_j, \varphi_j * f\rangle)_{j=0}^\infty)$, where Λ is a Banach limit just like in Example 7.1.

Observe that

$$2^{js}\left|\langle g_j, \varphi_j * f\rangle\right| \le 2^{js}\|g_j\|_{p'}\|\varphi_j * f\|_p \le C\,2^{js}\|\varphi_j * f\|_p \le C\,\|f\|_{s,p;\infty},$$

which shows that $\lambda \in (B_\infty^{s,p})'$. Moreover, if $2^{js}\|\varphi_j * f\|_p \to 0$ as $j \to \infty$, then $\lambda(f) = 0$, and this is the case for $f \in \mathcal{S}$ and in fact for $f \in B_q^{s,p}$ for any $q < \infty$ by the definition of the norm.

Let us then see that, in general, $\lambda(f_0) \neq 0$ for some $f_0 \in B_\infty^{s,p}$. For any given $f_0 \in B_\infty^{s,p}$, we can clearly fix the $g_j \in L^{p'}$ (which embeds in $(L^p)'$ as a norming subspace) in such a way that $\|g_j\|_{p'} = 1$ and $\langle g_j, \varphi_j * f_0\rangle = \|\varphi_j * f_0\|_p$.

Now consider $f_0 := \mathcal{S}'\text{-}\sum_{j=0}^\infty 2^{-js+jn(1/p-1)}\varphi_j$. Then

$$2^{is}\varphi_i * f_0 = 2^{in(1/p-1)}\sum_{j=-1}^{1} 2^{-js+jn(1/p-1)}\varphi_i * \varphi_{i+j}.$$

But $\varphi_i * \varphi_{j+i} = 2^{in}(\phi_0 * \phi_j)(2^i\cdot)$ for $i \ge 1$, where $\phi_j := 2^{(j-1)n}\varphi_1(2^{j-1}\cdot)$, and the function in parentheses is non-vanishing as is seen by inspection of the Fourier transforms. Then, from the elemtary dilation property $\|2^{in}g(2^i\cdot)\|_p = 2^{in(1-1/p)}\|g\|_p$ of the L^p-norm, we conclude that that $2^{is}\|\varphi_i * f_0\|_p$ is a non-zero constant c for all $i \ge 1$. This shows first of all that $f_0 \in B_\infty^{s,p}$, and moreover that $\lambda(f_0) = c \neq 0$.

As in Example 7.1, we then define $Lf := \lambda(f)g$, $g \in B^{s,p}_\infty$, and observe that $L|_{\mathcal{S}} = 0$, whereas the entire operator L on $B^{s,p}_\infty$ is translation-variant and fails to commute with convolutions.

Note that this time the present construction does not give non-trivial translation-invariant extensions, since we cannot take g to be a non-zero constant function, as it must be an element of $B^{s,p}_\infty$. Of course, it cannot give such extensions, since it was shown in Prop. 3.10 that the translation-invariant extension $T \in \mathcal{L}(B^{s,p}_q)$ of the convolution operator $k*$ (acting on $\mathcal{S}$) is unique when $p < \infty$, even if $q = \infty$.

8. Appendix: Comparison of multipliers on Besov and Bôchner spaces

As has been explained in Chapter 0, Bourgain [**10**] and Burkholder [**13**] demonstrated that the Mihlin-style condition

$$m \in C^1(\mathbb{R} \setminus \{0\}), \qquad m(\xi),\ \xi m'(\xi) \text{ bounded} \tag{8.1}$$

is sufficient for m to be a Fourier multiplier on $L^p(\mathbb{R}; X)$, $p \in \,]1, \infty[$, if and only if X is a UMD-space; on the other hand, Amann [**1**] and Weis [**85**] showed that the somewhat stronger condition

$$m \in C^2(\mathbb{R}), \qquad m(\xi),\ (1 + |\xi|)m'(\xi),\ (1 + |\xi|)^2 m''(\xi) \text{ bounded} \tag{8.2}$$

is sufficient for m to be a Fourier multiplier on $B^{s,p}_q(\mathbb{R}; X)$, $s \in \mathbb{R}$, $p, q \in [1, \infty]$, for any Banach space X whatsoever.

In light of these results only, one might be tempted to ask whether the UMD-condition is only required to deal with the possible discontinuity of the multiplier at the origin, and the lack of estimates for the second derivative, and whether we could still have a theory of sufficiently smooth Mihlin-type multipliers on more general Banach spaces. Moreover, one could also ask whether the weaker assumption (8.1) implies any boundedness on the Besov scale.

A simple answer to both questions is provided in the following propositions by investigating the Hilbert transform, whose multiplier $m(\xi) = -\mathbf{i}\,\mathrm{sgn}(\xi)$ satisfies the condition (8.1) but not (8.2).

Proposition 8.3. *Let X be a Banach space, and suppose that every $m \in C^\infty(\mathbb{R})$, for which*

$$(1 + |\xi|)^k D^k m(\xi) \text{ is bounded for every } k \in \mathbb{N},$$

is a Fourier multiplier on $L^p(\mathbb{R}; X)$ for some $p \in \,]1, \infty[$. Then X is a UMD space.

This rather simple result is probably folklore; its novelty is doubtful, but no explicit reference is in my knowledge.

Proof. Fix a function $m \in C^\infty(\mathbb{R})$ such that $m(\xi) = \mathrm{sgn}(\xi)$ for $|\xi| \geq 1$. If $\hat{f} \in \mathcal{S}(\mathbb{R}; X)$ has a compact support contained in $]-\infty, -1[\, \cup\,]1, \infty[$, then

$m(\xi)\hat{f}(\xi) = \operatorname{sgn}(\xi)\hat{f}(\xi)$. Let us denote the set of all such f by $\hat{\mathcal{D}}_1(\mathbb{R};X)$. Thus, by assumption, $\|Hf\|_p \leq \|f\|_p$ for all $f \in \hat{\mathcal{D}}_1(\mathbb{R};X)$, where C is independent of f.

Let then $f \in \mathcal{S}(\mathbb{R};X)$ be any Schwartz function whose Fourier transform is compactly supported away from the origin; the set of such f's is denoted by $\hat{\mathcal{D}}_0(\mathbb{R};X)$. Then $\mathcal{F}[f(\epsilon^{-1}\cdot)] = \epsilon\hat{f}(\epsilon\cdot)$, and we see that $f(\epsilon^{-1}\cdot) \in \hat{\mathcal{D}}_1(\mathbb{R};X)$ for a sufficiently small $\epsilon > 0$. Thus $\|H[f(\epsilon^{-1}\cdot)]\|_p \leq C\,\|f(\epsilon^{-1}\cdot)\|_p$. But using the fact that the Hilbert transform commutes with dilations and $\|f(\epsilon^{-1}\cdot)\|_p = \epsilon^{1/p}\,\|f\|_p$, we can cancel the dependence on ϵ from both sides, to the result $\|Hf\|_p \leq \|f\|_p$ which is now proved for all $f \in \hat{\mathcal{D}}_0(\mathbb{R};X)$. This is a dense subset of $L^p(\mathbb{R};X)$, and so we have proved the boundedness of the Hilbert transform on $L^p(\mathbb{R};X)$—this is equivalent to X being UMD. □

Proposition 8.4. *Let X be a Banach space. The Hilbert transform is bounded on $B^{s,p}_q(\mathbb{R};X)$ if and only if X is a UMD-space and $p \in\,]1,\infty[$.*

Proof. We use Theorem 3.15, which gives both necessary and sufficient condition for $k \in \mathcal{S}'(\mathbb{R})$ to be a *convolutor* (i.e., to induce a bounded convolution operator) on $B^{s,p}_q(\mathbb{R};X)$.

Denoting by $k = \text{p.v.-}1/\pi x$ the underlying distribution of the Hilbert transform, note that the functions $k * \chi_i$ form a bounded set in $L^1(\mathbb{R})$ for $i = 1, 2, \ldots$, since the $\chi_i \in H^1(\mathbb{R})$ are dilates of each other (hence have the same norm) and the Hilbert transform is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. Thus whether or not H is bounded on $B^{s,p}_q(\mathbb{R};X)$ depends on the boundedness of $(k * \chi_0)*$ on $L^p(\mathbb{R};X)$.

If X is UMD and $p \in\,]1,\infty[$, then $\chi_0 *$ is bounded on $L^p(\mathbb{R};X)$ as a convolution with an integrable function, and $k* = H$ is bounded by the characterization of UMD-spaces.

Conversely, suppose that $(k * \chi_0)*$ is bounded on $L^p(\mathbb{R};X)$. If $f \in \mathcal{S}(\mathbb{R};X)$ has its Fourier transform supported in the neighbourhood of the origin where $\hat{\chi}_0 = 1$, then $(k * \chi_0) * f = k * f = Hf$, and so $\|Hf\|_p \leq C\,\|f\|_p$ for all such f by assumption. By the same dilation argument that was used in the proof of the previous proposition, this already implies $\|Hf\|_p \leq C\,\|f\|_p$ for all $f \in \mathcal{S}(\mathbb{R};X)$ with compactly supported Fourier transform. This is impossible for $p \in \{1,\infty\}$ even in the scalar case (and hence for any Banach space, since the scalar field is contained as an isometric subspace); moreover, for $p \in\,]1,\infty[$, we are lead to the UMD condition on X. □

From these two results we find that the sufficiency of the multiplier condition (8.1) forces equally strong requirements on the Banach space X, no matter whether we consider multipliers on the Besov or on the Bôchner scale. The nice property of the Besov spaces is the fact that we can obtain bounded multiplier transformations by slightly strengthening the assumptions to (8.2). On the Bôchner spaces, however, the passing from (8.1) to (8.2) does not give any pay-off in the requirements on the space X, since the fact that the L^p norms essentially commute with dilations shows that the boundedness of the Hilbert transform can be deduced even from the formally weaker multiplier theorem with assumptions (8.2). The Besov spaces, on the other hand, are inhomogeneous with respect to dilations.

CHAPTER 4

Sharp Fourier-embeddings and Mihlin-type multiplier theorems

The theorems on singular convolution operators from the previous chapters are combined with new Fourier embedding results to prove strong multiplier theorems on all the function spaces considered so far. The results improve on known theorems even in the scalar case.

The results of this paper have been submitted in the form of the article [**46**].

1. Introduction

Recall the two sets of conditions:

$$|x|^{|\alpha|_1} |D^\alpha m(x)| \leq \kappa \qquad \text{for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha|_\infty \leq 1, \text{ and all } x \in \mathbb{R}^n \setminus \{0\} \tag{1.1}$$

due to S. G. Mihlin [**62, 63**] in 1956, and

$$r^{|\alpha|_1} \left(\frac{1}{r^n} \int_{r<|x|<2r} |D^\alpha m(x)|^2 \, \mathrm{d}x \right)^{1/2} \leq \kappa \qquad \text{for all } \alpha \in \mathbb{N}^n \text{ with } |\alpha|_1 \leq \ell := \lfloor n/2 \rfloor + 1,\ r > 0, \tag{1.2}$$

due to L. Hörmander [**43**] in 1960—both sufficient for m to be a Fourier multiplier on $L^p(\mathbb{R}^n)$ for $p \in \,]1, \infty[$.

In Hörmander's assumptions, the uniformity in x in Mihlin's condition is relaxed to L^2-averages on annuli; moreover, the set of multi-indices α for which the estimate is required has slightly changed. Although Hörmander's result improves on that of Mihlin in certain respects, it is readily observed that it does not contain the original result itself. In fact, as soon as $n \geq 2$, Hörmander does require, among others, an estimate for the derivative $\partial^2 m/\partial x_1^2$, whereas in Mihlin's conditions one always needs to differentiate at most once w.r.t. any *single* coordinate, even though the total order of differentiation w.r.t. *all* the coordinates does get higher, in general, in his assumptions.

If we ignore the difference between uniform and quadratic estimates in the two conditions, and moreover consider $n \geq 3$, we find that Mihlin's theorem outperforms if we want to minimize the order of required derivative conditions in the ∞-norm, whereas Hörmander does better if measured in the 1-norm.

For us, this simple remark is the key observation and the main inspiration of the present chapter.

Our aim is to find strong estimates for multipliers by taking as the starting point the Mihlin-type approach of minimizing the required smoothness (measured with a continuous parameter) in each coordinate direction. Such a heavily coordinate-dependent procedure might well seem objectionable. For if R is a rotation of the space, it is well-known that m is a Fourier multiplier on $L^p(\mathbb{R}^n)$ if and only if $m(R\,\cdot)$ is, and the operator norms agree—a fact which would suggest making the conditions of a multiplier theorem rotation-invariant. (This is the case with Hörmander's condition (1.2).) However, the justification of our approach lies perhaps in strength rather than in elegance. In fact, our attempt to minimize the smoothness in the ∞-norm turns out to yield a solution which simultaneously minimizes it in the 1-norm.

Let us give some examples of the results we are able to prove:

Theorem 1.3. *Let X and Y be Banach spaces with Fourier-type $t \in\,]1,2]$, and m be an $\mathcal{L}(X,Y)$-valued function on $\mathbb{R}^n$.*

(1) *If the estimates*

$$r^{|\alpha|_1}\left(\frac{1}{r^n}\int_{I(r)}\|D^\alpha m(\xi)\|^t_{\mathcal{L}(X,Y)}\,\mathrm{d}\xi\right)^{1/t}\le\kappa$$

hold for all $r\in[1,\infty[$, with $I(r):=\{r<|\xi|<2r\}$ for $r>1$ and $I(1):=\{|\xi|\le 1\}$, and for all $\alpha\in\mathbb{N}^n$ satisfying $|\alpha|_\infty\le 1$ and $|\alpha|_1\le\lfloor n/t\rfloor+1$, then m is a Fourier-multiplier from the Besov space $B^{s,p}_q(\mathbb{R}^n;X)$ (for definition, see the beginning of Sect. 5) to $B^{s,p}_q(\mathbb{R}^n;Y)$ for all $s\in\mathbb{R}$ and $p,q\in[1,\infty]$.

(2) *If the estimate in part (1) holds for the same α's, and for all $r\in\,]0,\infty[$, with $I(r):=\{r<|\xi|<2r\}$, and moreover m is a Fourier-multiplier from $L^{\tilde p}(\mathbb{R}^n;X)$ to $L^{\tilde p}(\mathbb{R}^n;Y)$ for some $\tilde p\in\,]1,\infty[$, then m is also a Fourier-multiplier from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n;Y)$ for all $p\in\,]1,\infty[$ and from the atomic Hardy space $H^1(\mathbb{R}^n;X)$ to $H^1(\mathbb{R}^n;Y)$.*

(3) *If X and Y are UMD-spaces, and the operator collection*

$$\left\{|\xi|^{|\alpha|_1}\,D^\alpha m(\xi):\ \xi\in\mathbb{R}^n\setminus\{0\}\right\}$$

is R-bounded for all $\alpha\in\mathbb{N}^n$ such that $|\alpha|_\infty\le 1$ and $|\alpha|_1\le\lfloor n/t\rfloor+1$, then m is a Fourier-multiplier from $L^p(\mathbb{R}^n;X)$ to $L^p(\mathbb{R}^n;Y)$ for all $p\in\,]1,\infty[$ and from $H^1(\mathbb{R}^n;X)$ to $H^1(\mathbb{R}^n;Y)$.

Recall that a Banach space X has *Fourier-type* $t\ge 1$ if the Hausdorff–Young inequality $\|\hat f\|_{t'}\le C\,\|f\|_t$ holds for X-valued functions f (with a finite C independent of f). All Banach spaces have Fourier-type 1 (which is hence called the *trivial* Fourier-type) and all Hilbert spaces (and only Hilbert spaces [**55**]) have

Fourier-type 2. X is said to have a *non-trivial* Fourier-type if it has some Fourier-type $t > 1$. Although the above theorem assumes a non-trivial Fourier-type for both X and Y, we also give versions in the text where this is not required.

For the notion of R-boundedness, see Sect. 2 of Ch. 1 and Sect. 3 of Ch. 2.

NOTES ON PROOF. The proofs of the various statements will be given in different parts of the text. Part (1) is a weaker version of Cor. 5.8, with operator-norm estimates in place of strong ones. The L^p assertion of part (2) is contained in Theorem 7.2 and the H^1 assertion in Theorem 9.4. [The above mentioned results in the body of the chapter are actually formulated in a slightly different style from Theorem 1.3 above, but the equivalence is explained in Sect. 4.] The L^p assertion of part (3) is Cor. 8.16, and the H^1 assertion follows from this L^p assertion, combined with part (2), since the assumption in part (3) is stronger than that in part (2). □

Specializing to the particular case of scalar-valued functions (a Hilbert space would work equally well), we have

THEOREM 1.4. *Let m be a function on $\mathbb{R}^n$ satisfying*

$$r^{|\alpha|_1}\left(\frac{1}{r^n}\int_{r<|\xi|<2r}|D^\alpha m(\xi)|^2\,\mathrm{d}\xi\right)^{1/2}\leq\kappa$$

for all $r \in]0,\infty[$ and all $\alpha \in \mathbb{N}^n$ satisfying $|\alpha|_\infty \leq 1$ and $|\alpha|_1 \leq \lfloor n/2\rfloor + 1$. Then m is a Fourier-multiplier on $L^p(\mathbb{R}^n)$ for all $p \in]1,\infty[$ and on $H^1(\mathbb{R}^n)$.

NOTES ON PROOF. The L^p assertion is Cor. 7.4, whereas the H^1 assertion is contained in Cor. 9.6. Another proof is presented in Chapter 5; a reader who is interested in this particular theorem rather than the most general form of our multiplier results is advised to turn to this simplified treatment. □

The main benefit of our approach shows up clearly in Theorem 1.4: The assumptions of this theorem consist of the *intersection* of the assumptions of HÖRMANDER and those of MIHLIN, yet they are sufficient to get the same conclusion. Exactly the same effect is present in the Banach space version of the multiplier theorem, i.e., Theorem 1.3. Part (1) improves a recent result of M. GIRARDI and L. WEIS [**35**] (Cor. 4.13) where the same estimate as above is required for all $|\alpha|_1 \leq \lfloor n/t\rfloor + 1$. Similarly, part (3) simultaneously improves results of GIRARDI and WEIS [**36**] (Cor. 4.4) and of Ž. ŠTRKALJ and WEIS [**80**] (Theorem 4.4); in fact, its assumptions contain the intersection of the assumptions of the above mentioned authors. While these results are valid for operator-valued multipliers acting on vector-valued functions, it seems that the sufficiency in Theorem 1.4 of assuming only the intersection of the assumptions of HÖRMANDER and MIHLIN is new even in this scalar-valued setting.

It should be emphasized that the theorems formulated above are not the most general results we are going to prove but rather corollaries giving sufficient "classical style" conditions for verifying the assumptions of the general theorems. In particular, the form of the assumptions, where the set of the required derivatives is controlled by both ∞-norm and 1-norm estimates, only occurs in this classical-style version. In the general form of our results, where the required smoothness is measured with a generalized smoothness parameter, the conditions are expressed entirely in consistence with the ∞-norm paradigm. Once the required technicalities are grasped, the power of this approach will reveal itself.

To understand this, let us sketch the idea behind our notion of smoothness: The conditions for derivatives will be replaced by estimates for certain differences δ_h^α (built so as to approximate $h^\alpha D^\alpha$; defined in Sect. 2). We impose smoothness conditions on the "dyadic pieces" m_μ (defined in Sect. 4) of the multiplier m by requiring that $\|\delta_h^\alpha m_\mu\|_t$, for all $\alpha \in \mathbb{N}^n$ with $|\alpha|_\infty \leq d$, should behave like $h^{\alpha\gamma}$ (for some $\gamma \in]0,1]$) when h gets small. Our smoothness index is then $\Gamma := \gamma \cdot d$. As it turns out (in Sect. 4), this kind of smoothness conditions always hold provided we have Hörmander-type smoothness (as in (1.2)) with $\ell = n\Gamma$, but the converse is false with a substantial difference.

Now the power of our approach lies in the fact that the infimum, say $\tilde{\Gamma}$, of the admissible smoothness index Γ in our conditions will be exactly $\tilde{\Gamma} = \tilde{\ell}/n$, when $\tilde{\ell}$ is the critical index for Hörmander-type theorems (i.e. $\tilde{\ell} = n/2$ in the scalar-valued setting, and more generally $\tilde{\ell} = n/t$, where t is the Fourier-type; hence, in fact, $\tilde{\Gamma} = 1/t$). Thus the smoothness required by Hörmander-type results already implies our required smoothness conditions, and so our results always improve on versions of Hörmander's theorem; that they improve Mihlin's result is rather clear, since the assumptions are of the same type but weaker.

Besides using the ∞-norm to measure smoothness, there is another philosophical aspect which we wish to point out, and which could be described by the equality

$$\begin{aligned} &\text{a Fourier multiplier theorem} \\ &\quad = \text{a theorem for convolution operators} + \text{a Fourier embedding theorem.} \end{aligned} \tag{1.5}$$

This is a well-known "recipe", having its roots in the equivalence of the two descriptions $\widehat{Tf} = m\hat{f}$ and $Tf = k * f$ (where $m = \hat{k}$, the Fourier transform in the sense of distributions), dating back to Hörmander [**43**] and followed by many authors after him. Yet the general vector-valued setting with which we deal here gives new reasons for stressing the relation (1.5), and in particular the fact that the two "terms" on the right-hand side represent essentially distinct tasks, as will be elaborated below.

This decoupling is clear in this chapter, where we only deal with the latter of the two terms. As concerns the theorems for convolution operators, we apply the results from the previous Chapters 1–3. The main new results of this chapter are really those concerning the Fourier embeddings, although the multiplier theorems (which are actually obtained from (1.5) by a simple addition) are those which are more likely to be of interest in applications and which are labelled "Theorems" in the text. Classical Fourier embeddings are of little use to us here, since the function spaces they deal with tend to be rotation-invariant, and it is exactly giving away this invariance that constitutes one of the main ingredients of our approach.

The coherence with which our approach will apply to the different function spaces is also largely due to the exploitation of the decoupling of the two right-hand terms in (1.5): the differences between the various spaces, as well as between the scalar-valued and vector-valued situations, mostly show up in the theorems for convolution operators which we only cite here. On the other hand, the Fourier embeddings are only aware of the range space of the functions in terms of its Fourier-type. Taking this into account is usually only an additional statement in the assumptions of a theorem, and thus very easily dealt with.

The chapter is organised as follows. Sect. 2 explains notation and contains some preliminary considerations that pave the way for our main Fourier embedding results, which are stated and proved in Sect. 3. In Sect. 4 we describe those features of our approach to the multiplier theorems that are common to all the function spaces we will consider. The results obtained so far are first applied to a concrete situation in Sect. 5 where we consider the multipliers on Besov spaces. A comparison of our results with recent related work of M. Girardi and L. Weis [**35**] is contained in Sect. 6. We then turn to consider the Lebesgue–Bôchner spaces: in Sect. 7 we give sufficient conditions to have the boundedness of a multiplier operator on all L^p, $p \in\,]1,\infty[$, provided the boundedness on one $L^{\tilde{p}}$ is known *a priori*. Sect. 8 then takes up the task of proving the boundedness without such *a priori* knowledge. (In this case, the underlying Banach spaces are required to have the UMD property, and the assumptions on the multiplier m involve the notion of R-boundedness.) Finally, we consider multipliers on Hardy spaces in Sect. 9.

Some of the more technical proofs are postponed to two Appendices (Sections 10 and 11).

Remark 1.6. Although our approach yields a simultaneous improvement of the multiplier theorems of Mihlin and Hörmander, it should be noted that these results have been generalized in various other directions also, many of which are not covered by our results. One of the earliest such generalization is due to P. I. Lizorkin who was able to relax Mihlin's assumption $|x|^{|\alpha|_1}\,|D^\alpha m(x)| \le \kappa$ to $|x^\alpha D^\alpha m(x)| \le \kappa$ (for the same multi-indices α). See Triebel's book [**82**], Sect. 2.2.4, for several remarks and references to this kind of developments.

Concerning the vector-valued situation, F. Zimmermann [**89**] observed that whereas the UMD-condition of the Banach space X is sufficient to extend Mihlin's theorem to scalar-valued multipliers on $L^p(\mathbb{R}^n; X)$, Lizorkin's theorem requires additional geometric structure, i.e., it does not hold for all UMD-spaces. An operator-valued version of Lizorkin's theorem is proved by Štrkalj and

Weis [**80**]. However, the Lizorkin-type generalizations of Mihlin's theorem fall outside the scope of our treatment.

2. Preliminaries

First a word about notation. Some of the computations we encounter seem to be most conveniently handled by adopting a slighly non-standard notation. Thus the product symbol should be understood in the most formal sense, i.e., if $I := \{i_1, \ldots, i_k\}$, then "$\prod_{i\in I} \mathfrak{E}_i$" means: write the expressions $\mathfrak{E}_{i_1}$ through $\mathfrak{E}_{i_k}$ in a sequence, and only then interpret the result. In particular, we will often use short-hand notations like

$$\left(\prod_{i\in I}\int_{a_i}^{b_i} \mathrm{d}x_i\right) F(x) := \int_{a_{i_1}}^{b_{i_1}} \mathrm{d}x_{i_1} \cdots \int_{a_{i_k}}^{b_{i_k}} \mathrm{d}x_{i_k} F(x).$$

where $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and $x = (x_1, \ldots, x_n)$. Thus no integrals should be evaluated until all the "products" have been formally expanded. The expression above is *not* meant to be equal to $\left(\prod_{i\in I}(b_i - a_i)\right) F(x)$, which the conventional use of the product symbol would suggest.

For $0 \neq \alpha \in \{0,1\}^n$, we denote by $\mathbb{N}^\alpha$ the set of $|\alpha|$-tuples of natural numbers ($\mathbb{N} := \{0, 1, 2, \ldots\}$), but with the components labelled by those indices i for which $\alpha_i = 1$, rather than the first $|\alpha|$ positive integers as usual. E.g., if $n = 3$ and $\alpha = (0, 1, 1)$, then $\mathbb{N}^\alpha$ consists of all pairs $\nu = (\nu_2, \nu_3)$, where $\nu_2, \nu_3 \in \mathbb{N}$.

For convenience, and since confusion seems unlikely, $|\cdot|$ denotes various different norms which should be clear from the context: thus $|\alpha| := |\alpha|_1$ when $\alpha \in \mathbb{N}^n$ serves as a multi-index, $|x| := |x|_2$ when $x \in \mathbb{R}^n$ serves as a point in the domain of definition of the functions to be considered, $|f(x)| := |f(x)|_X$ when f is a function with values in the Banach space X, and moreover $|E|$ is the Lebesgue measure of a measurable subset $E \subset \mathbb{R}^n$. Note in particular that although we use two different norms, $|\alpha|_1$ and $|\alpha|_\infty$ for multi-indices, the short-hand $|\alpha|$ always refers to the former.

The symbols $\lfloor \ell \rfloor$, $[\![\ell]\!]$ and $\lceil \ell \rceil$ denote the greatest integer at most ℓ, the greatest integer strictly less than ℓ, and the smallest integer at least ℓ, respectively.

All our Banach spaces have scalar field $\mathbb{C}$. All function spaces have domain $\mathbb{R}^n$, and so there is no need to indicate this explicitly in the notation. On the other hand, the range spaces will vary, and so we write $B_q^{s,p}(X)$, $L^p(X)$ and $H^p(X)$ for the Besov, Lebesgue–Bôchner and Hardy spaces, respectively, of X-valued functions. If the range space is completely irrelevant for a particular statement, we might drop it from the notation. $\|f\|_p$ denotes the p-integral norm of f, whatever the range space. $\mathcal{S}$ is the Schwartz space of smooth, rapidly decreasing, scalar-valued functions, and $\mathcal{S}'(X) := \mathcal{L}(\mathcal{S}, X)$ is the space of X-valued tempered distributions on $\mathbb{R}^n$. The same notational conventions apply to it as to the above mentioned function spaces.

All derivatives are taken in the sense of distributions unless otherwise mentioned.

We will need the following translation and difference operators which are well-defined on arbitrary functions (or even distributions) on $\mathbb{R}^n$. The first two notions are fairly common (up to irrelevant sign conventions), whereas the third will be extremely useful in deriving the kind of Mihlin-type theorems we have in mind.

$$\tau_h f(x) := f(x-h), \qquad \Delta_h f := f - \tau_h f, \qquad \delta_h^\alpha := \prod_{i=1}^n \Delta_{h_i \mathfrak{e}_i}^{\alpha_i} = \sum_{\theta \le \alpha} \binom{\alpha}{\theta} (-1)^{|\theta|} \tau_{\theta h}$$

Here and in the sequel the notation is as follows: $\mathfrak{e}_i$ is the ith standard unit-vector of $\mathbb{R}^n$. By $\theta \le \alpha$ we mean that $\theta_i \le \alpha_i$ for all i. The product of $\theta \in \mathbb{N}^n$ and $h \in \mathbb{R}^n$ is the point $y \in \mathbb{R}^n$ with $y_i := \theta_i h_i$. $\Delta_{h_i \mathfrak{e}_i}^{\alpha_i}$ is simply an α_i-fold application of the operator $\Delta_{h_i \mathfrak{e}_i}$, with $\Delta_{h_i \mathfrak{e}_i}^0$ equal to the identity, as usual.

Several authors have proved results where the derivatives in Hörmander's theorem are replaced by differences. However, in the need of difference-based substitutes of higher than the first order derivatives, the guiding principle seems to have been to consider differences like $\Delta_h^N f$, instead of our $\delta_h^\alpha f$. But such an approach is almost destined to lead to Hörmander rather than Mihlin-type results, since

$$\Delta_h \approx h \cdot \nabla, \qquad \Delta_h^N \approx (h \cdot \nabla)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!} h^\alpha D^\alpha,$$

and so the use of Δ_h^N implicitly contains reference to all derivatives of a given order $|\alpha| = |\alpha|_1 = N$, at least if one allows the h vary arbitrarily. In particular, as soon as one takes $h_1 \ne 0$, say, one is immediately forced to have the derivative D_1^N, so there is no way to get the mixed derivatives without also taking the pure ones. On the other hand, the difference operators δ_h^α are built to approximate each partial derivative D^α individually. This last claim is given more quantitative content in the following.

Lemma 2.1. *Let $f \in \mathcal{S}'$, $h \in \mathbb{R}^n$, $\alpha \in \mathbb{N}^n$, and $f, D^\alpha f \in L^1_{\mathrm{loc}}$. Then*

$$\delta_h^\alpha f(x) = \left(\prod_{i=1}^n \prod_{j=1}^{\alpha_i} \int_0^1 \mathrm{d}t_{i,j} \right) h^\alpha D^\alpha f\Big(x - \sum_{i=1}^n \sum_{j=1}^{\alpha_i} t_{i,j} h_i \mathfrak{e}_i\Big) \qquad \textit{for a.e. } x.$$

Proof. If f is a smooth enough function, this follows (for all x) from a repeated application of the fundamental theorem of calculus. The general case is handled by pairing the distribution $\delta_h^\alpha f$ with an arbitrary test function $\psi \in \mathcal{D}(\mathbb{R}^n)$ to the result: (For short, we denote the multiple integral in the statement of the lemma simply by $\int \mathrm{d}t$, and the multiple summation simply by Σth.)

$$\begin{aligned}\langle \delta_h^\alpha f, \psi\rangle = \langle f, \delta_{-h}^\alpha \psi\rangle &= \left\langle f, \int \mathrm{d}t (-h)^\alpha D^\alpha \psi(\cdot + \Sigma th)\right\rangle \\ &= \int \mathrm{d}t(-h)^\alpha \langle f, D^\alpha \psi(\cdot + \Sigma th)\rangle = \int \mathrm{d}t\, h^\alpha \langle D^\alpha f, \psi(\cdot + \Sigma th)\rangle\end{aligned}$$

The only non-trivial step above is bringing the distribution f inside the integral; that this is legitimate is verified by showing that the Riemann sums of the integral converge in the topology of $\mathcal{S}$. We can now substitute in place of ψ a sequence of functions ψ_n which converge to the Dirac mass at a point x. Then $\langle \delta_h^\alpha f, \psi_n\rangle \to \delta_h^\alpha f(x)$ at every Lebesgue point of $\delta_h^\alpha f$, and for every such x we have $\langle D^\alpha f, \psi_n(x + \Sigma th)\rangle \to D^\alpha f(x - \Sigma th)$ whenever $x - \Sigma th$ is a Lebesgue point of $D^\alpha f$. Thus, for a fixed h, the integrand converges pointwise for a.e. $t = (t_{i,j})_{j=1,\dots,\alpha_i}^{i=1,\dots,n}$, and the assertion follows from Lebesgue's convergence theorem. □

Corollary 2.2. *For f as in Lemma 2.1 we have $\|\delta_h^\alpha f\|_q \le |h^\alpha| \, \|D^\alpha f\|_q$ for all $q \in [1, \infty]$, and when $q = \infty$, more precisely*

$$|\delta_h^\alpha f(x)| \le |h^\alpha| \operatorname{ess\,sup}\{|D^\alpha f(y)| : |x_i - y_i| \le \alpha_i \, |h_i|\} \qquad \textit{for a.e. } x \in \mathbb{R}^n.$$

Next we are going to see how the norms of the differences $\delta_h^\alpha f$ can be used to control the size of the Fourier transform $\hat{f}$.

3. Basic embeddings

In this section we prove our basic Fourier embedding results. Although they do not apply to all situations we encounter, and we will need to derive several modifications for different purposes in the subsequent sections, the essence of the matter is contained here. In a sense, although there are other results which we will call "Theorems", Proposition 3.2 below is the most important result of this whole chapter. It is the basis on which our multiplier theorems will be built.

For the proof, we need to introduce a decomposition of the domain $\mathbb{R}^n$. For $\alpha \in \{0,1\}^n$, $\rho \in \,]0,\infty[^n$ and $j \in \mathbb{N}^\alpha$, we define the sets

$$\begin{aligned} E(\alpha, \rho) &:= \{x : \ |x_i| \le \rho_i \text{ if } \alpha_i = 0, \ |x_i| > \rho_i \text{ if } \alpha_i = 1\}, \\ E(\alpha, \rho, j) &:= \{x : \ |x_i| \le \rho_i \text{ if } \alpha_i = 0, \ 2^{j_i}\rho_i < |x_i| \le 2^{j_i+1}\rho_i \text{ if } \alpha_i = 1\}. \end{aligned}$$

Then obviously

$$\mathbb{R}^n = \bigcup_{\alpha \in \{0,1\}^n} E(\alpha, \rho), \qquad E(\alpha, \rho) = \bigcup_{j \in \mathbb{N}^\alpha} E(\alpha, \rho, j).$$

The main point of introducing the sets $E(\alpha, \rho, j)$ is the following observation which we exploit below (the idea comes from Strömberg and Torchinsky [**81**]): For $0 < a \le |v| \le b < 1$, we have $\left|1 - e^{\mathrm{i}2\pi v}\right| \ge c = c(a, b) > 0$, and thus for $2^{j_i}\rho_i < |x_i| \le 2^{j_i+1}\rho_i$, it holds $\left|1 - e^{\mathbf{i}2\pi x_i/b2^{j_i}\rho_i}\right| \ge c(b) > 0$ whenever $b > 2$. Hence, we have an estimate of the form

$$(3.1) \qquad c \le \left|\prod_{i:\alpha_i=1}(1 - e^{\mathbf{i}2\pi \mathbf{e}_i\cdot x/b2^{j_i}\rho_i})\right| \le C \qquad \text{for } x \in E(\alpha,\rho,j),$$

where $c > 0$ is some constant depending only on $b > 2$ and the dimension n of $\mathbb{R}^n$, and clearly we can take $C = 2^n$. It is the first inequality in the above estimate that is used below, but the second inequality shows that we do not lose anything but an immaterial constant in making such an estimate.

PROPOSITION 3.2. *Let X have Fourier type t, let $q \in [1,t']$ with $q < \infty$, and $d \in \mathbb{Z}_+$. Then we have, for all $r \in]0,\infty[$ and all $f \in L^t(X)$, the estimate*

$$(3.3) \quad \left\|\hat{f}\right\|_q^q \le Cr^{(1-q/t')n}\left\|f\right\|_t^q + C\sum_{0\neq\alpha\in\{0,1\}^n} r^{(1-q/t')(n-|\alpha|)}\left(\prod_{i:\alpha_i=1}\int_0^{1/r}\frac{\mathrm{d}h_i}{h_i}h_i^{q/t'-1}\right)\left\|\delta_h^{d\alpha}f\right\|_t^q,$$

where C is finite and independent of f and r.

REMARK 3.4. It will be seen in the proof that the first term on the RHS of (3.3) actually controls the size of $\hat{f}$ in an approximate r-neighbourhood of the origin, whereas the second term gives a bound for $\hat{f}$ outside this region.

The parameter r is introduced so as to obtain a more flexible assertion; when applying the result to different functions, the sharpest bounds of $\|\hat{f}\|_q^q$ will be obtained by different choices of r. Namely, when we have the strict inequality $q < t'$, which is the case in all our applications, the factor $r^{(1-q'/t)(n-|\alpha|)}$ increases as a function of r, whereas the integrals involving $\int_0^{1/r}$ obviously decrease. There will be a delicate balance determining the optimal value of r for a particular function.

Requiring the RHS of (3.3) to be finite is a smoothness condition on f. In fact, it is clear that the h-integrals without the factor $\|\delta_h^{\alpha}f\|_t^q$ in the integrand would be divergent at the origin. To make these integrals converge, it is required that $\|\delta_h^{\alpha}f\|_t$ gets small as $h \to 0$, i.e., that the differences of f at near-by points should not differ appreciably in the L^t-norm.

The parameter d could be taken to be 1 for most of our purposes; however, the flexibility offered by this extra degree of freedom in the statement of the proposition will be essential in certain applications below.

PROOF. Consider $\rho \in]0,\infty[^n$ and $0 \neq \alpha \in \{0,1\}^n$. Applying the estimate (3.1), HÖLDER's inequality, and the HAUSDORFF–YOUNG inequality, in this order, we obtain

$$\begin{aligned}
\int_{E(\alpha,\rho)} \left|\hat{f}(x)\right|^q \mathrm{d}x &= \sum_{j\in\mathbb{N}^\alpha} \int_{E(\alpha,\rho,j)} \left|\hat{f}(x)\right|^q \mathrm{d}x \\
&\le C \sum_{j\in\mathbb{N}^\alpha} \int_{E(\alpha,\rho,j)} \left|\prod_{i:\alpha_i=1} (1-e^{\mathbf{i}2\pi\mathfrak{e}_i\cdot x/b2^{j_i}\rho_i})^d \hat{f}(x)\right|^q \mathrm{d}x \\
&\le C \sum_{j\in\mathbb{N}^\alpha} \left\| x \mapsto \prod_{i:\alpha_i=1} (1-e^{\mathbf{i}2\pi\mathfrak{e}_i\cdot x/b2^{j_i}\rho_i})^d \hat{f}(x)\right\|_{t'}^q |E(\alpha,\rho,j)|^{1-q/t'} \\
&\le C \sum_{j\in\mathbb{N}^\alpha} \left\| \prod_{i:\alpha_i=1} \Delta^d_{\mathfrak{e}_i/b2^{j_i}\rho_i} f\right\|_t^q \cdot \prod_{i:\alpha_i=0} (2\rho_i)^{1-q/t'} \cdot \prod_{i:\alpha_i=1} (2\cdot 2^{j_i}\rho_i)^{1-q/t'}.
\end{aligned}$$

We take a logarithmic average over $\rho_i \in [r, 2r]$ and then make a change-of-variable $h_i = 1/b2^{j_i}\rho_i$, followed by some rearrangement:

$$\begin{aligned}
&\left(\prod_{i=1}^n \int_r^{2r} \frac{\mathrm{d}\rho_i}{\rho_i}\right) \int_{E(\alpha,\rho)} \left|\hat{f}(x)\right|^q \mathrm{d}x \\
&\le C \sum_{j\in\mathbb{N}^\alpha} r^{(1-q/t')(n-|\alpha|)} \left(\prod_{i:\alpha_i=1} \int_r^{2r} \frac{\mathrm{d}\rho_i}{\rho_i} (2^{j_i}\rho_i)^{1-q/t'}\right) \left\|\prod_{i:\alpha_i=1} \Delta^d_{\mathfrak{e}_i/b2^{j_i}\rho_i} f\right\|_t^q \\
&= C r^{(1-q/t')(n-|\alpha|)} \sum_{j\in\mathbb{N}^\alpha} \left(\prod_{i:\alpha_i=1} \int_{(2b\cdot 2^{j_i}r)^{-1}}^{(b\cdot 2^{j_i}r)^{-1}} \frac{\mathrm{d}h_{i_1}}{h_{i_1}} h_i^{-1+q/t'}\right) \left\|\prod_{i:\alpha_i=1} \Delta^d_{h_i\mathfrak{e}_i} f\right\|_t^q \\
&= C r^{(1-q/t')(n-|\alpha|)} \left(\prod_{i:\alpha_i=1} \int_0^{1/br} \frac{\mathrm{d}h_i}{h_i} h_i^{q/t'-1}\right) \left\|\delta_h^{d\alpha} f\right\|_t^q.
\end{aligned}$$

Concerning $\alpha = 0$, we have

$$\int_{E(0,\rho)} \left|\hat{f}(x)\right|^q \mathrm{d}x \le \left\|\hat{f}\right\|_{t'}^q |E(0,\rho)|^{1-q/t'} \le C \left\|f\right\|_t^q \cdot \prod_{i=1}^n \rho_i^{1-q/t'},$$

and again we take the logarithmic average as above.

Note that we have

$$\int_{\mathbb{R}^n} \left|\hat{f}(x)\right|^q \mathrm{d}x = \sum_{\alpha\in\{0,1\}^n} \int_{E(\alpha,\rho)} \left|\hat{f}(x)\right|^q \mathrm{d}x$$

for any $\rho \in]0,\infty[^n$. Taking the logarithmic average only multiplies the LHS by a constant, so we get the conclusion. $\square$

Remark 3.5. Suppose we had only proved Prop. 3.2 with $r = 1$. Let us see what happens when we substitute $f(\cdot/r)$ in place of f. Then $\|\hat{f}\|_q^q$ becomes $r^{n(q-1)}\|\hat{f}\|_q^q$. Using $\delta_h^{d\alpha}[f(\cdot/r)] = [\delta^\alpha_{h/r} f](\cdot/r)$, we find that $\|\delta_h^\alpha f\|_q^t$ becomes

$\|\delta^{\alpha}_{h/r}f\|^t_q r^{nq/t}$. It remains to make the change-of-variable $y := h/r$, and we find that the general form of Prop. 3.2 derives from this simple scaling argument.

The following variant of Prop. 3.2, which gives more precise information on the size of $\hat{f}$ outside a bounded region, will also be significant to us:

PROPOSITION 3.6. *Under the assumptions of Prop.* 3.2*, we also have for all* $R > 0$ *and all* $r \in \,]0, R]$

$$
(3.7)\quad \int_{|x|>R} |\hat{f}(x)|^q \,\mathrm{d}x \le C \sum_{0\neq\alpha\in\{0,1\}^n} r^{(1-q/t')(n-|\alpha|)} \times \sum_{j:\alpha_j\neq 0} \int_0^{1/R} \frac{\mathrm{d}h_j}{h_j} h_j^{q/t'-1} \Bigg(\prod_{i\neq j:\alpha_i\neq 0} \int_0^{1/r} \frac{\mathrm{d}h_i}{h_i} h_i^{q/t'-1} \Bigg) \big\|\delta_h^{d\alpha} f\big\|_t^q ,
$$

with C *indepent of* f, r *and* R.

PROOF. When $|\rho|_\infty \le R/\sqrt{n}$, we have $E(0,\rho)^c \supset \{x : |x|_\infty > R/\sqrt{n}\} \supset \{x : |x| > R\}$. Thus, in the proof of Prop. 3.2, we only need to replace the average $\prod_{i=1}^n \int_r^{2r} \mathrm{d}\rho_i/\rho_i$ by the expression $\sum_{j:\alpha_j\neq 0} \int_{R/2\sqrt{n}}^{R/\sqrt{n}} \mathrm{d}\rho_j/\rho_j \prod_{i\neq j} \int_r^{2r} \mathrm{d}\rho_i/\rho_i$ when $\alpha \neq 0$, and we do not need to consider $E(0,\rho)$ at all. Recall (from the discussion preceding the statement of Prop. 3.2) that the auxiliary parameter b that was used in the proof of Prop. 3.2 was quite arbitrary, subject only to the condition $b > 2$. Thus we can take $b \ge 2\sqrt{n}$ so that $2\sqrt{n}/bR \le 1/R$. □

4. Approach to multiplier theorems

The embedding results of the previous section will be applied to the dyadic parts of the multiplier m, in the way to be explained now.

Let $\hat{\varphi}_0 \in \mathcal{D}$ have range $[0,1]$, be supported in $\bar{B}(0,1)$ and equal to unity in $\bar{B}(0,2^{-1})$. Let $\hat{\phi}_0 := \hat{\varphi}_0 - \hat{\varphi}_0(2\cdot)$ and $\hat{\phi}_\mu := \hat{\phi}_0(2^{-\mu}\cdot)$ for $\mu \in \mathbb{Z}$. Finally, let $\hat{\varphi}_\mu := \hat{\phi}_\mu$ for $\mu = 1, 2, \ldots$

The two families $(\varphi_\mu)_{\mu=0}^\infty$ and $(\phi_\mu)_{-\infty}^\infty$ (defined above in terms of their Fourier transforms) provide *resolutions of unity*, which are basic to our study. Given a multiplier m, we consider its dyadic parts $m_\mu := \hat{\varphi}_\mu m$ or $m_\mu := \hat{\phi}_\mu m$. (Of course, we will need to specify in concrete situations which decomposition we use, but for the moment we can proceed on a general level, assuming that m is decomposed, as above, into the pieces m_μ where either $\mu \in \mathbb{N}$ or $\mu \in \mathbb{Z}$.) Note that many authors denote our $\hat{\varphi}_\mu$ by φ_μ, which might be a little confusing. In our notation, one should keep in mind that the quantities $\hat{\varphi}_\mu, \hat{\phi}_\mu$ with the hat are the ones living in the frequency space.

We refer to $(\varphi_\mu)_{\mu=0}^\infty$ as the *inhomogeneous* and to $(\phi_\mu)_{-\infty}^\infty$ as the *homogeneous* resolution of unity, and to the decompositions of m induced by these resolutions as the inhomogeneous and homogeneous decompositions, respectively. We also apply the words homogeneous and inhomogeneous to some related quantities; the former always means that the vicinity of the origin is decomposed to ever finer pieces, similar in all length scales, while the latter refers to the fact that a certain neighbourhood of the origin is treated as one block, and only the rest of the space is dyadically decomposed.

We denote $k := \check{m}$ (inverse Fourier transform in the sense of distributions) and $k_\mu := \check{m}_\mu$, which equals $\phi_\mu * k$ or $\varphi_\mu * k$, depending on the decomposition we use. In either case, k_μ is the convolution of a Schwartz function with a tempered distribution, thus an infinitely differentiable funtion with at most polynomial growth.

Strong multiplier theorems can be obtained, as we will see, by simply requiring that every m_μ in place of f makes the RHS of (3.3) finite (plus appropriate uniformity in μ). However, it is also useful to be able to check this condition in terms of estimates of more classical appearence. Our basic condition for doing this will be

$$\left\|\delta_h^\alpha m_\mu\right\|_q \leq \kappa 2^{\mu(n/q-|\alpha|\gamma)} h^{\alpha\gamma} \text{ for all } |\alpha|_\infty \leq d \text{ and } h \in \mathbb{R}_+^n, \tag{4.1}$$

where $d \in \mathbb{Z}_+$, $\gamma \in \,]0,1]$. It should be emphasized that the relevant smoothness parameter in this condition is $\Gamma := d \cdot \gamma$, rather than either of d or γ alone. This is not so clear *a priori*, but will be clarified when using the conditions below. For many but not all of our results, it would suffice to consider $d = 1$, but keeping the d in the expressions allows us to state more general assertions, rather than reformulating everything for the situations where $d = 1$ does not work.

Let us connect the condition (4.1) to the kinds of expressions we encountered in connection with the Fourier embeddings. For the present, we only give the following simple result, and other variants will appear when they are needed in the sequel. Even this lemma already gives a reason for the usefulness of the conditions we introduced in (4.1): they can be used to have control on the right-hand sides of the inequalities in Prop. 3.2 and Prop. 3.6, which in turn give, by the very statements of these results, estimates on the kernels $k_\mu = \check{m}_\mu$.

Lemma 4.2. *Suppose $\left\|\delta_h^{d\alpha} f\right\|_t \leq 2^{\mu(n/t-\Gamma|\alpha|)} h^{\alpha\Gamma}$ for all $|\alpha|_\infty \leq 1$, where $d \in \mathbb{Z}_+$, $t \in [1,2]$ is a Fourier-type for the underlying space X, $q \in [1,t']$ with $q < \infty$, and $\Gamma > 1/t - 1/q'$. Then the right-hand side of (3.3), with $r = 2^{-\mu}$, is at most $C2^{\mu n/q'}$, and the RHS of (3.7), with $r = 2^{-\mu} \leq R$, at most*

$$C 2^{\mu q((n-1)/q'+1/t-\Gamma)} R^{q(1/t-1/q'-\Gamma)}.$$

Consequently,

$$\left\|\hat{f}\right\|_q \leq C 2^{\mu n/q'}, \quad \int_{|x|>R} \left|\hat{f}(x)\right|^q \mathrm{d}x \leq C 2^{\mu q((n-1)/q'+1/t-\Gamma)} R^{q(1/t-1/q'-\Gamma)}. \tag{4.3}$$

Note in particular that with $q = 1$ (hence $q' = \infty$), we obtain a bound for $\left\|\hat{f}\right\|_1$ independent of μ.

PROOF. The assertions concerning the RHS's of (3.3) and (3.7) follow by a direct computation. Then the estimate for $\|\hat{f}\|_q^q$ follows from Prop. 3.2 and the estimate for the integral over $|x| > R$ from Prop. 3.6 when $R \geq 2^{-\mu}$. For $R < 2^{-\mu}$, the second claim in (4.3) follows from the first, since then $(2^\mu R)^{1/t-1/q'-\Gamma} \geq 1$. $\square$

In the rest of this section, we relate our condition (4.1) (which involves the somewhat unconventional difference operators δ_h^α) to more familiar conditions found in the literature. As an intermediate notion between (4.1) and the assumptions of HÖRMANDER or MIHLIN, there are conditions due to D. S. KURTZ and R. L. WHEEDEN [**54**], which have been generalized (to allow for a continuous smoothness parameter) by J.-O. STRÖMBERG and A. TORCHINSKY [**81**]. One assumes Hörmander-type bounds

$$
\begin{aligned}
&\left(\int_{I(r)} |D^\alpha m(\xi)|^q \,\mathrm{d}\xi\right)^{1/q} \leq \kappa r^{n/q-|\alpha|},\\
&\left(\int_{I(r)} |D^\alpha m(\xi) - D^\alpha m(\xi-\zeta)|^q \,\mathrm{d}\xi\right)^{1/q} \leq \kappa r^{n/q-|\alpha|}\left(\frac{|\zeta|}{r}\right)^\epsilon \text{ for } |\zeta| \leq r/2,
\end{aligned}
\tag{4.4}
$$

where $\epsilon \in [0, 1[$ and

- in the homogeneous version, the estimates (4.4) are assumed for all $r \in]0, \infty[$ and $I(r) := \{\xi :\ r < |\xi| < 2r\}$; and
- in the inhomogeneous version, we only consider $r \in [1, \infty[$ and $I(r)$ is the same as above for $r > 1$ but $I(1) := \{\xi :\ |\xi| \leq 1\}$.

STRÖMBERG and TORCHINSKY considered, in the homogeneous situation, conditions which they called $M(q, \ell)$, defined by the requirement that (4.4) hold with $\epsilon = \ell - \lfloor \ell \rfloor$ for all $\alpha \in \mathbb{N}^n$ satisfying $|\alpha| \leq \lfloor \ell \rfloor$. If $\ell = \lfloor \ell \rfloor$ is an integer, then $\epsilon = 0$, and the second condition in (4.4) is implied by the first one, with $c\kappa$ in place of κ, and c numerical. Note that HÖRMANDER's original condition (1.2) is $m \in M(2, \ell)$ with $\ell = \lfloor n/2 \rfloor + 1$.

In order to see the relation of these conditions to ours, it is useful to consider, a little more generally, the condition (4.4) for all $\alpha \in \mathcal{I}$, where $\mathcal{I} \subset \mathbb{N}^n$ has the property that $\theta \in \mathcal{I}$ whenever $\theta \leq \alpha \in \mathcal{I}$. We say that such an $\mathcal{I}$ is *stable*. Note that both the Hörmander-type set $\{\alpha : |\alpha|_1 \leq \ell\}$ and the Mihlin-type set $\{\alpha : |\alpha|_\infty \leq 1\}$ are stable, so that the notion of stability unifies the treatment of the two sets of conditions.

Whenever $\mathcal{I} \subset \mathbb{N}^n$ is stable, it is easy to see that (4.4) for all $\alpha \in \mathcal{I}$ is equivalent to

$$
\|D^\alpha m_\mu\|_q \leq \kappa 2^{\mu(n/q-|\alpha|)}, \qquad \|D^\alpha(m_\mu - \tau_h m_\mu)\|_q \leq \kappa 2^{\mu(n/q-|\alpha|-\epsilon)}\,|h|^\epsilon. \tag{4.5}
$$

for the same α's. (The constants κ are multiplied by numerical factors when passing from the conditions (4.4) to (4.5) or back, but this is irrelevant.) Observe that, directly from (4.4), we only get (4.5) for $|h| \leq c2^\mu$, but for $|h| > c2^\mu$ we get the second estimate in (4.5) from the first one with an application of the triangle inequality. Also note that (4.5) includes both the homogeneous and the inhomogeneous version, with the appropriate meaning of m_μ in either case.

In the following lemma, we see how pure difference estimates like in (4.1) can be checked in terms of the more classical conditions considered above, not only for differences of the same order as the assumed derivatives, but in fact for differences of any order. Of course there is a price to be paid, which is the change in the parameter γ. Recall that the relevant parameter in (4.1) is the product $\Gamma := d \cdot \gamma$.

The proof of the following lemma, being somewhat lengthy and technical, is postponed to the appendix (Sect. 10).

Lemma 4.6. *Let $\mathcal{I} \subset \mathbb{N}^n$ be stable. Then the estimates* (4.5) *for $\alpha \in \mathcal{I}$ with $|\alpha| \le \lfloor \ell \rfloor$, and with $\epsilon = \ell - \lfloor \ell \rfloor$, imply for $h \in \mathbb{R}^n_+$*

$$(4.7)\quad \|\delta_h^\alpha m_\mu\|_q \le c\kappa 2^{\mu(n/q - |\alpha|\tilde{\gamma})} h^{\alpha\tilde{\gamma}}, \text{ for } \alpha \in \mathcal{I} \text{ with } |\alpha| \le \lceil \ell \rceil, \text{ where } \tilde{\gamma} := \frac{\ell}{\lceil \ell \rceil},$$

and, more generally, for any $L \in \mathbb{N}$ with $L \ge \lceil \ell \rceil$ (and $h \in \mathbb{R}^n_+$)

$$(4.8)\qquad \|\delta_h^\alpha m_\mu\|_q \le c\kappa 2^{\mu(n/q - |\alpha|\gamma)} h^{\alpha\gamma}, \text{ for } \alpha \in \mathcal{I} \text{ with } |\alpha| \le L, \text{ where } \gamma := \frac{\ell}{L}.$$

Remark 4.9. If we measure smoothness "in the 1-norm", i.e., in terms of (the 1-norm of) the highest power of h in the conclusion of Lemma 4.6, we note that this is $L\gamma = \ell$, i.e., the same as the smoothness parameter of the conditions $M(q,\ell)$ (or its inhomogeneous version) with which we started.

From Lemma 4.6 and the preceding considerations we obtain:

Corollary 4.10. *Suppose m satifies* (4.4) [*or, what is equivalent,* (4.5)] *with $\epsilon = \ell - \lfloor \ell \rfloor$, for all $\alpha \in \mathbb{N}^n$ s.t. $|\alpha|_1 \le \lfloor \ell \rfloor$ and $|\alpha|_\infty \le d$, where $nd \ge \lceil \ell \rceil$.*

Then (4.1) (*with m_μ refering to the homogeneous or inhomogeneous version in consistence with the condition assumed for m*) *holds with $\gamma = \ell/nd$.*

Proof. It suffices to observe that $\mathcal{I} := \{\alpha \in \mathbb{N}^n : |\alpha|_\infty \le d\}$ is a stable collection, and moreover that $\alpha \in \mathcal{I}$ implies $|\alpha|_1 \le nd$. Then apply Lemma 4.6 with $L = nd$. $\square$

Remark 4.11. It is important to observe, as soon as $n \ge 2$, that the set of conditions (4.1), for some d and γ, is substantially weaker than the same set of conditions holding for all $|\alpha|_1 \le nd$. (We do require it for *one* α, namely $\alpha = (d, d, \dots, d)$, for which $|\alpha|_1 = nd$, though.) In fact, consider functions of the form $m(x) = \prod_{i=1}^n m_i(x_i)$, where each m_i is compactly supported outside the origin. Then the condition for $|\alpha|_1 \le nd$ resp. $|\alpha|_\infty \le d$ is essentially equivalent to

$$\prod_{i=1}^n h_i^{-\gamma\alpha_i} \left\|\Delta_{h_i}^{\alpha_i} m_i\right\|_q \le \kappa$$

for all $|\alpha|_1 \leq nd$ resp. all $|\alpha|_\infty \leq d$. For the latter condition, it suffices that $h^{-\gamma k} \left\|\Delta_h^k m_i\right\|_q \leq C$ for $k = 0, \ldots, d$, whereas the first condition would require this for $k = 0, \ldots, nd$.

With Cor. 4.10 and the above, we have obtained rather quantitative content for the idea that Hörmander-type estimates with 1-norm smoothness index ℓ imply our Mihlin-type conditions with ∞-norm smoothness index $\Gamma = \ell/n$, but the converse is false. Thus, roughly speaking, any result we prove with a Mihlin-type smoothness index $\Gamma = \ell/n$ is an improvement of a Hörmander-type theorem with required smoothness index ℓ.

We can now leave the general framework of the theory, and turn to apply the results obtained so far to the derivation of strong multiplier theorems in the setting of Besov spaces of vector-valued functions.

5. Multipliers on Besov spaces

We give our first multiplier theorems on the Besov spaces since, even though these are less natural and more difficult to define than most of the more classical function spaces, the technicality in the definitions actually gives substantial simplicity in the actual multiplier theory. Nevertheless, this simplified situation can serve as a toy model for the more relevant, and more difficult, case of the Lebesgue–Bôchner spaces which are treated after the Besov case.

The definition of the Besov spaces which we use reads as follows: a distribution $f \in \mathcal{S}'(X)$ is a member of $B_q^{s,p}(X)$ (where $s \in \mathbb{R}$, $p, q \in [1, \infty]$) if and only if

$$\|f\|_{s,p;q} := \left\| \left(2^{\mu s} \|\varphi_\mu * f\|_p \right)_{\mu=0}^{\infty} \right\|_{\ell^q}$$

is finite, and $\|\cdot\|_{s,p;q}$ is the norm of the space $B_q^{s,p}(X)$.

Thus the Besov spaces are defined in terms of the inhomogeneous dyadic decomposition. In consistence with this, the inhomogeneous decomposition is the only one we use when working with these spaces, and so m_μ and k_μ always refer to $\hat{\varphi}_\mu m$ and $\varphi_\mu * k$, respectively, in this section and the next one.

Before going to the new results, let us briefly sketch a historical perspective. A systematic treatment of the classical Besov space and related multiplier theory (where the word “classical” refers, above all, to scalar-valuedness) can be found in TRIEBEL’s book [**83**]. The study of operator-valued multipliers on Besov spaces of vector-valued distributions was initiated by H. AMANN [**1**] and L. WEIS [**85**]. In the latter work, which was recently expanded by M. GIRARDI and WEIS [**35**], rather general sufficient conditions were given, in which the Fourier-type of the underlying Banach spaces is taken into account to decrease the required smoothness with increasing Fourier-type, and the smoothness conditions are expressed in terms of the Besov space norms of the (inhomogeneous) dyadic pieces m_μ, $\mu \in \mathbb{N}$, of the multiplier.

As shown by Girardi and Weis in [**35**], their general theorem is strong enough to contain a Besov space version of Hörmander's multiplier theorem and some variants as special cases; however, it fails to imply Mihlin-type results in the sense in which we have used this word.

In this section, we are going to prove a multiplier theorem which contains both Hörmander and Mihlin-type theorems; and it also contains the above mentioned theorem of Girardi and Weis. The theorem is hence a rather strong one; in fact, Girardi and Weis [**35**] point out that their result is sharp in a certain sense, yet we are able to improve that. We will return to comment on this matter below.

Let us now define, for $f \in L^t(X)$, the following quantities (which should be compared with the right-hand sides of the inequalities established in Prop.'s 3.2 and 3.6):

$$\|f\|_{\mathcal{M}^t} := \|f\|_t + \sum_{0\neq\alpha\in\{0,1\}^n} \left(\prod_{i:\alpha_i=1} \int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t}\right) \left\|\delta_h^{d\alpha} f\right\|_t,$$

$$\|f\|_{\dot{\mathcal{M}}^t} := \inf_{r\in]0,\infty[} \|f(\cdot/r)\|_{\mathcal{M}^t},$$

$$\|f\|_{\mathcal{M}^t}^R := \sum_{0\neq\alpha\in\{0,1\}^n} \sum_{j:\alpha_j=1} \int_0^{1/R} \frac{\mathrm{d}h_j}{h_j} h_j^{-1/t} \left(\prod_{i\neq j,\alpha_i=1} \int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t}\right) \left\|\delta_h^{d\alpha} f\right\|_t,$$

where $d := 1$ if $t > 1$ and $d := 2$ if $t = 1$.

Remark 5.1. According to Rem. 3.5, $\|f(\cdot/r)\|_{\mathcal{M}^t}$ is equal to the right-hand side of the inequality in Prop. 3.2 when $q = 1$. Thus $\|\hat{f}\|_1 \leq C\,\|f\|_{\dot{\mathcal{M}}^t}$, as soon as X has Fourier-type t.

It follows easily that $\|f(\cdot/r)\|_{\mathcal{M}^t}$ is finite either for all $r \in]0,\infty[$ or for none. To see this using the expression in Prop. 3.2, just note that one can always estimate $\left\|\delta_h^{d\alpha} f\right\|_t \leq 2^{d(|\alpha|-|\theta|)} \left\|\delta_h^{d\theta} f\right\|_t$ and $\int_\epsilon^\infty \mathrm{d}h_i\, h_i^{-1-1/t} < \infty$, from which it follows readily that the behaviour of $\left\|\delta_h^{d\alpha} f\right\|_t$ is relevant only for small h, provided that $\|f\|_t$ is finite. Hence $\|f\|_{\mathcal{M}^t} < \infty$ iff $\|f\|_{\dot{\mathcal{M}}^t} < \infty$, but of course the actual values can be quite different.

We denote by $\mathcal{M}^t(X)$ the collection of those $f \in L^t(X)$ for which $\|f\|_{\mathcal{M}^t} < \infty$. It is not difficult to see that $\mathcal{M}^t(X)$ is a Banach space when equipped with this norm.

By Prop. 3.6, we have

$$\int_{|x|>R} \left|\hat{f}(x)\right| \,\mathrm{d}x \leq C\,\|f\|_{\mathcal{M}^t}^R$$

for all $R > 1$. Below, in the case where the Besov index p in $B_q^{s,p}$ is infinite, we are concerned with the behaviour of $\|f\|_{\mathcal{M}^t}^R$ as $R \to \infty$.

We will use the more complete notation $\|f\|_{\mathcal{M}^t(X)}$ for $\|f\|_{\mathcal{M}^t}$ when the underlying space needs to be specified.

THEOREM 5.2. *Suppose that*

$$\|m_\mu(\cdot)x\|_{\dot{\mathcal{M}}^t(Y)} \le \kappa\,|x|_X \quad \textit{and} \quad \|m_\mu(\cdot)'y'\|_{\dot{\mathcal{M}}^u(X')} \le \kappa\,|y'|_{Y'}\,,$$

where X has Fourier-type u, and Y has Fourier-type t. Then m is a Fourier-multiplier from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p \in [1,\infty[$ and $q \in [1,\infty]$. The assertion remains true for $p = \infty$, if we also assume

$$\sup_{|x|_X \le 1} \|m_\mu(\cdot)x\|_{L^t(Y)} < \infty, \quad \textit{and} \quad \sup_{|x|_X \le 1} \|m_\mu(\cdot)x\|^R_{\mathcal{M}^t(Y)} \underset{R\to\infty}{\longrightarrow} 0. \tag{5.3}$$

(*The last two conditions are not required to be uniform in μ.*)

Recall that Y' has Fourier-type t if and only if Y has.

PROOF. From the Remark 5.1 it follows that

$$\|k_\mu(\cdot)x\|_{L^1(Y)} \le C\kappa\,|x|_X\,, \qquad \|k_\mu(\cdot)'y'\|_{L^1(X')} \le C\kappa\,|y'|_{Y'}\,.$$

According to Theorem 4.10 of Chapter 3, this suffices for $p < \infty$.

For the general case, again according to Theorem 4.10 of Chapter 3, we should verify that, for all μ, the convergence

$$\int_{E_m} |k_\mu(s)x|_Y \,\mathrm{d}s \underset{m\to\infty}{\longrightarrow} 0$$

takes place uniformly in $|x|_X \le 1$, when either E_m is a decreasing sequence with $|E_m| \to 0$, or else $E_m = \{x : |x| > m\}$.

To see that this is the case for the multipliers of Theorem 5.2, observe that

$$\int_E |k_\mu(s)x|_Y \,\mathrm{d}s \le \|k_\mu(\cdot)x\|_{t'}\,|E|^{1/t} \le C\,\|m_\mu(\cdot)x\|_t\,|E|^{1/t}\,,$$

and this tends to 0 uniformly in $|x|_X \le 1$ as $|E| \to 0$ by the assumption.

As for the estimate for large sets going to infinity, we have from Rem. 5.1, for $R > 1$,

$$\sup_{|x|_X \le 1} \int_{|s|>R} |k_\mu(s)x|_Y \,\mathrm{d}s \le C \sup_{|x|_X \le 1} \|m_\mu(\cdot)x\|^R_{\mathcal{M}^t} \underset{R\to\infty}{\longrightarrow} 0 \tag{5.4}$$

by assumption. $\square$

REMARK 5.5. When $\|m_\mu(\cdot)x\|_{\mathcal{M}^t} < \infty$, it is clear that we have $\|m_\mu(\cdot)x\|_t < \infty$ and $\|m_\mu(\cdot)x\|^R_{\mathcal{M}^t} \to 0$ as $R \to \infty$ for each $x \in X$ individually; hence what is new in the additional assumption (5.3) is the uniformity in $|x|_X \le 1$. This condition is trivially fulfilled if we replace the strong estimate $\|m_\mu(\cdot)x\|_{\dot{\mathcal{M}}^t(Y)} \le \kappa\,|x|_X$ and its dual in the assumption of Theorem 5.2 by the single operator-norm condition $\|m_\mu(\cdot)\|_{\dot{\mathcal{M}}^t(\mathcal{L}(X,Y))} \le \kappa$. (Now both X and Y are required to have Fourier-type t.)

REMARK 5.6. We can also stay with strong conditions if we slightly strengthen the smoothness requirement $\|m_\mu(\cdot)x\|_{\dot{\mathcal{M}}^t(Y)} \le \kappa\,|x|_X$ to

(5.7)

$$\left\|\delta^{d\alpha}_h m_\mu(\cdot)x\right\|_{L^t(Y)} \le \kappa 2^{\mu(n/t-\Gamma|\alpha|)} h^{\alpha\Gamma}\,|x|_X\,, \qquad \Gamma > 1/t, \qquad \text{for all } |\alpha|_\infty \le 1.$$

That (5.7) implies $\dot{\mathcal{M}}^t(m_\mu(\cdot)x) \leq c\kappa\, |x|_X$ and the convergence statement in (5.3) is contained in Lemma 4.2.

The first estimate in (5.3) holds if we require that

$$\|m_\mu(\cdot/r)x\|_{\mathcal{M}^t(Y)} \leq \kappa\, |x|_X$$

for some fixed $r = r(\mu)$, possibly depending on μ but not on $x \in X$, instead of allowing for $r = r(\mu, x)$ as in the requirement $\|m_\mu(\cdot)x\|_{\dot{\mathcal{M}}^t(Y)} \leq \kappa\, |x|_X$. (This follows from the fact that $\|f\|_t \leq r^{-n/t}\, \|f(\cdot/r)\|_{\mathcal{M}^t}$.) Under the assumption (5.7), we can take $r(\mu) = 2^{-\mu}$ (cf. Lemma 4.2).

Using Cor. 4.10 and the above remarks, it is straightforward to derive from Theorem 5.2 corollaries of more classical appearance, but stronger than the usual forms of such results. We consider separately the cases of non-trivial ($t > 1$) and trivial ($t = 1$) Fourier-type, since writing down the classical-style condition for the latter requires slightly different treatment, although both cases are contained in a uniform manner in the general conditions of Theorem 5.2.

We first consider non-trivial Fourier-type:

Corollary 5.8. *Let X and Y have Fourier-type $t \in\,]1, 2]$, and suppose that*

$$\|D^\alpha m_\mu(\cdot)x\|_t \leq 2^{\mu(n/t - |\alpha|)}\, |x|_X\,, \qquad \|D^\alpha m_\mu(\cdot)'y'\|_t \leq 2^{\mu(n/t-|\alpha|)}\, |y'|_{Y'} \tag{5.9}$$

for all $\alpha \in \mathbb{N}^n$ satisfying $|\alpha|_\infty \leq 1$ and $|\alpha|_1 \leq \lfloor n/t \rfloor + 1$. Then m is a Fourier multiplier from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$.

Proof. By Cor. 4.10, the assumptions imply $\|\delta^\alpha_h m_\mu(\cdot)x\|_t \leq c\kappa 2^{\mu(n/t - |\alpha|\gamma)} h^{\alpha\gamma}$ (and the dual condition) for $|\alpha|_\infty \leq 1$, where $\gamma = (\lfloor n/t \rfloor + 1)/n > (n/t)/n = 1/t$. Then Rem. 5.6 completes the argument. (Now $d = 1$.) $\square$

With operator-norm estimates in place of strong ones, and these estimates required for all $|\alpha|_1 \leq \lfloor n/t \rfloor + 1$ (instead of just those with also satisfy $|\alpha|_\infty \leq 1$), this result is shown in Girardi and Weis [**35**] (Cor. 4.13) as a corollary of their general multiplier theorem.

Now we consider the situation with no Fourier-type (nor any other geometric assumption) imposed on the Banach spaces X and Y. In this case, we must modify the assumptions of Cor. 5.8 a little, since the bound $|\alpha|_\infty \leq 1$ would imply that $|\alpha|_1 \leq n$, and so there would be no α for which the upper limit in the condition $|\alpha|_1 \leq \lfloor n/t \rfloor + 1 = n + 1$ (when $t = 1$) is achieved. We need to include some α's for which this bound is reached, but in order to keep the number of such multi-indices at a minimum, we introduce the following set:

$$\mathcal{I}_n := \{\alpha \in \{0, 1, 2\}^n :\ \alpha_i = 2 \text{ for at most one } i = 1, \ldots, n\}.$$

Now the condition in terms of derivative estimates reads as follows:

Corollary 5.10. *Let X and Y be any Banach spaces, and suppose that* (5.9) *holds with $t = 1$ for all $\alpha \in \mathcal{I}_n$. Then m is a Fourier multiplier from $B^{s,p}_q(X)$ to $B^{s,p}_q(Y)$ for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$.*

PROOF. The aim is to check the condition (5.7) with $d = 2$. Cor. 4.10 is not directly applicable, but it is easy to modify the same ideas to get the desired result:

Given $\alpha \in \{0,1\}^n \setminus \{0\}$, consider the $|\alpha|_1$ multi-indices $\theta^j := \alpha + \mathfrak{e}_j$, where $\alpha_j = 1$. It is evident that $\sum \theta^j = (|\alpha| + 1)\alpha$, $\theta^j \leq 2\alpha$ and $\theta^j \in \mathcal{I}_n$.

Now we have

$$\left\|\delta_h^{2\alpha} m_\mu(\cdot)x\right\|_1 \leq 2^{|\alpha|-1} \left\|\delta_h^{\theta^j} m_\mu(\cdot)x\right\|_1 \leq c \left\|D^{\theta^j} m_\mu(\cdot)x\right\|_1 h^{\theta^j} \leq c\kappa 2^{\mu(n-\left|\theta^j\right|)} h^{\theta^j} |x|_X .$$

Multiplying the $|\alpha|$ estimates for all the different θ^j's, we further obtain

$$\left\|\delta_h^{2\alpha} m_\mu(\cdot)x\right\|_1^{|\alpha|} \leq \left(c\kappa 2^{\mu(n-(|\alpha|+1))}\right)^{|\alpha|} h^{\sum \theta^j} |x|_X^{|\alpha|},$$
$$\left\|\delta_h^{2\alpha} m_\mu(\cdot)x\right\|_1 \leq c\kappa 2^{\mu(n-|\alpha|(|\alpha|+1)/|\alpha|)} h^{\alpha(|\alpha|+1)/|\alpha|} |x|_X .$$

Since $(|\alpha|+1)/|\alpha| \geq (n+1)/n (=: \Gamma) > 1$, we obtain the estimate (5.7) as desired (for it is clear that the verification of the dual condition using the dual assumption is exactly the same). □

Other classical-looking variants (e.g., with derivative and difference conditions combined, as in the STRÖMBERG–TORCHINSKY conditions (4.4)) may be obtained in a similar and rather obvious fashion from the general result, Theorem 5.2. But the examples given above already show that our approach can be used to reprove, and in fact, improve, the corollaries derived by GIRARDI and WEIS [**35**] from their general multiplier theorem. However, the relation between the two general theorems is not yet clarified, and this is the problem we now take up.

6. Relation to Girardi–Weis conditions

The purpose of this section is to compare the conditions of our Theorem 5.2 with those of GIRARDI and WEIS ([**35**], Theorem 4.8). Their sufficient condition for m to be a Fourier multiplier from $B_q^{s,p}(X)$ to $B_q^{s,p}(Y)$ for all $s \in \mathbb{R}$, $p, q \in [1, \infty]$, is

$$\inf_{r \in]0,\infty[} \left\|m_\mu(\cdot/r)\right\|_{B_1^{n/t,t}(\mathcal{L}(X,Y))} \leq \kappa$$

for all $\mu \in \mathbb{N}$, where t is a Fourier-type for both X and Y. Comparing this with Rem. 5.5, in order to show that Theorem 5.2 implies the GIRARDI–WEIS theorem, we should show that $B_1^{n/t,t}(\mathcal{L}(X,Y)) \hookrightarrow \mathcal{M}^t(\mathcal{L}(X,Y))$.

In order to prove this fact, we exploit a characterization of the Besov spaces in terms of maximal functions. We first need the following auxiliary operators:

DEFINITION 6.1. We define the maximal functions

$$\varphi_\mu^* f(x) := \sup_{y \in \mathbb{R}^n} \frac{|\varphi_\mu * f(y)|}{1 + (2^\mu |y-x|)^a},$$

where $a > 0$ is fixed and "sufficiently large".

Of course, the definition depends on the choice of $a > 0$. However, the exact value is immaterial, as long as the value is large enough. Namely, we then have the estimate (see [**83**], Eq. (2.3.6/22))

$$\|\varphi_\mu * f\|_p \le \left\|\varphi_\mu^* f\right\|_p \le c\,\|\varphi_\mu * f\|_p\,. \tag{6.2}$$

(The first inequality above is obvious since $\varphi_\mu^* f$ dominates $\varphi_\mu * f$ even pointwise.) This leads to the following characterization of the Besov spaces: For all $s \in \mathbb{R}$, $p, q \in [1, \infty]$, we have $f \in B_q^{s,p}(X)$ if and only if

$$\left\|\left(2^{\mu s}\left\|\varphi_\mu^* f\right\|_p\right)_{\mu=0}^{\infty}\right\|_{\ell^q}$$

is finite, and this quantity is equivalent to the norm $\|f\|_{B_q^{s,p}}$. (Although [**83**] treats spaces of scalar-valued functions, it is not difficult to see that the proofs of the results quoted above do not depend on this in any way, and so the results immediately generalize to the vector-valued situation.)

Now let us see how the differences of $\varphi_\mu * f$ can be controlled in terms of $\varphi_\mu^* f$:

Lemma 6.3. *The following inequality holds:*

$$\|\delta_h^\alpha(\varphi_\mu * f)\|_p \le C_\alpha \prod_{i:|h_i|\le 2^{-\mu}} (2^\mu\,|h_i|)^{\alpha_i}\left\|\varphi_\mu^* f\right\|_p$$

Proof. We apply Cor. 2.2 to $\varphi_\mu * f$. To estimate the maximum, let $\hat{\psi}_0 \in \mathcal{D}$ be 1 on the support of $\hat{\varphi}_0$, and $\hat{\psi}_\mu := \hat{\psi}_0(2^{-\mu}\cdot)$. Then $\varphi_\mu * f = \psi_\mu * (\varphi_\mu * f)$ and

$$\begin{aligned}
&\max_{|x_i-y_i|\le\alpha_i|h_i|} |D^\alpha(\varphi_\mu * f)(y)| \\
&\quad\le \max_y \int |D^\alpha\psi_\mu(v)(\varphi_\mu * f)(y-v)|\,\mathrm{d}v \\
&\quad= \max_y \int 2^{\mu(|\alpha|+n)}\,|D^\alpha\psi_0(2^\mu v)(\varphi_\mu * f)(y-v)|\,\mathrm{d}v \\
&\quad= 2^{\mu|\alpha|} \max_{|u_i|\le\alpha_i 2^\mu|h_i|} \int \left|D^\alpha\psi_0(v)(\varphi_\mu * f)(x + 2^{-\mu}u - 2^{-\mu}v)\right|\,\mathrm{d}v \\
&\quad\le 2^{\mu|\alpha|}(\varphi_\mu^* f)(x) \max_u \int |D^\alpha\psi_0(v)|\,(1+|u-v|)^a)\,\mathrm{d}v \\
&\quad\le 2^{\mu|\alpha|}(\varphi_\mu^* f)(x) C_\alpha \qquad \text{provided } |h_i| \le 2^{-\mu} \text{ when } \alpha_i \ne 0.
\end{aligned}$$

We get $\|\delta_h^\alpha(\varphi_\mu * f)\|_p \le C_\alpha\,|(2^\mu h)^\alpha|\,\left\|\varphi_\mu^* f\right\|_p$. The proof is completed by observing that

$$\|\delta_h^\alpha(\varphi_\mu * f)\|_p = \left\|\delta_h^{\alpha-\beta}\delta_h^\beta(\varphi_\mu * f)\right\|_p \le 2^{|\alpha|-|\beta|}\left\|\delta_h^\beta(\varphi_\mu * f)\right\|_p$$

for $0 \le \beta \le \alpha$, which permits us to reduce the considerations to those components of h which are smaller than $2^{-\mu}$. □

Proposition 6.4. $B_1^{n/p,p} \hookrightarrow \mathcal{M}^p$.

Proof. Since $\|f\|_p \leq \|f\|_{n/p,p;1}$, it suffices to estimate the integrals in the definition of $\|f\|_{\mathcal{M}^p}$. We derive an estimate for each of the terms $\varphi_\mu * f$, $\mu \in \mathbb{N}$, whose sum is f. Lemma 6.3 is used in the first step:

$$\begin{aligned}
&\left(\prod_{i:\alpha_i=1}\int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/p}\right)\left\|\delta_h^{d\alpha}(\varphi_\mu * f)\right\|_p \\
&\leq C\left(\prod_{i:\alpha_i=1}\int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/p}\right)\prod_{i:h_i\leq 2^{-\mu},\alpha_i=1}(2^\mu h_i)^d \left\|\varphi_\mu^* f\right\|_p \\
&\leq \left(\int_0^{2^{-\mu}} h^{-1-1/p+d}2^{\mu d}\,\mathrm{d}h + \int_{2^{-\mu}}^\infty h^{-1-1/p}\,\mathrm{d}h\right)^{|\alpha|}\left\|\varphi_\mu^* f\right\|_p \\
&= \left((d-1/p)^{-1}2^{\mu/p} + p2^{\mu/p}\right)^{|\alpha|}\left\|\varphi_\mu^* f\right\|_p = C2^{\mu|\alpha|/p}\left\|\varphi_\mu^* f\right\|_p .
\end{aligned}$$

Now the sum of the pieces is estimated by

$$\left(\prod_{i:\alpha_i=1}\int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/p}\right)\left\|\delta_h^{d\alpha} f\right\|_p \leq C\sum_{\mu=0}^\infty 2^{\mu|\alpha|/p}\left\|\varphi_\mu^* f\right\|_p \leq C\sum_{\mu=0}^\infty 2^{\mu n/p}\left\|\varphi_\mu^* f\right\|_p ,$$

and this is equivalent to $\|f\|_{n/p,p;1}$. □

Remark 6.5. There is no converse to this result, i.e., $B_1^{n/p,p}$ is a strict subspace of $\mathcal{M}^p$. In fact, this is easily verified by considering functions of the product form $m(x) = \prod_{i=1}^n m_i(x_i)$; the smoothness requirement imposed on each m_i by the condition $m \in B_1^{n/p,p}$ is much heavier than that following from $m \in \mathcal{M}^p$. (Cf. Rem. 4.11.)

As was explained above, it follows from Prop. 6.4 that Theorem 5.2 implies the multiplier theorem of Girardi and Weis [**35**] (excluding some weak-to-weak-type continuity properties considered by these authors, which we have not treated). Rem. 6.5 shows that the converse is not true, and Corollaries 5.8 and 5.10 actually show that the difference is substantial: Namely, we have $B_1^{n/t,t} \hookrightarrow W^{k,t}$ for $k < n/t$, where $W^{k,t}$ is the Sobolev space of order k of L^t-type. Thus the Girardi–Weis conditions require that the multiplier m should have its distributional derivatives up to the order $k := \llbracket n/t \rrbracket$ locally in L^t; in particular, this should be the case for $\partial^k m/\partial x_1^k$. On the other hand, Cor. 5.8 contains no reference to derivatives $\partial^\nu m/\partial x_1^\nu$ for $\nu > 1$, all higher order derivatives being mixed ones. It is impossible to derive such Mihlin-type results from the rotation-invariant approach in [**35**].

To conclude, a word of explanation is in order, since it was pointed out in [**35**] that the Girardi–Weis theorem is *sharp*. But the sharpness is understood in the following sense: one cannot replace the Besov space $B_1^{n/t,t}$ by a larger space $B_1^{n/v,v}$ for $v > t$, i.e., v larger than any Fourier-type t for X and Y. This means, roughly speaking, that their Hörmander-type, rotation-invariant assumptions are optimal, whereas our improvement relies on giving away the rotation-invariance of the conditions.

7. Multipliers on Lebesgue–Bôchner spaces with *a priori* estimates

We now move from the Besov spaces to the L^p scale, and we first consider the problem of concluding the boundedness of a multiplier operator on the whole scale of spaces $L^p(X)$, $p \in]1,\infty[$, provided we know the boundedness for some $\tilde{p}$ in this range. Recall that when X and Y are Hilbert spaces (thus, in particular, in the classical setting of scalar valued functions), m gives a bounded Fourier multiplier from $L^2(X)$ to $L^2(Y)$ if and only if $m \in L^\infty(\mathcal{L}(X,Y))$, so that this is a relevant problem in many cases. In a more general Banach space setting, the hard task is determining the boundedness even for the single $\tilde{p}$; this will be considered in the following section.

In connection with the L^p spaces (as well as with the Hardy spaces later on), we always use the homogeneous dyadic decomposition; i.e., m_μ and k_μ now refer to $\hat{\phi}_\mu m$ and $\phi_\mu * k$, respectively.

We first give a lemma which allows one to check the classical Hörmander condition in terms of our multiplier conditions.

LEMMA 7.1. *Suppose* $\|\delta_h^\alpha m_\mu\|_t \leq \kappa 2^{\mu(n/t-\gamma|\alpha|)} h^{\alpha\gamma}$ *for all* $|\alpha|_\infty \leq d$ *and* $\mu \in \mathbb{Z}$, *where* $\gamma \in]0,1]$, $\Gamma := d \cdot \gamma > 1/t$ *and the underlying space has Fourier-type* t. *Then*

$$\sum_{\mu=-\infty}^{\infty} \int_{|x|>2|s|} |k_\mu(x-s) - k_\mu(x)| \,\mathrm{d}x \leq C\kappa,$$

PROOF. This is a very particular case of Lemma 11.5, proved in the appendix. It is actually possible to give a slightly shorter proof for this particular case, but this is left to the interested reader. □

From the work of HÖRMANDER [**43**] we know that the condition

$$\int_{|x|>2|s|} |k(x-s) - k(x)| \,\mathrm{d}x \leq C$$

is sufficient to guarantee that $k*$ is bounded on all L^p with $p \in]1,\infty[$, provided it is bounded on one such space $L^{\tilde{p}}$. Vector-valued generalizations of this result are also well-known (first proved by A. BENEDEK, A. P. CALDERÓN and R. PANZONE [**5**]), and thus Lemma 7.1 combined with standard arguments gives the following:

THEOREM 7.2. *Let m be a Fourier multiplier from $L^{\tilde{p}}(X)$ to $L^{\tilde{p}}(Y)$, where $\tilde{p} \in]1,\infty[$. Suppose*

$$\|\delta_h^\alpha m_\mu(\cdot)x\|_t \leq \kappa 2^{\mu(n/t-\gamma|\alpha|)} h^{\alpha\gamma} |x|_X\,, \qquad \|\delta_h^\alpha m_\mu(\cdot)'y'\|_t \leq \kappa 2^{\mu(n/t-\gamma|\alpha|)} h^{\alpha\gamma} |y'|_{Y'}$$

for all $|\alpha|_\infty \leq d$ *and* $\mu \in \mathbb{Z}$, *where* $\gamma \in]0,1]$, $\Gamma := d \cdot \gamma > 1/t$ *and* t *is a Fourier-type for* X *and* Y.

Then m *is a multiplier from* $L^p(X)$ *to* $L^p(Y)$ *for all* $p \in]1,\infty[$. *More precisely, the conditions on* $m_\mu(\cdot)x$ *suffice for the conclusion for* $p \in]1,\tilde{p}]$, *and those on* $m_\mu(\cdot)y'$ *for* $p \in [\tilde{p},\infty[$.

One can easily derive corollaries of Theorem 7.2, where the difference estimates are checked in terms of conditions for derivatives, in much the same way as we did in connection with the Besov spaces. The considerations are, however, so similar, that this really wouldn't add any novelty here. Instead, we specialize to the scalar-valued, or more generally, a Hilbert space setting, where we can remove the *a priori* boundedness assumption.

Corollary 7.3. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be Hilbert spaces and* m *be an* $\mathcal{L}(\mathcal{H}_1,\mathcal{H}_2)$*-valued function for which* $m(\cdot)$ *and* $m(\cdot)'$ *are strongly locally integrable and satisfy*

$$\|\delta_h^\alpha m_\mu(\cdot)x\|_2 \leq \kappa 2^{\mu(n/2-\gamma|\alpha|)} h^{\alpha\gamma} |x|_{\mathcal{H}_1}, \qquad \|\delta_h^\alpha m_\mu(\cdot)^* y\|_2 \leq \kappa 2^{\mu(n/2-\gamma|\alpha|)} h^{\alpha\gamma} |y|_{\mathcal{H}_2}$$

for all $\alpha \in \{0,1\}^n$ *and* $\mu \in \mathbb{Z}$, *where* $\gamma > 1/2$. *Then* m *is a Fourier multiplier from* $L^p(\mathcal{H}_1)$ *to* $L^p(\mathcal{H}_2)$ *for all* $p \in]1,\infty[$.

Proof. In view of the facts that Hilbert spaces have Fourier type 2 and can be identified with their duals by Riesz' representation theorem, all we need to show in order to get the conclusion by Theorem 7.2 is the boundedness of the multiplier operator induced by m from $L^2(\mathcal{H}_1)$ to $L^2(\mathcal{H}_2)$. This, in turn, amounts to showing that m is essentially bounded. But according to Lemma 4.2 (with $t = 2, d = 1, f(\cdot) = m_\mu(\cdot)x$), we have $\|\check{m}_\mu(\cdot)x\|_1 \leq C\kappa |x|_{\mathcal{H}_1}$, and hence $\|m_\mu\|_\infty = \sup_{|x|_{\mathcal{H}_1} \leq 1} \|m_\mu(\cdot)x\|_\infty \leq C\kappa$. Since $m(\xi)$ is, at every $\xi \in \mathbb{R}^n \setminus \{0\}$, a sum of at most two non-zero terms $m_\mu(\xi)$ and $m_{\mu-1}(\xi)$, we also get the essential boundedness of m. □

As a very special case, we obtain the following "intersection result", already mentioned in the Introduction:

Corollary 7.4 ("Mihlin ∩ Hörmander"). *Let* m *be a locally integrable function on* $\mathbb{R}^n$ *satisfying*

$$r^{|\alpha|} \left(\frac{1}{r^n} \int_{r<|x|<2r} |D^\alpha m(x)|^2 \, \mathrm{d}x \right)^{1/2} \leq \kappa$$

for all $\alpha \in \{0,1\}^n$ *s.t.* $|\alpha| \leq \ell := \lfloor n/2 \rfloor + 1$, *and all* $r \in]0,\infty[$. *Then* m *is a Fourier multiplier on* L^p *for all* $p \in]1,\infty[$.

Proof. The condition implies that $\|D^\alpha m_\mu\|_2 \leq c\kappa 2^{\mu(n/2-|\alpha|)}$, and then

$$\|\delta_h^\alpha m_\mu\|_2 \leq c\kappa 2^{\mu(n/2-|\alpha|)} h^\alpha \text{ for } \alpha \text{ s.t. } |\alpha|_\infty \leq 1 \text{ and } |\alpha|_1 \leq \ell.$$

Then Cor. 4.10 shows that we have $\|\delta_h^\alpha m_\mu\|_2 \leq c\kappa 2^{\mu(n/2-|\alpha|\gamma)} h^{\alpha\gamma}$, where $\gamma = \ell/n = (\lfloor n/2 \rfloor + 1)/n > 1/2$. Then Cor. 7.3 completes the argument. □

EXAMPLE 7.5. When $n = 3$, we have $\ell = 2$. Thus Cor. 7.4 requires estimates for the derivatives

$$\partial/\partial x,\ \partial/\partial y,\ \partial/\partial z,\ \partial^2/\partial x\partial y,\ \partial^2/\partial y\partial z,\ \partial^2/\partial z\partial x.$$

In addition to these, MIHLIN requires $\partial^3/\partial x\partial y\partial z$, whereas HÖRMANDER needs to add $\partial^2/\partial x^2$, $\partial^2/\partial y^2$ and $\partial^2/\partial z^2$.

Now we go on to the situation without *a priori* boundedness of the multiplier (nor the Hilbert space structure with PLANCHEREL's theorem to give a trivial device to show it).

8. Multipliers on Bôchner spaces without *a priori* estimates

This is the problem which forms the heart of matter, i.e., giving reasonable sufficient conditions for Fourier multipliers between L^p spaces of vector-valued functions, without having the known boundedness on one $L^{\tilde{p}}$ to begin with. Its solution has taken much hard effort and created lots of beautiful mathematics in the past decades. The class of Banach spaces X for which MIHLIN's theorem is valid on $L^p(\mathbb{R}^1; X)$ for *scalar-valued* multipliers m was identified, by D. L. BURKHOLDER and J. BOURGAIN, with the class of those X with the UMD property (unconditionality of martingale differences) in the 80's. (See [**15**] for an exposition of these matters and extensive references to the related work, including earlier history.) F. ZIMMERMANN [**89**] extended these results to $L^p(\mathbb{R}^n; X)$.

However, even after these deep results, the case of operator-valued m required another fundamental idea which turned out to be the notion of *R-boundedness*, systematically studied by PH. CLÉMENT, B. DE PAGTER, F. A. SUKOCHEV and H. WITVLIET [**21**] and by WEIS [**87**] but implicit already in the work of BOURGAIN [**12**]. An operator-valued multiplier theorem on $L^p(X)$, with X UMD, was first obtained by WEIS [**87**]. By now, several results in this direction are known, the sharpest ones so far proved by GIRARDI and WEIS [**36**], and in Chapter 2.

Chapter 2 has its main emphasis on singular convolution operators, and the multiplier theorems are obtained as an application, in the spirit of (1.5). Taking the same ideas as the starting point, we now aim at stronger versions of these multiplier theorems, Theorems 8.7 and 8.13. (See also Cor. 8.16 for a more "classical style" statement.) This is achieved by combining the convolution results from Chapter 2 with a sharper embedding lemma, which we next prove. It is a variant of Prop. 3.2 with a logarithmic weight; observe that $w(x) := \log(2 + |x|)$ satisfies the assumptions of the lemma.

In analogy with the treatment of the Besov spaces, let us define

$$\|f\|_{\mathcal{M}^t_w} := \|f\|_t + \sum_{0\neq\alpha\in\{0,1\}^n} \sum_{k:\alpha_k=1} \left(\prod_{i:\alpha_i=1} \int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} \right) w(h_k^{-1}) \left\| \delta_h^{d\alpha} f \right\|_t ,$$

where $d := 1$ if $t > 1$ and $d := 2$ if $t = 1$, as before.

LEMMA 8.1. *Let the underlying space have Fourier-type t, and let $w(x) \equiv w(|x|)$ be radial, increasing, and such that $w(2x) \le cw(x)$. Then*

$$\int_{\mathbb{R}^n} \left|\hat{f}(x)\right| w(x)\,\mathrm{d}x \le C \, \|f\|_{\mathcal{M}^t_w} .$$

PROOF. Adapting the proof of Prop. 3.2 to the present situation, we now get

$$\begin{aligned}
&\int_{E(\alpha,\rho)} \left|\hat{f}(x)\right| w(x)\,\mathrm{d}x \\
&\qquad \le C \sum_{j\in\mathbb{N}^\alpha} \left\| x \mapsto \prod_{i:\alpha_i=1} (1-e^{\mathbf{i}2\pi \mathfrak{e}_i\cdot x/b2^{j_i}\rho_i})^d \hat{f}(x) \right\|_{t'} \left(\int_{E(\alpha,\rho,j)} w^t(x)\,\mathrm{d}x \right)^{1/t} \\
&\qquad \le C \sum_{j\in\mathbb{N}^\alpha} \left\| \prod_{i:\alpha_i=1} \Delta^d_{\mathfrak{e}_i/b2^{j_i}\rho_i} f \right\|_t \prod_{i:\alpha_i=0} \rho_i^{1/t} \cdot \prod_{i:\alpha_i=1} (2^{j_i}\rho_i)^{1/t} \cdot \sum_{k:\alpha_k=1} w(2^{j_k}\rho_k).
\end{aligned}$$

Logarithmic averaging gives

$$\begin{aligned}
&\left(\prod_{i:\alpha_i=1} \int_r^{2r} \frac{\mathrm{d}\rho_i}{\rho_i} \right) \int_{E(\alpha,\rho)} \left|\hat{f}(x)\right| w(x)\,\mathrm{d}x \\
&\qquad\qquad \le C r^{(n-|\alpha|)/t} \left(\prod_{i:\alpha_i=1} \int_0^{1/r} \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} \right) \sum_{k:\alpha_k=1} w(h_k^{-1}) \, \|\delta^\alpha_h f\|_t \, .
\end{aligned}$$

For $\alpha = 0$ we have

$$\int_{E(0,\rho)} \left|\hat{f}(x)\right| w(x)\,\mathrm{d}x \le \left\|\hat{f}\right\|_{t'} \left(\int_{E(0,\rho)} w^t(x)\,\mathrm{d}x \right)^{1/t} \le C \, \|f\|_t \, w(|\rho|) \, |\rho|^{n/t} ,$$

and the proof is completed by taking the logarithmic average again, and fixing $r := 1$, say. $\square$

COROLLARY 8.2. *Let* $\left\|\delta^{d\alpha}_h f\right\|_t \le h^{\alpha\Gamma}$, *where* $\Gamma > 1/t$, *and* $w(x) = \log(2+|x|)$. *Then we have* $\left\|\hat{f}w\right\|_1 \le C$.

PROOF. We combine the elementary estimate

$$\begin{aligned}
&\left(\prod_{i:\alpha_i=1} \int_0^1 \frac{\mathrm{d}h_i}{h_i} h_i^{-1/t} \right) \log(2+h_k^{-1}) \left\|\delta^{d\alpha}_h f\right\|_t \\
&\qquad\qquad \le \left(\int_0^1 \frac{\mathrm{d}h}{h} h^{-1/t+\Gamma} \right)^{|\alpha|-1} \int_0^1 \frac{\mathrm{d}h_k}{h_k} h_k^{-1/t+\Gamma} \log(2+h_k^{-1}) \le C.
\end{aligned}$$

with Lemma 8.1. $\square$

The point of considering the logarithmically weighted conditions is the following result from Chapter 2. The assumptions in the following involve several concepts which we define only after the statement of the result.

PROPOSITION 8.3 (Prop. 4.6 of Chapter 2). *Given an $\mathcal{L}(X,Y)$-valued function m, denote by*

$$M(\cdot) := (m_\mu(2^\mu\cdot))_{\mu=-\infty}^{\infty} = \hat{\phi}_0(\cdot)(m(2^\mu\cdot))_{\mu=-\infty}^{\infty} \tag{8.4}$$

the sequence of the dyadic pieces of m, dilated so that they are all supported on the support of $\hat{\phi}_0$. Its inverse Fourier transform is

$$K(\cdot) := \check{M}(\cdot) = (2^{-n\mu}k_\mu(2^{-\mu}\cdot))_{\mu=-\infty}^{\infty} = (\phi_0 * 2^{-n\mu}k(2^{-\mu}\cdot))_{\mu=-\infty}^{\infty}.$$

Let X and Y be UMD-spaces and $p \in \,]1,\infty[$, $w(x) := \log(2+|x|)$, and the estimate

$$\|K(\cdot)'g\, w(\cdot)\|_{L^1(\mathrm{Rad}(L^{p'}(X')))} \leq C\, \|g\|_{\mathrm{Rad}(L^{p'}(Y'))}, \tag{8.5}$$

hold for all $g \in (\varepsilon_\mu)_{\mu=-\infty}^{\infty} \otimes L^{p'}(Y')$.

Then m is a Fourier multiplier from $L^p(X)$ to $L^p(Y)$.

Let us explain the notation adopted in Prop. 8.3. First, $(\varepsilon_\mu)_{\mu=-\infty}^{\infty}$ is the Rademacher system of independent random variables (on some probability space Ω) with $\mathbb{P}(\varepsilon_\mu = 1) = \mathbb{P}(\varepsilon_\mu = -1) = 1/2$. Then $g \in (\varepsilon_\mu)_{-\infty}^{\infty} \otimes L^{p'}(Y')$ (where $\otimes$ designates the algebraic tensor product) can be identified with a finitely non-zero sequence $(g_\mu)_{-\infty}^{\infty}$, with $g_\mu \in L^{p'}(Y')$. By $K(\cdot)'g$ (where $'$ denotes the Banach adjoint) we mean the sequence

$$K(\xi)'g = \big(2^{-n\mu}k_\mu(2^{-\mu}\xi)'g_\mu\big)_{\mu=-\infty}^{\infty};$$

the operator $2^{-n\mu}k_\mu(2^{-\mu}\xi)'$, which belongs to $\mathcal{L}(Y',X')$, is understood to be canonically extended to $\mathcal{L}(L^{p'}(Y'),L^{p'}(X'))$. Thus, for every ξ, $K(\xi)'g$ is a finitely non-zero sequence with entries in $L^{p'}(X')$.

For a Banach space Z, the space $\mathrm{Rad}(Z)$ (which appears in Prop. 8.3 with $Z = L^{p'}(X')$ and $L^{p'}(Y')$) is the completion of the set of finitely non-zero Z-sequences in the norm

$$\big\|(g_\mu)_{\mu=-\infty}^{\infty}\big\|_{\mathrm{Rad}(Z)} := \mathbb{E}\left|\sum_{\mu=-\infty}^{\infty} \varepsilon_\mu g_\mu\right|_Z \approx \left(\mathbb{E}\left|\sum_{\mu=-\infty}^{\infty} \varepsilon_\mu g_\mu\right|_Z^v\right)^{1/v}, \tag{8.6}$$

where $\mathbb{E}$ is the expectation related to the random variables ε_μ. The $\approx$ sign indicates the boundedness of the ratio of the last two quantities by absolute constants, for any $v \in \,]0,\infty[$. (This is KAHANE's inequality.)

We recall from Chapter 2 the following useful properties of $\mathrm{Rad}(Z)$ in this connection:

- $L^v(\mathrm{Rad}(Z)) \approx \mathrm{Rad}(L^v(Z))$ (isomorphism of spaces) for any $v \in \,]0,\infty[$,
- $\mathrm{Rad}(Z)$ has Fourier-type t if and only if Z has,
- $\mathrm{Rad}(Z)' \approx \mathrm{Rad}(Z')$ when Z is UMD (weaker assumptions would suffice).

Now it should be clear what is meant by $\|M(\cdot)'g\, w(\cdot)\|_{L^1(\mathrm{Rad}(L^{p'}(Y')))}$ and by $\|g\|_{\mathrm{Rad}(L^{p'}(Y'))}$ in the assumption (8.5).

Prop. 8.3 will now be combined with the preceding Fourier embedding results, so as to check the condition (8.5) more directly in terms of the smoothness of the multiplier m. We are going to derive two types of theorems: one with smoothness expressed in terms of the norm of $\mathcal{M}_w^t(\mathcal{L}(\mathrm{Rad}(X), \mathrm{Rad}(Y)))$ (this version is Theorem 8.7); and the other with somewhat stronger smoothness conditions of the form (4.1), but relaxing the uniform operator topology to strong topology estimates (this is Theorem 8.13). It seems to us that the latter version is the more useful of the two, since the conditions (4.1) are so much simpler than the $\mathcal{M}_w^t$ condition (8.8); however, with the first version we are able to recover and improve the multiplier theorem in [**36**].

The proof of this result is almost a repetition of that of Cor. 3.11 in [**36**]; we simply apply our new Fourier embeddings instead of the ones used in [**36**].

THEOREM 8.7. *Let X and Y be UMD-spaces and $w(\cdot) := \log(2 + |\cdot|)$. Let M as in* (8.4) *satisfy*

$$\|M(\cdot)\|_{\mathcal{M}_w^t(\mathcal{L}(\mathrm{Rad}(X),\mathrm{Rad}(Y)))} =: \kappa < \infty, \tag{8.8}$$

where $t \in]1, 2]$ is a Fourier-type for both X and Y.

Then m is a Fourier-multiplier from $L^p(X)$ to $L^p(Y)$ for all $p \in]1, \infty[$.

PROOF. Since $\|T\|_{\mathcal{L}(U,V)} = \|T'\|_{\mathcal{L}(V',U')} = \|\tilde{T}\|_{\mathcal{L}(L^p(U),L^p(V))}$, where $\tilde{T}$ is the canonical extension of T to the indicated space, it follows that

$$\begin{aligned}\|M(\cdot)\|_{\mathcal{M}_w^t(\mathcal{L}(\mathrm{Rad}(X),\mathrm{Rad}(Y)))} &= \|M(\cdot)\|_{\mathcal{M}_w^t(\mathcal{L}(L^t(\mathrm{Rad}(X)),L^t(\mathrm{Rad}(Y))))} \\ &= \|M(\cdot)'\|_{\mathcal{M}_w^t(\mathcal{L}(\mathrm{Rad}(Y'),\mathrm{Rad}(X')))} = \|M(\cdot)'\|_{\mathcal{M}_w^t(\mathcal{L}(L^t(\mathrm{Rad}(Y')),L^t(\mathrm{Rad}(X'))))} \,.\end{aligned}$$

For $g \in \mathrm{Rad}(L^t(Y')) \approx L^t(\mathrm{Rad}(Y'))$, we can apply Lemma 8.1 to the function $M(\cdot)'g \in L^t(\mathrm{Rad}(L^t(Y')))$, to the result

$$\begin{aligned}\|K(\cdot)'g\, w(\cdot)\|_{L^1(L^t(\mathrm{Rad}(X')))} &\le C\,\|M(\cdot)'g\|_{\mathcal{M}_w^t(L^t(\mathrm{Rad}(X')))} \\ &\le C\kappa\,\|g\|_{L^t(\mathrm{Rad}(Y'))}\,,\end{aligned} \tag{8.9}$$

where we used the fact that Y having Fourier-type t implies the same property for Y', $\mathrm{Rad}(Y')$, and finally for $L^t(\mathrm{Rad}(Y'))$.

Similarly, for every $v \in \mathrm{Rad}(Y')$, we have

$$\|K(\cdot)'v\, w(\cdot)\|_{L^1(\mathrm{Rad}(Y'))} \le C\,\|M(\cdot)'v\|_{\mathcal{M}_w^t(\mathrm{Rad}(Y'))} \le C\kappa\,\|v\|_{\mathrm{Rad}(Y')}\,.$$

If $g \in L^1(\mathrm{Rad}(Y'))$ and we apply the previous estimate to every point evaluation $g(s) \in \mathrm{Rad}(Y')$ in place of v, integrate over $s \in \mathbb{R}^n$ and apply FUBINI's theorem, we obtain

$$\|K(\cdot)'g\, w(\cdot)\|_{L^1(L^1(\mathrm{Rad}(X')))} \le C\kappa\,\|g\|_{L^1(\mathrm{Rad}(Y'))}\,. \tag{8.10}$$

Now we can apply an interpolation theorem ([**6**], Theorem 5.1.2) to conclude from (8.9) and (8.10) that

$$\|K(\cdot)'g\, w(\cdot)\|_{L^1(L^{p'}(\mathrm{Rad}(X')))} \le C\kappa\,\|g\|_{L^{p'}(\mathrm{Rad}(Y'))} \qquad \text{for } p' \in]1, t]$$

and then Prop. 8.3 shows that m is a multiplier from $L^p(X)$ to $L^p(Y)$ for $p \in [t', \infty[$.

Considerations which are directly analogous to the ones above, with $M(\cdot)$ in place of $M(\cdot)'$, can be used to show the dual estimates

$$\|K(\cdot)f\, w(\cdot)\|_{L^1(L^p(\mathrm{Rad}(Y)))} \le C\kappa \|f\|_{L^p(\mathrm{Rad}(X))} \qquad \text{for } p \in]1,t]$$

and to conclude that $m(\cdot)'$ is a multiplier from $L^{p'}(Y')$ to $L^{p'}(X')$ for $p' \in [t', \infty[$. (Observe that X and Y are reflexive when they are UMD, so that the situation is completely symmetric to the previous one.)

A well-known duality argument now shows that m is a multiplier from $L^p(X)$ to $L^p(Y)$ for $p \in]1,t]$, and interpolation can be used to cover the remaining values of the index p. □

Remark 8.11. Theorem 8.7 covers the L^p-multiplier theorem (Theorem 4.1 in [**36**]) of Girardi and Weis, where the assuption (8.8) is replaced by the condition $M \in B_t^A(\mathcal{L}(X,Y))$. Here B_t^A is a modified Besov space, where membership is determined by the finiteness of the norm

$$\|f\|_{B_t^A} := \sum_{\mu=0}^{\infty} (\mu+1) 2^{\mu n/t} \|\varphi_\mu * f\|_t\,;$$

thus B_t^A is slightly smaller than $B_1^{n/t,t}$, but $B_1^{s,t} \hookrightarrow B_t^A$ whenever $s > n/t$.

In order to show that Theorem 8.7 implies the multiplier theorem of Girardi and Weis, we should prove that $\mathcal{M}_w^t(\mathcal{L}(X,Y)) \hookrightarrow B_t^A(\mathcal{L}(X,Y))$ for $w(\cdot) := \log(2+|\cdot|)$, as earlier. But an easy modification of the proof of Prop. 6.4 shows that

$$\|f\|_{\mathcal{M}_w^t} \le C \sum_{\mu=0}^{\infty} (\mu+1) 2^{\mu n/t} \left\|\varphi_\mu^* f\right\|_t,$$

and Eq. (6.2) then completes the argument.

We then come to the second version of our L^p multiplier theorem. In addition to the differences in comparison to Theorem 8.7 which were already mentioned, we also wish to express the assumptions without reference to the Rademacher classes $\mathrm{Rad}(X)$ etc. but rather in terms of expressions similar to the notion of R-boundedness in the first papers [**21, 87**] systematically dealing with this matter.

We first give a lemma explaining the relation between the various forms of the conditions.

Lemma 8.12. *Suppose that*

$$\mathbb{E}\left\| \sum_{\mu=-\infty}^{\infty} \varepsilon_\mu \delta_h^\alpha [m_\mu(2^\mu \cdot)] x_\mu \right\|_{L^t(Y)} \le \kappa \mathbb{E} \left| \sum_{\mu=-\infty}^{\infty} \varepsilon_\mu x_\mu \right|_X h^{\alpha\gamma}$$

for all finitely non-zero sequences $(x_\mu)_{-\infty}^{\infty} \subset X$.

Then, for M as in (8.4), we have

$$\|\delta_h^\alpha M(\cdot)f\|_{L^t(\mathrm{Rad}(L^t(Y)))} \le C\kappa \|f\|_{\mathrm{Rad}(L^t(X))} h^{\alpha\gamma}$$

for all $f \in (\varepsilon_\mu)_{-\infty}^{\infty} \otimes L^t(X)$, and also

$$\|\delta_h^\alpha m_\mu(\cdot)x\|_{L^t(Y)} \le \kappa |x|_X 2^{\mu(n/t-|\alpha|\gamma)} h^{\alpha\gamma}$$

for all $\mu \in \mathbb{Z}$.

Proof. If $x := (x_\mu)_{-\infty}^{\infty}$, our assumption can be written $\|\delta_h^\alpha M(\cdot)x\|_{\mathrm{Rad}(L^t(Y))} \le \kappa \|x\|_{\mathrm{Rad}(X)} h^{\alpha\gamma}$. For f as in the statement of the lemma, we can apply the previous inequality to each $f(s)$, $s \in \mathbb{R}^n$. Then the first claim follows by taking the power of t, integrating over $s \in \mathbb{R}^n$ and using Fubini's theorem. (Note that we can take the (equivalent) t-norm in $\mathrm{Rad}(Z)$.)

To see the second assertion, simply choose all but one of the x_μ's vanish in the assumption. This gives $\|\delta_h^\alpha[m_\mu(2^\mu\cdot)]x\|_{L^t(Y)} \le \kappa |x|_X h^{\alpha\gamma}$. Then note that

$$\|\delta_h^\alpha[m_\mu(2^\mu\cdot)]x\|_{L^t(Y)} = \|(\delta_{2^\mu h}^\alpha m_\mu)(2^\mu\cdot)x\|_{L^t(Y)} = 2^{-\mu n/t} \|\delta_{2^\mu h}^\alpha m_\mu(\cdot)x\|_{L^t(Y)}.$$

The assertion follows by taking a new variable $y := 2^\mu h$. □

Now we come to the theorem. The point of the two alternative conditions is that we require a randomized condition for either the multiplier or its pointwise adjoint, but once this is satisfied, a simple non-randomized bound suffices for the other.

Theorem 8.13. *Let X and Y be UMD-spaces and m be an $\mathcal{L}(X,Y)$-valued function. Suppose, for every $\alpha \in \{0,1\}^n$ and $h \in \mathbb{R}_+^n$, either*

$$\begin{aligned} &\mathbb{E}\left\|\sum_{\mu=-\infty}^{\infty} \varepsilon_\mu \delta_h^\alpha[m_\mu(2^\mu\cdot)]x_\mu\right\|_{L^t(Y)} \le \kappa \mathbb{E}\left|\sum_{\mu=-\infty}^{\infty} \varepsilon_\mu x_\mu\right|_X h^{\alpha\gamma}, \\ &\|\delta_h^\alpha m_\nu(\cdot)'y'\|_{L^t(X')} \le \kappa 2^{\nu(n/t-|\alpha|\gamma)} |y'|_{Y'} h^{\alpha\gamma} \end{aligned} \tag{8.14}$$

for all finitely non-zero $(x_\mu) \subset X$, $y' \in Y'$ and $\nu \in \mathbb{N}$, or

$$\begin{aligned} &\|\delta_h^\alpha m_\nu(\cdot)x\|_{L^t(Y)} \le \kappa 2^{\nu(n/t-|\alpha|\gamma)} |x|_X h^{\alpha\gamma}, \\ &\mathbb{E}\left\|\sum_{\mu=-\infty}^{\infty} \varepsilon_\mu \delta_h^\alpha[m_\mu(2^\mu\cdot)']y'_\mu\right\|_{L^t(X')} \le \kappa \mathbb{E}\left|\sum_{\mu=-\infty}^{\infty} \varepsilon_\mu y'_\mu\right|_{Y'} h^{\alpha\gamma} \end{aligned} \tag{8.15}$$

for all $x \in X$, $\nu \in \mathbb{N}$, and all finitely non-zero $(y'_\mu) \subset Y'$. In both cases, we assume that $t \in]1,2]$ is a Fourier-type for both X and Y, and $\gamma > 1/t$.

Then m is a Fourier multiplier from $L^p(X)$ to $L^p(Y)$ for all $p \in]1,\infty[$.

Proof. Assume first the set of conditions (8.14). Then by Cor. 8.2 (with $M(\cdot)f$ in place of $f(\cdot)$) and Lemma 8.12 we have

$$\|K(\cdot)f\, w\|_{L^1(\mathrm{Rad}(L^t(Y)))} \le C\kappa \|f\|_{\mathrm{Rad}(L^t(X))}$$

for $f \in (\varepsilon_\mu)_{-\infty}^{\infty} \otimes L^t(X)$, and Prop. 8.3 shows that the adjoint $m(\cdot)'$ is a multiplier from $L^{t'}(Y')$ to $L^{t'}(X')$. Now we can use Theorem 7.2 to conclude that $m(\cdot)'$ is actually a multiplier from $L^p(Y')$ to $L^p(X')$ for all $p \in]1,\infty[$, and the same result for m itself follows by duality.

The case of the other set of conditions (8.15) is similar; now we obtain directly the boundedness from $L^{t'}(X)$ to $L^{t'}(Y)$, and then with Theorem 7.2 from $L^p(X)$ to $L^p(Y)$ for all $p \in]1,\infty[$. $\square$

Using the results in Sect. 4, one finds that Theorem 8.13 implies and improves Theorem 4.21 of Chapter 2. Since the way of deriving such corollaries is very much similar to the corresponding task in the earlier sections, we only give a simple example to illustrate how to work with the randomized estimates that constitute the main new ingredient in this section.

Corollary 8.16. *Let X and Y be UMD-spaces with Fourier-type $t \in]1,2]$, let m be an $\mathcal{L}(X,Y)$-valued function, and suppose that the collection*

$$\left\{ |\xi|^{|\alpha|} D^\alpha m(\xi) : \ \alpha \in \mathbb{N}^n \text{ with } |\alpha|_\infty \le 1 \text{ and } |\alpha|_1 \le \lfloor n/t \rfloor + 1, \ \xi \in \mathbb{R}^n \setminus \{0\} \right\}$$

is R-bounded. Then m is a Fourier multiplier from $L^p(X)$ to $L^p(Y)$ for all $p \in]1,\infty[$.

Proof. The assumption means, by the definition of R-boundedness, that

$$\mathbb{E} \left| \sum_\mu \varepsilon_\mu |\xi_\mu|^{|\alpha|} D^\alpha m(\xi_\mu) x_\mu \right|_Y \le \kappa \mathbb{E} \left| \sum_\mu \varepsilon_\mu x_\mu \right|_X ,$$

for every choice of the $\xi_\mu \in \mathbb{R}^n \setminus \{0\}$, and so in particular

$$\begin{aligned} &\mathbb{E} \left| \sum_{\mu=-\infty}^{\infty} \varepsilon_\mu |\xi|^{|\alpha|} D^\alpha [\hat{\phi}_0(\cdot) m(2^\mu \cdot)](\xi) x_\mu \right|_Y \\ &\qquad \le \sum_{\theta \le \alpha} \mathbb{E} \left| \sum_\mu \varepsilon_\mu 2^{\mu|\theta|} |\xi|^{|\theta|} D^\theta m(2^\mu \xi) \cdot |\xi|^{|\alpha|-|\theta|} D^{\alpha-\theta} \hat{\phi}_0(\xi) x_\mu \right|_Y \\ &\qquad \le \kappa C \mathbb{E} \left| \sum_\mu \varepsilon_\mu x_\mu \right|_X \end{aligned}$$

for all finitely non-zero sequences $(x_\mu)_{-\infty}^{\infty} \subset X$. The last estimate used the assumed R-boundedness (with $\xi_\mu = 2^\mu \xi$), and the uniform boundedness of the quantities $|\xi|^{|\alpha|-|\theta|} \hat{\phi}_0(\xi)$, for $\theta \le \alpha \in \{0,1\}^n$.

The estimate is written more compactly as $\|D^\alpha M(\xi) x\|_{\mathrm{Rad}(Y)} \le C\kappa \|x\|_{\mathrm{Rad}(X)}$, with $M(\cdot)$ defined in (8.4). It follows, since $M(\cdot)$ is compactly supported, that $\|D^\alpha M(\cdot) x\|_{L^t(\mathrm{Rad}(Y))} \le C\kappa \|x\|_{\mathrm{Rad}(X)}$. This estimate holding for all $|\alpha|_\infty \le 1$,

$|\alpha|_1 \leq \lfloor n/t \rfloor + 1$, implies, by Cor. 4.10, that we have

$$\|\delta_h^\alpha M(\cdot)x\|_{L^t(\mathrm{Rad}(Y))} \leq C\kappa \|x\|_{\mathrm{Rad}(X)} h^{\alpha\gamma}$$

for all $|\alpha|_\infty \leq 1$, with $\gamma = (\lfloor n/t \rfloor + 1)/n > 1/t$, which is the first condition in (8.14).

The second, non-randomized, estimate in (8.14) follows easily by similar reasoning. In fact, even the second estimate in (8.15) is true, since the R-boundedness of a set $\mathcal{T} \subset \mathcal{L}(X,Y)$ also implies the R-boundedness of the collection $\mathcal{T}' := \{T' : T \in \mathcal{T}\}$ of its adjoint operators whenever X is a B-convex space, in particular, when X is UMD (see Lemma 3.4 of Chapter 2; but since the non-randomized estimate suffices for the adjoint operators, we need not resort to this). □

9. Multipliers on Hardy spaces

The Hardy spaces H^p, $p \in]0,1]$, provide a natural continuation of the scale of the L^p spaces, $p \in]1,\infty[$. In the scalar-valued situation there exist various equivalent characterizations of these spaces; not all of the equivalences remain valid in the vector-valued situation, and we use here the definition in terms of the *atomic decomposition*, see Chapter 1.

Multipliers on H^p (of scalar-valued functions) have been treated by various authors, but we only mention the rather general theory (also covering weighted spaces, which we do not consider) developed by J.-O. Strömberg and A. Torchinsky [**81**], since their approach generalizes easily to the vector-valued context. These authors formulated a scale of Hörmander-type conditions, already stated in (4.4), such that one obtains boundedness on any desired H^p by choosing the smoothness index in these conditions large enough.

The multiplier problem on the real-variable Hardy spaces of vector-valued functions was considered in Chapter 1, where it was observed that the scalar-valued results from [**81**] essentially go through in the vector-valued setting as such, only assuming the appropriate Fourier-type of the underlying space. This success is largely due to the generality of the conditions considered by Strömberg and Torchinsky, allowing for different values of the exponent q in (4.4) and making use of the Hausdorff–Young inequality, instead of restricting to $q = 2$ and the use of Plancherel's theorem. This latter approach, followed by many authors, has the unfortunate restriction of generalizing only to the class of Banach spaces of Fourier-type 2, which is, as shown by S. Kwapień [**55**], exactly the class of Hilbert spaces.

Strömberg and Torchinsky considered the following condition which a convolution kernel k may or may not satisfy:

DEFINITION 9.1. Define the set of conditions

$$\Big(\int_{r<|x|<2r} |D^\alpha k(x)|^q \,\mathrm{d}x\Big)^{1/q} \le \kappa r^{-n/q'-|\alpha|},$$

$$\Big(\int_{r<|x|<2r} \big|D^\beta k(x) - D^\beta k(x-z)\big| \,\mathrm{d}x\Big)^{1/q} \le \kappa r^{-n/q'-[\![\ell]\!]} \varrho(\frac{|z|}{r}; \ell - [\![\ell]\!]),$$

which should hold for all $|\alpha| \le [\![\ell]\!]$ and $|\beta| = [\![\ell]\!]$, for all $r \in \,]0,\infty[$ and all $z \in \mathbb{R}^n$ with $|z| \le r/2$, all derivatives being classical and continuous. We denote by $[\![\ell]\!]$ the greatest integer which is strictly less than ℓ. Moreover, $\varrho(x;\epsilon) := x^\epsilon$ if $\epsilon \in \,]0,1[$ and $\varrho(x;1) := x\log(x^{-1})$.

When k satisfies these conditions, we denote this by $k \in K(q,\ell)$. (The notation in [**81**] is $k \in \tilde{M}(q,\ell)$.) We also say that $k \in K(q,\ell)$ if $k = \sum_{-\infty}^{\infty} k_\mu$ is the dyadic decomposition of k in the Fourier side, and $\sum_{-\nu}^{\nu} k_\mu$ satisfy $K(q,\ell)$ uniformly (i.e., with the same κ) in $\nu \in \mathbb{N}$.

The significance of these conditions lies in the following result which, in the scalar case, is implicitly contained in [**81**] (where only multiplier theorems are formulated, although convolution theorems are proved on the way, in accordance with (1.5)). The vector-valued version is Theorem 5.6 of Chapter 1.

PROPOSITION 9.2 ([**81**], Ch. XI). *Suppose that k is an $\mathcal{L}(X,Y)$-valued kernel with $k(\cdot)x \in K(q,\ell)$ uniformly in $|x|_X \le 1$, where $q \in \,]1,\infty[$ and $\ell > 0$; and moreover that $f \mapsto k * f$ is bounded from $L^q(X)$ to $L^q(Y)$.*

Then $f \mapsto k * f$ is bounded from $H^p(X)$ to $H^p(Y)$ for all $p \in \,](1+\ell/n)^{-1},1]$, i.e., for all $p \in \,]0,1]$ s.t. $\ell > n(p^{-1}-1)$.

The approach of STRÖMBERG and TORCHINSKY to the H^p-multiplier theorems in [**81**] was essentially to check the condition $k \in K(q,\ell)$ in terms of a multiplier condition $m \in M(\tilde{q},\tilde{\ell})$ (which is of Hörmander or 1-norm type). This approach was also followed in Chapter 1. The purpose of the present section is to check the condition $k \in K(q,\ell)$ through the use of our Mihlin or ∞-norm type conditions of the form (4.1). This is done in the following lemma, whose proof is given in the Appendix.

LEMMA 9.3. *Let m be an $\mathcal{L}(X,Y)$-valued function, suppose*

$$\|\delta_h^\alpha m_\mu(\cdot)x\|_t \le \kappa 2^{\mu(n/t-|\alpha|\gamma)} h^{\alpha\gamma} |x|_X$$

for all $|\alpha|_\infty \le d$ and some $\gamma \in \,]0,1]$, where $\Gamma := d\cdot\gamma = \ell + (n-1)/q' + 1/t$, for certain $\ell > 0$ and $q \in \,]1,\infty[$, and Y has Fourier-type t.

Then $k(\cdot)x \in K(q,\ell)$ uniformly in $|x|_X \le 1$, where $k = \check{m}$.

PROOF. This is contained in Lemmata 11.4 and 11.5. □

Now we are ready for the theorem:

Theorem 9.4. *Let m be a Fourier multiplier from $L^{\tilde p}(X)$ to $L^{\tilde p}(Y)$ for some $\tilde p \in \,]1,\infty[$. Let Y have Fourier-type $t \in [1,2]$, and suppose that*

$$\|\delta_h^\alpha m_\mu(\cdot)x\|_t \le \kappa 2^{\mu(n/t-|\alpha|\gamma)} h^{\alpha\gamma} |x|_X$$

for all $\mu \in \mathbb{Z}$, all $|\alpha|_\infty \le d$ and some $\gamma \in \,]0,1]$, where $\Gamma := d\cdot\gamma > 1/t$.

Then m is a Fourier multiplier from $H^p(X)$ to $H^p(Y)$ for all

$$p \in \left]\frac{n}{n+\Gamma-1/t}, 1\right],$$

i.e., for all $p \in \,]0,1]$ such that $\Gamma > t^{-1} + n(p^{-1}-1)$.

Proof. Fix some $p \in \,]0,1]$ s.t. $\Gamma > t^{-1} + n(p^{-1}-1)$. Then choose $q \in \,]1,\infty[$ small enough, i.e., $q' \in \,]1,\infty[$ large enough, so that $q < \tilde p$ and

$$\Gamma > t^{-1} + n(p^{-1}-1) + (n-1)/q'.$$

Denote by $\ell > n(p^{-1}-1)$ the number for which the equality

$$\Gamma = t^{-1} + \ell + (n-1)/q'$$

holds. According to Lemma 9.3, we then have $k \in K(q,\ell)$.

On the other hand, the assumptions in combination with Theorem 7.2 imply that m is a Fourier multiplier from $L^{\tilde q}(X)$ to $L^{\tilde q}(Y)$ for all $\tilde q \in \,]1,\tilde p]$, in particular, from $L^q(X)$ to $L^q(Y)$. Thus $f \mapsto k * f$ is bounded from $L^q(X)$ to $L^q(Y)$, and $k \in K(q,\ell)$, so Prop. 9.2 shows that $f \mapsto k * f$ is also bounded from $H^p(X)$ to $H^p(Y)$, since $\ell > n(p^{-1}-1)$. Since p was arbitrary in the range we asserted, this completes the proof. □

Remark 9.5. The minimal conditions in Theorem 9.4 which guarantee the boundedness from $H^1(X)$ to $H^1(Y)$ (i.e., $\Gamma > t^{-1}$) are just the same as those in Theorem 7.2 which guarantee the boundedness from $L^p(X)$ to $L^p(Y)$ for $p \in \,]1,\tilde p]$.

As before, one can check the assumptions of Theorem 9.4 in terms of various more or less classical conditions. The procedure should be familiar by now, and we content ourselves by giving a corollary in the scalar-valued case, for comparison with the multiplier theory of Strömberg and Torchinsky [**81**]. (We should note that these authors consider the multiplier problem on rather general weighted Hardy spaces, which we have not treated, and we only make a comparison in the intersection covered by their approach as well as ours, i.e., the unweighted H^p spaces of scalar-valued functions.)

Corollary 9.6. *Let m be a measurable function on $\mathbb{R}^n$ such that*

$$\|\delta_h^\alpha m_\mu\|_t \le \kappa 2^{\mu(n/t-|\alpha|\gamma)} h^{\alpha\gamma}$$

for all $\mu \in \mathbb{Z}$ and all $|\alpha|_\infty \le d$, where $\Gamma := d\cdot\gamma > t^{-1} + n(p^{-1}-1)$, $t \in [1,2]$, and $p \in \,]0,1]$. Then m is a Fourier multiplier on H^p.

Proof. This is very similar to the proof of Cor. 7.3. □

The unweighted version (Corollary on p. 164 of [**81**]) of the multiplier theorem of Strömberg and Torchinsky concludes the boundedness on H^p from the assumption $m \in M(t, \ell)$ where $\ell > n/t + n(p^{-1} - 1)$. If we denote the infima of the admissible Γ (in Cor. 9.6) and ℓ (in the above mentioned result of Strömberg and Torchinsky) by $\tilde{\Gamma}$ and $\tilde{\ell}$, respectively, it is clear (for $n \geq 2$ and $p \in\,]0, 1[$) that

$$\tilde{\Gamma} \equiv \frac{1}{t} + n(\frac{1}{p} - 1) < \tilde{\ell} \equiv \frac{n}{t} + n(\frac{1}{p} - 1) < n\tilde{\Gamma}.$$

Since $n\tilde{\Gamma} > \tilde{\ell}$ is the largest 1-norm possible if the ∞-norm is $\tilde{\Gamma}$, we find that our Cor. 9.6 does not recover the result of Strömberg and Torchinsky. But the converse is false, also, since the largest ∞-norm is $\tilde{\ell} > \tilde{\Gamma}$ when the 1-norm is allowed to be $\tilde{\ell}$, and so we find that our result wins if measured in the ∞-norm, and Strömberg's and Torchinsky's if measured in the 1-norm.

A very concrete illustration of this is obtained if we consider the minimum number of classical derivatives for checking (by methods of Sect. 4) either of the conditions:

Example 9.7. Take $n = 2$ and $p = 2/3$, so that $n(p^{-1} - 1) = 1$, and moreover $t = 2$. The smallest integral Γ satisfying $\Gamma > 1/t + n(p^{-1} - 1) = 1.5$ is $\Gamma = 2$, while the smallest integral ℓ for which $\ell > n/t + n(p^{-1} - 1) = 2$ is $\ell = 3$. Now let us compare the required derivatives in our conditions ($|\alpha|_\infty \leq 2$) and the Strömberg–Torchinsky conditions ($|\alpha|_1 \leq 3$). Common to both are the derivatives

$$\partial/\partial x,\ \partial/\partial y,\ \partial^2/\partial x^2,\ \partial^2/\partial x\partial y,\ \partial^2/\partial y^2, \partial^3/\partial x^2\partial y,\ \partial^3/\partial x\partial y^2,$$

in addition, we need $\partial^4/\partial x^2\partial y^2$, whereas Strömberg and Torchinsky need $\partial^3/\partial x^3$ and $\partial^3/\partial y^3$.

We conclude that our ∞-norm method no longer outperforms for the Hardy spaces H^p, $p \in\,]0, 1[$, but it still yields results that apply to certain situations not covered by the 1-norm approach. Neither approach is superior in general, but one or the other might be better suited to a particular situation.

For the case of H^1, on the other hand, the situation is just the same as it was with the Besov and Bôchner spaces: The minimization of the ∞-norm simultaneously minimizes the 1-norm, and our Mihlin-type result is a genuine improvement (in the unweighted situation) of the Hörmander-type result of Strömberg and Torchinsky.

10. Appendix: Proof of Lemma 4.6

We first give a simple but useful lemma, which shows the monotonicity of our conditions as a function of the smoothness index γ.

Lemma 10.1. *Suppose* $\|\delta_h^\alpha f\|_q \leq \kappa 2^{\mu(n/q - |\alpha|\gamma_\alpha)} h^{\alpha\gamma_\alpha}$ *for all* $\alpha \in \mathcal{I}$, *a stable collection. Then* $\|\delta_h^\alpha f\|_q \leq c\kappa 2^{\mu(n/q - |\alpha|\tilde{\gamma})} h^{\alpha\tilde{\gamma}}$ *for* $\tilde{\gamma} \in [0, \min_{\alpha\in\mathcal{I}} \gamma_\alpha]$.

Proof. If $h_i \le 2^\mu$ when $\alpha_i \ne 0$, we can simply estimate $2^{-\mu|\alpha|\gamma_\alpha} h^{\alpha\gamma_\alpha} \le 2^{-\mu|\alpha|\tilde\gamma} h^{\alpha\tilde\gamma}$. Otherwise, let $\theta \le \alpha$ (thus $\theta \in \mathcal{I}$ if $\alpha \in \mathcal{I}$) be defined by $\theta_i := \alpha_i$ if $h_i \le 2^\mu$ and $\theta_i := 0$ otherwise. Then

$$\begin{aligned}\|\delta_h^\alpha f\|_q &\le 2^{|\alpha|-|\theta|} \left\|\delta_h^\theta f\right\|_q \le 2^{|\alpha|-|\theta|} \kappa 2^{\mu(n/q-|\theta|\gamma_\theta)} h^{\theta\gamma_\theta} \cdot 1 \\ &\le c\kappa 2^{\mu n/q} 2^{-\mu|\theta|\tilde\gamma} h^{\theta\tilde\gamma} \cdot (2^{-\mu} h)^{(\alpha-\theta)\tilde\gamma} = c\kappa 2^{\mu(n/q-|\alpha|\tilde\gamma)} h^{\alpha\tilde\gamma},\end{aligned}$$

so the claim follows in this case, too. □

Now we are ready to prove the first assertion of Lemma 4.6.

Proof of (4.7). By Cor. 2.2, the first condition in (4.5) immediately gives the estimate $\|\delta_h^\alpha f\|_q \le \kappa 2^{\mu(n/q-|\alpha|)} h^\alpha$ for $\alpha \in \mathcal{I}$ with $|\alpha| \le \lfloor \ell \rfloor$. If ℓ is an integer, this is all we claimed. Otherwise, we get $\|\delta_h^\alpha f\|_q \le \kappa 2^{\mu(n/q-|\alpha|\gamma)} h^{\alpha\gamma}$ for the same α's and with $\gamma = \ell/\lceil \ell \rceil < 1$ from Lemma 10.1.

Finally, for non-integral ℓ, consider $|\alpha| = \lceil \ell \rceil = \lfloor \ell \rfloor + 1$. Then we can write (possibly in several ways) $\alpha = \beta + \mathfrak{e}_i$, where $|\beta| = \lfloor \ell \rfloor$. (Since $\beta \le \alpha$, we always have $\beta \in \mathcal{I}$ as long as $\alpha \in \mathcal{I}$.) Thus

$$\begin{aligned}\|\delta_h^\alpha f\|_q = \left\|\delta_h^\beta \delta_h^{\mathfrak{e}_i} f\right\|_q &= \left\|\delta_h^\beta (f - \tau_{h_i \mathfrak{e}_i} f)\right\|_q \le h^\beta \left\|D^\beta (f - \tau_{h_i \mathfrak{e}_i} f)\right\|_q \\ &\le h^\beta \kappa 2^{\mu(n/q-|\beta|-\epsilon)} h_i^\epsilon = h^\alpha h_i^{\epsilon-1} \kappa 2^{\mu(n/q-|\alpha|+1-\epsilon)}.\end{aligned}$$

Now we multiply all such estimates for different choices of $\mathfrak{e}_i$ s.t. $\alpha_i \ne 0$; moreover, the estimate with $\mathfrak{e}_i$ is taken α_i times to the product. This gives

$$\begin{aligned}\|\delta_h^\alpha f\|_q^{|\alpha|} &\le \left(\kappa 2^{\mu(n/q-|\alpha|+1-\epsilon)} h^\alpha\right)^{|\alpha|} \prod_{i=1}^n h_i^{(\epsilon-1)\alpha_i} \\ &= \left(\kappa 2^{\mu(n/q-|\alpha|+1-\epsilon)}\right)^{|\alpha|} h^{\alpha|\alpha|} h^{\alpha(\epsilon-1)} = \left(\kappa 2^{\mu(n/q-|\alpha|\gamma)} h^{\alpha\gamma}\right)^{|\alpha|},\end{aligned}$$

where

$$\gamma = 1 + \frac{\epsilon-1}{|\alpha|} = 1 + \frac{\epsilon-1}{\lceil \ell \rceil} = \frac{\lceil \ell \rceil - 1 + \epsilon}{\lceil \ell \rceil} = \frac{\ell}{\lceil \ell \rceil},$$

and thus the assertion is established. □

In order to prove the second assertion (4.8) of Lemma 4.6 (where the conclusion involves differences of possibly much higher order than does the assumption), we need a convenient representation, provided by the following lemma, of higher order multi-indices as linear combinations of lower order ones. The short proof, which replaces my original cumbersome argument, was pointed to me by E. Saksman.

Lemma 10.2. *Suppose $\alpha \in \mathbb{N}^n$, $L \in \mathbb{Z}_+$, and $L < |\alpha|_1$. Then there exist $M, m_j \in \mathbb{Z}_+$, $\theta^j \in \mathbb{N}^n$ with $\theta^j \le \alpha$ and $|\theta^j|_1 = L$ ($j = 1, \ldots, k \in \mathbb{Z}_+$), such that*

$$M\alpha = \sum_{j=1}^k m_j \theta^j.$$

Proof. The case $|\alpha|_1 = L+1$ is immediate from

$$(|\alpha|-1)\alpha = \sum_{i:\alpha_i\neq 0} \alpha_i(\alpha - \mathfrak{e}_i).$$

The general case then follows by induction. □

Remark 10.3. In the situation of Lemma 10.2, we have $M|\alpha|_1 = L\sum_{j=1}^k m_j$. This follows by simply taking the 1-norms of both sides of the asserted representation formula for α, and using the fact that all entries are non-negative, so that the 1-norm is additive.

We are now able to prove the following result. Then (4.8) will be a consequence of (4.7), which is its own special case.

Lemma 10.4. *Suppose we have the estimates*

$$\|\delta_h^\alpha m_\mu\|_q \leq \kappa 2^{\mu(n/q-|\alpha|\tilde{\gamma})} h^{\alpha\tilde{\gamma}}$$

for all $h\in\mathbb{R}_+^n$, $\alpha\in\mathbb{N}^n$ with $|\alpha|\leq \tilde{L}\in\mathbb{N}$ and some $\tilde{\gamma}\in\,]0,1]$.

Let $\tilde{L} < L \in \mathbb{N}$. Then we also have

$$\|\delta_h^\alpha m_\mu\|_q \leq c\kappa 2^{\mu(n/q-|\alpha|\gamma)} h^{\alpha\gamma}, \qquad \gamma := \tilde{\gamma}\tilde{L}/L$$

for all $h\in\mathbb{R}_+^n$ and $\alpha\in\mathbb{N}^n$ with $|\alpha|\leq L$.

Proof. For $|\alpha|\leq\tilde{L}$, the assertion follows from Lemma 10.1. For $|\alpha|>\tilde{L}$, take θ^j, m_j as in Lemma 10.2 (with $\tilde{L}$ in place of L in the assumptions). Note that $\theta^j\leq\alpha\in\mathcal{I}$ implies $\theta^j\in\mathcal{I}$. For each such θ^j we estimate

$$\|\delta_h^\alpha m_\mu\|_q \leq 2^{|\alpha|-|\theta^j|}\left\|\delta_h^{\theta^j} m_\mu\right\|_q \leq c\kappa 2^{\mu(n/q-|\theta^j|\tilde{\gamma})} h^{\theta^j\tilde{\gamma}}.$$

We raise this estimate to the power of m_j and multiply all the estimates with different j's, to the result (we use Rem. 10.3 and also $|\theta^j| = \tilde{L}$)

$$\|\delta_h^\alpha m_\mu\|_q^{M|\alpha|/\tilde{L}} \leq \left(c\kappa 2^{\mu(n/q-\tilde{L}\tilde{\gamma})}\right)^{M|\alpha|/\tilde{L}} h^{\tilde{\gamma}\sum m_j\theta^j},$$

$$\|\delta_h^\alpha m_\mu\|_q \leq c\kappa 2^{\mu(n/q-\tilde{L}\tilde{\gamma})} h^{\alpha\tilde{\gamma}\tilde{L}/|\alpha|} = c\kappa 2^{\mu(n/q-|\alpha|(\tilde{\gamma}\tilde{L}/|\alpha|))} h^{\alpha\tilde{\gamma}\tilde{L}/|\alpha|}.$$

For $|\alpha|\leq L$, we have $\tilde{\gamma}\tilde{L}/|\alpha|\geq\tilde{\gamma}\tilde{L}/L=\gamma$, and hence the conclusion follows from Lemma 10.1. □

Proof of (4.8). This follows by applying Lemma 10.4 (with $\tilde{\gamma}=\ell/\lceil\ell\rceil$ and $\tilde{L}=\lceil\ell\rceil$) to the estimates in (4.7). □

11. Appendix: Variations on the theme of Fourier embeddings

In all of this section, m_μ and k_μ refer to the homogeneous dyadic decomposition; i.e., $m_\mu = \hat{\phi}_\mu m$ and $k_\mu = \phi_\mu * k$, where $\mu \in \mathbb{Z}$.

LEMMA 11.1. *Suppose* $\left\|\delta_h^\theta f\right\|_t \le 2^{\mu(n/t-\gamma|\theta|)} h^{\theta\gamma}$ *for* $\theta \le d\alpha$, *where* $\gamma \in\,]0,1]$, *and moreover* $\operatorname{supp} f \subset \{x : |x| \le c2^\mu\}$. *Then* $\left\|\delta_h^\theta(f\cdot x^\beta)\right\|_t \le c2^{\mu(n/t+|\beta|-\gamma|\theta|)} h^{\theta\gamma}$ *for* $\theta \le d\alpha$.

PROOF. Consider the multi-index $\zeta \le \theta$ defined by $\zeta_i := \theta_i$ if $h_i \le 2^\mu$, $\zeta_i := 0$ otherwise. Then, since $\left\|\delta_h^\theta(fx^\beta)\right\|_t \le 2^{|\theta|-|\zeta|}\left\|\delta_h^\zeta(fx^\beta)\right\|_t$ and $1 \le 2^{-\mu(|\theta|-|\zeta|)} h^{\theta-\zeta}$, it suffices to prove the claim when $h_i \le 2^\mu$. (Observe that those h_i for which $\theta_i = 0$ are completely immaterial.) We then have

$$\left\|\delta_h^\theta(f\cdot x^\beta)\right\|_t \le \sum_{\vartheta\le\theta}\binom{\theta}{\vartheta}\left\|\delta_h^{\theta-\vartheta} f\right\|_t \left\|\tau_{(\theta-\vartheta)h}\delta_h^\vartheta x^\beta\right\|_{\infty,\operatorname{supp}\delta_h^{\theta-\vartheta} f},$$

where the t-norms are estimated by $2^{\mu(n/t-\gamma|\theta|+\gamma|\vartheta|)} h^{(\theta-\vartheta)\gamma}$ and the ∞-norms by $c2^{\mu(|\beta|-|\vartheta|)} h^\vartheta \le c2^{\mu|\beta|}2^{-\mu|\vartheta|\gamma} h^{\vartheta\gamma}$. □

LEMMA 11.2. *Let the assumptions of Lemma* 11.1 *hold. Let* $g_s(x) := e^{\mathbf{i}2\pi s\cdot x} - 1$, *where* $|s| \le 2^{-\mu}$. *Then* $\left\|\delta_h^\theta(f\cdot x^\beta\cdot g_s)\right\|_t \le c2^{\mu(n/t+|\beta|-\gamma|\theta|+1)}|s|\, h^{\theta\gamma}$ *for* $\beta \in \mathbb{N}^n$ *and* $\theta \le d\alpha$.

PROOF. We make the same reduction as in the proof of Lemma 11.1. Again,

$$\begin{aligned}\left\|\delta_h^\theta(f\cdot x^\beta\cdot g_s)\right\|_t \le{}& \left\|\delta_h^\theta(fx^\beta)\right\|_t \left\|\tau_{\theta h} g_s\right\|_{\infty,\operatorname{supp}\delta_h^\theta f}\\ &+ \sum_{0\ne\vartheta\le\theta}\binom{\theta}{\vartheta}\left\|\delta_h^{\theta-\vartheta}(fx^\beta)\right\|_t \left\|\tau_{h(\theta-\vartheta)}\delta_h^\vartheta g_s\right\|_\infty.\end{aligned}$$

Here $|g(x)| \le 2\pi\,|x|\cdot|s| \le c2^\mu\,|s|$ on the set where we evaluate the ∞-norm, and $\left\|\delta_h^\vartheta g_s\right\|_\infty \le h^\vartheta\left\|D^\vartheta g_s\right\|_\infty = c\,|s|^{|\vartheta|}\, h^\vartheta \le c2^{-\mu|\vartheta|}2^\mu\,|s|\,h^\vartheta \le c2^{-\mu|\vartheta|\gamma}h^{\vartheta\gamma}2^\mu\,|s|$. Combining these observations with the estimate

$$\left\|\delta_h^{\theta-\vartheta}(fx^\beta)\right\|_t \le 2^{\mu(n/t+|\beta|-\gamma(|\theta|-|\vartheta|))} h^{(\theta-\vartheta)\gamma}$$

from Lemma 11.1, we find that every term above is bounded by

$$c2^{\mu(n/t+|\beta|-\gamma|\theta|+1)}\,|s|\,h^{\theta\gamma},$$

as we claimed. □

Lemma 11.3. *Let the assumptions of Lemmata* 11.1 *and* 11.2 *hold for all* $\alpha \in \{0,1\}^n$, *and* $\Gamma := d \cdot \gamma > 1/t - 1/q'$. *Then*

$$\left\| D^\beta \hat{f} \right\|_q \leq C 2^{\mu(n/q'+|\beta|)},$$

$$\left(\int_{|x|>R} \left| D^\beta \hat{f}(x) \right|^q \,\mathrm{d}x \right)^{1/q} \leq C 2^{\mu((n-1)/q'+|\beta|+1/t-\Gamma)} R^{1/t-1/q'-\Gamma},$$

$$\left\| D^\beta \tau_s \hat{f} - D^\beta \hat{f} \right\|_q \leq C 2^{\mu(n/q'+|\beta|+1)} |s|,$$

and

$$\left(\int_{|x|>R} \left| D^\beta \hat{f}(x-s) - D^\beta \hat{f}(x) \right|^q \,\mathrm{d}x \right)^{1/q} \leq C 2^{\mu((n-1)/q'+|\beta|+1+1/t-\Gamma)} |s| R^{1/t-1/q'-\Gamma}.$$

Proof. This follows from an application of Lemma 4.2 to the conclusions of Lemmata 11.1 and 11.2. Note that Lemma 4.2 applies to the integrals over $|x| > R$ only when $R \geq 2^{-\mu}$, but for $R < 2^{-\mu}$ the second and the fourth estimate above, respectively, are consequences of the first and the third estimate. □

Lemma 11.4. *For all* $\mu \in \mathbb{Z}$, *let* $\left\| \delta_h^\theta m_\mu \right\|_t \leq 2^{\mu(n/t-\gamma|\theta|)}$ *for all* $|\theta|_\infty \leq d$, *where* $\gamma \in \,]0,1]$ *and* $\Gamma := d \cdot \gamma > (n-1)/q' + |\beta| + 1/t$. *Then, if* $q > 1$ *or* $|\beta| > 0$, *we have*

$$\sum_{\mu=-\infty}^{\infty} \left(\int_{|x|>R} \left| D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q} \leq C R^{-(n/q'+|\beta|)}.$$

Proof. We estimate $\left(\int_{|x|>R} \left| D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q}$ in two ways for different ways depending on μ.

Case $2^\mu > R^{-1}$. From Lemma 11.3 we have the upper bound

$$C 2^{\mu((n-1)/q'+|\beta|+1/t-\Gamma)} R^{1/t-1/q'-\Gamma},$$

and summing these up we arrive at

$$\sum_{\mu:2^\mu>R^{-1}} \left(\int \left| D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q} \leq C R^{-((n-1)/q'+|\beta|+1/t-\Gamma)+(1/t-1/q'-\Gamma)} = C R^{-n/q'-|\beta|}.$$

Case $2^\mu \leq R^{-1}$. Now that R is “small”, we estimate the quantity of interest by the integral over the whole space, and then

$$\left\| D^\beta k_\mu \right\|_q \leq C 2^{\mu(n/q'+|\beta|)}.$$

Thus we obtain

$$\sum_{\mu:2^\mu \le R^{-1}} \left(\int_{|x|>R} \left| D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q} \le C R^{-(n/q'+|\beta|)}.$$

It is here that we required the assumption $q > 1$ (i.e., $1/q' > 0$) or $|\beta| > 0$. □

LEMMA 11.5. *For all $\mu \in \mathbb{Z}$, let $\left\| \delta_h^\theta m_\mu \right\|_t \le 2^{\mu(n/t-\gamma|\theta|)}$ for all $\theta \in \{0, 1 \ldots, d\}^n$, where $\gamma \in \,]0,1]$ and $\Gamma := d \cdot \gamma > (n-1)/q' + |\beta| + 1/t$. Then for all $R \ge 2|s| > 0$, we have*

$$\sum_{\mu=-\infty}^{\infty} \left(\int_{|x|>R} \left| D^\beta k_\mu(x-s) - D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q}$$
$$\le C R^{-(n/q'+|\beta|)} \varrho\Big(\frac{|s|}{R}; \Gamma - (n-1)/q' - |\beta| - 1/t\Big)$$

where $\varrho(x;\eta) := x^\eta$ if $\eta \in \,]0,1[$, $\varrho(x;\eta) := x \log(x^{-1})$ if $\eta = 1$, and $\varrho(x;\eta) := x$ if $\eta > 1$.

PROOF. We divide the estimation of

$$\left(\int_{|x|>R} \left| D^\beta k_\mu(x-s) - D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q}$$

into three cases, depending on the size of μ.

Case $2^\mu \ge |s|^{-1}$. Here we estimate the integral by

$$\left(\int_{|x|>R/2} \left| D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q} \le C 2^{\mu((n-1)/q'+|\beta|+1/t-\Gamma)} R^{1/t-1/q'-\Gamma},$$

and summing over the appropriate range of μ, we have

$$\sum_{\mu:2^\mu \ge |s|^{-1}} \left(\int_{|x|>R} \left| D^\beta k_\mu(x-s) - D^\beta k_\mu(x) \right|^q \,\mathrm{d}x \right)^{1/q}$$
$$\le C\,|s|^{-((n-1)/q'+|\beta|+1/t-\Gamma)}\, R^{1/t-1/q'-\Gamma}$$
$$= C R^{-(n/q'+|\beta|)} \left(\frac{|s|}{R} \right)^{\Gamma-(n-1)/q'-|\beta|-1/t}.$$

Case $2^\mu \le R^{-1}$. Now R is "small", and we simply estimate the integral over $|x| > R$ by

$$\left\| D^\beta \tau_s k_\mu - D^\beta k_\mu \right\|_q \le C 2^{\mu(n/q'+|\beta|+1)} |s|.$$

The contribution of all the terms with $2^\mu \le R^{-1}$ is then estimated by

$$\sum_{\mu:2^\mu \le R^{-1}} \left(\int_{|x|>R} \left| D^\beta k_\mu(x-s) - D^\beta k_\mu(x) \right|^q \, \mathrm{d}x \right)^{1/q}$$
$$\le CR^{-(n/q'+|\beta|+1)} |s| = CR^{-(n/q'+|\beta|)} \frac{|s|}{R}.$$

Case $R^{-1} < 2^\mu < |s|^{-1}$. Here we apply directly the estimate in Lemma 11.3 to the result

$$\left(\int_{|x|>R} \left| D^\beta k_\mu(x-s) - D^\beta k_\mu(x) \right|^q \, \mathrm{d}x \right)^{1/q}$$
$$\le C 2^{\mu((n-1)/q'+|\beta|+1+1/t-\Gamma)} |s| R^{1/t-1/q'-\Gamma}.$$

Then, if $(n-1)/q' + |\beta| + 1 + 1/t - \Gamma > 0$, we have

$$\begin{aligned} \sum_{\mu:R^{-1}<2^\mu<|s|^{-1}} &\left(\int_{|x|>R} \left| D^\beta k_\mu(x-s) - D^\beta k_\mu(x) \right|^q \, \mathrm{d}x \right)^{1/q} \\ &\le C |s|^{-((n-1)/q'+|\beta|+1+1/t-\Gamma)+1} R^{1/t-1/q'-\Gamma} \\ &= CR^{-(n/q'+|\beta|)} \left(\frac{|s|}{R} \right)^{\Gamma-(n-1)/q'-|\beta|-1/t}, \end{aligned}$$

and if $(n-1)/q' + |\beta| + 1 + 1/t - \Gamma < 0$, we instead obtain the estimate

$$CR^{-((n-1)/q'+|\beta|+1+1/t-\Gamma)+(1/t-1/q'-\Gamma)} |s| = CR^{-(n/q'+|\beta|)} \frac{|s|}{R}.$$

Finally, if $(n-1)/q' + |\beta| + 1 + 1/t - \Gamma = 0$, an upper bound will be

$$C \log \frac{R}{|s|} \cdot |s| R^{1/t-1/q'-\Gamma} = C \log \frac{R}{|s|} \cdot |s| R^{-(n/q'+|\beta|+1)}$$
$$\le CR^{-(n/q'+|\beta|)} \cdot \frac{|s|}{R} \log \frac{R}{|s|}.$$

Combining the estimates in all cases, we have the assertion. □

CHAPTER 5

Epilogue: On the "Hörmander $\cap$ Mihlin" theorem

1. Introduction and preliminary considerations

Here we restate and reprove the following result, which was obtained in Chapter 4 as a corollary of results of more general, but also more technical nature. The underlying ideas of the proof are the same as in the general setting considered in Chapter 4 but the details are simplified.

THEOREM 1.1. *Let $q \in \,]1,2]$, and suppose that*

$$\left(\frac{1}{r^n}\int_{r<|\xi|<2r} |D^\alpha m(\xi)|^q \,\mathrm{d}\xi\right)^{1/q} \leq Cr^{-|\alpha|} \tag{1.2}$$

for all $r > 0$ and all $\alpha \in \{0,1\}^n$ satisfying $|\alpha| \leq \lfloor n/q \rfloor + 1$. Then m is a Fourier multiplier on $L^p(\mathbb{R}^n)$ for all $p \in \,]1,\infty[$.

When $q = 2$, the results of both HÖRMANDER and MIHLIN are seen to be special cases of Theorem 1.1.

The idea of proof is first to verify that the assumptions in fact imply the boundedness of m, thus the boundedness of the corresponding Fourier multiplier operator on $L^2(\mathbb{R}^n)$ according to PLANCHEREL's theorem. Then it is shown that the corresponding kernel $k = \check{m}$ satisfies HÖRMANDER's integral condition, which gives us the boundedness on all $L^p(\mathbb{R}^n)$, $p \in \,]1,\infty[$.

To begin the proof of the theorem, we recall the dyadic decomposition of the multiplier:

Decomposition of the multiplier. Let $\hat{\varphi} \in \mathcal{D}(\mathbb{R}^n)$ be identically 1 in $\bar{B}(0,1)$, identically 0 outside $\bar{B}(0,2)$, and with range in $[0,1]$. Let $\hat{\phi}_0(\xi) := \hat{\varphi}(\xi) - \hat{\varphi}(2\xi)$ and $\hat{\phi}_j(\xi) := \hat{\phi}_0(2^{-j}\xi)$ for $j \in \mathbb{Z}$. Then $\hat{\phi}_j$ is supported in the annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$, and

$$\sum_{j=-\infty}^{\infty} \hat{\phi}_j(\xi) \equiv 1, \qquad \text{for } \xi \neq 0.$$

The multiplier m is decomposed into the pieces $m_j(\xi) := \hat{\phi}_j(\xi)m(\xi)$. The assumption (1.2) implies the following estimates, which will be useful in the sequel:

LEMMA 1.3. *Under the assumption* (1.2), *the following estimates holds for all $j \in \mathbb{Z}$, all $|y| \leq 2^{-j}$, and all $\alpha \in \{0,1\}^n$ such that $|\alpha| \leq \lfloor n/q \rfloor + 1$, with $C < \infty$ independent of j:*

$$\left\|D^\alpha m_j\right\|_q \leq C2^{j(n/q-|\alpha|)}, \qquad \left\|D^\alpha[(e^{\mathrm{i}2\pi y\cdot\xi} - 1)m_j(\xi)]\right\|_q \leq C2^{j(n/q-|\alpha|+1)}\,|y|\,.$$

Proof. By a direct computation

$$\left(\int |D^\alpha m_j(\xi)|^q \,\mathrm{d}\xi\right)^{1/q}$$
$$\leq \sum_{\theta\leq\alpha}\left(\int_{2^{j-1}\leq|\xi|\leq 2^j} \left|D^\theta m(\xi)\right|^q \left|2^{-j(|\alpha|-|\theta|)}D^{\alpha-\theta}\hat{\phi}_0(2^{-j}\xi)\right|^q \mathrm{d}\xi\right)^{1/q}$$
$$\leq \sum_{\theta\leq\alpha} 2^{jn/q}\left(\frac{1}{2^{jn}}\int_{2^{j-1}\leq|\xi|\leq 2^j}\left|D^\theta m(\xi)\right|^q \mathrm{d}\xi\right)^{1/q} 2^{-j(|\alpha|-|\theta|)}C_{\alpha-\theta}$$
$$\leq \sum_{\theta\leq\alpha} C2^{jn/q}2^{-j|\theta|}2^{-j(|\alpha|-|\theta|)}C_{\alpha-\theta} \leq C2^{j(n/q-|\alpha|)}.$$

To establish the second estimate, note first that

$$D^\alpha[(e^{\mathbf{i}2\pi y\cdot\xi}-1)m_j(\xi)] = (e^{\mathbf{i}2\pi y\cdot\xi}-1)D^\alpha m_j(\xi) + \sum_{0\neq\theta\leq\alpha}(\mathbf{i}2\pi y)^\theta e^{\mathbf{i}2\pi y\cdot\xi}D^{\alpha-\theta}m_j(\xi).$$

When $|\xi|\leq 2^{j+1}$, we have

$$\left|e^{\mathbf{i}2\pi y\cdot\xi}-1\right| \leq 2\pi\,|y\cdot\xi| \leq 2\pi 2^{j+1}\,|y| = c2^j\,|y|\,;$$

hence

$$\left\|(e^{\mathbf{i}2\pi y\cdot\xi}-1)D^\alpha m_j(\xi)\right\|_q \leq c2^j\,|y|\,\|D^\alpha m_j\|_q \leq C2^{j(n/q-|\alpha|+1)}\,|y|\,.$$

For $0\neq\theta\leq\alpha$ we get

$$\left\|(\mathbf{i}2\pi y)^\theta e^{\mathbf{i}2\pi y\cdot\xi}D^{\alpha-\theta}m_j(\xi)\right\|_q \leq c\,|y|^\theta\,\left\|D^{\alpha-\theta}m_j\right\|_q \leq C\,|y|\,|y|^{|\theta|-1}\,2^{j(n/q-|\alpha|+|\theta|)};$$

hence for $|y|\leq 2^{-j}$ we have $|y|^{|\theta|-1}\leq 2^{-j|\theta|+j}$, and thus

$$\left\|(\mathbf{i}2\pi y)^\theta e^{\mathbf{i}2\pi y\cdot\xi}D^{\alpha-\theta}m_j(\xi)\right\|_q \leq C2^{j(n/q-|\alpha|+1)}\,|y|\,.$$

Now the assertion is clear after summing the estimates just obtained. □

2. From multipliers to kernels: Fourier embeddings

The parts m_j into which the multiplier m was divided are L^q functions with compact support; hence their inverse Fourier transforms $k_j := \check{m}_j$ are infinitely smooth functions in $L^{q'}(\mathbb{R}^n)$. In the following, we see how the derivative conditions satisfied by the m_j can be exploited to get more precise integrability conditions on the kernels k_j.

As a reminder, we first present a classical embedding theorem which is, however, insufficient for our purposes:

Proposition 2.1. *For $q\in[1,2]$, we have $\mathcal{F}W^{N,q}\hookrightarrow L^1$ for $N>n/q$, and more precisely, for any $r>0$,*

$$\left\|\hat{f}\right\|_1 \leq Cr^{n/q}\,\|f\|_q + Cr^{n/q-N}\sum_{|\alpha|=N}\|D^\alpha f\|_q$$

Proof. We divide the integration domain into regions where different estimates are applied:

$$\begin{aligned}\int_{\mathbb{R}^n} \left|\hat{f}(x)\right| \mathrm{d}x &= \int_{|x|<r} \left|\hat{f}(x)\right| \mathrm{d}x + \sum_{j=0}^{\infty} \int_{2^j r \le |x| < 2^{j+1} r} \left|\hat{f}(x)\right| \mathrm{d}x \\ &\le C r^{n/q} \left\|\hat{f}\right\|_{q'} + C \sum_{j=0}^{\infty} \sum_{|\alpha|=N} \frac{1}{(2^j r)^N} \int_{2^j r \le |x| < 2^{j+1} r} \left|x^\alpha \hat{f}(x)\right| \mathrm{d}x \\ &\le C r^{n/q} \left\|\hat{f}\right\|_{q'} + C \sum_{j=0}^{\infty} (2^j r)^{-N} (2^j r)^{n/q} \left\|x^\alpha \hat{f}(x)\right\|_{q'} \\ &\le C r^{n/q} \|f\|_q + C r^{n/q-N} \sum_{|\alpha|=N} \|D^\alpha f\|_q \sum_{j=0}^{\infty} 2^{j(n/q-N)}.\end{aligned}$$

In the first estimate, Hölder's inequality (applied to $\chi_{\bar{B}(0,r)}$ and $\hat{f}$) was used for the first term, and the inequality

$$|x|^N \le C \sum_{|\alpha|=N} |x^\alpha| \tag{2.2}$$

for the second. In the remaining estimates, we used Hölder's inequality again, and finally the Hausdorff–Young inequality $\left\|\hat{f}\right\|_{q'} \le \|f\|_q$ for $q \in [1, 2]$. The series in the last step in summable since $n/q - N < 0$, and hence the assertion is obtained. □

Remarks. There are good reasons to suspect that the result above is not the sharpest possible: In estimating the integral of $\hat{f}$, we only used our assumptions concerning the derivatives of f of orders 0 and N, but none of the intermediate ones.

The crucial estimate above was (2.2). While this is the best one can say for a general $x \in \mathbb{R}^n$, there is a lot of redundancy on the right-hand side if we have more detailed knowledge concerning the direction of x. E.g., if we know that x is on (or close to) the diagonal $x_1 = \ldots = x_n$, then the single term $x_1 \cdots x_n$ can be used to bound $|x|^n$. If, on the other hand, some of the x_i's (nearly) vanish, the terms involving these x_i's are of little use on the right hand side of (2.2). This suggests that instead of the annular decomposition of $\mathbb{R}^n$ which was used in the proof of Prop. 2.1, one should use a more refined decomposition so as to be able to keep track of the size of the individual components x_i of x, and not just the length $|x|$.

Decomposition of space. Let $\alpha \in \{0,1\}^n$ and $\varrho \in\ ]0,\infty[^n$. We define the set $E(\alpha, \varrho) \subset \mathbb{R}^n$ to consist of those $x \in \mathbb{R}^n$, for which x_i is "small" if $\alpha_i = 0$ and "large" if $\alpha_i = 1$:

$$E(\alpha, \varrho) := \{x \in \mathbb{R}^n : |x_i| \le \varrho_i \text{ if } \alpha_i = 0,\ |x_i| > \varrho_i \text{ if } \alpha_i = 1\}.$$

We further want to know more precisely how large the “large” components are. Let $\mathbb{N}^\alpha := \{j \in \mathbb{N}^n : j_i = 0 \text{ if } \alpha_i = 0\}$, and

$$E(\alpha, \varrho, j) := \left\{x \in E(\alpha, \varrho) : 2^{j_i}\varrho_i < |x_i| \le 2^{j_i+1}\varrho_i \text{ if } \alpha_i = 1\right\}.$$

Now obviously, for any $\varrho \in]0, \infty[^n$,

$$\mathbb{R}^n = \bigcup_{\alpha \in \{0,1\}^n} E(\alpha, \varrho), \qquad E(\alpha, \varrho) = \bigcup_{j \in \mathbb{N}^\alpha} E(\alpha, \varrho, j),$$

where the unions are disjoint.

The significance of the sets $E(\alpha, \varrho, j)$ lies in the fact that we have

$$2^{j\cdot\beta}\varrho^\beta \le \left|x^\beta\right| \le 2^{|\beta|+j\cdot\beta}\varrho^\beta \qquad \forall\ x \in E(\alpha, \varrho, j),\ \beta \le \alpha; \tag{2.3}$$

moreover

LEMMA 2.4. *If* $0 \ne \alpha \in \{0,1\}^n$, $x \in E(\alpha, \varrho, j)$ *and* $N \le |\alpha|$, *then*

$$\sum_{\beta \le \alpha, |\beta| = N} \left|x^\beta\right| \ge (2^{|j|}\varrho^\alpha)^{N/|\alpha|}$$

PROOF. Let $2^{j\cdot\gamma}\varrho^\gamma$ be the largest among the numbers $2^{j\cdot\beta}\varrho^\beta$, for $\beta \le \alpha$, $|\beta| = N$. Multiplying the inequalities $2^{j\cdot\gamma}\varrho^\gamma \ge 2^{j\cdot\beta}\varrho^\beta$ for all $\binom{|\alpha|}{N}$ multi-indices $\beta \le \alpha$, $|\beta| = N$, we get

$$\begin{aligned}(2^{j\cdot\gamma}\varrho^\gamma)^{\binom{|\alpha|}{N}} &\ge \prod_{\substack{\beta \le \alpha \\ |\beta| = N}} 2^{j\cdot\beta}\varrho^\beta \\ &= \prod_{\substack{\beta \le \alpha \\ |\beta| = N}} \prod_{i:\beta_i = 1} 2^{j_i}\varrho_i = \prod_{i:\alpha_i = 1} (2^{j_i}\varrho_i)^{\binom{|\alpha|-1}{N-1}} = (2^{|j|}\varrho^\alpha)^{\binom{|\alpha|-1}{N-1}},\end{aligned}$$

and division by the binomial coefficient $\binom{|\alpha|}{N}$ gives

$$2^{j\cdot\gamma}\varrho^\gamma \ge (2^{|j|}\varrho^\alpha)^{N/|\alpha|}.$$

Now the assertion is clear from (2.3) and the fact that the sum of all the terms $\left|x^\beta\right|$ is certainly not less than the single term $|x^\gamma|$. □

Now we are ready to get sharp estimates on the integral of $\hat{f}$ on each of the regions $E(\alpha, \varrho)$. We denote $\iota := (1, \ldots, 1) \in \mathbb{N}^n$.

LEMMA 2.5. *Let* $q \in]1, 2]$, *and* $N = \lfloor n/q \rfloor + 1$. *Then*

$$\int_{E(\alpha,\varrho)} \left|\hat{f}(x)\right| \mathrm{d}x \le C\varrho^{\iota/q-\alpha} \left\|D^\alpha f\right\|_q \qquad \forall\ \alpha \in \{0,1\}^n,$$

and also

$$\int_{E(\alpha,\varrho)} \left|\hat{f}(x)\right| \mathrm{d}x \le C\varrho^{\iota/q-\alpha N/|\alpha|} \sum_{\substack{\beta \le \alpha \\ |\beta| = N}} \left\|D^\beta f\right\|_q \qquad \forall\ \alpha \in \{0,1\}^n \text{ with } |\alpha| > N.$$

Proof. For $\alpha = 0$, we immediately get

$$\int_{E(0,\varrho)} \left|\hat{f}(x)\right| \, \mathrm{d}x \le |E(0,\varrho)|^{1/q} \left\|\hat{f}\right\|_{q'} \le C\varrho^{\iota/q} \|f\|_q \, .$$

Concerning $0 \neq \alpha \in \{0,1\}^n$, we estimate

$$\begin{aligned}
\int_{E(\alpha,\varrho)} \left|\hat{f}(x)\right| \, \mathrm{d}x &= \sum_{j\in\mathbb{N}^\alpha} \int_{E(\alpha,\varrho,j)} \left|\hat{f}(x)\right| \, \mathrm{d}x \\
&\le \sum_{j\in\mathbb{N}^n} \frac{1}{2^{|j|}\varrho^\alpha} \int_{E(\alpha,\varrho,j)} \left|x^\alpha \hat{f}(x)\right| \, \mathrm{d}x \\
&\le C \sum_{j\in\mathbb{N}^\alpha} (2^{|j|}\varrho^\alpha)^{-1} \, |E(\alpha,\varrho,j)|^{1/q} \left\|x^\alpha \hat{f}(x)\right\|_{q'} \\
&\le C \sum_{j\in\mathbb{N}^\alpha} (2^{|j|}\varrho^\alpha)^{-1} (2^{|j|}\varrho^\iota)^{1/q} \, \|D^\alpha f\|_q \\
&= C \, \|D^\alpha f\|_q \, \varrho^{\iota/q-\alpha} \sum_{j\in\mathbb{N}^\alpha} 2^{|j|(1/q-1)},
\end{aligned}$$

and the first assertion follows from the summability of the series, since we have $1/q - 1 < 0$ by assumption.

The second assertion is established by a modification of the argument that lead to the first one, using Lemma 2.4:

$$\begin{aligned}
\int_{E(\alpha,\varrho)} \left|\hat{f}(x)\right| \, \mathrm{d}x &= \sum_{j\in\mathbb{N}^\alpha} \int_{E(\alpha,\varrho,j)} \left|\hat{f}(x)\right| \, \mathrm{d}x \\
&\le \sum_{j\in\mathbb{N}^\alpha} \frac{1}{(2^{|j|}\varrho^\alpha)^{N/|\alpha|}} \sum_{\substack{\beta\le\alpha \\ |\beta|=N}} \int_{E(\alpha,\varrho,j)} \left|x^\beta \hat{f}(x)\right| \, \mathrm{d}x \\
&\le C \sum_{j\in\mathbb{N}^\alpha} 2^{-|j|N/|\alpha|} \varrho^{-\alpha N/|\alpha|} \sum_{\substack{\beta\le\alpha \\ |\beta|=N}} |E(\alpha,\varrho,j)|^{1/q} \left\|x^\beta \hat{f}(x)\right\|_{q'} \\
&\le C\varrho^{-\alpha N/|\alpha|} \sum_{j\in\mathbb{N}^\alpha} 2^{-|j|N/|\alpha|} \sum_{\substack{\beta\le\alpha \\ |\beta|=N}} (2^{|j|}\varrho^\iota)^{1/q} \, \|D^\alpha f\|_q \\
&= C\varrho^{\iota/q-\alpha N/|\alpha|} \sum_{\substack{\beta\le\alpha \\ |\beta|=N}} \left\|D^\beta f\right\|_q \sum_{j\in\mathbb{N}^\alpha} 2^{|j|(1/q-N/|\alpha|)}.
\end{aligned}$$

The series is summable since

$$N/|\alpha| - 1/q \ge N/n - 1/q > (n/q)/n - 1/q = 0,$$

and hence the second assertion is established. □

Next we add the pieces together to estimate the total integral of $\left|\hat{f}(x)\right|$; we also require a more precise estimate for the integral over the exterior of a ball $\bar{B}(0,R)$. The first assertion of the following proposition is our promised improvement of the classical embedding result in Prop. 2.1

PROPOSITION 2.6. *For all $q \in]1,2]$ and $0 < r < R$, we have the estimates*

$$\int_{\mathbb{R}^n} \left|\hat{f}(x)\right| \,\mathrm{d}x \le C \sum_{\substack{\alpha\in\{0,1\}^n \\ |\alpha|\le N}} \|D^\alpha f\|_q \, r^{n/q-|\alpha|}$$

and

$$\int_{|x|>R} \left|\hat{f}(x)\right| \,\mathrm{d}x \le C \left(\frac{r}{R}\right)^{\delta} \sum_{\substack{0\ne\alpha\in\{0,1\}^n \\ |\alpha|\le N}} \|D^\alpha f\|_q \, r^{n/q-|\alpha|},$$

where $N = \lfloor n/q \rfloor + 1$ and $\delta = N/n - 1/q > 0$.

PROOF. The first estimate follows directly from Lemma 2.5, after adding the estimates for the integrals of $\left|\hat{f}(x)\right|$ over each of the sets $E(\alpha,\varrho)$, with the choice $\varrho := (r,\dots,r)$.

We prove the second estimate with $\sqrt{n}R$ in place of R. If $|x| > \sqrt{n}R$, then $|x_i| > R$ for some $i \in \{1,\dots,n\}$. Let ϱ^i be the multi-index whose ith component is R, while the others are r. Then clearly $x \in E(\alpha,\varrho(i))$ for some (possibly several) $i \in \{1,\dots,n\}$ and $\alpha \in \{0,1\}^n$ such that $\alpha_i \ne 0$. Thus

$$\begin{aligned}
\int_{|x|>\sqrt{n}R} \left|\hat{f}(x)\right| \,\mathrm{d}x &\le \sum_{0\ne\alpha\in\{0,1\}^n} \sum_{i:\alpha_i\ne 0} \int_{E(\alpha,\rho(i))} \left|\hat{f}(x)\right| \,\mathrm{d}x \\
&\le \sum_{\substack{0\ne\alpha\in\{0,1\}^n \\ |\alpha|<N}} C \|D^\alpha f\|_q \sum_{i:\alpha_i=1} \varrho(i)^{\iota/q-\alpha} \\
&\qquad + \sum_{\substack{\beta\in\{0,1\}^n \\ |\beta|=N}} C \left\|D^\beta f\right\|_q \sum_{\substack{\alpha\in\{0,1\}^n \\ \alpha\ge\beta}} \sum_{i:\alpha_i=1} \varrho(i)^{\iota/q-\alpha N/|\alpha|},
\end{aligned}$$

where Lemma 2.5 was used to get the second inequality.

Next we observe that $\varrho(i)^\iota = r^{n-1}R$, and, for $\alpha_i = 1$, that $\varrho(i)^\alpha = r^{|\alpha|-1}R$. Thus

$$\varrho(i)^{\iota/q-\alpha} = r^{(n-1)/q-(|\alpha|-1)} R^{1/q-1} = r^{n/q-|\alpha|} \left(\frac{r}{R}\right)^{1-1/q},$$

and

$$\varrho(i)^{\iota/q-\alpha N/|\alpha|} = r^{(n-1)/q-(|\alpha|-1)N/|\alpha|} R^{1/q-N/|\alpha|} = r^{n/q-N} \left(\frac{r}{R}\right)^{N/|\alpha|-1/q},$$

and here $N/|\alpha|-1/q \geq N/n-1/q > (n/q)/n-1/q = 0$; also $1-1/q \geq N/n-1/q$. Hence, for $r \leq R$, we have

$$\int_{|x|>\sqrt{n}R} \left|\hat{f}(x)\right| \, \mathrm{d}x \leq C \left(\frac{r}{R}\right)^{N/n-1/q} \sum_{\substack{0 \neq \alpha \in \{0,1\}^n \\ |\alpha| \leq N}} \|D^\alpha f\|_q \, r^{n/q-|\alpha|},$$

and so the proposition is proven. □

3. Kernel estimates and conclusion

Combining the estimates in Lemma 1.3 with Prop. 2.6, we immediately derive the following kernel estimates:

$$\|k_j\|_1 \leq \sum_{\substack{\alpha \in \{0,1\}^n \\ |\alpha| \leq N}} C r^{n/q-|\alpha|} 2^{j(n/q-|\alpha|)} \leq C, \tag{3.1}$$

after choosing $r := 2^{-j}$, as we may. Similarly, for $|y| \leq 2^{-j}$, we get

$$\begin{aligned} \|k_j(\cdot - y) - k_j\|_1 &= \left\|\mathcal{F}^{-1}[(e^{\mathbf{i}2\pi y\cdot\xi} - 1)m_j(\xi)]\right\|_1 \\ &\leq \sum_{\substack{\alpha \in \{0,1\}^n \\ |\alpha| \leq N}} C r^{n/q-|\alpha|} 2^{j(n/q-|\alpha|+1)} |y| \leq C 2^j |y| \end{aligned} \tag{3.2}$$

with the same choice of r.

Moreover, for the exterior integral we have

$$\int_{|x|>R} |k_j(x)| \, \mathrm{d}x \leq C \left(\frac{r}{R}\right)^\delta \sum_{\substack{\alpha \in \{0,1\}^n \\ |\alpha| \leq N}} r^{n/q-|\alpha|} 2^{j(n/q-|\alpha|)} \leq C(2^j R)^{-\delta}, \tag{3.3}$$

with $r = 2^{-j}$, once again.

As a consequence of (3.1), we obtain $\|m_j\|_\infty \leq \|k_j\|_1 \leq C$, and hence $\|m\|_\infty \leq 2C$. This gives the boundedness of $\widehat{Tf} = m\hat{f}$ on $L^2(\mathbb{R}^n)$ by Plancherel's theorem.

Next we estimate the Hörmander-type integrals

$$\int_{|x|>2|y|} |k_j(x-y) - k_j(x)| \, \mathrm{d}x.$$

For $2^j \leq |y|^{-1}$, we simply ignore the range of integration to estimate this by

$$\|k_j(\cdot) - k_j\|_1 \leq C 2^j |y|$$

according to (3.2). For $2^j > |y|$, we make the estimate by

$$2 \int_{|x|>|y|} |k_j(x)| \, \mathrm{d}x \leq C(2^j |y|)^{-\delta}.$$

Using the two estimates directly above, we obtain

$$\sum_{j=-\infty}^{\infty} \int_{|x|>2|y|} |k_j(x-y) - k_j(x)| \, \mathrm{d}x \le \sum_{j:2^j \le |y|^{-1}} C2^j |y| + \sum_{j:2^j > |y|^{-1}} C(2^j |y|)^{-\delta} \le C,$$

since both geometric series are summable.

Thus $k = \sum_{-\infty}^{\infty} k_j$ satisfies Hörmander's condition, and the operator $f \mapsto k * f = \mathcal{F}^{-1}(m\hat{f})$ is bounded on $L^2(\mathbb{R}^n)$. Hence the boundedness on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ follows from the well-known theory of singular integrals.

Bibliography

[1] H. Amann. Operator-valued Fourier multipliers, vector-valued Besov spaces, and applications. *Math. Nachr.* **186** (1997), 5–56.
[2] W. Arendt, S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.* **240** (2002), 311–343.
[3] ———. Operator-valued multiplier theorems characterizing Hilbert spaces. Preprint, 2001.
[4] W. Beckner, A. P. Calderón, R. Fefferman, P. W. Jones (eds.). *Conference on Harmonic Analysis in Honor of Antoni Zygmund* (Chicago, 1981). Wadsworth, 1983.
[5] A. Benedek, A. P. Calderón, R. Panzone. Convolution operators on Banach space valued functions. *Proc. Nat. Acad. Sci. U.S.A.* **48** (1962), 356–365.
[6] J. Bergh, J. Löfström. *Interpolation Spaces: An Introduction.* Grundlehren der math. Wissenschaften 223, Springer-Verlag, Berlin–Heidelberg–New York, 1976.
[7] E. Berkson, T. A. Gillespie. Spectral decompositions and harmonic analysis on UMD spaces. *Studia Math.* **112** #1 (1994), 13–49.
[8] O. Blasco. Hardy spaces of vector-valued functions: duality. *Trans. Amer. Math. Soc.* **308** (1988), 495–507.
[9] S. Blunck. Maximal regularity of discrete and continuous time evolution equations. *Studia Math.* **146** #2 (2001), 157–176.
[10] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. *Ark. f. Mat.* **21** (1983), 163–168.
[11] ———. Extension of a result of Benedek, Calderón and Panzone. *Ark. Mat.* **22** (1984), 91–95.
[12] ———. Vector-valued singular integrals and the H^1-BMO duality. In J.-A. Chao, W. A. Woyczyński (eds.), *Probability theory and harmonic analysis*, Marcel Dekker, New York, 1986, 1–19.
[13] D. L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. *Ann. Prob.* **9** (1981), 997–1011.
[14] ———. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. In [**4**], 270–286.
[15] ———. Martingales and Fourier analysis in Banach spaces. In A. Dold, B. Eckmann (eds.), *Probability and Analysis*, Lect. Notes in Math. 1206, Springer (1985), 61–108.
[16] A. P. Calderon, A. Zygmund. On the existence of certain singular integrals. *Acta Math.* **88** (1952), 85–139.
[17] P. Cannarsa, V. Vespri. On maximal L^p regularity for the abstract Cauchy problem. *Boll. Unione Mat. Ital.* **5**-B (1986), 165–175.
[18] I. Chueshov, I. Lasiecka. Inertial manifolds for von Karman plate equations. *Appl. Math. Optim.* **46** (2002), 179–206.
[19] Ph. Clément, G. Gripenberg, S.-O. Londen. Schauder estimates for equations with fractional derivatives. *Trans. Amer. Math. Soc.* **352** (2000), 2239–2260.
[20] Ph. Clément, S.-O. Londen, G. Simonett. Quasilinear evolutionary equations and continuous interpolation spaces. *J. Differ. Equations*, to appear.

[21] Ph. Clément, B. de Pagter, F. A. Sukochev, H. Witvliet. Schauder decompositions and multiplier theorems. *Studia Math.* **138** (2000), 135–163.

[22] Ph. Clément, J. Prüss. An operator-valued transference principle and maximal regularity on vector-valued L_p-spaces. In [**59**].

[23] R. R. Coifman, C. Fefferman. Weighted norm inequalities for maximal functions and singular integrals. *Studia Math.* **51** (1974), 241–250.

[24] T. Coulhon, D. Lamberton. Regularité L^p pour les équations d'évolution. In B. Beauzamy, B. Maurey, G. Pisier (eds.), *Sémin. d'analyse fonctionelle, Paris* 1984/1985, Publ. Math. Univ. Paris VII **26** (1985), 155–165.

[25] R. Denk, M. Hieber, J. Prüss. R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Memoirs of the Amer. Math. Soc.*, to appear.

[26] J. Diestel, J. J. Uhl, Jr. *Vector Measures.* Math. Surveys 15, AMS, Providence, RI, 1977.

[27] G. Dore. L^p-regularity for abstract differential equations. In *Functional Analysis and Related Topics*, Lect. Notes in Math. 1540, Springer (1993), 25–38.

[28] ———. Maximal regularity in L^p spaces for an abstract Cauchy problem. *Adv. Differ. Equ.* **5** (2000) 293–322.

[29] G. Dore, A. Venni. On the closedness of a sum of two closed operators. *Math. Z.* **196** (1987), 189–201.

[30] K.-J. Engel, R. Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Graduate Texts in Math. 194, Springer-Verlag, New York–Berlin–Heidelberg, 2000.

[31] C. Fefferman, E. M. Stein. H^p spaces of several variables. *Acta Math.* **129** (1972) 137–193.

[32] A. Fröhlich. The Stokes operator in weighted L^q-spaces I: Weighted estimates for the Stokes resolvent problem in a half space. *J. Math. Fluid Mech.*, to appear.

[33] ———. The Stokes operator in weighted L^q-spaces II: Weighted resolvent estimates and maximal L^p-regularity. *Math. Ann.*, to appear.

[34] J. García-Cuerva, J. L. Rubio de Francia. *Weighted Norm Inequalities and Related Topics.* North-Holland Math. studies 116, Elsevier Science Publishers, Amsterdam, 1985.

[35] M. Girardi, L. Weis. Operator-valued Fourier multiplier theorems on Besov spaces. *Math. Nachr.*, to appear.

[36] ———. Operator-valued Fourier multiplier theorems on $L_p(X)$ and geometry of Banach spaces. *J. Funct. Anal.*, to appear.

[37] ———. Criteria for R-boundedness of operator families. *Recent Contributions to Evolution Equations*, Lecture Notes in Math., Marcel Dekker, to appear.

[38] ———. Integral operators with operator-valued kernels. In preparation.

[39] D. Goldberg. A local version of real Hardy spaces. *Duke Math. J.* **46** #1 (1979), 27–42.

[40] R. Haller, H. Heck, A. Noll. Mikhlin's theorem for operator-valued Fourier multipliers in n variables. *Math. Nachr.* **244** (2002), 110–130.

[41] M. Hieber. Operator-valued Fourier multipliers. In J. Escher, G. Simonett (eds.), *Topics in Nonlinear Analysis, The Herbert Amann Anniversary Volume*, Birkhäuser, 1999.

[42] M. Hieber, J. Prüss. Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations. *Commun. Partial Differ. Equations* **22** (1997), 1647–1669.

[43] L. Hörmander. Estimates for translation invariant operators in L^p spaces. *Acta Math.* **104** (1960), 93–140.

[44] T. Hytönen. R-boundedness and multiplier theorems. Master's thesis, Helsinki Univ. Techn. Inst. Math. Research Report C16, 2001.

[45] ———. Convolutions, multipliers and maximal regularity on vector-valued Hardy spaces. Submitted.

[46] ———. Fourier-embeddings and Mihlin-type multiplier theorems. Submitted.

[47] T. Hytönen, L. Weis. Singular convolution integrals with operator-valued kernel. Submitted.

[48] ______. Singular integrals on Besov spaces. Manuscript, 2002.

[49] N. Kalton, G. Lancien. A solution to the problem of L^p-maximal regularity. *Math. Z.* **235** (2000), 559–568.

[50] N. Kalton, L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.* **321** #2 (2001), 319–345.

[51] Y. Katznelson. *An Introduction to Harmonic Analysis.* Dover, 1976.

[52] P. C. Kunstmann, L. Weis. Perturbation theorems for maximal L_p-regularity. *Ann. Sc. Norm. Super. Pisa Cl. Sci. IV. Ser.* **30** (2001), 415–435.

[53] ______. Maximal L_p-regularity for parabolic problems, Fourier multiplier theorems and H^∞ functional calculus. In preparation.

[54] D. S. Kurtz, R. L. Wheeden. Results on weighted norm inequalities for multipliers. *Trans. Amer. Math. Soc.* **255** (1979), 343–362.

[55] S. Kwapień. Isomorphic characterizations of inner product spaces by orthogonal series with vector valued coefficients. *Studia Math.* **44** (1972), 583–595.

[56] D. Lamberton. Equations d'évolution linéaires associées à des semi-groupes de contraction dans les espaces L^p. *J. Func. Anal.* **71** (1987), 252–262.

[57] C. Le Merdy. Counterexamples on L_p-maximal regularity. *Math. Z.* **230** (1999), 47–62.

[58] J. Lindenstrauss, L. Tzafriri. On the complemented subspaces problem. *Israel J. Math.* **9** (1971), 263–269.

[59] G. Lumer, L. Weis (eds.). *Evolution Equations and Their Applications in Physical and Life Sciences.* Proc. of the 6th Int. Conference on Evolution Equations, Bad Herrenalb, 1998. Marcel Dekker, 2000.

[60] J. Marcinkiewicz. Sur les multiplicateurs des séries de Fourier. *Studia Math.* **8** (1939), 78–91.

[61] T. R. McConnell. On Fourier multiplier transformations of Banach-valued functions. *Trans. Amer. Math. Soc.* **285** #2 (1984), 739–757.

[62] S. G. Mihlin (С. Г. Михлин). О мультипликаторах интегралов Фурье (On the multipliers of Fourier integrals; Russian). *Докл. Акад. наук СССР* **109** (1956), 701–703.

[63] ______. Интегралы Фурье и кратные сингулярные интегралы (Fourier integrals and multiple singular integrals; Russian, with English summary). *Вестник Ленинград. ун-та, сер. матем. мех. и астр.* **7** (1957), 143–155.

[64] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.* **165** (1972), 207–226.

[65] J. Peetre. Sur la transformation de Fourier des fonctions à valeurs vectorielles, *Rend. Sem. Mat. Univ. Padova* **62** (1969), 15–26.

[66] A. Pełczyński, M. Wojciechowski. Molecular decompositions and embedding theorems for vector-valued Sobolev spaces with gradient norm. *Studia Math.* **107** #1 (1993), 61–100.

[67] A. Pietsch, J. Wenzel. *Orthonormal Systems and Banach Space Geometry.* Encyclopedia of mathematics and its applications 70, Cambridge Univ. Press, 1998.

[68] P. Portal. Discrete time analytic semi-groups and the geometry of Banach spaces. *Semigroup Forum*, to appear.

[69] J. Prüss. *Evolutionary Integral Equations and Applications.* Monographs in Math. 87, Birkhäuser, Basel, 1993.

[70] M. Riesz. Sur les fonctions conjuguées. *Math. Z.* **27** (1927), 218–244.

[71] J. L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. In *Probability in Banach spaces*, Lect. Notes in Math. 1221, 195–222. Springer, 1986.

[72] J. Schwartz. A remark on inequalities of Calderon–Zygmund for vector-valued functions. *Commun. Pure Appl. Math.* **14** (1961) 785–799.

[73] L. Schwartz. *Théorie des distributions*. Publications de l'Institute de Mathématique de l'Université de Strasbourg, Hermann, Paris, 1966.

[74] L. de Simon. Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rend. Sem. Mat. Univ. Padova* **34** (1964) 547–558.

[75] P. E. Sobolevskiĭ. Coerciveness inequalities for abstract parabolic equations. *Soviet Math. Dokl.* **5** #4 (1964), 894–897. (Translation by R. N. Goss from *Dokl. Akad. Nauk SSSR* **157** (1964), 52–57.)

[76] E. M. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton Math. Ser. 30. Princeton Univ. Press, NJ, 1970.

[77] ______. *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton Math. Ser. 43. Princeton Univ. Press, NJ, 1993.

[78] E. M. Stein, G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces*. 2nd ed., Princeton Math. Ser. 32, Princeton Univ. Press, NJ, 1975.

[79] Ž. Štrkalj. R-Beschränktheit, Summensätze abgeschlossener Operatoren und operatorwertige Pseudodifferentialoperatoren. Dissertation, Univ. Karlsruhe Fakultät f. Mathematik, 2000.

[80] Ž. Štrkalj, L. Weis. On operator-valued Fourier multiplier theorems. Univ. Karlsruhe Fakultät f. Mathematik Preprint Nr. 36, 2000; submitted.

[81] J.-O. Strömberg, A. Torchinsky. *Weighted Hardy Spaces*. Lect. Notes in Math. 1381. Springer, 1989.

[82] H. Triebel. *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Math. Library 18, North-Holland, Amsterdam–New York–Oxford, 1978.

[83] ______. *Theory of Function Spaces*. Monographs in Mathematics, Birkhäuser, Basel–Boston–Stuttgart, 1983.

[84] A. Venni. Marcinkiewicz and Mihlin multiplier theorems, and R-boundedness. Preprint, 2001.

[85] L. Weis. Stability theorems for semi-groups via multiplier theorems. In M. Demuth, B.-W. Schulze (eds.), *Differential equations, asymptotic analysis, and mathematical physics*, Potsdam, 1996. Akademie Verlag, Berlin, 1997, 407–411.

[86] ______. A new approach to maximal L_p-regularity. In [**59**].

[87] ______. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.* **319** (2001), 735–758.

[88] H. Witvliet. *Unconditional Schauder decompositions and multiplier theorems.* Dissertation, TU Delft, 2000.

[89] F. Zimmermann. On vector-valued Fourier multiplier theorems. *Studia Math.* **93** (1989), 201–222.

[90] A. Zygmund. *Trigonometric Series*, Vol. II. Cambridge Univ. Press, 1979.

(continued from the back cover)

A452 Kalle Mikkola
Infinite-Dimensional Linear Systems, Optimal Control and Algebraic Riccati Equations
October 2002

A451 Marko Huhtanen
Combining normality with the FFT techniques
September 2002

A450 Nikolai Yu. Bakaev
Resolvent estimates of elliptic differential and finite element operators in pairs of function spaces
August 2002

A449 Juhani Pitkäranta
Mathematical and historical reflections on the lowest order finite element models for thin structures
May 2002

A448 Teijo Arponen
Numerical solution and structural analysis of differential-algebraic equations
May 2002

A447 Timo Salin
Quenching-rate estimate for a reaction diffusion equation with weakly singular reaction term
April 2002

A446 Tuomas Hytönen
R-Boundedness is Necessary for Multipliers on H^1
February 2002

A445 Philippe Clément , Stig-Olof Londen , Gieri Simonett
Quasilinear Evolutionary Equations and Continuous Interpolation Spaces
March 2002

A444 Tuomas Hytönen
Convolutions, Multipliers and Maximal Regularity on Vector-Valued Hardy Spaces
December 2001

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A457 Timo Salin
On a Refined Asymptotic Analysis for the Quenching Problem
March 2003

A456 Ville Havu , Harri Hakula , Tomi Tuominen
A benchmark study of elliptic and hyperbolic shells of revolution
January 2003

A455 Yaroslav V. Kurylev , Matti Lassas , Erkki Somersalo
Maxwell's Equations with Scalar Impedance: Direct and Inverse Problems
January 2003

A454 Timo Eirola , Marko Huhtanen , Jan von Pfaler
Solution methods for R-linear problems in C^n
October 2002

A453 Marko Huhtanen
Aspects of nonnormality for iterative methods
September 2002

ISBN 951-22-6456-0
ISSN 0784-3143
Printed by Otamedia
Espoo, 2003