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PSEUDODIFFERENTIAL CALCULUS ON COMPACT HOMOGENEOUS SPACES

Ville Turunen

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

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Abstract: *Pseudodifferential operators on a compact Lie group G are projected to pseudodifferential operators on an orientable compact homogeneous space G/K. Starting with a pseudodifferential operator on a compact homogeneous space G/K with torus K, we extend the operator to act on G; a special example of such a homogeneous space is the two-sphere $\mathbb{S}^2$ as the base space for the Hopf fibration.*

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Ville.Turunen@hut.fi

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Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email:math@hut.fi http://www.math.hut.fi/

1 Introduction

In this article we treat pseudodifferential analysis on orientable homogeneous spaces G/K, where G is a compact Lie group with a closed subgroup K. This research continues the work in [11], where such analysis on compact Lie groups was studied. Apart from pure theoretical interests, there are applications which call for the present treatise: e.g. Dirichlet boundary value problems in a domain diffeomorphic to the unit ball of $\mathbb{R}^3$ may be considered within the framework of harmonic analysis on the two-sphere $\mathbb{S}^2 \subset \mathbb{R}^3$. Taylor (see [7]) has characterized pseudodifferential operators on the spheres $\mathbb{S}^n$ by studying the smoothness of certain operator-valued functions on a large group of symmetries, but this result cannot be used for our purposes here.

We explain how a pseudodifferential operator on a compact Lie group G can be "projected" to a pseudodifferential operator on orientable compact homogeneous spaces G/K in a way respecting the algebraic structures. The other way round, given a pseudodifferential operator on G/K when K is a torus we construct an "extended" pseudodifferential operator on G; the "projection" of this "extension" in turn returns the original operator. "Extended" operators can be used to calculate asymptotic expansions for operators on G/K using operator-valued symbolic calculus on G (see [8], [11]).

Vector space notation

The space of the continuous linear operators between topological vector spaces X and Y is denoted by $\mathcal{L}(X,Y)$, and we write $\mathcal{L}(X) := \mathcal{L}(X,X)$; the dual space of X is $X' := \mathcal{L}(X,\mathbb{C})$. If X is a nuclear Fréchet space, $X \otimes X'$ stands for the complete locally convex tensor product.

2 Pseudodifferential operators on $\mathbb{R}^p \times \mathbb{T}^q$

For general treatments of pseudodifferential calculus on the Euclidean spaces or manifolds, see e.g. [3] or [9]. Periodic pseudodifferential operators, i.e. pseudodifferential operators on tori expressed utilizing Fourier series, were introduced in [1], and their complete symbolic calculus is presented in [12].

Let $\mathbb{T}^q = \mathbb{R}^q/\mathbb{Z}^q$ be the q-dimensional torus group. In the sequel we shall identify $\mathbb{R}^0$ and $\mathbb{Z}^0$ with the set $\{0\}$, and $\mathbb{R}^p \times \mathbb{T}^0$ is identified with $\mathbb{R}^p$. Let $\mathcal{S}(\mathbb{R}^p \times \mathbb{T}^q) = \{f \in C^\infty(\mathbb{R}^p \times \mathbb{T}^q) \mid \forall y \in \mathbb{T}^q : (x \mapsto f(x,y)) \in \mathcal{S}(\mathbb{R}^p)\}$ be endowed with the natural Fréchet space structure of the test functions. In this space, we define the *Fourier transform* $f \mapsto \hat{f}$ by

$$\hat{f}(\xi) = \int_{\mathbb{R}^p \times \mathbb{T}^q} f(x)\, e^{-i2\pi x\cdot\xi}\, dx_1 \cdots dx_{p+q},$$

where $\xi \in \mathbb{R}^p \times \mathbb{Z}^q$. Let $e_\xi(x) = e^{i2\pi x\cdot\xi}$, and let $A \in \mathcal{L}(\mathbb{S}'(\mathbb{R}^p \times \mathbb{T}^q))$; then $e_\xi \in \mathcal{S}'(\mathbb{R}^p \times \mathbb{T}^q)$, and we can define the *symbol* $\sigma_A : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q) \to \mathbb{C}$ of A:

$$\sigma_A(x,\xi) := e_\xi(x)^{-1}(Ae_\xi)(x), \tag{1}$$

and it is clear that σ_A is C^∞-smooth with respect to the variable $x \in \mathbb{R}^p$. Then A can be retrieved from its symbol σ_A by

$$(Af)(x) = \int_{\mathbb{R}^p} \sum_{\xi_{p+1},\ldots,\xi_{p+q} \in \mathbb{Z}} \sigma_A(x,\xi)\ \hat{f}(\xi)\ e^{i2\pi x\cdot\xi}\ d\xi_1 \cdots d\xi_p. \tag{2}$$

The symbol class $S^m(\mathbb{R}^p \times \mathbb{T}^q)$ consists of those C^∞-smooth functions $\sigma_A : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q) \to \mathbb{C}$ for which

$$\sup_{x \in \mathbb{R}^p \times \mathbb{T}^q} |\partial_\xi^{\alpha'} \triangle_\xi^{\alpha''} \partial_x^\beta \sigma_A(x,\xi)| \leq C_{A\alpha\beta m}\ \langle\xi\rangle^{m-|\alpha|} \tag{3}$$

for every multi-index $\alpha = \alpha' + \alpha'', \beta \in \mathbb{N}_0^{p+q}$; here $\alpha = \alpha' + \alpha''$, $\alpha' = (\alpha_1, \ldots, \alpha_p, 0, \ldots, 0)$, and $\langle\xi\rangle = (1 + \sum_{j=1}^{p+q} \xi_j^2)^{1/2}$. Here $\triangle_\xi^\alpha$ is the αth forward difference operator defined by

$$(\triangle_\xi^\alpha \sigma)(\xi) := \sum_{0 \leq \gamma \leq \alpha} \binom{\alpha}{\gamma} (-1)^{|\alpha-\gamma|}\ \sigma(\xi+\gamma), \tag{4}$$

$|\alpha| = 1$ implies $(\triangle_\xi^\alpha \sigma)(\xi) := \sigma(\xi+\alpha) - \sigma(\xi)$. Operator $A \in \mathcal{L}(\mathcal{S}(\mathbb{R}^p \times \mathbb{T}^q))$ is called *a pseudodifferential operator of order* $m \in \mathbb{R}$, $A \in \Psi^m(\mathbb{R}^p \times \mathbb{T}^q) = \mathrm{Op}S^m(\mathbb{R}^p \times \mathbb{T}^q)$, if $\sigma_A \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$.

3 Analysis on closed manifolds

Let M be a C^∞-smooth, closed (i.e. compact, without a boundary) orientable manifold. The test function space $\mathcal{D}(M)$ is the space of $C^\infty(M)$ endowed with the usual Fréchet space topology. Its dual $\mathcal{D}'(M) = \mathcal{L}(\mathcal{D}(M), \mathbb{C})$ is the space of distributions, endowed with the weak-$*$-topology. The duality is expressed by the brackets $\langle\phi, f\rangle = f(\phi)$ ($\phi \in \mathcal{D}(M)$, $f \in \mathcal{D}'(M)$). Embedding $\mathcal{D}(M) \hookrightarrow \mathcal{D}'(M)$ is interpreted by

$$\langle\phi, \psi\rangle := \int_M \phi(x)\ \psi(x)\ dx.$$

The Schwartz kernel theorem states that $\mathcal{L}(\mathcal{D}(M))$ is isomorphic to $\mathcal{D}(M) \otimes \mathcal{D}'(M)$; the isomorphism is given by

$$\langle A\phi, f\rangle = \langle K_A, f \otimes \phi\rangle, \tag{5}$$

where $A \in \mathcal{L}(\mathcal{D}(M))$, $\phi \in \mathcal{D}(M)$, $f \in \mathcal{D}'(M)$, and distribution $K_A \in \mathcal{D}(M) \otimes \mathcal{D}'(M)$ is called the *Schwartz kernel* of A. Then A can uniquely be extended (by duality) to $A \in \mathcal{L}(\mathcal{D}'(M))$, and it is customary to write informally

$$(Af)(x) = \int_M K_A(x,y)\ f(y)\ dy$$

instead of $\phi \mapsto \langle\phi, Af\rangle$ ($\phi \in \mathcal{D}(M)$). Recall that $L^2(M) = H^0(M)$, $\mathcal{D}'(M) = \cup_{s\in\mathbb{R}} H^s(M)$ and $\mathcal{D}(M) = \cap_{s\in\mathbb{R}} H^s(M)$, where $H^s(M)$ is the (L^2-type) Sobolev space of order $s \in \mathbb{R}$.

An operator $A \in \mathcal{L}(\mathcal{D}(M))$ is *a pseudodifferential operator of order* $m \in \mathbb{R}$ *on* M, $A \in \Psi^m(M)$, if $(M_\phi A M_\psi)_\kappa \in \Psi^m(\mathbb{R}^{\dim(M)})$ for every chart (U,κ) of M and for every $\phi,\psi \in C_0^\infty(U)$, where M_ϕ is the multiplication operator $f \mapsto \phi f$, and

$$(M_\phi A M_\psi)_\kappa f := (M_\phi A M_\psi(f \circ \kappa)) \circ \kappa^{-1} \quad (f \in C^\infty(\kappa U)).$$

We sometimes treat write $M_\phi A M_\psi \in \Psi^m(\mathbb{R}^{\dim(M)})$, thus omitting the subscript κ and leaving the chart mapping implicit. Equivalently, pseudodifferential operators can be characterized by commutators (see [11]): $A \in \mathcal{L}(\mathcal{D}(M))$ belongs to $\Psi^m(M)$ if and only if $(A_k)_{k=0}^\infty \subset \mathcal{L}(H^m(M), H^0(M))$ for every sequence of smooth vector fields $(D_k)_{k=1}^\infty$ on M, where $A_0 = A$ and $A_{k+1} = [D_{k+1}, A_k]$.

A smooth *left transformation group* is

$$(G, M, m),$$

where G is a Lie group, M is a C^∞-manifold and $m : G \times M \to M$ is a C^∞-mapping called a left *action*, satisfying $m(e,p) = p$ and $m(x, m(y,p)) = m(xy, p)$ for every $x, y \in G$ and $p \in M$, where $e \in G$ is the neutral element of the group. The action is *free*, if $m(x,p) = p$ implies $x = e$. It is evident how one defines a *right* transformation group (G, M, m) with a *right* action $m : M \times G \to M$.

A smooth *fiber bundle* is

$$(E, B, F, p_{E\to B}),$$

where E, B, F are C^∞-manifolds and $p_{E\to B} \in C^\infty(E,B)$ is a surjective mapping such that there exists an open cover $\mathcal{U} = \{U_j \mid j \in J\}$ of B and diffeomorphisms $\phi_j : p^{-1}(U_j) \to U_j \times F$ satisfying $\phi_j(x) = (p_{E\to B}(x), \psi_j(x))$ for every $x \in p_{E\to B}^{-1}(U_j)$. The spaces E, B, F are called the *total space*, the *base space*, and the *fiber* of the bundle, respectively. The cover $\mathcal{U}$ is called a *locally trivializing cover* of the bundle. Sometimes the mapping $p_{E\to B}$ is called the fiber bundle.

A *principal fiber bundle* is

$$(E, B, F, p_{E\to B}, m),$$

where $(E, B, F, p_{E\to B})$ is a smooth fiber bundle with cover $\mathcal{U}$ and mappings ϕ_j, ψ_j as above and (F, E, m) is a smooth right transformation group with a free action satisfying $p_{E\to B}(m(x,y)) = p_{E\to B}(x)$ for every $(x,y) \in E \times F$ and $\psi_j(m(x,y)) = \psi_j(x)y$ for every $(x,y) \in p_{E\to B}^{-1}(U_j) \times F$.

4 Harmonic analysis on compact Lie groups

Let G be a compact Lie group. Let μ_G be the normalized Haar measure of G. The starting point of harmonic analysis on G is the *left regular representation* of G, which is the homomorphism $\pi_L : G \to \mathcal{L}(L^2(G))$ defined by

$$(\pi_L(y)f)(x) = f(y^{-1}x) \tag{6}$$

for almost every $x \in G$; equivalently we could begin with the *right regular representation* $\pi_R : G \to \mathcal{L}(L^2(G))$ defined by

$$(\pi_R(y)f)(x) = f(xy) \tag{7}$$

for almost every $x \in G$.

The *Fourier transform* of a distribution $f \in \mathcal{D}'(G)$ is said to be the operator $\pi(f) \in \mathcal{L}(\mathcal{D}(G))$ defined by

$$\pi(f)g = f * g, \tag{8}$$

i.e. the left convolution by f. Let $A \in \mathcal{L}(\mathcal{D}(G))$ with the Schwartz kernel K_A. The *symbol of* A is the mapping $\sigma_A : G \to \mathcal{L}(\mathcal{D}(G))$ defined by $\sigma_A(x) = \pi(s_A(x))$, where $K_A(x,y) = (s_A(x))(xy^{-1})$ in the sense of distributions. Then we denote $A = \mathrm{Op}(\sigma_A)$, and we have

$$\begin{aligned}(Af)(x) &= (\sigma_A(x)f)(x) \\ &= \mathrm{Tr}\,(\sigma_A(x)\ \pi(f)\ \pi_L(x)^*) \quad (f \in \mathcal{D}(G),\ x \in G).\end{aligned}$$

In the sequel Δ is the bi-invariant Laplacian of G (i.e. the left and right translation invariant Laplacian, or the Laplacian corresponding to the bi-invariant Riemannian metric of G), and we define $\Xi := (I - \Delta)^{1/2}$; then Ξ^m is a Sobolev space isomorphism $H^s(G) \to H^{s-m}(G)$, and it is also bi-invariant.

In the notation of [11], let us define

$$Q^\alpha \pi(s) = \pi(y \mapsto \check{q}_\alpha(y)\ s(y)),$$

where if $s \in \mathcal{D}'(G)$, and $q_\alpha \in C^\infty(G)$ ($\alpha \in \mathbb{N}_0^{\dim(G)}$) satisfies

$$q_\alpha(\exp(x)) = \frac{1}{\alpha!} x^\alpha$$

when x belongs to a small neighbourhood of $0 \in \mathfrak{g}$, the origin of the Lie algebra $\mathfrak{g}$ of G; technical details can be found in [11], where we presented the following characterization of pseudodifferential operators:

Definition. *An operator $A \in \mathcal{L}(\mathcal{D}(G))$ belongs to $\Psi^m(G)$ if and only if $\sigma_A \in S^m(G) = \cap_{k=0}^\infty S_k^m(G)$; here $\sigma_B \in S_0^m(G)$ if and only if*

$$\|\Xi^{|\alpha|-m} Q^\alpha \partial_x^\beta \sigma_B(x)\|_{\mathcal{L}(L^2(G))} \leq C_{B\alpha\beta m} \tag{9}$$

uniformly in $x \in G$ for every $\alpha, \beta \in \mathbb{N}_0^{\dim(G)}$; $\sigma_B \in S_{k+1}^m(G)$, if

$$\sigma_B \in S_k^m(G), \tag{10}$$

$$[\sigma_{\partial_j}, \sigma_B] \in S_k^m(G), \tag{11}$$

$$(Q^\gamma \sigma_{\partial_j})\sigma_A \in S_k^{m+1-|\gamma|}(G) \tag{12}$$

and

$$(Q^\gamma \sigma_A)\sigma_{\partial_j} \in S_k^{m+1-|\gamma|}(G) \tag{13}$$

for every $j \in \{1, \ldots, \dim(G)\}$ and $\gamma \in \mathbb{N}_0^{\dim(G)}$ with $|\gamma| > 0$, where $\{\partial_j \mid 1 \leq j \leq \dim(G)\}$ is a basis for the vector space of the right-invariant vector fields on G.

5 Harmonic analysis on compact homogeneous spaces

Let (G, E, m) be a smooth left transformation group. The manifold M is called a *homogeneous space* if the action $m : G \times M \to M$ is *transitive*, i.e. for every $p, q \in M$ there exists $x \in G$ such that $m(x, p) = q$.

Let us give another, equivalent definition for a homogeneous space: Let G be a Lie group with a closed subgroup K. The *homogeneous space* G/K is the set of classes $xK = \{xk \mid k \in K\}$ $(x \in G)$ endowed with the topology co-induced by $x \mapsto xK$ and equipped with the unique C^∞-manifold structure such that the mapping $(x, yK) \mapsto xyK$ belongs to $C^\infty(G \times (G/K), G/K)$ and such that there is a neighbourhood $U \subset G/K$ of $eK \in G/K$ and a mapping $\psi \in C^\infty(U, G)$ satisfying $\psi(xK)K = xK$. The group G acts smoothly from the left on the manifold G/K by $(x, yK) \mapsto x^{-1}yK$. Actually a smooth homogeneous space M is diffeomorphic to G/G_p, where $G_p = \{x \in G \mid m(x, p) = p\}$.

Notice also that $(G, G/K, K, x \mapsto xK, (x, k) \mapsto xk)$ has a structure of a principal fiber bundle (see [2]).

From now on we assume the Lie group G to be compact. We can regard functions (or distributions) constant on the cosets xK $(x \in G)$ as functions (or distributions) on G/K; it is obvious how one embeds the spaces $\mathcal{D}(G/K)$ and $\mathcal{D}'(G/K)$ into the spaces $\mathcal{D}(G)$ and $\mathcal{D}'(G)$, respectively. Let us define $P_{G/K} \in \mathcal{L}(\mathcal{D}(G))$ by

$$(P_{G/K}f)(x) = \int_K f(xk)\ d\mu_K(k). \tag{14}$$

Hence $P_{G/K}f \in C^\infty(G/K)$, and $P_{G/K}$ extends uniquely to the orthogonal projection of $L^2(G)$ onto the subspace $L^2(G/K)$. Let us consider operators $A \in \mathcal{L}(\mathcal{D}(G))$ with the symbol satisfying

$$\sigma_A(xk) = \sigma_A(x) \quad (x \in G,\ k \in K); \tag{15}$$

this condition is equivalent to

$$s_A(xk)(y) = s_A(x)(y)$$

in the sense of distributions, or

$$K_A(xk, yk) = K_A(x, y).$$

Then A maps the space $\mathcal{D}(G/K)$ into itself. Of course, for a general $A \in \mathcal{L}(\mathcal{D}(G))$ this is not true, but then we can define an operator $A_{G/K} \in \mathcal{L}(\mathcal{D}(G))$ by

$$s_{A_{G/K}} = (P_{G/K} \otimes \mathrm{id}) s_A. \tag{16}$$

Recall that $\sigma_A \in C^\infty(G, \mathcal{L}(H^m(G), H^0(G)))$ when $A \in \Psi^m(G)$, so that then

$$\sigma_{A_{G/K}}(x) = \int_K \sigma_A(xk)\ d\mu_K(k) \tag{17}$$

exists as a weak integral (Pettis integral), see [4].

Suppose we are given symbols of pseudodifferential operators A_1, A_2 on G satisfying the K-invariance (15). If we look at the asymptotic expansion formulae for $\sigma_{A_1 A_2}$, $\sigma_{A_1^*}$ and $\sigma_{A_1^t}$ in [11], we see that all the terms there are K-invariant in the same sense. Moreover, for an elliptic K-invariant symbol the terms in the asymptotic expansion for a parametrix are also K-invariant.

Theorem 1 and its corollary show how to 'project' pseudodifferential operators on G to pseudodifferential operators on G/K:

Theorem 1. *Let G be a compact Lie group with a closed Lie subgroup K. If $A \in \Psi^m(G)$, then $A_{G/K} \in \Psi^m(G)$.*

Proof. First, notice that $P_{G/K}$ is left-invariant, and hence

$$(\partial_x^\beta \otimes M_{\tilde{q}_\alpha})(P_{G/K} \otimes \mathrm{id}) s_A = (P_{G/K} \otimes \mathrm{id})(\partial_x^\beta \otimes M_{\tilde{q}_\alpha}) s_A$$

for a right-invariant partial differential operator ∂_x^β and a multiplication $M_{\tilde{q}_\alpha}$ for every $\alpha, \beta \in \mathbb{N}_0^{\dim(G)}$. Therefore

$$\mathrm{Op}(Q^\alpha \partial_x^\beta \sigma_{A_{G/K}}) = \left(\mathrm{Op}(Q^\alpha \partial_x^\beta \sigma_A)\right)_{G/K}.$$

Since $A \in \Psi^m(G)$, we have

$$\|Q^\alpha \partial_x^\beta \sigma_A(x)\|_{\mathcal{L}(H^{m-|\alpha|}(G), H^0(G))} \leq C_{A\alpha\beta m},$$

and so the mapping $k \mapsto Q^\alpha \partial_x^\beta \sigma_A(xk)$ belongs to $C^\infty(K, \mathcal{L}(H^{m-|\alpha|}(G), H^0(G)))$ for every $x \in G$. Then

$$\begin{aligned}
\|Q^\alpha \partial_x^\beta \sigma_{A_{G/K}}(x)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} &= \left\| \int_K Q^\alpha \partial_x^\beta \sigma_A(xk)\ d\mu_K(k) \right\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} \\
&\leq \int_K \|Q^\alpha \partial_x^\beta \sigma_A(xk)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)}\ d\mu_K(k) \\
&\leq \sup_{k \in K} \|Q^\alpha \partial_x^\beta \sigma_A(xk)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} \\
&\leq \sup_{y \in G} \|Q^\alpha \partial^\beta \sigma_A(y)\|_{\mathcal{L}(H^{m-|\alpha|}, H^0)} \\
&\leq C_{A\alpha\beta m}.
\end{aligned}$$

This proves that $\sigma_{A_{G/K}} \in \mathrm{Op}S_0^m(G)$. Let $B \in \mathcal{L}(\mathcal{D}(G))$ be any right-invariant (left convolution) pseudodifferential operator. Then $\sigma_B(x) = B$ for each $x \in G$ and $x \mapsto s_B(x)$ is a constant mapping $G \to \mathcal{D}'(G)$, $B = B_{G/K}$, and

$$(\mathrm{Op}(\sigma_A \sigma_B))_{G/K} = \mathrm{Op}(\sigma_{A_{G/K}} \sigma_B)$$

and

$$(\mathrm{Op}(\sigma_B \sigma_A))_{G/K} = \mathrm{Op}(\sigma_B \sigma_{A_{G/K}}).$$

Assume that we have proven $\sigma_{C_{G/K}} \in S^r_k(G)$ for every $C \in \Psi^r(G)$, for every $r \in \mathbb{R}$. Using Lemma 6, Theorem 9 and Proposition 11 in [11], we hence get

$$\mathrm{Op}([\sigma_{\partial_j}, \sigma_{A_{G/K}}]) = \mathrm{Op}([\sigma_{\partial_j}, \sigma_A])_{G/K} \in \mathrm{Op}S^m_k(G),$$

$$\mathrm{Op}((Q^\gamma \sigma_{\partial_j})\sigma_{A_{G/K}}) = \mathrm{Op}((Q^\gamma \sigma_{\partial_j})\sigma_A)_{G/K} \in \mathrm{Op}S^{m+1-|\gamma|}_k(G)$$

and

$$\mathrm{Op}((Q^\gamma \sigma_{A_{G/K}})\sigma_{\partial_j}) = \mathrm{Op}((Q^\gamma \sigma_A)\sigma_{\partial_j})_{G/K} \in \mathrm{Op}S^{m+1-|\gamma|}_k(G);$$

this means that $\sigma_{A_{G/K}} \in S^m_{k+1}(G)$, and then by induction we get $\sigma_{A_{G/K}} \in S^m(G) = \cap_{k=0}^{\infty} S^m_k(G)$ □

Corollary 2. *Let G/K be orientable. Then $A_{G/K}|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$ for every $A \in \Psi^m(G)$.*

Proof. Let

$$\Psi^m(G)_{G/K} = \{A_{G/K} \mid A \in \Psi^m(G)\}$$

and

$$\Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)} = \{A_{G/K}|_{\mathcal{D}(G/K)} \ : \ A \in \Psi^m(G)\}.$$

By Theorem 1 we know that $\Psi^m(G)_{G/K} \subset \Psi^m(G)$. Let D be a smooth vector field on G/K. Since $(G, G/K, K, x \mapsto xK, (x,k) \mapsto xk)$ is a principal fiber bundle, there exists a smooth vector field $X = X_{G/K}$ on G such that $X|_{\mathcal{D}(G/K)} = D$ (see [5]). Then

$$[D, \Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)}] = [X, \Psi^m(G)_{G/K}]|_{\mathcal{D}(G/K)} \subset \Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)},$$

and this combined with $\Psi^m(G)_{G/K}|_{\mathcal{D}(G/K)} \subset \mathcal{L}(H^m(G/K), H^0(G/K))$ yields the conclusion due to the commutator characterization of pseudodifferential operators on closed manifolds □

Hence at least sometimes a pseudodifferential operator on G/K has a non-unique extension to a pseudodifferential operator on G. If $B_j \in \Psi^{m_j}(G/K)$ has an extension $C_j = (C_j)_{G/K} \in \Psi^{m_j}(G)$ (i.e. $C_j|_{\mathcal{D}(G/K)} = B_j$), then $C_j^* \in \Psi^{m_j}(G)$ is an extension of the adjoint operator $B_j^* \in \Psi^{m_j}(G/K)$, and $B_1 B_2 \in \Psi^{m_1+m_2}(G/K)$ has an extension $C_1 C_2 \in \Psi^{m_1+m_2}(G)$; and if C_1 is elliptic with a parametrix $D \in \Psi^{-m_1}(G)$, then $D = D_{G/K}$ and $B_1 \in \Psi^{m_1}(G/K)$ is elliptic with a parametrix $D|_{\mathcal{D}(G/K)} \in \Psi^{-m_1}(G/K)$.

6 Harmonic analysis on G/K, K a torus

In the sequel we always assume that the subgroup K of G is a torus, $K \cong \mathbb{T}^q$.

Example of special interest: Let $\mathbb{B}^n$ be the unit ball of the Euclidean space $\mathbb{R}^n$, and $\mathbb{S}^{n-1}$ its boundary, the $(n-1)$-sphere. The two-sphere $\mathbb{S}^2$ can be considered as the base space of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$, where the fibers are diffeomorphic to the unit circle $\mathbb{S}^1 \subset \mathbb{R}^2$. In the context of harmonic analysis, $\mathbb{S}^3$ is diffeomorphic to the compact non-commutative Lie group $G = \mathrm{SU}(2)$, having a maximal torus $K \cong \mathbb{S}^1 \cong \mathbb{T}^1$. Then the homogeneous space G/K is diffeomorphic to $\mathbb{S}^2$, so that the canonical projection $p_{G\to G/K} : x \mapsto xK$ is interpreted as the Hopf fiber bundle $G \to G/K$; in the sequel we treat the two-sphere $\mathbb{S}^2$ always as the homogeneous space G/K. Notice that also $\mathbb{S}^2 \cong \mathrm{SO}(3)/\mathbb{T}^1$.

In [6] a subalgebra of $\Psi^m(\mathbb{S}^2)$ was described in terms of so called spherical symbols. Functions $f \in \mathcal{D}(\mathbb{S}^2)$ can be expanded in series

$$f(\phi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \hat{f}(l)_m \; Y_l^m(\phi,\theta), \tag{18}$$

where $(\phi,\theta) \in [0,2\pi] \times [0,\pi]$ are the spherical coordinates, the functions Y_l^m the spherical harmonics with Fourier coefficients

$$\hat{f}(l)_m := \int_0^{\pi} \int_0^{2\pi} f(\phi,\theta) \; \overline{Y_l^m(\phi,\theta)} \sin(\theta) \; d\phi \; d\theta. \tag{19}$$

Let us define

$$(Af)(\phi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a(l) \; \hat{f}(l)_m \; Y_l^m(\phi,\theta), \tag{20}$$

where $a : \mathbb{N}_0 \to \mathbb{C}$ is a rational function; in [6], Svensson states that $A \in \Psi^m(\mathbb{S}^2)$ if and only if

$$|a(l)| \le C_{A,m}(l+1)^m. \tag{21}$$

Let us present another proof for a special case of Theorem 1 and Corollary 2.

Theorem 3. *Let G be a compact Lie group with a torus subgroup K. If $A \in \Psi^m(G)$, then $A_{G/K} \in \Psi^m(G)$ and the restriction $A_{G/K}|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$.*

Proof. Let $\dim(G) = p + q$, $K \cong \mathbb{T}^q$. Let $\mathcal{V} = \{V_i \mid i \in \mathcal{I}\}$ be a locally trivializing open cover of G/K for the principal fiber bundle $(G, G/K, K, x \mapsto xK, (x,k) \mapsto xk)$; Let $\mathcal{U} = \{U_j \mid 1 \le j \le N\}$ be an open cover of G/K such that for every $j_1, j_2 \in \{1,\ldots,N\}$ there exists $V_i \in \mathcal{V}$ containing $U_{j_1} \cup U_{j_2}$ whenever $U_{j_1} \cap U_{j_2} \ne \emptyset$. Notice that we can always refine any open cover on a finite-dimensional manifold to get a new cover satisfying this additional requirement (proving this is easy, see an analogous treatment for partitions of unity in [10]). Then each $U_i \cup U_j$ $(1 \le i,j \le N)$ is a chart neighbourhood on G/K, and furthermore there exist diffeomorphisms $\phi_{ij} : (U_i \cup U_j) \times K \to p_{G\to G/K}^{-1}(U_i \cup U_j)$ such that $p_{G\to G/K}(\phi_{ij}(x,k)) = x$ for every $x \in U_i \cup U_j$ and

$k \in K$. To simplify notation, we treat the neighbourhood $U_i \cup U_j \subset G/K$ as a set $U_i \cup U_j \subset \mathbb{R}^p$, and $p^{-1}_{G\to G/K}(U_i \cup U_j) \subset G$ as a set $(U_i \cup U_j) \times \mathbb{T}^q \subset \mathbb{R}^p \times \mathbb{T}^q$.

Let $\{(U_j, \psi_j) \mid 1 \leq j \leq N\}$ be a partition of unity subordinate to $\mathcal{U}$, and let $A_{ij} = M_{\psi_i} A M_{\psi_j} \in \Psi^m(G)$. With the localized notation we consider $A_{ij} \in \Psi^m(\mathbb{R}^p \times \mathbb{T}^q)$, so that it has the symbol $\sigma_{A_{ij}} \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$. Then

$$\begin{aligned}\sigma_{(A_{G/K})_{ij}}(x,\xi) &= \sigma_{(A_{ij})_{G/K}}(x,\xi) \\ &= \int_{\mathbb{T}^q} \sigma_{A_{ij}}(x_1,\ldots,x_p,x_{p+1}+z_1,\ldots,x_{p+q}+z_q;\xi)\ dz_1\ \cdots\ dz_q,\end{aligned}$$

and it is now easy to check that $\sigma_{(A_{G/K})_{ij}} \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$. This yields $(A_{G/K})_{ij} \in \Psi^m(G)$, thus

$$A_{G/K} = \sum_{i,j} (A_{G/K})_{ij} \in \Psi^m(G)$$

□

Theorem 4. *Let G be a compact Lie group with a torus subgroup K. Let $B \in \Psi^m(G/K)$. Then there exists an operator $A = A_{G/K} \in \Psi^m(G)$ such that $A|_{\mathcal{D}(G/K)} = B$.*

Proof. Let $K \cong \mathbb{T}^q$, $\dim(G) = p + q$, and let $\{(U_j, \psi_j) \mid 1 \leq j \leq N\}$ be the same partition of unity as in the proof of Theorem 3. Let $B_{ij} = M_{\psi_i} B M_{\psi_j} \in \Psi^m(G/K)$. With the localized notation we consider $B_{ij} \in \Psi^m(\mathbb{R}^p)$, so that it has the symbol $\sigma_{B_{ij}} : \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{C}$, and the mapping $(x,\xi) \mapsto \sigma_{B_{ij}}(x,\xi)$ is zero when $x \in \mathbb{R}^p \setminus (U_i \cup U_j)$. We use Lemma 5 in Appendix to construct a pseudodifferential operator $A_{ij} \in \Psi^m(\mathbb{R}^p \times \mathbb{T}^q)$ such that $\sigma_{A_{ij}} : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q) \to \mathbb{C}$,

$$\sigma_{A_{ij}}(x; P\xi, 0, \ldots, 0) = \sigma_{B_{ij}}(Px; P\xi),$$

where $Py = (y_1, \ldots, y_p)$ $(y \in \mathbb{R}^{p+q})$. Hence $A = A_{G/K} = \sum_{i,j} A_{ij} \in \Psi^m(G)$ and $A|_{\mathcal{D}(G/K)} \in \Psi^m(G/K)$. Let $f = \sum_k f_k \in C^\infty(G/K) \subset C^\infty(G)$, $f_k = f\psi_k$; then

$$\begin{aligned}(Af)(x) &= \sum_{i,j,k} (A_{ij} f_k)(x) \\ &= \sum_{i,j,k} \int_{\mathbb{R}^p} \sum_{\xi_{p+1},\ldots,\xi_{p+q}\in\mathbb{Z}} \sigma_{A_{ij}}(x,\xi)\ \hat{f}_k(\xi)\ e^{i2\pi x\cdot\xi}\ d\xi_1 \cdots d\xi_p \\ &= \sum_{i,j,k} \int_{\mathbb{R}^p} \sigma_{A_{ij}}(x; P\xi, 0, \ldots, 0)\ \hat{f}_k(P\xi, 0, \ldots, 0)\ e^{i2\pi(Px)\cdot(P\xi)}\ d\xi_1 \cdots d\xi_p \\ &= \sum_{i,j,k} \int_{\mathbb{R}^p} \sigma_{B_{ij}}(Px; P\xi)\ \hat{f}_k(P\xi, 0, \ldots, 0)\ e^{i2\pi(Px)\cdot(P\xi)}\ d\xi_1 \cdots d\xi_p \\ &= \sum_{i,j,k} (B_{ij} f_k)(Px) \\ &= (Bf)(xK)\end{aligned}$$

□

7 Discussion

Theorem 4 combined with Lemma 5 provides just one way of extending operators, unfortunately destroying ellipticity: this is due to the apparent non-ellipticity of the symbol χ in Lemma 5. Let us discuss this problem and provide other extensions.

Let us extend the identity operator $I \in \Psi^0(\mathbb{R}^p)$ using the process suggested by Lemma 5. Of course, it would be desirable if $I \in \Psi^0(\mathbb{R}^p)$ could be extended to the identity in $\Psi^0(\mathbb{R}^{p+q})$, but now $\sigma_I(x,\xi) \equiv 1$, and thereby its extension $A \in \Psi^0(\mathbb{R}^{p+q})$ has the non-elliptic homogeneous symbol $\sigma_A = \chi \in S^0(\mathbb{R}^{p+q})$.

Given an elliptic symbol $\sigma_B \in S^m(\mathbb{R}^p)$ we can occasionally modify the construction in Lemma 5 to get an extended elliptic symbol in $S^m(\mathbb{R}^{p+q})$. Sometimes the following trick helps: Let $\sigma_{A_1} \in S^m(\mathbb{R}^{p+q})$ be an extension of σ_{B_1} as in Lemma 5,

$$\sigma_{A_1}(x,\xi) = \chi_1(\xi)\ \sigma_{B_1}(x_1,\ldots,x_p;\xi_1,\ldots,\xi_p),$$

where $\chi_1 \in S^0(\mathbb{R}^{p+q})$ is a homogeneous symbol satisfying $\chi_1|_{(U\times\mathbb{R}^q)\setminus\mathbb{B}(0,1)} \equiv 0$, $\chi_1|_{\mathbb{R}^p\times V} \equiv 1$, where $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^q$ are neighborhoods of zeros. Take any elliptic symbol $\sigma_{B_2} \in S^m(\mathbb{R}^q)$, and modify Lemma 5 to construct an extension $\sigma_{A_2} \in S^m(\mathbb{R}^{p+q})$ such that

$$\sigma_{A_2}(x,\xi) = \chi_2(\xi)\ \sigma_{B_2}(x_p,\ldots,x_{p+q};\xi_p,\ldots,\xi_{p+q})$$

for a homogeneous symbol $\chi_2 \in S^0(\mathbb{R}^{p+q})$ satisfying $\chi_2|_{(U\times\mathbb{R}^q)\setminus\mathbb{B}(0,1)} \equiv 1$, $\chi_2|_{(\mathbb{R}^p\times V)\setminus\mathbb{B}(0,1)} \equiv 0$. Then $\sigma_{A_1} + \sigma_{A_2} \in S^m(\mathbb{R}^{p+q})$ is an extension for σ_{B_1} (modulo infinitely smoothing operators). For instance, if $B_1 = I \in \Psi^0(\mathbb{R}^p)$, let $B_2 = I \in \Psi^0(\mathbb{R}^q)$ and $\chi_2(\xi) = 1 - \chi_1(\xi)$ (for $|\xi| > 1$), then $A_1 + A_2 = I \in \Psi^0(\mathbb{R}^{p+q})$ (modulo infinitely smoothing operators).

It may happen that any extension process for an elliptic symbol $\sigma_B \in S^m(\mathbb{R}^p)$ constructs a non-elliptic symbol in $S^m(\mathbb{R}^{p+q})$. Consider, for instance, a case where $B \in \Psi^m(\mathbb{R}^2)$ is an elliptic convolution operator and $\xi \mapsto f(\xi) \equiv \sigma_B(x,\xi)$ is homogeneous outside the unit ball $\mathbb{B}(0,1) \subset \mathbb{R}^2$. If the mapping $f|_{\mathbb{S}^1} : \mathbb{S}^1 \to \mathbb{C}\setminus\{0\}$ is not homotopic to a constant mapping (i.e. $f|_{\mathbb{S}^1}$ has a non-zero winding number) then no extension $\sigma_A \in S^m(\mathbb{R}^3)$ of σ_B can be elliptic.

Multiplications on G/K have already been extended to multiplications G via $x \mapsto xK$, and $A = A_{G/K}$ for any left convolution operator (multiplier) $A \in \mathcal{L}(\mathcal{D}(G))$ (in fact, then $\sigma_A(x) = A$ for every $x \in G$). Sometimes on G/K we have operators that resemble convolution operators. Suppose we are given a left convolution operator $A \in \Psi^m(\mathrm{SU}(2))$. Then the restriction $B = A|_{\mathcal{D}(\mathbb{S}^2)} \in \Psi^m(\mathbb{S}^2)$ is of the form

$$(Bf)(\phi,\theta) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(\sum_{n=-l}^{l} a(l)_{mn}\ \hat{f}(l)_n\right)\ Y_l^m(\phi,\theta), \tag{22}$$

where the coefficients $a(l)_{mn} \in \mathbb{C}$ can be calculated from the data

$$\{BY_l^m \mid l \in \mathbb{N}_0,\ m \in \{-l, -l+1, \dots, l-1, l\}\}.$$

It is even true that the original operator A can be retrieved from the coefficients $a(l)_{mn}$. In fact, any operator $B \in \mathcal{L}(\mathcal{D}(\mathbb{S}^2))$ of the form (22) can be extended to a unique left convolution operator belonging to $\mathcal{L}(\mathcal{D}(\mathrm{SU}(2)))$. Now a natural question arises: given a pseudodifferential operator $B \in \Psi^m(\mathbb{S}^2)$ of the form (22), does its extension to the left convolution operator belong to $\Psi^m(\mathrm{SU}(2))$? This is an open problem. An interesting special case is

$$(Bf)(x) = \int_{\mathbb{S}^2} \kappa(x \cdot y)\ f(y)\ dy, \tag{23}$$

where $\kappa \in \mathcal{D}'(\mathbb{S}^2)$, $(x, y) \mapsto x \cdot y$ is the scalar product of $\mathbb{R}^3$, and the integration is with respect to the angular part of the Lebesgue measure of $\mathbb{R}^3$. Then

$$(Bf)(\phi, \theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} c_l\ \hat{\kappa}(l)_0\ \hat{f}(l)_m\ Y_l^m(\phi, \theta)$$

for some normalizing constants c_l depending only on $l \in \mathbb{N}_0$.

8 Appendix

Lemma 5. *Let $\chi \in C^\infty(\mathbb{R}^{p+q})$ be homogeneous of order 0 in $\mathbb{R}^{p+q} \setminus \mathbb{B}(0,1)$, i.e. $\chi(\xi) = \chi(\xi/\|\xi\|)$ when $\|\xi\| \geq 1$. Furthermore, assume that χ satisfies $\chi|_{(U \times \mathbb{R}^q) \setminus \mathbb{B}(0,1)} \equiv 0$, $\chi|_{\mathbb{R}^p \times V} \equiv 1$, where $U \subset \mathbb{R}^p$ and $V \subset \mathbb{R}^q$ are neighborhoods of zeros. Let $\sigma_B \in S^m(\mathbb{R}^p)$ and*

$$\sigma_A(x, \xi) := \chi(\xi)\ \sigma_B(Px, P\xi),$$

where $P(x_1, \dots, x_{p+q}) = (x_1, \dots, x_p)$. Then $\sigma_A \in S^m(\mathbb{R}^{p+q})$. Moreover, $\sigma_A|_{(\mathbb{R}^p \times \mathbb{R}^q) \times (\mathbb{R}^p \times \mathbb{Z}^q)} \in S^m(\mathbb{R}^p \times \mathbb{T}^q)$.

Proof. We shall first prove that

$$|(\partial_\xi^\gamma \chi)(\xi)| \leq C_{\gamma r}\ \langle P\xi \rangle^{-r}\ \langle \xi \rangle^{r - |\gamma|} \tag{24}$$

for every $r \in \mathbb{R}$ and for every $\gamma \in \mathbb{N}_0^{p+q}$. It is trivial that $(x, \xi) \mapsto \chi(\xi)$ belongs to $S^0(\mathbb{R}^{p+q})$. If $r \geq 0$ then obviously (24) is true. Since we are not interested in the behaviour of the symbols when $\|\xi\|$ is small, we assume that $\|\xi\| > 1$ from here on. There exists $r_0 \in (0,1)$ such that $\chi(\xi) = 0$ when $\|P\xi\| < r_0$. Let $r < 0$ and $\xi \in \mathrm{supp}(\chi)$. Then $\|P\xi\| \geq r_0\|\xi\|$, and thus

$$\begin{aligned}
|(\partial_\xi^\gamma \chi)(\xi)| &\leq C_\gamma\ \langle \xi \rangle^{-|\gamma|} \\
&= C_\gamma\ \langle P\xi \rangle^{-r}\ \langle P\xi \rangle^{r}\ \langle \xi \rangle^{-|\gamma|} \\
&\leq C_\gamma\ \langle P\xi \rangle^{-r}\ \langle r_0 \xi \rangle^{r}\ \langle \xi \rangle^{-|\gamma|} \\
&\leq C_\gamma\ r_0^r\ \langle P\xi \rangle^{-r}\ \langle \xi \rangle^{r - |\gamma|}.
\end{aligned}$$

Hence the inequality (24) is proven. Now

$$
\begin{aligned}
|\partial_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| &\le \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \; |(\partial_\xi^\gamma \chi)(\xi)| \; |(\partial_\xi^{\alpha-\gamma} \partial_x^\beta \sigma_B)(Px, P\xi)| \\
&\le \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \; C_{\gamma r_\gamma} \; \langle P\xi\rangle^{-r_\gamma} \; \langle \xi\rangle^{r_\gamma - |\gamma|} \; C_{B(\alpha-\gamma)\beta m} \; \langle P\xi\rangle^{m-|\alpha-\gamma|} \\
&\le C_{B\alpha\beta m\chi} \; \langle \xi\rangle^{m-|\alpha|},
\end{aligned}
$$

if we choose $r_\gamma = m - |\alpha-\gamma|$. Thereby $\sigma_A \in S^m(\mathbb{R}^{p+q})$. Clearly we can consider this symbol as a function $\sigma_A : (\mathbb{R}^p \times \mathbb{T}^q) \times (\mathbb{R}^p \times \mathbb{R}^q) \to \mathbb{C}$ and study its restriction $\sigma_A|_{(\mathbb{R}^p\times\mathbb{T}^q)\times(\mathbb{R}^p\times\mathbb{Z}^q)}$ we claim that this restriction belongs to $S^m(\mathbb{R}^p \times \mathbb{T}^q)$. Indeed, Taylor expansion of a function $\sigma \in C^\infty(\mathbb{R}^q)$ yields

$$
\begin{aligned}
\triangle_\xi^\gamma \sigma(\xi) &= \sum_{\delta\le\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|} \; \sigma(\xi+\delta) \\
&= \sum_{\delta\le\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|} \\
&\quad \times \left(\sum_{|\rho|<|\gamma|} \frac{1}{\rho!}\delta^\rho \; (\partial_\xi^\rho \sigma)(\xi) + \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \; (\partial_\xi^\rho \sigma)(\xi+\theta_\delta \delta) \right) \\
&= \sum_{|\rho|<|\gamma|} \frac{1}{\rho!} \; (\partial_\xi^\rho \sigma)(\xi) \sum_{\delta\le\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|} \delta^\rho \\
&\quad + \sum_{\delta\le\gamma} \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \; (\partial_\xi^\rho \sigma)(\xi+\theta_\delta \delta) \\
&= \sum_{\delta\le\gamma} \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \; (\partial_\xi^\rho \sigma)(\xi+\theta_\delta \delta),
\end{aligned}
$$

because

$$
\sum_{\delta\le\gamma} \binom{\gamma}{\delta} (-1)^{|\gamma-\delta|} \delta^\rho = \triangle_\xi^\gamma \xi^\rho|_{\xi=0} = 0
$$

whenever $|\rho| < |\gamma|$. Therefore

$$
\begin{aligned}
|\triangle_\xi^\gamma \sigma(\xi)| &\le \sum_{\delta\le\gamma} \sum_{|\rho|=|\gamma|} \frac{1}{\rho!}\delta^\rho \; |(\partial_\xi^\rho \sigma)(\xi+\theta_\delta \delta)| \\
&\le c_\gamma \sup_{\eta\in S_\gamma, |\rho|=|\gamma|} |(\partial_\xi^\rho \sigma)(\xi+\eta)|,
\end{aligned}
$$

where S_γ is the hyper-rectangle $\prod_{j=1}^q [0,\gamma_j]$. Let $\alpha' = (P\alpha, 0, \ldots, 0)$, $\alpha'' = \alpha - \alpha'$; then

$$
\begin{aligned}
|\partial_\xi^{\alpha'} \triangle_\xi^{\alpha''} \partial_x^\beta \sigma_A(x,\xi)| &\le C_\alpha \sup_{\eta\in S_{\alpha''}, |\rho|=|\alpha''|} |\partial_\xi^{\alpha'+\rho} \partial_x^\beta \sigma_A(x, \xi+\eta)| \\
&\le C_\alpha \; C_{A\alpha\beta m} \sup_{\eta\in S_\alpha} \langle \xi+\eta\rangle^{m-|\alpha|}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C_\alpha\, C_{A\alpha\beta m}\, 2^{|m-|\alpha||} \sup_{\eta\in S_\alpha} \langle\eta\rangle^{|m-|\alpha||}\, \langle\xi\rangle^{m-|\alpha|} \\
&\leq C_\alpha\, C_{A\alpha\beta m}\, 2^{|m-|\alpha||}\, \langle\alpha\rangle^{|m-|\alpha||}\, \langle\xi\rangle^{m-|\alpha|} \\
&= C'_{A\alpha\beta m}\, \langle\xi\rangle^{m-|\alpha|};
\end{aligned}
$$

notice the application of the Peetre inequality

$$\langle\xi+\eta\rangle^s \leq 2^{|s|}\, \langle\xi\rangle^s\, \langle\eta\rangle^{|s|}.$$

Hence $\sigma_A|_{(\mathbb{R}^p\times\mathbb{T}^q)\times(\mathbb{R}^p\times\mathbb{Z}^q)} \in S^m(\mathbb{R}^p\times\mathbb{T}^q)$ □

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