Helsinki University of Technology Institute of Mathematics Research Reports
Teknillisen korkeakoulun matematiikan laitoksen tutkimusraporttisarja
Espoo 1999 A415

SMOOTH OPERATOR-VALUED SYMBOL ANALYSIS

Ville Turunen

TEKNILLINEN KORKEAKOULU
TEKNISKA HÖGSKOLAN
HELSINKI UNIVERSITY OF TECHNOLOGY
TECHNISCHE UNIVERSITÄT HELSINKI
UNIVERSITE DE TECHNOLOGIE D'HELSINKI

Ville Turunen: Smooth Operator-Valued Symbol Analysis; Helsinki University of Technology Institute of Mathematics Research Reports A415 (1999).

Abstract: *In this paper periodic pseudodifferential operators, i.e. pseudodifferential operators on the torus* $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ *are studied by exploiting the Fourier series representation. Using a generalized freezing coefficients technique, we represent a periodic pseudodifferential operator as a smooth convolution operator-valued function on* $\mathbb{T}^n$*; consequently, we transform the traditional symbol inequalities into analogous operator norm inequalities. It is known that pseudodifferential operators are characterized by the Sobolev orders of an infinite family of commutators, i.e. by the continuity properties in the Sobolev scale; furthermore, the Sobolev order of a pseudodifferential operator is bounded by the order of the pseudodifferential symbol. Within this theme, also order and continuity properties are studied. Then we examine a non-commutative operator-valued Fourier transform of distributions on a compact non-commutative Lie group. Applying this, we introduce and study the symbols of continuous linear operators (pseudodifferential operators) acting on the distribution space.*

AMS subject classifications: Primary 35S05, 43A30, secondary 22E30, 42A20, 43A75, 43A77, 46E40, 46F05, 47A05, 47A56, 47G30, 58G15.

Keywords: Pseudodifferential operators, Fourier series, symbol analysis, operator-valued functions, non-commutative Fourier transform, Fourier transforms of distributions, quasidifference operators, asymptotic expansions, Schwartz kernels, commutators.

ISBN 951-22-4554-X
ISSN 0784-3143
Libella Painopalvelu, Espoo, 1999

Helsinki University of Technology
Department of Engineering Physics and Mathematics
Institute of Mathematics
P.O. Box 1100, 02015 HUT, Finland
email: *math@hut.fi*
downloadables: *http://www.math.hut.fi/*

author's email: *Ville.Turunen@hut.fi*

1 Introduction

In its present form, the theory of pseudodifferential operators originates from the works of Hörmander and others in the 1960's. These operators form algebras naturally generalizing the behaviour of linear partial differential operators on manifolds, and the theory is an indispensable tool in the analysis of elliptic equations. The starting point in the theory is the Fourier transform in $\mathbb{R}^n$, and there is a one-to-one correspondence between the operators and so called symbol functions, which can be seen as weights for the inverse Fourier transform; the problems in a non-commutative operator algebra are transfered to a commutative function algebra. The behaviour of symbols is restricted by symbol inequalities, imitating essential properties of partial differential operators.

In this paper, following Agranovich [1], we examine periodic pseudodifferential operators, or pseudodifferential operators on the torus $\mathbb{T}^n$: We do not appeal to the common analysis on a manifold using the local charts and Hörmander's theory on $\mathbb{R}^n$, but instead we replace the Fourier integrals by the Fourier series, so that a global analysis is possible. Again, any operator possesses a unique symbol function, and we have natural global symbol inequalities.

First, we briefly present some basic background of periodic pseudodifferential operators. In the second section, we introduce the method of freezing the symbol of a linear operator on $C^\infty(\mathbb{T}^n)$. This procedure realizes the operator as a mapping on $\mathbb{T}^n$, with the values in translation invariant operators (convolutions). Then we lift the traditional symbol inequalities to corresponding norm inequalities for the new operator symbols. The motivation behind this treatment is its applicability in the non-commutative harmonic analysis (see Coda section). In the third part of the work, various relations between the Sobolev continuity and the pseudodifferential order of operators are studied.

Eventually, we briefly present the outlines of the non-commutative symbol analysis; we will give details in a future paper.

1.1 Pseudodifferential prerequisites

For general calculus of pseudodifferential operators, see e.g. [9] or [16].

Periodic pseudodifferential operators. Let $\mathbb{T}^n$ be the n-dimensional torus, i.e. $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$; it is natural to identify $\mathbb{T}^n$ with the n-fold Cartesian product of the interval $[0,1) \subset \mathbb{R}$. Function $\sigma_A : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ is called a *periodic symbol of order* $m \in \mathbb{R}$, if it is C^∞-smooth and if it satisfies the *periodic symbol inequalities*

$$\left|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)\right| \leq C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}; \tag{1}$$

here α, β are multi-indices in $\mathbb{N}_0^n$, $\mathbb{N}_0 = \{0,1,2,\ldots\}$; the common partial differential operators are $\partial_x^\beta = (\partial/\partial x_1)^{\beta_1} \cdots (\partial/\partial x_n)^{\beta_n}$; likewise, $\triangle_\xi^\alpha$ is the partial difference operator $\triangle_{\xi_1}^{\alpha_1}\cdots\triangle_{\xi_n}^{\alpha_n}$, where

$$\triangle_{\xi_j} f(x) = f(x+\delta_j) - f(x)$$

(with $\delta_j = (\delta_{jk})_{k=1}^n \in \mathbb{Z}^n$, δ_{jk} being the Kronecker delta). Moreover, $|\xi|^2 = |(\xi_1,\ldots,\xi_n)|^2 = \sum_{j=1}^n \xi_j^2$, $|\alpha| = |(\alpha_1,\ldots,\alpha_n)| = \sum_{j=1}^n \alpha_j$, and $C_{\alpha\beta}$ are constants depending on α and β. The class of such symbols is denoted by $S^m(\mathbb{T}^n)$. The symbol $\sigma_A \in S^m(\mathbb{T}^n)$ defines a linear operator $A : C^\infty(\mathbb{T}^n) \to C^\infty(\mathbb{T}^n)$ by

$$(Af)(x) = \sum_{\xi\in\mathbb{Z}^n} \sigma_A(x,\xi)\hat{f}(\xi)e^{i2\pi x\cdot\xi}, \tag{2}$$

where $x\cdot\xi = \sum_{j=1}^n x_j\xi_j$, and $\hat{f} : \mathbb{Z}^n \to \mathbb{C}$ is the Fourier transform of $f \in C^\infty(\mathbb{T}^n)$,

$$\hat{f}(\xi) = \int_{\mathbb{T}^n} f(x)e^{-i2\pi x\cdot\xi}\,dx.$$

Then A is called a *periodic pseudodifferential operator of order* $m \in \mathbb{R}^n$, denoted by $A \in \mathrm{Op}S^m(\mathbb{T}^n)$. Periodic pseudodifferential operators are continuous in the test function space (which is the set $C^\infty(\mathbb{T}^n)$ equipped with the natural Fréchet structure). Note that

$$\sigma_A(x,\xi) = e^{-i2\pi x\cdot\xi}(Ae_\xi)(x),$$

where $e_\xi(x) = e^{i2\pi x\cdot\xi}$.

Example: Any partial differential operator A on $\mathbb{T}^n$, given by

$$(Au)(x) = \sum_{|\alpha|\le m} c_\alpha(x)\partial_x^\alpha u(x),$$

with smooth coefficient functions c_α, is a periodic pseudodifferential operator of order m. Its symbol is $\sigma_A(x,\xi) = \sum_{|\alpha|\le m} c_\alpha(x)(i2\pi\xi)^\alpha$.

Pseudodifferential operators on $\mathbb{R}^n$. The pseudodifferential operators on Euclidean spaces are defined analogously: *Symbol of order (or degree)* $m \in \mathbb{R}$ *on* $\mathbb{R}^n$ is a C^∞-function $\sigma_A : \mathbb{R}^n\times\mathbb{R}^n \to \mathbb{C}$ satisfying the *symbol inequalities*

$$\left|\partial_\xi^\alpha\partial_x^\beta\sigma_A(x,\xi)\right| \le C_{\alpha\beta}(1+|\xi|)^{m-|\alpha|}. \tag{3}$$

uniformly in x. The set of these symbols is $S^m(\mathbb{R}^n)$, and σ_A defines a *pseudodifferential operator* $A : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ *of order* $m \in \mathbb{R}$ by

$$(Af)(x) = \int_{\mathbb{R}^n} \sigma_A(x,\xi)\hat{f}(\xi)e^{i2\pi x\cdot\xi}\,d\xi; \tag{4}$$

here $\mathcal{S}(\mathbb{R}^n)$ is the space of the Schwartz test functions, and $\hat{f}$ is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$. The set of mth order pseudodifferential operators is now denoted by $\mathrm{Op}S^m(\mathbb{R}^n)$, and these operators are continuous in the Schwartz topology. By duality, pseudodifferential operators on $\mathbb{R}^n$ extend to continuous linear operators on the space of tempered distributions; again, one recovers the symbol from the operator by

$$\sigma_A(x,\xi) = e^{-i2\pi x\cdot\xi}(Ae_\xi)(x).$$

Pseudodifferential operators on manifolds. On a compact smooth manifold M, the class $\mathrm{Op}S^m(\mathbb{R}^n)$ gives naturally rise to a class $\Psi^m(M)$ via charts. Then a pseudodifferential operator $A \in \Psi^m(M)$ is continuous on the test function space $\mathcal{D}(M)$. In fact, it turns out that $\mathrm{Op}S^m(\mathbb{T}^n) = \Psi^m(\mathbb{T}^n)$ (see [2], [3], [12], [13], [20]). Periodic pseudodifferential operators offer obvious technical advantages over the differential geometric approach involved in the direct application of the definition of $\Psi^m(\mathbb{T}^n)$.

Pseudodifferential operators form an involutive algebra. More precisely, if $A_j \in \Psi^{m_j}(M)$ $(j = 1, 2)$, $\lambda \in \mathbb{C}$, then $A_1 + A_2 \in \Psi^{\max\{m_1,m_2\}}(M)$, $\lambda A_j \in \Psi^{m_j}(M)$, $A_j^* \in \Psi^{m_j}(M)$ and $A_1A_2 \in \Psi^{m_1+m_2}(M)$. An interesting nuance of the theory is that the operators satisfy a kind of generalized Leibniz property for commutators: that is,

$$[A_1, A_2] = A_1A_2 - A_2A_1 \in \Psi^{m_1+m_2-1}(M). \tag{5}$$

1.2 Continuity of pseudodifferential operators

Any $A \in \mathrm{Op}S^m(\mathbb{R}^n)$ extends to a bounded linear operator between the Sobolev spaces $H^s(\mathbb{R}^n)$ and $H^{s-m}(\mathbb{R}^n)$ for every $s \in \mathbb{R}$. On a compact manifold M, Sobolev spaces $H^s(M)$ can be defined via charts, as usual. A linear operator A defined on $\mathcal{D}(M)$ is said to be of *Sobolev order* $m \in \mathbb{R}$, if it extends to a continuous operator between $H^s(M)$ and $H^{s-m}(M)$ for any $s \in \mathbb{R}$; pseudodifferential operators in $\Psi^m(M)$ are an example of this. The infimum of the Sobolev orders (possibly $-\infty$) of A is said to be the *true Sobolev order* of A [23]. The collection $\{H^s(M)\}_{s\in\mathbb{R}}$ is called the *Sobolev scale on* M. If $s < t$, then the inclusion $H^t(M) \subset H^s(M)$ is compact. Moreover, $\cap_{s\in\mathbb{R}} H^s(M) = C^\infty(M)$, and $\cup_{s\in\mathbb{R}} H^s(M) \cong \mathcal{D}'(M) = \mathcal{L}(\mathcal{D}(M), \mathbb{C})$.

Sobolev spaces on $\mathbb{T}^n$ admit a particularly simple description: the Hilbert space $H^s(\mathbb{T}^n)$ is the completion of the test function space $C^\infty(\mathbb{T}^n)$ in the Sobolev norm $\|\cdot\|_{H^s(\mathbb{T}^n)}$ given by the Sobolev inner product

$$(f, g)_{H^s(\mathbb{T}^n)} = \sum_{\xi\in\mathbb{Z}^n} (1 + |\xi|)^{2s} \hat{f}(\xi)\overline{\hat{g}(\xi)}.$$

Using this definition and the periodic symbol inequalities, one can directly prove the Sobolev continuity of periodic pseudodifferential operators (see [18] or [20]; for another proof, see [22]).

Example: The canonical Sobolev space isomorphism $\varphi_m : H^s(\mathbb{T}^n) \to H^{s+m}(\mathbb{T}^n)$ given by

$$(\varphi_m u)(x) = \sum_{\xi\in\mathbb{Z}^n} (1 + |\xi|)^{-m} \hat{u}(\xi) e^{i2\pi x\cdot\xi} \tag{6}$$

belongs to $\Psi^m(\mathbb{T}^n)$.

Commutator characterization. Actually, the Sobolev continuity of the commutators with differential operators characterizes the pseudodifferential operators completely; see [4], [7], [5], [6], [20] (see also [17]. More precisely, a linear operator A_0 on $C^\infty(M)$ is a pseudodifferential operator of order $m \in \mathbb{R}$, if and only if $A_0 \in \mathcal{L}(H^s(M), H^{s-m}(M))$ $(s \in \mathbb{R})$ and

$$A_{k+1} = [A_k, D_{k+1}] \in \mathcal{L}\left(H^s(M), H^{s-m+k-\sum_{j=1}^{k} \deg(D_j)}(M)\right)$$

for every sequence of differential operators $(D_k)_{k=0}^{\infty}$, each D_k being of order $\deg(D_k) \in \mathbb{N}_0$ with smooth coefficients; multiplication operators are considered as differential operators of order 0.

2 Operator symbols

Freezing symbols. When examining a non-translation-invariant elliptic partial differential equation on a domain in a Euclidean space, the method of freezing coefficients provides us with the techniques of constant coefficient case. This means that we study the set of translation invariant (that is, constant coefficient) partial differential operators obtained by fixing a point in the domain and evaluating the coefficient functions there; the set is parametrized by the points of the domain. More precisely, the operator A given by

$$(Au)(x) = \sum_{|\alpha| \leq k} c_\alpha(x)(\partial_x^\alpha u)(x)$$

gives rise to an operator-valued mapping Σ_A defined on the domain by

$$(\Sigma_A(x_0)u)(x) = \sum_{|\alpha| \leq k} c_\alpha(x_0)(\partial_x^\alpha u)(x);$$

here $\Sigma_A(x_0)$ is a translation-invariant operator approximating A near x_0.

Accordingly, we define the *freezing* of an operator $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$ at a point $x_0 \in \mathbb{T}^n$ to be the formal convolution operator $\Sigma_A(x_0)$ with the symbol

$$\sigma_{\Sigma_A(x_0)}(x, \xi) = \sigma_A(x_0, \xi).$$

Provided that $\Sigma_A(x) \in \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$ for every $x \in \mathbb{T}^n$, we call the mapping

$$\Sigma_A : \mathbb{T}^n \to \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$$

the *operator symbol of* A; there should be no chance of confusing this with the symbol function $\sigma_A : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$.

The idea of the freezing is to study the family $\{\Sigma_A(x)\}_{x \in \mathbb{T}^n}$ of convolution operators (parametrized by the points of the manifold $\mathbb{T}^n$) instead of a complicated operator A. A natural question arises: as the function σ_A gives rise to the mapping Σ_A, does there exist an analogue of the traditional symbol inequalities (1) in the realm of operator symbols? Now we pursue this end.

Operations on operator symbols. If $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$, its symbol function $\sigma_A : \mathbb{T}^n \times \mathbb{Z}^n \to \mathbb{C}$ is trivially C^∞-smooth, and a simple calculation yields

$$\sigma_{[\partial_{x_j},A]}(x,\xi) = \partial_{x_j}\sigma_A(x,\xi).$$

Hence if A has the operator symbol Σ_A, we define its partial derivative $\partial_{x_j}\Sigma_A : \mathbb{T}^n \to \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$ by

$$\left(\partial_{x_j}\Sigma_A\right)(x) = \Sigma_{[\partial_{x_j},A]}(x). \tag{7}$$

Let X be a Banach space and $f : \mathbb{T}^n \to X$. If the limit

$$f^{(\delta_j)}(x) = \lim_{h\to 0} \frac{f(x+h\delta_j) - f(x)}{h}$$

exists for exist in X, we call it the jth partial differential of f at $x \in \mathbb{T}^n$, and the definitions of $f^{(\beta)}(x)$ $(\beta \in \mathbb{N}_0^n)$ and $C^\infty(\mathbb{T}^n, X)$ become evident. If X is a space of bounded linear operators between two suitable Sobolev spaces and $f = \Sigma_A$ and if the difference quotient limit exists, then $f^{(\delta_j)}(x) = (\partial_{x_j} f)(x)$.

Just as the commutators with partial derivatives inspired the differentiation of the operator symbol, similarly the commutators with multiplications give rise to operations on operator symbol. Namely,

$$\sigma_{[A,e_{\xi_k}]}(x,\xi) = e_{\xi_k}(x)\triangle_{\xi_k}\sigma_A(x,\xi);$$

hence we define operators Q^α $(\alpha \in \mathbb{N}_0)$ acting on $\mathcal{L}(\mathcal{D}(\mathbb{T}^n))$ by

$$\sigma_{Q^\alpha A}(x,\xi) = \triangle_\xi^\alpha \sigma_A(x,\xi).$$

Convention. In order to make things clearer in the sequel, without losing generality, we prove some of the results on $\mathbb{T}^1$ instead of $\mathbb{T}^n$; the generalizations are, however, straightforward.

2.1 Minor observations

Lemma 2.1 *Let $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^1))$ be a convolution operator $f \mapsto k * f$ (that is $\sigma_A(x,\xi) = \hat{k}(\xi)$) such that $A \in \mathcal{L}(H^s(\mathbb{T}^1), H^{s-m}(\mathbb{T}^1))$. Then*

$$\|A\|_{\mathcal{L}(H^s(\mathbb{T}^1),H^{s-m}(\mathbb{T}^1))} = \sup_{\xi\in\mathbb{Z}^1}(1+|\xi|)^{-m}|\hat{k}(\xi)|.$$

The proof of this auxiliary result is simple and thus omitted. But does it even follow from $AH^s(\mathbb{T}^1) \subset H^{s-m}(\mathbb{T}^1)$ that $A \in \mathcal{L}(H^s(\mathbb{T}^1), H^{s-m}(\mathbb{T}^1))$? Indeed, Corollary 2.2 yields the affirmative answer.

The proof of the next well-known theorem can be found in [10, p. 290–291].

Theorem 2.1 (Abel–Dini) *Let d_j be positive numbers and $D_N = \sum_{j=1}^N d_j$. Assume that $(D_N)_{N=1}^\infty$ is divergent. Then $\sum_{j=1}^\infty d_j/D_j^m$ diverges exactly when $m \le 1$.*

Corollary 2.1 *If $(p_j)_{j=1}^\infty$ is a monotone sequence of positive real numbers diverging to infinity, then there is a non-negative sequence $(c_j)_{j=1}^\infty$ such that the series $\sum_{j=1}^\infty c_j$ converges, but $\sum_{j=1}^\infty p_j c_j$ diverges.*

Proof. (A modification of [10, p. 302].) Define $d_1 = p_1$ and $d_{j+1} = p_{j+1} - p_j$. Then, in the notation of Abel–Dini theorem, $D_N = \sum_{j=1}^{N} d_j = p_N \to \infty$, and $\sum_{j=1}^{\infty} d_j/D_j = 1 + \sum_{j=1}^{\infty}(p_{j+1} - p_j)/p_{j+1}$ diverges. Let us define $c_j = (p_{j+1} - p_j)/(p_{j+1}\, p_j)$. Then $\sum_{j=1}^{\infty} c_j$ converges, because $1/p_j \to 0$:

$$\sum_{j=1}^{\infty} c_j = \sum_{j=1}^{\infty} \left(\frac{1}{p_j} - \frac{1}{p_{j+1}} \right) = \frac{1}{p_1}.$$

Clearly, $\sum_{j=1}^{\infty} p_j c_j = \sum_{j=1}^{\infty}(p_{j+1} - p_j)/p_{j+1}$ diverges □

Hence, the following result concerning convolutions can be said:

Lemma 2.2 *If $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^1))$ is a convolution operator $f \mapsto k * f$, and if $\forall C \in \mathbb{R}\ \exists \xi \in \mathbb{Z}^1 : |\hat{k}(\xi)| > C(1 + |\xi|)^m$, then $AH^s(\mathbb{T}^1) \not\subset H^{s-m}(\mathbb{T}^1)$ $(s \in \mathbb{R})$.*

Proof. If $\forall C > 0\ \exists \xi \in \mathbb{Z}^1 : |\hat{k}(\xi)| > C(1 + |\xi|)^m$, there is a subsequence of $((1 + |\xi|)^{-2\beta} |\hat{k}(\xi)|^2)_{\xi \in \mathbb{Z}^1}$ that converges to ∞ as $|\xi| \to \infty$. Corollary 2.1 then provides the existence of a sequence $(\hat{u}(\xi))_{\xi \in \mathbb{Z}^1}$ for which $\sum_\xi (1 + |\xi|)^{2s} |\hat{u}(\xi)|^2$ converges, but $\sum_\xi (1 + |\xi|)^{2(s-m)} |\hat{k}(\xi)\hat{u}(\xi)|^2$ diverges. Thus $u \in H^s(\mathbb{T}^1)$, and it is mapped to $Au \notin H^{s-m}(\mathbb{T}^1)$ □

Corollary 2.2 *Let A be a convolution operator with $\sigma_A(x, \xi) = \hat{k}(\xi)$. Then the following conditions are equivalent:*

(1) $|\hat{k}(\xi)| = \mathcal{O}((1 + |\xi|)^m)$;

(2) $A \in \mathcal{L}(H^s(\mathbb{T}^1), H^{s-m}(\mathbb{T}^1))$;

(3) $AH^s(\mathbb{T}^1) \subset H^{s-m}(\mathbb{T}^1)$.

Proof. Implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial. And if A maps $H^s(\mathbb{T}^1)$ into $H^{s-m}(\mathbb{T}^1)$, we obtain condition (1) by Lemma 2.2 □

2.2 Operator symbol inequalities

Next we present a novel characterization of periodic pseudodifferential operators.

Theorem 2.2 *Let $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^1))$ with $\Sigma_A(x) \in \mathcal{L}(\mathcal{D}(\mathbb{T}^1))$ for every $x \in \mathbb{T}^1$. Then $A \in \Psi^m(\mathbb{T}^1)$ if and only if*

$$\| Q^\alpha \partial_x^\beta \Sigma_A(x) \|_{\mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1))} \le C_{\alpha\beta}, \tag{8}$$

where $C_{\alpha\beta}$ $(\alpha, \beta \in \mathbb{N}_0^1)$ are the same constants as in the traditional symbol inequalities (1). In fact, $A \in \Psi^m(\mathbb{T}^1)$ is equivalent even to that

$$Q^\alpha \Sigma_A \in C^\infty(\mathbb{T}^1, \mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1)))$$

for every $\alpha \in \mathbb{N}_0^1$.

Proof. Let $A \in \Psi^m(\mathbb{T}^1)$, $\alpha, \beta \in \mathbb{N}_0^1$ and $u \in H^s(\mathbb{T}^1)$. Let $x, x_0 \in \mathbb{T}^1$, and let $|x - x_0|$ denote their distance. Using the Lagrange Mean Value Theorem and applying the symbol inequality $|\triangle_\xi^\alpha \partial_x^{\beta+1} \sigma_A(x,\xi)| \leq C_{00}(1+|\xi|)^{m-|\alpha|}$, we obtain

$$\begin{aligned}
&||Q^\alpha \partial_x^\beta \Sigma_A(x)u - Q^\alpha \partial_x^\beta \Sigma_A(x_0)u||_{H^{s-(m-|\alpha|)}(\mathbb{T}^1)} \\
= &\left(\sum_\xi (1+|\xi|)^{2(s-(m-|\alpha|))} |\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi) - \triangle_\xi^\alpha \partial_x^\beta \sigma_A(x_0,\xi)|^2 |\hat{u}(\xi)|^2 \right)^{1/2} \\
= &\left(\sum_\xi (1+|\xi|)^{2(s-(m-|\alpha|))} |(x-x_0) \triangle_\xi^\alpha \partial_x^{\beta+1} \sigma_A(x_\xi,\xi)|^2 |\hat{u}(\xi)|^2 \right)^{1/2} \\
\leq &\left(\sum_{|\xi|>N} (1+|\xi|)^{2(s-(m-|\alpha|))} |x-x_0|^2 C_{\alpha(\beta+1)}^2 (1+|\xi|)^{2(m-\alpha)} |\hat{u}(\xi)|^2 \right)^{1/2} \\
= &\ |x-x_0| \ C_{\alpha(\beta+1)} \ ||u||_{H^s(\mathbb{T}^1)}.
\end{aligned}$$

In the same way we see that

$$||Q^\alpha \partial_x^\beta \Sigma_A(x)||_{\mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1))} \leq C_{\alpha\beta},$$

and

$$Q^\alpha \Sigma_A \in C^\infty(\mathbb{T}^1, \mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1))).$$

Now assume that $Q^\alpha \Sigma_A \in C^\infty(\mathbb{T}^1, \mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1)))$. Then the compact space $\mathbb{T}^1$ is mapped via the continuous mapping to a compact set $Q^\alpha \Sigma_A^{(\beta)}(\mathbb{T}^1) = Q^\alpha \partial_x^\beta \Sigma_A(\mathbb{T}^1) \subset \mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1))$, so that this set is also operator norm bounded, i.e.

$$||Q^\alpha \partial_x^\beta \Sigma_A(x)||_{\mathcal{L}(H^s(\mathbb{T}^1), H^{s-(m-|\alpha|)}(\mathbb{T}^1))} \leq c_{\alpha\beta} \tag{9}$$

for every $x \in \mathbb{T}^1$.

Assuming the inequality (9), due to Lemma 2.1, we have

$$|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \leq c_{\alpha\beta} \ (1+|\xi|)^{m-|\alpha|};$$

the proof is thus completed □

3 On Sobolev and pseudodifferential orders

In the sequel, we tackle some problems concerning the determination of Sobolev and pseudodifferential orders of operators. For general information on Sobolev orders, see [23].

3.1 Sobolev orders

Lemma 3.1 *Let X, Y, Z be Banach spaces, $Y \subset Z$ so that the relative topology of Y inherited from Z is coarser than the original topology of Y. If $A \in \mathcal{L}(X, Z)$ maps X into Y, then $A \in \mathcal{L}(X, Y)$.*

Proof. Suppose $\|u-u_n\|_X \to 0$ and $\|v-Au_n\|_Y \to 0$. Since the imbedding of Y into Z is continuous, we have $\|v-Au_n\|_Z \to 0$; thereby $v = Au$, since $A \in \mathcal{L}(X,Z)$. The claim follows now by the Closed Graph Theorem □

Recall that $A \in \Psi^m(\mathbb{T}^n)$ belongs to $\mathcal{L}(H^s(\mathbb{T}^n), H^{s-m}(\mathbb{T}^n))$ for every $s \in \mathbb{R}$. By the previous lemma, we know even that if $A \in \Psi^m(\mathbb{T}^n)$ maps $H^q(\mathbb{T}^n)$ into $H^{q-r}(\mathbb{T}^n)$ for some fixed $q, r \in \mathbb{R}$, then $A \in \mathcal{L}(H^q(\mathbb{T}^n), H^{q-r}(\mathbb{T}^n))$ (See also Corollary 2.2 about convolutions).

Theorem 3.1 *If $A \in \mathcal{L}(\mathcal{D}(\mathbb{T}^n))$ is an operator of a finite Sobolev order and if A maps some $H^q(\mathbb{T}^n)$ into some $H^{q-r}(\mathbb{T}^n)$, then A has a Sobolev order $r+\varepsilon$ for every $\varepsilon > 0$. Furthermore, if $A \in \Psi^m(\mathbb{T}^n)$ and $r \geq m-1$, then r is a Sobolev order of A.*

Proof. Fix $\varepsilon > 0$, and assume that $s < q$ (the case $s > q$ is totally symmetric). Then, by choosing $p < s$ small enough, the interpolation theorems

$$\mathcal{L}(H^{q_1}, H^{q_2}) \cap \mathcal{L}(H^{p_1}, H^{p_2}) \subset \mathcal{L}([H^{q_1}, H^{p_1}]_\theta, [H^{q_2}, H^{p_2}]_\theta)$$

and

$$[H^{q_j}, H^{p_j}]_\theta = H^{\theta q_j + (1-\theta)p_j}$$

(here $0 < \theta < 1$; see [11, Theorems 5.1 and 7.7]) imply that

$$A \in \mathcal{L}(H^s(\mathbb{T}^n), H^{s-r-\varepsilon}(\mathbb{T}^n)).$$

Now suppose $r \geq m-1$. With the aid of the canonical Sobolev isomorphisms φ_t (see (6)),and recalling the generalized Leibniz property of commutators (5), we get

$$\varphi_{s-q} A \varphi_{q-s} - A = \varphi_{s-q}[A, \varphi_{q-s}] \in \Psi^{m-1}(\mathbb{T}^n).$$

On the other hand,

$$\varphi_{s-q} A \varphi_{q-s} H^s(\mathbb{T}^n) = \varphi_{s-q} A H^q(\mathbb{T}^n) \subset \varphi_{s-q} H^{q-r}(\mathbb{T}^n) = H^{s-r}(\mathbb{T}^n).$$

Thus $AH^s(\mathbb{T}^n) \subset H^{s-r}(\mathbb{T}^n)$. This completes the proof □

3.2 Pseudodifferential orders

Theorem 3.2 *If A is a pseudodifferential operator on $\mathbb{T}^1$, $\varepsilon > 0$ and*

$$|\partial_x^\beta \sigma_A(x,\xi)| \leq C_\beta (1+|\xi|)^{m-\varepsilon}$$

(where the constants C_β depend on $\beta \in \mathbb{N}_0^1$), then

$$|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \leq C_{\alpha\beta,x}(1+|\xi|)^{m+1/2-|\alpha|}$$

(where the constants $C_{\alpha\beta x}$ depend on $\alpha, \beta \in \mathbb{N}_0^1$ and $x \in \mathbb{T}^1$). In particular, if A is a convolution pseudodifferential operator $f \mapsto k * f$ with

$$|\hat{k}(\xi)| \leq C(1+|\xi|)^{m-\varepsilon},$$

then $A \in \Psi^{m+1}(\mathbb{T}^1)$.

Proof. Let $A \in \Psi^r(\mathbb{T}^1)$ for some unknown $r \in \mathbb{R}$. Let us define functions $\rho^m_{\beta x} : \mathbb{T}^1 \to \mathbb{C}$ ($\beta \in \mathbb{N}_0^1$, $x \in \mathbb{T}^1$) by

$$\rho^m_{\beta x}(y) = \sum_\xi (\partial_x^\beta \sigma_A)(x,\xi) e^{i2\pi y\cdot\xi}(1+|\xi|)^{-m-1}.$$

Then

$$\begin{aligned}\|\rho^m_{\beta x}\|^2_{H^{(1+\varepsilon)/2}(\mathbb{T}^1)} &= \sum_\xi (1+|\xi|)^{1+\varepsilon} |\partial_x^\beta \sigma_A(x,\xi)|^2 (1+|\xi|)^{-2(m+1)} \\ &= \sum_\xi (1+|\xi|)^{-(1+\varepsilon)} \left(\frac{|\partial_x^\beta \sigma_A(x,\xi)|}{(1+|\xi|)^{m-\varepsilon}} \right)^2 < \infty,\end{aligned}$$

i.e. $\rho^m_{\beta x}$ belongs to $H^{(1+\varepsilon)/2}(\mathbb{T}^1)$, hence being continuous, as we notice that $H^{(1+\varepsilon)/2}(\mathbb{T}^1) \subset C(\mathbb{T}^1)$. Then define functions $\rho^m_{\beta x\alpha} : \mathbb{T}^1 \to \mathbb{C}$ by

$$\rho^m_{\beta x\alpha}(y) = (e^{-i2\pi y} - 1)^\alpha \rho^m_{\beta x}(y).$$

Thereby $\rho^m_{\beta x\alpha}$ has exactly the same local Sobolev smoothness than $\rho^m_{\beta x}$, when $y \neq 0 \in \mathbb{T}^1$. At $y = 0$, the zero of the function $y \mapsto (e^{-i2\pi y} - 1)^\alpha$ has the multiplicity of order $|\alpha|$; this should affect the local Sobolev smoothness at that point. Indeed, since $A \in \Psi^r(\mathbb{T}^1)$, we have

$$\widehat{(\rho^m_{\beta x\alpha})}(\xi) = \triangle_\xi^\alpha \partial_x^\beta \left[\sigma_A(x,\xi)(1+|\xi|)^{-m-1}\right] = \mathcal{O}(|\xi|^{r-|\alpha|-m-1})$$

for every $\alpha \in \mathbb{N}_0^1$, so that it follows $\rho^m_{\beta x\alpha} \in C^\infty(\mathbb{T}^1 \setminus \{0\})$. Furthermore, the singularity of $\rho^m_{\beta x\alpha}$ at $y = 0$ is less severe as $|\alpha|$ grows; by the Leibniz formula, $\rho^m_{\beta x\alpha}$ belongs to $C^{|\alpha|}(\mathbb{T}^1)$, because $\rho^m_{\beta x} \in C^0(\mathbb{T}^1)$. Since $C^{|\alpha|}(\mathbb{T}^1) \subset H^{|\alpha|}(\mathbb{T}^1)$, we have $\widehat{(\rho^m_{\beta x\alpha})}(\xi) = o(|\xi|^{-|\alpha|})$ as $|\xi| \to \infty$, so that

$$|\triangle_\xi^\alpha \partial_x^\beta \sigma_A(x,\xi)| \le C_{\alpha\beta,x}(1+|\xi|)^{m+1-|\alpha|}$$

□

Integral operators. Hence for a convolution-type pseudodifferential operator we got an estimate for the pseudodifferential order by looking just at first symbol inequality (with $\alpha = 0 = \beta$). But there are more complicated operators that allow such an order estimation. For instance, consider integral operator A given by

$$(Au)(x) = \int_{\mathbb{T}^1} u(y)a(x,y)k(x-y)\ dy,$$

where $a \in C^\infty(\mathbb{T}^1 \times \mathbb{T}^1)$ and k is a distribution on $\mathbb{T}^1$; i.e. A is some kind of a distorted convolution. Such operators arise e.g. from boundary value problems, and it turns out that $A \in \mathrm{Op}S^m(\mathbb{T}^1)$, when $\hat{k} \in S^m(\mathbb{T}^1)$ (see [19]).

But the pseudodifferential order may be lower than that: The symbol σ_A has an asymptotic expansion ([19])

$$\sigma_A(x,\xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} (\partial_y^{(\alpha)} a(x,y))|_{y=x} \triangle_\xi^\alpha \hat{k}(\xi),$$

where $\partial_y^{(0)} = I$ and $\partial_y^{(\alpha+1)} = (\partial_y - \alpha)\partial_y^{(\alpha)}$ (Note that the main part of the expansion is $a(x,x)\hat{k}(\xi)$). The more derivatives of a vanish on the diagonal of $\mathbb{T}^1 \times \mathbb{T}^1$, the more smoothing A is; in an extreme case, if a vanishes in a neighbourhood on the diagonal, then A is infinitely smoothing ($A\mathcal{D}'(\mathbb{T}^1) \subset \mathcal{D}(\mathbb{T}^1)$, or $A \in \cap_{m\in\mathbb{R}} \Psi^m(\mathbb{T}^1)$).

We gave a rather simple proof for Theorem 3.2; one might hope that under the given assumptions using more sophisticated approach we could verify that $\sigma_A \in S^m(\mathbb{T}^1)$. For a convolution operator, we obtained $\sigma_A \in S^{m+1}(\mathbb{T}^1)$, but can we get rid of the "additional pseudodifferential order" $+1$ here? In the proof presented above this oddity was perhaps caused by the naive exploitation of the facts $C^k(\mathbb{T}^1) \subset H^k(\mathbb{T}^1)$ and $H^{k+1/2+\varepsilon}(\mathbb{T}^1) \subset C^k(\mathbb{T}^1)$. Moreover, in the general case there were x-dependent coefficients $C_{\alpha\beta,x}$. Could we eliminate this dependence somehow? Essentially the problem is the following: if $f \in H^s(\mathbb{R}^1)$ is a distribution with the support in some fixed bounded subset of $\mathbb{R}^1$, and if $f \in C^\infty(\mathbb{R} \setminus \{0\})$, does it then follow that $\|pf\|_{H^r(\mathbb{R}^1)} \leq C_{rs}\|f\|_{H^s(\mathbb{R}^1)}$, where $p(x) = x$ and $r = s+1$? At least the role of $\varepsilon > 0$ in Theorem 3.2 is justified by the following example.

Pathological example (by Gennadi Vainikko): Let $\sigma(x,\xi) = \sin([\ln|\xi|]^2)$ (when $|\xi| \geq 1$, $\xi \in \mathbb{R}$; the definition of σ for $|\xi| < 1$ is not interesting). Notice that we can give the symbol inequalities (1) on torus also for "interpolated periodic symbols" $\sigma_A : \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{C}$, by replacing the partial difference operators $\triangle_\xi^\alpha$ by the common partial differential operators ∂_ξ^α (see [19] or [22]). Now σ is independent of x, and it is bounded, resulting in that the corresponding operator maps $H^s(\mathbb{T}^1)$ into itself for every $s \in \mathbb{R}$. On the other hand, σ defines a periodic pseudodifferential operator of degree ε for any $\varepsilon > 0$, as it is easily verified — however, $\sigma \notin S^0(\mathbb{T}^1)$, because

$$\partial_\xi \sigma(x,\xi) = 2\frac{\ln|\xi|}{\xi} \cos([\ln|\xi|]^2), \quad (|\xi| > 1).$$

This example above displays essentially "the worst kind" phenomenon known to us. So let us finally guess:

Conjecture. If A is a periodic pseudodifferential operator on $\mathbb{T}^n$, $\varepsilon > 0$ and

$$|\sigma_A(x,\xi)| \leq C(1+|\xi|)^{m-\varepsilon},$$

then $\sigma_A \in S^m(\mathbb{T}^n)$.

4 Coda

We have examined various questions swirling around continuity and orders of pseudodifferential operators. Undoubtedly, the central theme is the representation of a pseudodifferential operator on a group as a smooth convolution-operator-valued mapping defined on the group. We have successfully translated the traditional symbol (function) inequalities to the corresponding symbol (operator) norm inequalities between Sobolev spaces.

Why we have introduced operator symbols, which at the first sight seem to be just as useful as the traditional symbol functions? What is the new catch? The groups in our study have been commutative (tori $\mathbb{T}^n$); But there is no essential restriction in generalizing and altering these operator symbol methods to the setting of non-commutative harmonic analysis on groups and homogeneous spaces, while attempts to introduce global symbol functions are usually prevented by the lack of commutativity. Indeed, these matters are studied in the sequel.

Another topic in this paper has been the problems about Sobolev and pseudodifferential orders. >From the computation point of view, not to speak of pure theoretical interest, effective methods for recognizing pseudodifferential operators and determining their orders would be nice. Possible ramifications of Theorem 3.2 would be important.

5 Non-commutative Fourier transform

For a general treatment on harmonic analysis, see [8]. Let π_L and π_R be the left and right regular representations of a compact Lie group, respectively. Let $\mathcal{D}(G)$ be the set $C^\infty(G)$ equipped with the standard test function Fréchet space topology. We call $\pi_L(f)$ and $\pi_R(f)$ the left and right Fourier transforms of a test function $f \in \mathcal{D}(G)$, respectively; these operators are defined by $\pi_L(f)g = f * g$ and $\pi_R(f)g = g * f$, and they both belong to $\mathcal{L}(L^2(G))$ and $\mathcal{L}(\mathcal{D}(G))$. In general, if $\pi : G \to \mathcal{L}(\mathcal{D}(G))$ (with some necessary additional assumptions), we formally define

$$\pi(f) = \int_G f(x)\pi(x)\ d\mu_G(x) \tag{10}$$

(i.e. $(\pi(f)g)(y) = \int_G f(x)(\pi(x)g)(y)\ d\mu_G(x)$); this is interpreted as a vector-valued integral (in appropriate spaces, depending on π). We call $\pi_L(f)$ *the* Fourier transform of f (see [15]). Recall that $\pi_L(f * g) = \pi_L(f)\pi_L(g)$.

The Fourier transform is an injective mapping, and its inverse transform is defined by

$$f(x) = \mathrm{Tr}\left(\pi_L^*(x)\pi_L(f)W_{G(P)}\right), \tag{11}$$

where $\pi_L^*(x) = \pi_L(x)^*$ and $W_{G(P)}$ is the Plancherel weight of the group (and Tr can be considered as some kind of an integral over the group dual). >From now on, let us use the standard decomposition of π_L into a direct sum of irreducible unitary representations $\pi_{L\xi}$ ($\xi \in \hat{G}$) of dimension $d(\xi)$, so that

$\{pi_{L\xi ij} \mid \xi \in \hat{G},\ 1 \leq i,j \leq d(\xi)\}$ is an orthogonal basis of $L^2(G)$; in this basis, $W_{G(P)}$ is a diagonal operator, more precisely $W_{G(P)}\pi_{L\xi ij} = d(\xi)\pi_{L\xi ij}$.

Let us define the space $\mathcal{D}(\hat{G})$ of *test operators* by

$$\mathcal{D}(\hat{G}) = \{\pi_L(f) \mid f \in \mathcal{D}(G)\};$$

this set is given the Fréchet topology induced from the test functions by the Fourier transform.

Let $\mathcal{D}'(G) = \mathcal{L}(\mathcal{D}(G), \mathbb{C})$ be the distribution space, equipped with the weak$*$-topology; we denote $\langle f, u\rangle_G = u(f)$, when f is a test function and u is a distribution. Then $\mathcal{D}(G)$ is considered as a subspace of $\mathcal{D}'(G)$, embedded by

$$\langle f, g\rangle_G = \int_G f(x)g(x)\ d\mu_G(x).$$

Notice that $\langle f, g\rangle_G = (\check{f} * g)(e)$, where $\check{f} = f \circ \iota$, $\iota(x) = x^{-1}$. Similarly, for operators $A, B \in \mathcal{D}(\hat{G})$ we define

$$\langle A, B\rangle_{\hat{G}} = \mathrm{Tr}(ABW_{G(P)}),$$

so that we obtain the Parseval formula

$$\langle f, g\rangle_G = \langle \pi_L^*(f), \pi_L(g)\rangle_{\hat{G}}.$$

This allows us to extend the dual brackets $\langle\cdot,\cdot\rangle_{\hat{G}}$ and the Fourier transform by the generalized Parseval formula

$$\langle \pi_L^*(f), \pi_L(u)\rangle_{\hat{G}} = \langle f, u\rangle_G \tag{12}$$

(also $\pi_L(u)f = u * f$; in the chosen coordinates, $\pi_L(u)$ $(u \in \mathcal{D}'(G))$ is of the same block diagonal form as the representation π_L); we denote the set of the "non-commutative distributions" by

$$\mathcal{D}'(\hat{G}) = \pi_L(\mathcal{D}'(G)).$$

Again, the Fourier transform is injective bringing in the canonical topology from the space $\mathcal{D}'(G)$. Notice that usually $\pi_L^*(f) \neq \pi_L(f)^*$ (unless f is real-valued), but $\pi_L^t(f) = \pi_L(f)^t$ always. In fact, $\pi_L^*(f) = \pi_L(\check{f})$ and $\pi_L(f)^* = \pi_L(\tilde{f})$, where $\tilde{f}(x) = \overline{f(x^{-1})}$.

The Fourier transform induces operators: Let $A \in \mathcal{L}(\mathcal{D}(G))$. Then the adjoint $A' \in \mathcal{L}(\mathcal{D}'(G))$ is defined by

$$\langle Af, u\rangle_G = \langle f, A'u\rangle_G.$$

Similarly $B \in \mathcal{L}(\mathcal{D}'(G))$ defines $B' \in \mathcal{L}(\mathcal{D}(G))$, and $A'' = A$, $B'' = B$. We define the operators $\pi_L(A), \pi_L^*(A) \in \mathcal{L}(\mathcal{D}(G))$ by

$$\pi_L(A)\pi_L(u) = \pi_L(Au), \quad \pi_L^*(A)\pi_L^*(u) = \pi_L^*(Au)$$

so that $\pi_L(A)' = \pi_L^*(A'), \pi_L^*(A)' = \pi_L(A') \in \mathcal{L}(\mathcal{D}'(G))$. Notice, that $\pi_L(A)\pi_L(B) = \pi_L(B)\pi_L(A)$, if $AB = BA$.

We denote $L^2(\hat{G}) = \pi_L(L^2(G))$ with the obvious Hilbert space structure. Recall that $\pi_L(f)\pi_L(g) = \pi_L(f * g)$.

Thus we have "Fourier transforms" (related to the symmetries) on three levels: First, the regular representation $\pi_L : G \to \mathcal{L}(L^2(G))$ embeds the group into an operator algebra; Second, the regular representation induces a transform of functions, $\pi_L : \mathcal{D}(G) \to \mathcal{D}(\hat{G})$; Third, this induces the transform of operators acting on functions, $\pi_L : \mathcal{L}(\mathcal{D}(G)) \to \mathcal{L}(\mathcal{D}(\hat{G}))$. These induced Fourier transforms were also extended by duality.

The Laplacian Δ of G is an unbounded diagonal (translation invariant) operator with respect to the regular representation; the positive diagonal operator $W_{G(S)}$ defined by

$$W_{G(S)}(\pi_L(x)) = ((I - \Delta)^{1/2}\pi_L)(x)$$

(alternatively $\sigma_{(I-\Delta)^{1/2}}(x) \equiv W_{G(S)}$; see below) is called the Sobolev weight of the group. Then a Sobolev space $H^s(G)$ is the completion of the test functions with respect to the inner product

$$(f, g)_{H^s(G)} = \mathrm{Tr}\left(W_{G(S)}^{2s}\pi_L(f)\pi_L(g)^* W_{G(P)}\right).$$

Then $L^2(G) = H^0(G)$, $\mathcal{D}(G) = \cap_{s\in\mathbb{R}} H^s(G)$ and $\mathcal{D}'(G) = \cup_{s\in\mathbb{R}} H^s(G)$ (as sets without topology). This definition of the Sobolev spaces coincides with the common (local) differential geometric definition.

5.1 Generalized Taylor polynomials

Remark. Notice that a left (right) convolution $f \mapsto g * f$ ($f \mapsto f * g$) is a right- (left-) translation invariant operator. Contradicting the traditions, we shall define the Lie algebra of a Lie group to be the set of right-invariant vector fields (instead left-invariant ones; this is just a matter of convenience).

We identify the Lie algebra of G also with the set of the right-invariant differential operators on G, and also with $\mathbb{R}^n$ and its canonical inner product space structure. Let U be a neighbourhood of $e \in G$ such that $\overline{U}$ still belongs to a chart neighbourhood and so that the exponential mapping has an inverse function on $\overline{U}$. Let $\{\partial_j = \xi_j\}_{j=1}^n$ be such a basis of the Lie algebra that the functions $p_k \in C^\infty(G)$ given on U by $p_k(\exp(\xi)) = \xi \cdot \xi_k$ satisfy $(\partial_j p_k)(e) = \delta_{jk}$ (a complex alternative for p_k could be given by $p_k(\exp(\xi)) = i^{-n}e^{i\xi\cdot\xi_k} - 1$). We define $q_\alpha \in C^\infty(G)$ $(\alpha \in \mathbb{N}_0^n)$ by

$$q_\alpha(x) = p_1(x)^{\alpha_1} \cdots p_n(x)^{\alpha_n},$$

and accordingly

$$D_\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

Then, as an asymptotic expansion (Taylor–Lie series [14])

$$f(x) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} q_\alpha(x)(D_\alpha f)(e)$$

for $f \in C^\infty(G)$, when x is near e. Notice the geometry involved in the definition of q_α. Via non-commutative Fourier transform we define "quasidifference operators" Q^α by

$$Q^\alpha(\pi_L(u)) = \pi_L(M_{q_\alpha} u),$$

where M_ϕ stands for the multiplication operator $u \mapsto \phi u$; that is, $Q^\alpha = \pi_L(M_{q_\alpha})$.

Remark. Notice the distinction between the quasidifferences Q^α and the operators Q^α (inspired by commutators) presented in the commutative analysis part of this paper.

6 Pseudodifferential operators

Symbol. If $A \in \mathcal{L}(\mathcal{D}(G))$, we can define $\sigma_A(x) = \pi_L(x)(A\pi_L^*)(x)$, where A operates on the canonical coordinate functions $\overline{\pi_{L\xi ji}(x)}$ of π_L^*, i.e. $(A\pi_L^*)(x) = ((A\overline{\pi_{L\xi ji}})(x))_{\xi ij}$, where $\pi_L(x) = (\pi_{L\xi ij}(x))_{\xi ij}$. Then

$$(Af)(x) = \mathrm{Tr}\left((A\pi_L^*)(x)\pi_L(f)W_{G(P)}\right) = \mathrm{Tr}\left(\pi_L^*(x)\sigma_A(x)\pi_L(f)W_{G(P)}\right). \quad (13)$$

This σ_A is called the *symbol of* A. The operator $\Sigma_A(x)$ with the symbol $\sigma_{\Sigma_A(x)}(y) = \sigma_A(x)$ is then right-invariant, obtained from A by "freezing coefficients"; we call Σ_A the *operator symbol of* A (we also denote $\Sigma_A(x) = \mathrm{Op}(\sigma_A(x))$ ($x \in G$ fixed), whereas $A = \mathrm{Op}(\sigma_A)$).

Remark. Notice that the traditional symbol of a pseudodifferential operator on $\mathbb{R}^n$ or $\mathbb{T}^n$ is just a "1×1-dimensional Jordan block" of the symbol just defined.

Remark. In the sequel (contrary to the commutative harmonic analysis in the beginning of this paper), we define partial differential operators ∂_x^β acting on the operator-valued functions by Banach space limits of difference quotients (see Appendix) instead of definition by commutators.

Amplitudes and symbol classes. We say that a symbol $\sigma_A : G \to \mathcal{D}(G)$ is *of order* $m \in \mathbb{R}$, if the *symbol inequalities*

$$\|A_{m\alpha\beta kx}\| \leq C_{\alpha\beta k} \quad (14)$$

hold uniformly in $x \in G$, where

$$\sigma_{A_{m\alpha\beta kx}}(y) = W_{G(S)}^{|\alpha|-m} Q^\alpha \partial_x^\beta ad_{\sigma_{\partial_{j_1}}} \circ \ldots \circ ad_{\sigma_{\partial_{j_k}}} \sigma_A(x);$$

here $1 \leq j_1, \ldots, j_k \leq n$ and $ad_X Y = [X, Y]$. The class of such operators A is denoted by $\mathrm{Op}\Sigma^m(G)$, and the corresponding class of symbols is denoted by $\Sigma^m(G)$.

If merely

$$||A_{m\alpha\beta 0x}|| \leq C_{\alpha\beta 0} \tag{15}$$

(i.e. the inequalities (14) with $k = 0$ fixed) hold, we write $\sigma_A \in \tilde{\Sigma}^m(G)$ and $A \in \mathrm{Op}\tilde{\Sigma}^m(G)$. (Apparently $\Sigma^m(G) \subset \tilde{\Sigma}^m(G)$. Notice that if $G = \mathbb{T}^n$, then $\Sigma^m(G) = \tilde{\Sigma}^m(G)$.)

In the similar manner one defines the orders of amplitudes $a : G \times G \to \mathcal{L}(\mathcal{D}'(G))$ and the spaces $\mathcal{A}^m(G)$ and $\tilde{\mathcal{A}}^m(G)$; i.e. $a \in \mathcal{A}^m(G)$, if it is of the block diagonal form (see the remark above) and if the *amplitude inequalities*

$$||A_{m\alpha\beta\gamma kxy}||_{\mathcal{L}(L^2(G))} \leq C_{\alpha\beta\gamma k}, \tag{16}$$

hold uniformly in $x, y \in G$, where

$$\sigma_{A_{m\alpha\beta\gamma kxy}}(z) \equiv W_{G(S)}^{|\alpha|-m} Q^\alpha \partial_x^\beta \partial_y^\gamma ad_{\sigma_{\partial_{j_1}}} \circ \ldots \circ ad_{\sigma_{\partial_{j_k}}} a(x,y)$$

uniformly in $x, y \in G$. Then

$$(\mathrm{Op}(a)u)(x) = \mathrm{Tr}\left(\pi_L^*(x) \int_G a(x,y) f(y) \pi_L(y)\ d\mu_G(y) W_{G(P)}\right) \tag{17}$$

defines formally a linear operator on $\mathcal{D}(G)$. Then the symbol $\sigma_{\mathrm{Op}(a)}$ of $\mathrm{Op}(a)$ is an amplitude of order m (see below). (Notice that the symbol of an operator is unique).

How does one practically calculate the norm of the convolution operator $A_{m\alpha\beta kx}$ in the symbol inequalities? Just like in the commutative case! This time, however, we take the supremum of the norms of the (finite-dimensional!) blocks of the symbol (just as we evaluated the supremum of the absolute values of $\hat{k}(\xi)$ in the torus case above).

Differential operators. Of course, partial differential operators of order m belong to $\mathrm{Op}\Sigma^m(G)$. But there are others, as we shall see...

Remark: Taylor [17] studies a related operator-valued function $y \mapsto A(y) = \pi_L(y) A(\pi_L^*(y))$; then $\sigma_{A(y)}(x) = \sigma_A(y^{-1}x)$.

Schwartz kernel. Let a be an amplitude. Then the Schwartz kernel $K_{\mathrm{Op}(a)}$ of $\mathrm{Op}(a)$ is formally the distribution

$$K_{\mathrm{Op}(a)}(x,y) = \mathrm{Tr}\left(\pi_L^*(x) a(x,y) \pi_L(y) W_{G(P)}\right),$$

as one easily calculates. Then $K_{[A,M_\phi]}(x,y) = (\phi(y) - \phi(x)) K_A(x,y)$ and $K_{[X_x,A]}(x,y) = (X_x - X_y) K_A(x,y)$, where $\phi \in C^\infty(G)$ and X is a vector field.

Commutators. Some useful commutator formulae are $[A, BC] = B[A,C] + [A,B]C$ and $[A^*, B] = -[A, B^*]^*$.

Quasidifferences. Let a be an amplitude. Then

$$a(x,y) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} q_\alpha(yx^{-1}) \partial_y^\alpha a(x,y)|_{y=x} \tag{18}$$

formally. Let b be an amplitude and let $\phi \in C^\infty(G)$. Suppose that $b(x,y) = \pi_L^*(\psi_{x,y})$, where $\psi_{x,y} \in H^s(G)$ with $s > n/2$ for every $x, y \in G$. By a "generalized integration by parts", we obtain formally

$$\begin{aligned}
(\mathrm{Op}(\phi(yx^{-1})b)f)(x) &= \int_G f(y) \mathrm{Tr}\left(b(x,y)\phi(yx^{-1})\pi_L(y)\pi_L^*(x)W_{G(P)}\right) \, d\mu_G(y) \\
&= \int_G f(y)\phi(yx^{-1})\langle b(x,y), \pi_L(y)\pi_L^*(x)\rangle_{\hat{G}} \, d\mu_G(y) \\
&= \int_G f(y)\phi(yx^{-1})\langle b(x,y), \pi_L(yx^{-1})\rangle_{\hat{G}} \, d\mu_G(y) \\
&= \int_G f(y)\phi(yx^{-1})\langle \psi_{x,y}, \delta_{yx^{-1}}\rangle_G \, d\mu_G(y) \\
&= \int_G f(y)\langle \psi_{x,y}, M_\phi \delta_{yx^{-1}}\rangle_G \, d\mu_G(y) \\
&= \int_G f(y)\langle M_\phi \psi_{x,y}, \delta_{yx^{-1}}\rangle_G \, d\mu_G(y) \\
&= \int_G f(y)\langle \pi_L^*(M_\phi)(b(x,y)), \pi_L(yx^{-1})\rangle_{\hat{G}} \, d\mu_G(y) \\
&= \int_G f(y) \mathrm{Tr}\left(\pi_L^*(M_\phi)(b(x,y))\pi_L(y)\pi_L^*(x)W_{G(P)}\right) \, d\mu_G(y).
\end{aligned}$$

(Above $\delta_{yx^{-1}} \in H^{-s}(G)$ $(-s < -n/2)$ denotes the Dirac delta distribution at the point $yx^{-1} \in G$.) Hence the amplitudes $(x,y) \mapsto \phi(y^{-1}x)b(x,y)$ and $(x,y) \mapsto \pi_L^*(M_\phi)b(x,y)$ define the same operator. Notice that $\pi_L^*(M_\phi)$ belongs to $\mathcal{L}(\mathcal{D}(\hat{G}))$, since $M_\phi \in \mathcal{L}(\mathcal{D}(G))$. Furthermore, the restriction $s > n/2$ above can be removed.

Asymptotic expansion for amplitudes. Let us define $Q^\alpha = \pi_L(M_{q_\alpha})$. Then formally

$$a(x,y) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} Q^\alpha \left(\partial_y^\alpha a(x,y)|_{y=x}\right). \tag{19}$$

Notice that this expansion has no longer dependence on the variable y; hence it is a formal asymptotic expansion of the symbol $x \mapsto \sigma_{\mathrm{Op}(a)}(x)$.

Asymptotic expansion for adjoint. Using the formula

$$\int_G (Af)(x)\overline{g(x)} \, d\mu_G(x) = \int_G f(x)\overline{(A^*g)(x)} \, d\mu_G(x)$$

we formally obtain $A^* = \mathrm{Op}(b)$, $b(x,y) = a(y,x)^*$, when $A = \mathrm{Op}(a)$. Especially, $b(x,y) = \sigma_A(y)^*$ is an amplitude of A^*. A question arises: is this an amplitude of order $m \in \mathbb{R}$, if $\sigma_A \in \tilde{\Sigma}^m(G)$? Indeed, it is: there is no doubt about the behaviour under differentiations, and the only problems might occur with quasidifferences. But if we define

$$\tilde{Q}^\alpha = \pi_L^*(M_{\tilde{q}_\alpha}),$$

we get

$$Q^\alpha(\sigma_A(y)^*) = \left(\tilde{Q}^\alpha(\sigma_A(y))\right)^*;$$

In the amplitude inequalities it clearly makes no difference if we replace Q^α by $\tilde{Q}^\alpha$ (recall that $\tilde{q}(x) = \overline{q(x^{-1})}$). Hence $b \in \mathcal{A}^m(G)$. Using the asymptotic expansion formula we thus obtain

$$\sigma_{A^*}(x) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} Q^\alpha \left(\partial_x^\alpha \sigma_A(x)^*\right). \tag{20}$$

Asymptotic expansion for composition. Since

$$\overline{(A^*\overline{\pi_L})(y)}\pi_L^*(y) = (\pi_L(y)((A^*\overline{\pi_L})(y))^t)^* = (\pi_L(y)(A^*\pi_L^*)(y))^* = \sigma_{A^*}(y)^*,$$

we obtain

$$\begin{aligned}
(BAf)(x) &= \operatorname{Tr}\left(\pi_L^*(x)\sigma_B(x)\int_G (Af)(y)\pi_L(y)\ d\mu_G(y) W_{G(P)}\right) \\
&= \operatorname{Tr}\left(\pi_L^*(x)\sigma_B(x)\int_G f(y)\overline{(A^*\overline{\pi_L})(y)}\ d\mu_G(y) W_{G(P)}\right) \\
&= \operatorname{Tr}\left(\pi_L^*(x)\sigma_B(x)\int_G \sigma_{A^*}(y)^* f(y)\pi_L(y) d\mu_G(y) W_{G(P)}\right);
\end{aligned}$$

hence we have $BA = \operatorname{Op}(c)$ with $c(x,y) = \sigma_B(x)\sigma_{A^*}(y)^*$, so that combining this with the previous asymptotic expansions we get an asymptotic expansion for $\sigma_{BA}(x)$, provided that a pointwise product of amplitudes of orders $m_1, m_2 \in \mathbb{R}$ is an amplitude of order $m_1 + m_2$. Since traditional derivatives obey the familiar Leibniz formula, this problem amounts to whether there is a Leibniz-like property for quasidifferences:

Generalized Leibniz formula for quasidifferences. On $\mathbb{Z} = \hat{\mathbb{T}}$ the generalized Leibniz formula for the forward difference operator is

$$(\triangle(\hat{f}\hat{g}))(\xi) = (\triangle\hat{f})(\xi)\hat{g}(\xi) + \hat{f}(\xi)(\triangle\hat{g})(\xi) + (\triangle\hat{f})(\xi)(\triangle\hat{g})(\xi),$$

and this is the Fourier transform of the function $x \mapsto (e_{-1}(x) - 1)(f * g)(x)$, where $e_{-1}(x) = e^{-i2\pi x}$. For a non-commutative group, it would be nice to find analogous functions $e \in C^\infty(G)$ for which $e(y)e(y^{-1}x) = e(x)$ globally (i.e. a one-dimensional representation of the group). However, it is enough to require the Leibniz formula to be only approximative: Indeed, let $|\alpha| = 1$

and $e_\alpha = \check{q}_\alpha = q_\alpha \circ \iota \in C^\infty(G)$. Let $a(x,y) = \pi_L(a_{xy})$ and $b(x,y) = \pi_L(b_{xy})$ with $a_{xy}, b_{xy} \in \mathcal{D}'(G) \cap C^\infty(G \setminus \{e\})$. Then (Notice that $\pi_L^*(M_\phi) = \pi_L(M_{\check{\phi}})$)

$$Q^\alpha(a(x,y)b(x,y)) = \pi_L(e_\alpha(a_{xy} * b_{xy})).$$

Furthermore,

$$\begin{aligned} e_\alpha(z)(a_{xy} * b_{xy})(z) &= ((e_\alpha a_{xy}) * b_{xy})(z) + (a_{xy} * (e_\alpha b_{xy}))(z) \\ &+ \int_G \left(e_\alpha(z) - e_\alpha(zw^{-1}) - e_\alpha(w)\right) a_{xy}(zw^{-1}) b_{xy}(w)\ d\mu_G(w); \end{aligned}$$

here we have to study the integral operator R_{xy} ("remainder") given by

$$(R_{xy}f)(z) = \int_G f(w) \mathsf{a}_\alpha(z,w) a_{xy}(zw^{-1})\ d\mu_G(w),$$

where $\mathsf{a}_\alpha(z,w) = e_\alpha(z) - e_\alpha(zw^{-1}) - e_\alpha(w)$, $\mathsf{a}_\alpha \in C^\infty(G \times G)$, $\mathsf{a}_\alpha(z,z) = 0$ (vanishing on the diagonal) (we shall encounter this type of operators in the sequel, too). Now

$$\sigma_{R_{xy}}(z) \sim \sum_{\beta > 0} \frac{1}{\beta!} (\partial_w^\beta \mathsf{a}_\alpha(z,w))|_{w=z} Q^\beta a(x,y),$$

so that

$$(R_{xy} b_{xy})(z) \sim \operatorname{Tr}\left(\pi_L^*(z) \sum_{\beta > 0} \frac{1}{\beta!} (\partial_w^\beta \mathsf{a}_\alpha(z,w))|_{w=z} (Q^\beta a(x,y)) b(x,y) W_{G(P)}\right).$$

Let us define $c_{\alpha\beta}(z) = (\partial_w^\beta \mathsf{a}_\alpha(z,w))|_{w=z}$. Then

$$\begin{aligned} Q^\alpha(a(x,y)b(x,y)) &\sim (Q^\alpha a(x,y)) b(x,y) + a(x,y)(Q^\alpha b(x,y)) \\ &+ \sum_{\beta > 0} \frac{1}{\beta!} \pi_L^*(M_{c_{\alpha\beta}}) \left((Q^\beta a(x,y)) b(x,y)\right). \end{aligned}$$

We call this the Leibniz formula for the quasidifference Q^α ($|\alpha| = 1$).

Sobolev continuity. If $\sigma_A \in \tilde{\Sigma}^{-m}(G)$ with "$m \in \mathbb{R}$ small enough", then $K_A \in C(G \times G)$. Then it follows that both A and A^* are compact operators on $L^2(G)$. This would yield inductively the $L^2(G)$-continuity (and compactness) of any $A \in \mathrm{Op}\tilde{\Sigma}^{-\varepsilon}(G)$, $\varepsilon > 0$: Indeed, then $\sigma_A, \sigma_{A^*} \in \tilde{\Sigma}^{-\varepsilon}(G)$, so that $A^*A \in \mathrm{Op}\tilde{\Sigma}^{-2\varepsilon}(G)$. Noticing that $\|Af\|_{L^2(G)}^2 = (A^*Af, f)_{L^2(G)} \leq \|A * A\|_{\mathcal{L}(L^2(G))} \|f\|_{L^2(G)}^2$, we reach a conclusion: $\mathrm{Op}\tilde{\Sigma}^{-\varepsilon}(G) \subset \mathcal{L}(H^s(G))$ (and the operators here are compact). More generally, $\mathrm{Op}\tilde{\Sigma}^{m-\varepsilon}(G) \subset \mathcal{L}(H^s(G), H^{s-m}(G))$

Subalgebra. Notice that if

$$(Af)(x) = \int_G f(y)\mathtt{a}(x,y)k(xy^{-1})\ d\mu_G(y),$$

where $\mathtt{a} \in C^\infty(G \times G)$ and $k \in \mathcal{D}'(G)$ (with sing supp$(k) \subset \{e\}$), then

$$\sigma_A(x) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!}(\partial_y^\alpha \mathtt{a}(x,y))|_{y=x} Q^\alpha \pi_L(k).$$

Let $A \in \cup_{m\in\mathbb{R}} \mathrm{Op}\Sigma^m(G)$ (or $\cup_{m\in\mathbb{R}} \mathrm{Op}\tilde{\Sigma}^m(G)$) have the symbol of the form

$$\sigma_A(x) \sim \sum_{\alpha \geq 0} \phi_\alpha(x) k_\alpha$$

where $\phi_\alpha \in C^\infty(G)$, and $\mathrm{Op}(k_\alpha)$ is a convolution operator (for every $\alpha \in \mathbb{N}_0^n$). This kind of operators form an involutive subalgebra Op^m of $\cup_{m\in\mathbb{R}} \mathrm{Op}\Sigma^m(G)$ (or subalgebra $\tilde{Op}^m$ of $\cup_{m\in\mathbb{R}} \mathrm{Op}\tilde{\Sigma}^m(G)$, respectively), as it is easily verified. If A is elliptic and belongs to Op^m ($\tilde{Op}^m$), then its parametrix belongs to Op^m ($\tilde{Op}^m$). Furthermore, $Op^m \subset \mathcal{L}(H^s(G), H^{s-m}(G))$.

Ellipticity. A symbol $\sigma_A \in \tilde{\Sigma}^m(G)$ ($\Sigma^m(G)$) is called an elliptic symbol of order m, if $x \mapsto \sigma_{A+P}(x)^{-1}$ is a symbol of order $-m$ for some infinitely smoothing operator P. Then there exists $\sigma_B \in \tilde{\Sigma}^{-m}(G)$ such that $\sigma_{BA}(x) \sim I \sim \sigma_{AB}(x)$, and from this (using the asymptotic expansion for operator composition) we obtain an asymptotic expansion for σ_B; B is called a parametrix of A.

Pseudolocality. Operators in $\mathrm{Op}\tilde{\Sigma}^m(G)$ are pseudolocal. This can be proved just like the corresponding theorem in the commutative case (see [18]).

Pseudodifferential operators. Now if $\phi \in C^\infty(G)$ and $|\beta| = 1$, we have $[M_\phi, \mathrm{Op}\Sigma^m(G)] \subset \mathrm{Op}\Sigma^{m-1}(G)$ and $[\partial_x^\beta, \mathrm{Op}\Sigma^m(G)] \subset \mathrm{Op}\Sigma^m(G)$. By the Sobolev continuity result above, we get $\mathrm{Op}\Sigma^m(G) \subset \Psi^{m+\varepsilon}(G)$ for every $\varepsilon > 0$. Moreover, we get $Op^m \subset \Psi^m(G)$.

7 Appendix: Operator-valued calculus on groups

Let X be a topological vector space. Mapping $F : \mathbb{R} \to X$ is said to be differentiable at the point $0 \in \mathbb{R}$, if the limit

$$\frac{dF}{dx}(0) = F'(0) = \lim_{h\to 0} \frac{1}{h}\left(F(h) - F(0)\right)$$

exists. If such a limit exists in every point $x \in \mathbb{R}$ and $F' : \mathbb{R} \to X$ so defined is a continuous function, we call F to be continuously differentiable, $F \in$

$C^1(\mathbb{R}, X)$. In the apparent manner one defines partial differential operators ∂_x^β and classes $C^k(M, X)$ $(k \in \mathbb{N}_0 \cup \{\infty\})$, where M is a C^∞-manifold.

Theorem 7.1 (Leibniz formula for operator-valued functions) *Let X be a Banach space and let $F, G \in C^1(\mathbb{R}, \mathcal{L}(X))$. Then*

$$\frac{d(FG)}{dx}(x) = F(x)G'(x) + F'(x)G(x).$$

References

[1] Agranovich, M. S.: *Spectral properties of elliptic pseudodifferential operators on a closed curve.* Funct. Anal. Appl. 13 (1979), 279-281.

[2] Agranovich, M. S.: *On elliptic pseudodifferential operators on a closed curve.* Trans. Moscow Math. Soc. 47 (1985), 23-74.

[3] Agranovich, M. S.: *Elliptic operators on closed manifolds* (in Russian). Itogi Nauki i Tehniki, Ser. Sovrem. Probl. Mat. Fund. Napravl. 63 (1990), 5-129. (English translation in Encyclopaedia Math. Sci. 63 (1994), 1-130.)

[4] Beals, R.: *Characterization of pseudo-differential operators and applications.* Duke Math. J. 44 (1977), 1, 45-57.

[5] Coifman, R. R. and Y. Meyer: *Au-delà des opérateurs pseudo-différentiels.* Astérisque 57, Société Math. de France. 1978.

[6] Cordes, H. O.: *On pseudodifferential operators and smoothness of special Lie group representations.* Manuscripta Math. 28 (1979), 51-69.

[7] Dunau, J.: *Fonctions d'un operateur elliptique sur une variete compacte.* J. Math. Pures et Appl. 56 (1977), 367-391.

[8] Hewitt E. and K. A. Ross: *Abstract Harmonic Analysis I-II.* Springer–Verlag. 1963.

[9] Hörmander, L.: *The Analysis of Linear Partial Differential Operators III.* Berlin: Springer-Verlag. 1985.

[10] Knopp, K.: *Theory and Application of Infinite Series.* Glasgow: Blackie & Son Ltd. 1948.

[11] Lions, J. L. and E. Magenes: *Non-homogeneous Boundary Value Problems and Applications. Vol. I.* Berlin: Springer-Verlag. 1972.

[12] McLean, W.: *Local and global descriptions of periodic pseudodifferential operators.* Math. Nachr. 150 (1991), 151-161.

[13] Melo, S. T.: *Characterization of pseudodifferential operators on the circle.* Proc. AMS 125 (1997), 5, 1407-1412.

[14] Olver, P.: *Applications of Lie Groups to Differential Equations.* Springer–Verlag. 1986.

[15] Stinespring, W. F.: *Integration theorems for gages and duality for unimodular groups.* Trans. AMS 90, 1959, 15–56.

[16] Taylor, M. E.: *Pseudodifferential operators.* Princeton: Princeton University Press. 1981.

[17] Taylor, M. E.: *Beals–Cordes-type characterizations of pseudodifferential operators.* Proc. AMS 125 (1997), 6, 1711-1716.

[18] Turunen, V.: *Symbol analysis of periodic pseudodifferential operators.* M.Sc. Thesis. Helsinki Univ. Techn. 1997.

[19] Turunen, V. and G. Vainikko: *On symbol analysis of periodic pseudodifferential operators.* Z. Anal. Anw. 17 (1998), 9-22.

[20] Turunen, V.: *Commutator characterization of periodic pseudodifferential operators.* Submitted. 1999.

[21] Turunen, V.: *Smooth operator-valued symbol analysis.* Preprint, Helsinki Univ. Techn. 1999.

[22] Vainikko, G.: *Periodic integral and pseudodifferential equations.* Helsinki Univ. Techn. 1996.

[23] Zaidman, S.: *Topics in pseudo-differential operators.* Harlow: Addison Wesley Longman Ltd. 1996.

(continued from the back cover)

A412 Marko Huhtanen
Ideal GMRES can be bounded from below by three factors, Jan 1999

A411 Juhani Pitkäranta
The first locking-free plane-elastic finite element: historia mathematica, Jan 1999

A410 Kari Eloranta
Bounded Triangular and Kagomé Ice, Jan 1999

A409 Jukka Tuomela and Teijo Arponen
On the numerical solution of involutive ordinary differential systems: Boundary value problems, Dec 1998

A408 Ville Turunen
Commutator Characterization of Periodic Pseudodifferential Operators, Dec 1998

A407 Jarmo Malinen
Discrete time Riccati equations and invariant subspaces of linear operators, Feb 1999

A406 Jarmo Malinen
Riccati Equations for H^∞ Discrete Time Systems: Part II, Feb 1999

A405 Jarmo Malinen
Riccati Equations for H^∞ Discrete Time Systems: Part I, Feb 1999

A404 Jarmo Malinen
Toeplitz Preconditioning of Toeplitz Matrices an Operator Theoretic Approach, Feb 1999

A403 Saara Hyvönen and Olavi Nevanlinna
Robust bounds for Krylov method, Nov 1998

A402 Saara Hyvönen
Growth of resolvents of certain infinite matrice, Nov 1998

A401 Jukka Tuomela
On the Numerical Solution of Involutive Ordinary Differential Systems: Boundary value problems, Dec 1998

A400 Seppo Hiltunen
Implicit functions from locally convex spaces to Banach spaces, Jan 1999

HELSINKI UNIVERSITY OF TECHNOLOGY INSTITUTE OF MATHEMATICS
RESEARCH REPORTS

The list of reports is continued inside. Electronical versions of the reports are available at *http://www.math.hut.fi/reports/* .

A417 Kirsi Peltonen
Examples of uniformly quasiregular mappings, Jul 1999

A416 Saara Hyvönen
On the Iterative Solution of Nonnormal Problems, Jul 1999

A415 Ville Turunen
A smooth operator valued symbol analysis, Aug 1999

A414 Marko Huhtanen
A stratification of the set of normal matrices, May 1999

A413 Cl´ement Ph., Gripenberg G. and Londen S-O
Regularity properties of solutions of functional evalution equations, May 1999

ISBN 951-22-4554-X
ISSN 0784-3143
Libella Painopalvelu, Espoo, 1999