ON SYMBOL ANALYSIS OF PERIODIC PSEUDODIFFERENTIAL OPERATORS

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Abstract. *This paper deals with periodic pseudodifferential operators on the unit circle* $\mathbb{T}^1 := \mathbb{R}^1/\mathbb{Z}^1$*. The main results are asymptotic expansions for adjoints and products of PPDOs.*

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1. Introduction

Boundary integral operators on planar curves can be treated as periodic pseudodifferential operators (PPDOs), or as pseudodifferential operators on the unit circle $\mathbb{T}^1 := \mathbb{R}^1/\mathbb{Z}^1$. On general n-dimensional manifolds one can locally apply the theory of the pseudodifferential operators on $\mathbb{R}^n$ (see e.g. [15], [13]). When the manifold is diffeomorphic to the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$ this approach is unnecessarily cumbersome; a more elementary, global treatment of PPDOs is based on the Fourier series representation of functions.

The historical account announcing PPDOs can be found in [2], and it is further developed in [3]. What is essential, is that the global definitions of PPDOs are equivalent to the local (differential geometric) approach treating $\mathbb{T}^n$ as a manifold, and using the theory of pseudodifferential operators on $\mathbb{R}^n$: this fundamental result is proved in [8] (see also [11] for an elementary special case on $\mathbb{T}^1$).

Standard operations with pseudodifferential operators (adjoints, products, commutators) can be characterized by their symbols. For pseudodifferential operators on $\mathbb{R}^n$ the symbol analysis is well-known (see [15], [13], [14]). For pseudodifferential operators on $\mathbb{T}^1$ the corresponding results have been developed by Elschner [6] in the special case of *classical* symbols. Symbols $\sigma(t,\xi)$ in [6] are defined on $\mathbb{T}^1 \times \mathbb{R}$, and the derivatives with respect to both arguments t and ξ are involved in the symbol analysis.

In this paper we shall show that Elschner's formulae hold for any periodic pseudodifferential symbol. Moreover, we give the corresponding formulae also when symbols $\sigma(t,n)$ are defined only on $\mathbb{T}^1 \times \mathbb{Z}$. In this case differences are used instead of derivatives with respect to the second argument ξ, and this causes further changes in the formulae. An early version of the proofs of these results was published in [16].

In our line of reasoning, we obtain some useful asymptotic expansions for periodic integral operators. The theory of PPDOs has been successfully applied in numerical analysis of boundary integral equations (see e.g. [4], [9], [5], [7], [10], and [17]).

2. Periodic Sobolev space H^λ

Our working spaces will be the Sobolev spaces H^λ on the compact circle group $\mathbb{T}^1 := \mathbb{R}^1/\mathbb{Z}^1 = \{x + \mathbb{Z}^1 | x \in \mathbb{R}^1\}$, but the study can be extended on any torus $\mathbb{T}^n$. The space of the test functions or the C^∞-smooth functions on $\mathbb{T}^1$ is denoted by C_1^∞. The natural topology of C_1^∞ is induced by the seminorms that one gets by demanding the following convergence: $u_n \to u$ if and only if $\forall k \in \mathbb{N}_0 = \{0,1,2,\ldots\} : u_n^{(k)} \to u^{(k)}$ uniformly. This makes C_1^∞ a non-normable Fréchet space.

An orthonormal basis of $L^2(\mathbb{T}^1)$ is given by the vectors $e^{i2\pi t\cdot n}$ $(n \in \mathbb{Z})$. Here the dot "$\cdot$" is a reminder of the dot product in higher dimensions. For $u \in L^2(\mathbb{T}^1)$ the Fourier coefficients are defined by $\hat{u}(n) = \int_0^1 e^{-i2\pi t\cdot n} u(t)\,dt$, and generally the Fourier coefficients of a linear map $u \in L(C_1^\infty, \mathbb{C})$ are given by $\hat{u}(n) := u(e_{-n})$, where $e_n(t) := e^{i2\pi t\cdot n}$.

Let us define the inner products $(\cdot,\cdot)_\lambda$ $(\lambda \in \mathbb{R})$ by

$$(u,v)_\lambda := \sum_{n\in\mathbb{Z}} (1+|n|)^{2\lambda} \hat{u}(n) \overline{\hat{v}(n)}.$$

(Here $(1+|n|)^{2\lambda}$ could be replaced by $\underline{n}^{2\lambda}$ where $\underline{n} := \max\{1, |n|\}$.) With the norm $\|u\|_\lambda := (u,u)_\lambda^{1/2}$ we get a Hilbert space H^λ called the Sobolev space. It consists of periodic functions and distributions u, represented by the formal series $\sum_n \hat{u}(n) e^{i2\pi t\cdot n}$ of the finite norm $\|u\|_\lambda$. Actually, it is known that in the presented topology of the test functions, $\bigcup_{\lambda\in\mathbb{R}} H^\lambda$ is the dual of C_1^∞, that is $\mathcal{L}(C_1^\infty, \mathbb{C}) \cong \bigcup_{\lambda\in\mathbb{R}} H^\lambda$.

The Sobolev space $H^{-\lambda}$ is the dual space of H^λ via the Banach duality product $\langle\cdot,\cdot\rangle$ defined by

$$\langle u, v\rangle := \sum_{n\in\mathbb{Z}} \hat{u}(n)\hat{v}(-n)$$

where $u \in H^\lambda$ and $v \in H^{-\lambda}$. Note that $\langle u, v\rangle = \int_0^1 u(t)v(t)\,dt$ when $\lambda = 0$. Accordingly, the L^2- (or H^0-) inner product $(u,v)_0 = \int_0^1 u(t)\overline{v(t)}\,dt$ builds the Hilbert duality product between H^λ and $H^{-\lambda}$. If A is a linear operator between two Sobolev spaces, we shall denote its Banach and Hilbert adjoints by A^* and $A^{*(H)}$, respectively.

3. Periodic pseudodifferential operators

Every operator $A \in \mathcal{L}(H^\lambda, H^\mu)$ on the Sobolev spaces is of the form

$$u(t) \mapsto Au(t) = \sum_{n\in\mathbb{Z}} \sigma_A(t,n)\hat{u}(n)e^{i2\pi t\cdot n} =: Op(\sigma_A)u(t) \tag{1}$$

where $\sigma_A(t,n) := e^{-i2\pi t\cdot n} A e^{i2\pi t\cdot n}$ is called the symbol of A. Indeed, the Fourier series representation $u(t) = \sum_n \hat{u}(n)e^{i2\pi t\cdot n}$ of $u \in H^\lambda$ converges in H^λ, and respectively the series $Au(t) = \sum_n \hat{u}(n)Ae^{i2\pi t\cdot n} = \sum_n \sigma_A(t,n)\hat{u}(n)e^{i2\pi t\cdot n}$ converges in H^μ.

A function $\sigma : \mathbb{T}^1\times\mathbb{Z} \to \mathbb{C}$ is called a symbol of degree (or order) α, denoted $\sigma \in \Sigma^\alpha$, if it is a C^∞-smooth function in the first argument, and if it satisfies for every $t \in \mathbb{T}^1$ and $n \in \mathbb{Z}$

$$\forall j,k \in \mathbb{N}_0\ \exists C_{jk} \in \mathbb{R} : \left|\partial_t^j \Delta_n^k \sigma(t,n)\right| \le C_{jk}\ (1+|n|)^{\alpha-k} \tag{2}$$

where

$$\partial_t := \frac{1}{i2\pi}\frac{\partial}{\partial t},$$

the forward difference operator $\Delta = \Delta_n$ is given by

$$\Delta\psi(n) := \psi(n+1) - \psi(n),$$

and $\partial_t^{j+1} := \partial_t\partial_t^j$, $\Delta_n^{k+1} := \Delta_n\Delta_n^k$.

A symbol $\sigma \in \Sigma^\alpha$ defines a periodic pseudodifferential operator (PPDO) $Op(\sigma) \in Op(\Sigma^\alpha)$ by formula (1) on C_1^∞. We denote $\Sigma^{-\infty} := \bigcap_{\alpha\in\mathbb{R}} \Sigma^\alpha$ and $Op(\Sigma^{-\infty}) := \bigcap_{\alpha\in\mathbb{R}} Op(\Sigma^\alpha)$.

An operator $Op(\sigma) \in Op(\Sigma^\alpha)$ maps C_1^∞ into itself, and as it turns out to be bounded with respect to the norms of H^λ and $H^{\lambda-\alpha}$ for every $\lambda \in \mathbb{R}$, and as C_1^∞ is dense in every H^λ, $Op(\sigma)$ can be extended to a linear map in $\mathcal{L}(H^\lambda, H^{\lambda-\alpha})$. From now on we consider PPDOs to be defined on the Sobolev spaces rather than on C_1^∞.

A formal integration of $\hat{u}(n) = \int_0^1 e^{-i2\pi t\cdot n} u(t)\,dt$ by parts (see (1)) suggests the notation

$$Op(\sigma)u(t) =: \int_0^1 u(s) \sum_{n\in\mathbb{Z}} \sigma(t,n) e^{i2\pi(t-s)\cdot n}\,ds. \tag{3}$$

This inspires a possible generalization of symbols. A function $a : \mathbb{T}^1 \times \mathbb{T}^1 \times \mathbb{Z} \to \mathbb{C}$ is by definition an amplitude of degree α, if it is C^∞-smooth in the first two arguments, and if it satisfies for every $t, s \in \mathbb{T}^1$ and $n \in \mathbb{Z}$

$$\forall j,k,l \in \mathbb{N}_0\ \exists C_{jkl} \in \mathbb{R} : \left|\partial_t^j \partial_s^k \Delta_n^l a(t,s,n)\right| \leq C_{jkl}\ (1+|n|)^{\alpha-l}. \tag{4}$$

The family of amplitudes of degree α is denoted by $\mathcal{A}^\alpha$. We also denote $\mathcal{A}^{-\infty} := \bigcap_{\alpha\in\mathbb{R}} \mathcal{A}^\alpha$. An amplitude $a \in \mathcal{A}^\alpha$ defines a linear operator $Op(a) \in Op(\mathcal{A}^\alpha)$ by

$$Op(a)u(t) = \int_0^1 u(s) \sum_{n\in\mathbb{Z}} a(t,s,n) e^{i2\pi(t-s)\cdot n}\,ds \tag{5}$$

where $u \in C_1^\infty$. Again, this is to be interpreted as a result of a formal integration by parts, being an abbreviation for

$$\begin{aligned} Op(a)u(t) = &\int_0^1 \sum_{p=0}^q \binom{q}{p} u^{(p)}(s) \sum_{n\neq 0} \left[\left(\frac{\partial}{\partial s}\right)^{q-p} a(t,s,n)\right] (i2\pi n)^{-q} e^{i2\pi(t-s)\cdot n}\,ds \\ &+ \int_0^1 u(s) a(t,s,0)\,ds \end{aligned}$$

where the series $\sum_{n\neq 0}[(\partial/\partial s)^{q-p} a(t,s,n)](i2\pi n)^{-q} e^{i2\pi(t-s)\cdot n}$ converges uniformly with $q > \alpha+1$. This operator is called an amplitude operator of degree α. Just as in the case of PPDOs, an amplitude operator $Op(a) \in Op(\mathcal{A}^\alpha)$ has a unique extension by density and continuity in $\mathcal{L}(H^\lambda, H^{\lambda-\alpha})$ ($\lambda \in \mathbb{R}$).

Often another definition for PPDOs is used: a linear operator A is called a PPDO, if there exists a function (prolongated symbol) $\sigma \in C^\infty(\mathbb{T}^1 \times \mathbb{R})$ such that $A = Op(\sigma|_{\mathbb{T}^1\times\mathbb{Z}})$ and for every $t \in \mathbb{T}^1$ and $\xi \in \mathbb{R}$

$$\forall j,k \in \mathbb{N}_0\ \exists c_{jk} \in \mathbb{R} : \left|\partial_t^j \partial_\xi^k \sigma(t,\xi)\right| \leq c_{jk}\ (1+|\xi|)^{\alpha-k}. \tag{6}$$

There is a standard linear interpolation procedure for prolongating a symbol $\sigma(t,n)$ to $\sigma(t,\xi)$ so that (2) would imply (6); see [17] for details. We remark that all the relevant information is contained already in definition (2) of symbols on $\mathbb{T}^1 \times \mathbb{Z}$. In a sense, the prolongation is arbitrary, and it is definitely not unique.

The prolongation process can also be modified for amplitudes to get $a(t,s,\xi)$ ($\xi \in \mathbb{R}$) from $a(t,s,n)$ ($n \in \mathbb{Z}$), with inequality

$$\forall j,k,l \in \mathbb{N}_0\ \exists c_{jkl} \in \mathbb{R} : |\partial_t^j \partial_s^k \partial_\xi^l a(t,s,\xi)| \leq c_{jkl}\ (1+|\xi|)^{\alpha-k}. \tag{7}$$

Amplitudes can be considered as a generalization of symbols, but it turns out that the family of amplitude operators coincides exactly with the family of PPDOs (see

Theorem 4.2). Nevertheless, the concept of amplitudes is highly justified as a tool in symbol analysis. Moreover, amplitudes literally manifest themselves in certain integral operators.

In many applications, $\sigma(t,\xi)$ is given from the very beginning on $\mathbb{T}^1\times\mathbb{R}$, and then it is natural to operate with prolongated symbols. Therefore we will present two versions of the symbol expansions — for symbols defined on $\mathbb{T}^1\times\mathbb{Z}$ and on $\mathbb{T}^1\times\mathbb{R}$.

4. Formulation of main results

In this section we show the equivalence of notions of amplitude operators and PPDOs, and we construct asymptotic expansions for the symbols of adjoints and products. Most of the proofs are stated in Section 5 after the presentation of the main results.

Let us first introduce some equivalence relations. We say that amplitudes a, a' are γ-equivalent ($\gamma\in\mathbb{R}$), $a \overset{\gamma}{\sim} a'$, if $a-a'\in\mathcal{A}^\gamma$; they are asymptotically equivalent, $a\sim a'$, if $a-a'\in\mathcal{A}^{-\infty}$. For the related operators we write $Op(a)\overset{\gamma}{\sim} Op(a')$ and $Op(a)\sim Op(a')$, respectively.

Theorem 4.1 is a prelude to asymptotic expansions, which are the main tools in the symbol analysis of PPDOs. The proof is omitted, since the result is well-known.

Theorem 4.1. *Let $(\alpha_j)_{j=0}^\infty\subset\mathbb{R}$ be a sequence such that $\alpha_j>\alpha_{j+1}\to_{j\to\infty}-\infty$, and $\sigma_j\in\Sigma^{\alpha_j}$ ($j\in\mathbb{N}_0$). Then there exists a symbol $\sigma\in\Sigma^{\alpha_0}$ such that*

$$\forall N\in\mathbb{N}: \sigma \overset{\alpha_N}{\sim} \sum_{j=0}^{N-1}\sigma_j.$$

Here the formal series $\sum_{j=0}^\infty\sigma_j$ is called an asymptotic expansion of the symbol $\sigma\in\Sigma^{\alpha_0}$, and we denote $\sigma\sim\sum_{j=0}^\infty\sigma_j$ (cf. $a\sim a'$ above; a different but related meaning). Respectively, $\sum_{j=0}^\infty Op(\sigma_j)$ is an asymptotic expansion of the operator $Op(\sigma)\in Op(\Sigma^{\alpha_0})$, denoted $Op(\sigma)\sim\sum_{j=0}^\infty Op(\sigma_j)$.

As there are two alternative definitions for the symbols, given by inequalities (2) and (6), we present asymptotic expansions for both cases. In the former case the shift in the difference operator $\triangle$ has to be compensated. This is done with the aid of operators $\partial_t^{(k)}$, defined by $\partial_t^{(0)}:=I$, and for $k\geq 1$

$$\partial_t^{(k)}:=\prod_{j=0}^{k-1}(\partial_t-jI),\quad \partial_t=\frac{1}{i2\pi}\frac{\partial}{\partial t}.$$

This definition is closely related to the Stirling numbers, which are introduced in Section 5. The proof of the following lemma is also given there.

Lemma 4.1. *Assume that $\sigma\in\Sigma^\alpha$ satisfies (6). Then*

$$\sum_{j=0}^\infty\frac{1}{j!}\triangle_\xi^j\partial_t^{(j)}\sigma(t,\xi)\sim\sum_{j=0}^\infty\frac{1}{j!}\left(\frac{\partial}{\partial\xi}\right)^j\partial_t^j\sigma(t,\xi). \tag{8}$$

Next we present an elementary result stating that amplitude operators are merely PPDOs, and we provide an effective way to calculate the symbol modulo $\Sigma^{-\infty}$ from an amplitude: this theorem has a fundamental status in the symbol analysis. Its proof is in Section 5. For a pseudodifferential analogue on $\mathbb{R}^n$, see [13].

Theorem 4.2. *For every amplitude $a \in \mathcal{A}^\alpha$ there exists a unique symbol $\sigma \in \Sigma^\alpha$ of the same degree α satisfying $Op(a) = Op(\sigma)$, and σ has the following asymptotic expansions:*

$$\sigma(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \Delta_n^j \partial_s^{(j)} a(t,s,n)|_{s=t}, \tag{9}$$

$$\sigma(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \partial_s^j a(t,s,\xi)|_{s=t}. \tag{10}$$

Integral operators that are PPDOs. As an example of the symbol analysis techniques developed so far, we study the periodic integral operators (PIOs). This subject is studied more thoroughly e.g. by Kelle and Vainikko in [7]. Let A be a linear operator defined on C_1^∞ by

$$Au(t) := \int_0^1 u(s)\mathsf{a}(t,s)\kappa(t-s)\,ds \tag{11}$$

where a is a C^∞-smooth 1-biperiodic function, and κ is a 1-periodic distribution. As usual, A is extended to appropriate Sobolev spaces.

Theorem 4.3. *The periodic integral operator A defined by (11) is a PPDO of degree α if and only if the Fourier coefficients of the distribution κ satisfy*

$$\forall k \in \mathbb{N}_0\ \exists C_k \in \mathbb{R}\ \forall n \in \mathbb{Z} : \left|\Delta_n^k \hat{\kappa}(n)\right| \leq C_k(1+|n|)^{\alpha-k}.$$

Thereby the symbol of A has the following asymptotic expansion:

$$\sigma_A(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \Delta_n^j \hat{\kappa}(n) \partial_s^{(j)} \mathsf{a}(t,s)|_{s=t}. \tag{12}$$

Let $\hat{\kappa}(n)$ be prolonged to function $\hat{\kappa}(\xi)$ ($\xi \in \mathbb{R}$) satisfying (6), i.e. $|\partial_\xi^k \hat{\kappa}(\xi)| \leq c_k(1+|\xi|)^{\alpha-k}$. Then the prolongated symbol $\sigma_A(t,\xi)$ has the following asymptotic expansion:

$$\sigma_A(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \hat{\kappa}(\xi) \partial_s^j \mathsf{a}(t,s)|_{s=t}. \tag{13}$$

Proof. An amplitude of A is right in the front of our eyes:

$$Au(t) = \int_0^1 u(s)\mathsf{a}(t,s)\kappa(t-s)\,ds = \int_0^1 u(s)\mathsf{a}(t,s) \sum_{n\in\mathbb{Z}} \hat{\kappa}(n) e^{i2\pi(t-s)\cdot n}\,ds = Op(a)u(t)$$

where $a(t,s,n) = \mathsf{a}(t,s)\hat{\kappa}(n)$. Certainly $\hat{\kappa}$ satisfies the presented inequality if and only if a is an amplitude of degree α. Accordingly, a yields asymptotic expansions (12) and (13) on the basis of (9) and (10) ∎

We are ready to give the main results of the paper. The proofs of the following theorems are given in Section 5. These results are analogous to the general pseudo-differential theory on $\mathbb{R}^n$, as presented in [15], [13], and [14].

Theorem 4.4. *Let A be a PPDO of degree α with its symbol prolongated to $\sigma_A(t,\xi)$ ($\xi \in \mathbb{R}$), and let $a(t,s,n)$ be an amplitude of A. Then the Banach adjoint A^* is in $Op(\Sigma^\alpha)$ with an amplitude $a^*(t,s,n) = a(s,t,-n)$. Moreover, the symbol of A^* has the following asymptotic expansions:*

$$\sigma_{A^*}(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \Delta_n^j \partial_t^{(j)} \sigma_A(t,-n), \tag{14}$$

$$\sigma_{A^*}(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \partial_t^j \sigma_A(t,-\xi). \tag{15}$$

Accordingly, the Hilbert adjoint $A^{*(H)}$ has an amplitude $a^{*(H)}(t,s,n) = \overline{a(s,t,n)}$, and asymptotic expansions for its symbol are given by

$$\sigma_{A^{*(H)}}(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \Delta_n^j \partial_t^{(j)} \overline{\sigma_A(t,n)}, \tag{16}$$

$$\sigma_{A^{*(H)}}(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{\partial}{\partial \xi}\right)^j \partial_t^j \overline{\sigma_A(t,\xi)}. \tag{17}$$

Hence by (14) and (16), $\sigma_A(t,-n)$ and $\overline{\sigma_A(t,n)}$ are the principal symbols of the Banach and Hilbert adjoints, respectively.

Theorem 4.5. *The product BA of $B \in Op(\Sigma^\beta)$ and $A \in Op(\Sigma^\alpha)$ is in $Op(\Sigma^{\alpha+\beta})$, and its symbol has the following asymptotic expansions:*

$$\sigma_{BA}(t,n) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left[\Delta_n^j \sigma_B(t,n)\right] \partial_t^{(j)} \sigma_A(t,n), \tag{18}$$

$$\sigma_{BA}(t,\xi) \sim \sum_{j=0}^{\infty} \frac{1}{j!} \left[\left(\frac{\partial}{\partial \xi}\right)^j \sigma_B(t,\xi)\right] \partial_t^j \sigma_A(t,\xi). \tag{19}$$

Without proofs asymptotic expansions (14), (15), (18), and (19) were announced in [17]. For classical PPDOs, formulae of the type (15), (17), and (19) were established by Elschner [6]. Elschner's proofs make use of the special structure of classical symbols.

According to (18), $\sigma_B \sigma_A$ is the principal symbol of both BA and AB, so that the commutator $AB - BA \in Op(\Sigma^{\alpha+\beta-1})$ when $A \in Op(\Sigma^\alpha)$, $B \in Op(\Sigma^\beta)$. This result is familiar from the general theory of pseudodifferential operators.

5. Proofs of main results

The rest of this paper is devoted to the proofs of the results stated in Section 4.

Lemma 5.1. *Assume that $\sigma \in \Sigma^\alpha$ and $\hat{\sigma}$ is its Fourier transform with respect to the first argument. Then*

$$\left|\partial_\xi^k \hat{\sigma}(m,\xi)\right| \leq c_{rk}\ (1+|m|)^{-r}(1+|\xi|)^{\alpha-k} \tag{20}$$

for every $k, r \in \mathbb{N}_0$. Respectively, for $a \in \mathcal{A}^\alpha$ we have with any $k, q, r \in \mathbb{N}_0$

$$\left|\partial_\xi^k \hat{a}(l,m,\xi)\right| \leq c_{qrk}\ (1+|l|)^{-q}(1+|m|)^{-r}(1+|\xi|)^{\alpha-k}. \tag{21}$$

Proof. We use the defining inequality (6). Clearly $|\partial_\xi^k \hat{\sigma}(0,\xi)| \leq c_{0k}\ (1+|\xi|)^{\alpha-k}$; now assume that $m \neq 0$ and integrate by parts:

$$\begin{aligned}\left|\partial_\xi^k \hat{\sigma}(m,\xi)\right| &= \left|\partial_\xi^k \int_0^1 e^{-i2\pi t\cdot m}\sigma(t,\xi)\,dt\right| \\ &= \left|m^{-r}\int_0^1 e^{-i2\pi t\cdot m}\partial_t^r\partial_\xi^k\sigma(t,\xi)\,dt\right| \leq |m|^{-r}c_{rk}\ (1+|\xi|)^{\alpha-k}.\end{aligned}$$

Collecting these results we deduce inequality (20). The proof for amplitudes is similar ∎

Working with pseudodifferential operators, some form of the elementary inequality of Peetre is needed. The version of the Sobolev norm in this paper suggests the following Peetre inequality:

$$\forall\lambda\in\mathbb{R}\ \forall\xi,\eta\in\mathbb{R} : (1+|\xi+\eta|)^\lambda \leq 2^{|\lambda|}(1+|\xi|)^{|\lambda|}(1+|\eta|)^\lambda. \tag{22}$$

The proof is easy and thus it is omitted.

Stirling numbers of the first kind $\alpha_j^{(k)}$, are defined for $0 \leq j \leq k$ $(k \geq 1)$ by

$$x^{(k)} = \prod_{i=0}^{k-1}(x-i) = \sum_{j=0}^{k}\alpha_j^{(k)}x^j$$

where $x \in \mathbb{R}$. It is natural to extend this definition by $x^{(0)} = \alpha_0^{(0)} = 1$ and $\alpha_j^{(k)} = 0$ when $j < 0$ or $j > k$. Stirling numbers of the second kind, $\beta_j^{(k)}$ $(0 \leq j \leq k)$, are a sort of dual for the first kind:

$$x^k = \sum_{j=0}^{k}\beta_j^{(k)}x^{(j)}.$$

Again, we set $\beta_j^{(k)} = 0$ for $j < 0$ and $j > k$. One notices that

$$\alpha_j^{(k)} = \frac{1}{j!}\left(\frac{d}{dx}\right)^j x^{(k)}\Big|_{x=0}, \quad \beta_j^{(k)} = \frac{1}{j!}\Delta^j\,\xi^k\big|_{\xi=0}.$$

From these equalities recursion formulae for the Stirling numbers can be obtained by applying the Leibniz formulae, but for the second kind there is also a closed form (see [1]). Some other definitions and properties of the Stirling numbers can be found in [1]. For instance, $(\alpha_j^{(k)})_{j,k=0}^N$ and $(\beta_j^{(k)})_{j,k=0}^N$ are inverse matrices of each other.

We have presented two alternative definitions for PPDOs. To build a bridge between the asymptotic expansions of these approaches (by Lemma 4.1), we have know how to approximate differences by derivatives. A finest account is by Steffensen in [12] where *Markoff's formulae* are presented (but there the Stirling numbers were not used):

Proposition 5.1. *For every $\varphi \in C^\infty(\mathbb{R})$, $\xi \in \mathbb{R}$, $N \in \mathbb{N}_0$, and $1 \le j < N$ there exist $\eta_\Delta \in [0, j]$ and $\eta_d \in [0, N-1]$ such that the following equalities hold:*

$$\frac{1}{j!}\Delta^j \varphi(\xi) - \sum_{k=j}^{N-1} \frac{1}{k!}\beta_j^{(k)} \varphi^{(k)}(\xi) = \frac{1}{N!}\beta_j^{(N)} \varphi^{(N)}(\xi + \eta_\Delta), \tag{23}$$

$$\frac{1}{j!}\varphi^{(j)}(\xi) - \sum_{k=j}^{N-1} \frac{1}{k!}\alpha_j^{(k)} \Delta^k \varphi(\xi) = \frac{1}{N!}\alpha_j^{(N)} \varphi^{(N)}(\xi + \eta_d). \tag{24}$$

Proof of Lemma 4.1. We apply Proposition 5.1 in order to translate differences into derivatives, and use the definition of the Stirling numbers of the second kind:

$$\sum_{k=0}^{\infty} \frac{1}{k!}\Delta_\xi^k \partial_t^{(k)} \sigma(t,\xi) \overset{\alpha-N}{\sim} \sum_{k=0}^{N-1} \frac{1}{k!}\left[\sum_{j=k}^{N-1} \frac{k!}{j!}\beta_k^{(j)} \left(\frac{\partial}{\partial \xi}\right)^j\right] \partial_t^{(k)} \sigma(t,\xi)$$
$$= \sum_{j=0}^{N-1} \frac{1}{j!}\left(\frac{\partial}{\partial \xi}\right)^j \left[\sum_{k=0}^{j} \beta_k^{(j)} \partial_t^{(k)}\right] \sigma(t,\xi) = \sum_{j=0}^{N-1} \frac{1}{j!}\left(\frac{\partial}{\partial \xi}\right)^j \partial_t^j \sigma(t,\xi).$$

Since there is no upper bound for N, we have completed the proof ∎

Proof of Theorem 4.2. As a bounded linear operator in Sobolev spaces, $Op(a)$ possesses the unique symbol $\sigma = \sigma_{Op(a)}$, but at the moment we do not yet know whether $\sigma \in \Sigma^\alpha$. The symbol is computed from

$$\sigma(t,n) = e^{-i2\pi t\cdot n} Op(a)\left(e^{i2\pi t\cdot n}\right) = e^{-i2\pi t\cdot n} \int_0^1 e^{i2\pi s\cdot n} \sum_{m\in\mathbb{Z}} a(t,s,m) e^{i2\pi(t-s)\cdot m}\, ds$$
$$= \int_0^1 e^{-i2\pi(t-s)\cdot n} \sum_{m\in\mathbb{Z}} a(t,s,m) e^{i2\pi(t-s)\cdot m}\, ds.$$

Note that here $n \in \mathbb{Z}$, but we can use a C^∞-smooth prolongation for amplitude a with respect to its third argument n. Now, according to the definition of amplitude operators, we may change the order of integration and summation, so that the Taylor formula can be applied as follows:

$$\sigma(t,n) = \sum_{m\in\mathbb{Z}} \int_0^1 a(t,s,m) e^{i2\pi(t-s)\cdot(m-n)}\, ds$$
$$= \sum_{m\in\mathbb{Z}} \hat{a}_2(t, m-n, m) e^{i2\pi t\cdot(m-n)} = \sum_{m\in\mathbb{Z}} \hat{a}_2(t, m, n+m) e^{i2\pi t\cdot m}$$
$$= \sum_{j=0}^{N-1} \frac{1}{j!}\left(\frac{\partial}{\partial n}\right)^j \sum_{m\in\mathbb{Z}} \hat{a}_2(t,m,n) m^j e^{i2\pi t\cdot m} + \sum_{m\in\mathbb{Z}} R_N(t,m,n,m) e^{i2\pi t\cdot m}$$

where $R_N(t,m,n,p)$ is the error term of the Taylor series representation of $\hat{a}_2(t,m,n+p)$. Here $\hat{a}_2(t,m,n) := \int_0^1 e^{-i2\pi s\cdot m} a(t,s,n)\, ds$. Notice that R_N is a C^∞-smooth function in the third argument n, a property that will be used soon. Let us define $E_N(t,n) := \sum_m R_N(t,m,n,m)e^{i2\pi t\cdot m}$. Notice that

$$\partial_s^j a(t,s,n) = \partial_s^j \sum_{m\in\mathbb{Z}} \hat{a}_2(t,m,n)e^{i2\pi s\cdot m} = \sum_{m\in\mathbb{Z}} \hat{a}_2(t,m,n)m^j e^{i2\pi s\cdot m},$$

which yields

$$\sigma(t,n) = \sum_{j=0}^{N-1} \frac{1}{j!}\left(\frac{\partial}{\partial n}\right)^j \partial_s^j a(t,s,n)|_{s=t} + E_N(t,n). \tag{25}$$

All we need is that $E_N \in \Sigma^{\alpha-N}$, and for this we have to study the remainder R_N. Using the Lagrange form of remainder term, (a close variant of) inequality (20), and Peetre inequality (22), we get

$$\begin{aligned}
&\left|\partial_t^{j'}\partial_n^{k'} R_N(t,m,n,m)\right| \\
&\le \frac{1}{N!}\left|m^N\right|(2\pi)^N \max_{\theta\in[0,1]}\left|\partial_t^{j'}\partial_n^{N+k'}\hat{a}_2(t,m,n+\theta m)\right| \\
&\le \frac{1}{N!}(1+|m|)^N(2\pi)^N \max_{\theta\in[0,1]}\left[c_{r,j',N+k'}\ (1+|m|)^{-r}(1+|n+\theta m|)^{\alpha-N-k'}\right] \\
&\le c_{r,j',k',N}\ (1+|n|)^{\alpha-N-k'}(1+|m|)^{N+|\alpha-N-k'|-r}
\end{aligned}$$

where r may be taken arbitrarily large. Therefore

$$|\partial_t^{j'}\triangle_n^{k'} R_N(t,m,n,m)| \le c'_{r,j',k',N}(1+|n|)^{\alpha-N-k'}(1+|m|)^{N+|\alpha-N-k'|-r}.$$

This results to

$$\left|\partial_t^j\triangle_n^k E_N(t,n)\right| \le c_{j,k,N}\ (1+|n|)^{\alpha-N-k},$$

and hence E_N is a symbol of degree $\alpha-N$. Consequently σ belongs to Σ^α by equation (25), and Theorem 4.1 provides asymptotic expansion (10). Lemma 4.1 applied on (10) yields then (9) ∎

Proof of Theorem 4.4. Assume that $u,v\in C_1^\infty$. We make use of the integral representation of the duality product and the definition of amplitude operators:

$$\begin{aligned}
\int_0^1 v(t)A^*u(t)\,dt = \langle A^*u,v\rangle := \langle u,Av\rangle &= \int_0^1 u(s)Av(s)\,ds \\
&= \int_0^1 u(s)\left\{\int_0^1 v(t)\sum_{n\in\mathbb{Z}} a(s,t,n)e^{i2\pi(s-t)\cdot n}\,dt\right\}ds \\
&= \int_0^1 v(t)\left\{\int_0^1 u(s)\sum_{n\in\mathbb{Z}} a(s,t,n)e^{i2\pi(s-t)\cdot n}\,ds\right\}dt.
\end{aligned}$$

Thus $A^* = Op(a^*)$ with $a^*(t,s,n) = a(s,t,-n)$. Especially $\sigma_A(s,-n)$ is an amplitude of A^*, so that by (9) asymptotic expansion (14) follows. To get expansion (15) we apply (8) on (14) ∎

Proof of Theorem 4.5. Of course, by going through the procedure

$$\begin{aligned}\sigma_{BA}(t,n) &= e^{-i2\pi t\cdot n}\left[BA\left(e^{i2\pi t\cdot n}\right)\right] = e^{-i2\pi t\cdot n}B\left(\sigma_A(t,n)e^{i2\pi t\cdot n}\right)\\ &= e^{-i2\pi t\cdot n}B\sum_{m\in\mathbb{Z}}\hat{\sigma}_A(m,n)e^{i2\pi t\cdot(m+n)} = \sum_{m\in\mathbb{Z}}\sigma_B(t,m+n)\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m},\end{aligned}$$

one gets the exact symbol of the composition of the PPDOs. Although this representation cannot be used effectively to approximate BA, it yields an asymptotic expansion. As in the proof of Theorem 4.2, the Taylor formula is applied.

$$\begin{aligned}\sigma_{BA}(t,n) &= \sum_{m\in\mathbb{Z}}\sigma_B(t,m+n)\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m}\\ &= \sum_{m\in\mathbb{Z}}\left[\sum_{j=0}^{N-1}\frac{1}{j!}\left(\frac{\partial}{\partial n}\right)^j\sigma_B(t,n)m^j + R_N(t,n,m)\right]\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m}\\ &= \sum_{j=0}^{N-1}\frac{1}{j!}\left[\left(\frac{\partial}{\partial n}\right)^j\sigma_B(t,n)\right]\sum_{m\in\mathbb{Z}}\hat{\sigma}_A(m,n)m^j e^{i2\pi t\cdot m}\\ &\quad + \sum_{m\in\mathbb{Z}}R_N(t,n,m)\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m}\\ &= \sum_{j=0}^{N-1}\frac{1}{j!}\left[\left(\frac{\partial}{\partial n}\right)^j\sigma_B(t,n)\right]\partial_t^j\sigma_A(t,n) + E_N(t,n)\end{aligned}$$

where $E_N(t,n) = \sum_m R_N(t,n,m)\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m}$. The pointwise product of symbols of degree α and β is a symbol of degree $\alpha+\beta$, so that the first term of the expansion, $\sigma_B(t,n)\sigma_A(t,n)$, is in $\Sigma^{\alpha+\beta}$. We only need to prove that the error term $E_N(t,n)$ is well-behaved, which means that $E_N \in \Sigma^{\alpha+\beta-N}$, or

$$\begin{aligned}\left|\partial_t^j\triangle_n^k E_N(t,n)\right| &= \left|\partial_t^j\triangle_n^k\sum_{m\in\mathbb{Z}}R_N(t,n,m)\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m}\right|\\ &\le c_{jkN}\,(1+|n|)^{\alpha+\beta-N-k}.\end{aligned}$$

Indeed, by inequality (20) we have with any $r\in\mathbb{R}$

$$\left|\partial_t^{j_A}\triangle_n^{k_A}\hat{\sigma}_A(m,n)e^{i2\pi t\cdot m}\right| \le c^A_{j_A,k_A,r}\,(1+|m|)^{j_A-r}(1+|n|)^{\alpha-k_A}.$$

The error term of the Taylor series, inequality (2) for σ_B, and Peetre inequality (22) give

$$\begin{aligned}\left|\partial_t^{j_R}\partial_n^{k_R}R_N(t,n,m)\right| &\le c^R_{j_R,N}\,|m|^N\max_{\theta\in[0,1]}\left|\partial_t^{j_R}\partial_n^{N+k_R}\sigma_B(t,n+\theta m)\right|\\ &\le c^R_{j_R,N}\,|m|^N\max_{\theta\in[0,1]}\left[C^B_{j_R(N+k_R)}(1+|n+\theta m|)^{\beta-N-k_R}\right]\\ &\le c^R_{j_R,k_R,N}\,(1+|m|)^{N+|\beta-N-k_R|}(1+|n|)^{\beta-N-k_R},\end{aligned}$$

and consequently

$$\left|\partial_t^{j_R} \triangle_n^{k_R} R_N(t,n,m)\right| \le C_{j_R,k_R,N}(1+|m|)^{N+|\beta-N-k_R|}(1+|n|)^{\beta-N-k_R}.$$

Take $j_A + j_R = j$ and $k_A + k_R = k$. By the discrete Leibniz formula

$$\triangle^k[u(n)v(n)] = \sum_{j=0}^{k} \binom{k}{j} \left[\triangle^j u(n)\right] \triangle^{k-j} v(n+j)$$

it holds that

$$\begin{aligned}\left|\partial_t^j \triangle_n^k E_N(t,n)\right| &\le c_{jkNr}\ (1+|n|)^{\alpha+\beta-N-k} \sum_{m\in\mathbb{Z}} (1+|m|)^{2N+|\beta|+j+k-r} \\ &\le c_{jkN}\ (1+|n|)^{\alpha+\beta-N-k}\end{aligned}$$

if r is chosen large enough. Hence $E_N \in \Sigma^{\alpha+\beta-N}$, and thus formula (19) is valid. Formula (18) is obtained by applying (8) on (19) ∎

References

[1] Abramowitz, M. and I. A. Stegun: *Handbook of mathematical functions.* 4th printing. United States Department of Commerce 1965.

[2] Agranovich, M. S.: *Spectral properties of elliptic pseudodifferential operators on a closed curve.* Funct. Anal. Appl. 13 (1979), 279 – 281.

[3] Agranovich, M. S.: *On elliptic pseudodifferential operators on a closed curve.* Trans. Moscow Math. Soc. 47 (1985), 23 – 74.

[4] Amosov, B. A.: *On the approximate solution of elliptic pseudodifferential equations on a smooth closed curve* (in Russian). Z. Anal. Anw. 9 (1990), 545 – 563.

[5] Berthold, D. and B. Silbermann: *Corrected collocation methods for periodic pseudodifferential equations.* Numer. Math. 70 (1995), 397 – 425.

[6] Elschner, J.: *Singular ordinary differential operators and pseudodifferential equations.* Berlin: Akademe-Verlag 1985.

[7] Kelle, O. and G. Vainikko: *A fully discrete Galerkin method of integral and pseudodifferential equations on closed curves.* Z. Anal. Anw. 14 (1995), 593 – 622.

[8] McLean, W.: *Local and global description of periodic pseudodifferential operators.* Math. Nachr. 150 (1991), 151 – 161.

[9] McLean, W., S. Prössdorf and W. L. Wendland: *A fully-discrete trigonometric collocation method.* J. Integral Equations Appl. 5 (1993), 103 – 129.

[10] Saranen, J. and G. Vainikko: *Trigonometric collocation methods with product integration for boundary integral equations on closed curves.* SIAM J. Numer. Anal. 33 (1996), 1577 – 1596.

[11] Saranen, J. and W. L. Wendland: *The Fourier series representation of pseudo-differential operators on closed curves.* Complex Variables 8 (1987), 55 – 64.

[12] Steffensen, J. F.: *Interpolation.* 2nd ed. New York: Chelsea Publishing Company 1950.

[13] Taylor, M.: *Pseudodifferential operators.* Princeton: Princeton University Press 1981.

[14] Taylor, M.: *Pseudodifferential operators and nonlinear PDE.* Boston – Basel – Berlin: Birkhäuser 1991.

[15] Treves, F.: *Introduction to pseudodifferential and Fourier integral operators.* New York: Plenum Press 1980.

[16] Turunen, V.: Symbol analysis of periodic pseudodifferential operators. Master's Thesis. Helsinki University of Technology, May 1997.

[17] Vainikko, G.: *Periodic integral and pseudodifferential equations.* Research report C13. Institute of Mathematics, Helsinki University of Technology 1996.