Institute of mathematics
Helsinki University of Technology
Arponen

Mat-1.3651 Numerical Linear Algebra (Numeerinen matriisilaskenta)

Final examination September 5, 2008

Please fill in clearly *on every sheet* the data on you and the examination. On *Examination code* mark course code, title and text mid-term or final examination. Study programmes are ARK, AUT, BIO, EST, ENE, GMA, INF, KEM, KJO, KTA, KON, MAK, MAR, PUU, RAK, TFY, TIK, TLT, TUO, YHD.

Calculators are not allowed nor needed. Time for the exam is 4 hours. You can answer in Finnish if you wish.

1. Let $A \in \mathbb{C}^{m\times m}$ be hermitian ($A = A^*$) and $S \in \mathbb{C}^{m\times m}$ be skew-hermitian ($S^* = -S$). Prove that
   (a) all eigenvalues of A are real.
   (b) if x and y are eigenvectors corresponding to different eigenvalues of A, then $x \perp y$.
   (c) all eigenvalues of S are pure imaginary.
2. Let $A \in \mathbb{C}^{m\times m}$. Show that A is unitarily similar to a diagonal matrix if and only if
$$\sum_{j=1}^{m} \sigma_j^2 = \sum_{j=1}^{m} |\lambda_j|^2,$$
   where λ_j are the eigenvalues and σ_j the singular values of A.
3. Let $A \in \mathbb{C}^{m\times n}$ be full rank, $m \geq n$ and $b \in \mathbb{C}^m$. Describe (the main steps, no implementation details needed) the following three ways to solve the Least Squares problem: find $x \in \mathbb{C}^n$ s.t. the residual $\|b - Ax\|_2 =$ minimum.
   (a) Solving by normal equations,
   (b) Solving by QR,
   (c) Solving by SVD.
4. Let A be a square matrix. Denote $A = L + D + U$ and deduce the classical Gauss-Seidel iteration $x_{k+1} = Gx_k + r$ for $Ax = b$. Show that if A is Hermitian and positive definite, then this iteration converges for all initial values x_0.
5. Suppose $A \in \mathbb{C}^{m\times m}$ is tridiagonal and Hermitian. Count the floating point operations of the QR iteration for A.