

Pekonen/Aalto

- 1) Tutustu Hölderin epäyhtälön $\sum_{k=1}^{\infty} x_k y_k \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \left(\sum_{k=1}^{\infty} |y_k|^q\right)^{1/q}$ todistukseen (liitteenä) jonoille $(x_k) \in \ell^p, (y_k) \in \ell^q, 1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} = 1$.
 - 2) Osoita, että edellisen tehtävän epäyhtälö on yhtälö jos ja vain jos löytyy vakio C siten, että $|y_k|^q = C |x_k|^p$ kaikilla $k \in \mathbb{N}$.
 - 3) Tutustu Minkowskin epäyhtälön $\left(\sum_{k=1}^{\infty} |x_k + y_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} + \left(\sum_{k=1}^{\infty} |y_k|^p\right)^{1/p}$ todistukseen (liitteenä) jonoille $(x_k), (y_k) \in \ell^p, 1 \leq p < \infty$.
 - 4) Osoita, että edellisen tehtävän epäyhtälö on yhtälö jos ja vain jos löytyy vakio $C \geq 0$ siten, että $x_k = C y_k$ kaikilla $k \in \mathbb{N}$.
 - 5) Selitä (kurssin Modu 1000 pohjalta) käsitteet
 - a) Lebesguen mitallinen joukko
 - b) yksinkertainen funktio $f = \sum_{k=1}^m \alpha_k \chi_{A_k}$
 - c) yksinkertaisen funktion f Lebesguen integraali $\int f d\mu = \sum_{k=1}^m \alpha_k \mu(A_k)$
 - d) Jos $f \geq 0$ mitallinen niin $\int f d\mu := \lim_{n \rightarrow \infty} \int f_n d\mu$ missä $f(x) = \lim_{n \rightarrow \infty} f_n(x)$ ja $0 \leq f_1 \leq f_2 \leq \dots \leq f$ f_n yksinkertaisia funktioita
 - e) Jos f mitallinen ja $\int |f| d\mu < \infty$ niin f on integroituva ja $\int f d\mu := \int f_+ d\mu - \int f_- d\mu$ $f_+(x) = \max\{f(x), 0\}, f_-(x) = \max\{-f(x), 0\}$
 - 6) Asetetaan $L^{(p)}(\Omega) = \{f: \Omega \rightarrow \mathbb{R} \mid f \text{ mitallinen, } \|f\|_p := (\int |f(x)|^p d\mu)^{1/p} < \infty\}$ ($\Omega \subset \mathbb{R}^n$ mitallinen)
- Osoita $L^{(p)}$ -avaruuden Hölderin epäyhtälö:
- $$\int |(fg)(x)| d\mu \leq \left(\int |f(x)|^p d\mu\right)^{1/p} \left(\int |g(x)|^q d\mu\right)^{1/q}$$
- kun $f \in L^{(p)}$ ja $g \in L^{(q)}$ ja $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.
- Vihje: valitse tehtävän 1 todistuksessa $\alpha = \frac{|f(x)|}{\|f\|_p}, \beta = \frac{|g(x)|}{\|g\|_q}$.

where $y = (\eta_j)$ and $\sum |\eta_j|^p < \infty$. If we take only real sequences [satisfying (1)], we get the *real space* l^p , and if we take complex sequences [satisfying (1)], we get the *complex space* l^p . (Whenever the distinction is essential, we can indicate it by a subscript \mathbf{R} or \mathbf{C} , respectively.)

In the case $p = 2$ we have the famous *Hilbert sequence space* l^2 with metric defined by

$$(3) \quad d(x, y) = \sqrt{\sum_{j=1}^{\infty} |\xi_j - \eta_j|^2}.$$

This space was introduced and studied by D. Hilbert (1912) in connection with integral equations and is the earliest example of what is now called a *Hilbert space*. (We shall consider Hilbert spaces in great detail, starting in Chap. 3.)

We prove that l^p is a metric space. Clearly, (2) satisfies (M1) to (M3) provided the series on the right converges. We shall prove that it does converge and that (M4) is satisfied. Proceeding stepwise, we shall derive

- (a) an auxiliary inequality,
- (b) the Hölder inequality from (a),
- (c) the Minkowski inequality from (b),
- (d) the triangle inequality (M4) from (c).

The details are as follows.

(a) Let $p > 1$ and define q by

$$(4) \quad \frac{1}{p} + \frac{1}{q} = 1.$$

p and q are then called **conjugate exponents**. This is a standard term. From (4) we have

$$(5) \quad 1 = \frac{p+q}{pq}, \quad pq = p+q, \quad (p-1)(q-1) = 1.$$

Hence $1/(p-1) = q-1$, so that

$$u = t^{p-1} \quad \text{implies} \quad t = u^{q-1}.$$

Let α and β be any positive numbers. Since $\alpha\beta$ is the area of the rectangle in Fig. 5, we thus obtain by integration the inequality

$$(6) \quad \alpha\beta \cong \int_0^\alpha t^{p-1} dt + \int_0^\beta u^{q-1} du = \frac{\alpha^p}{p} + \frac{\beta^q}{q}.$$

Note that this inequality is trivially true if $\alpha = 0$ or $\beta = 0$.

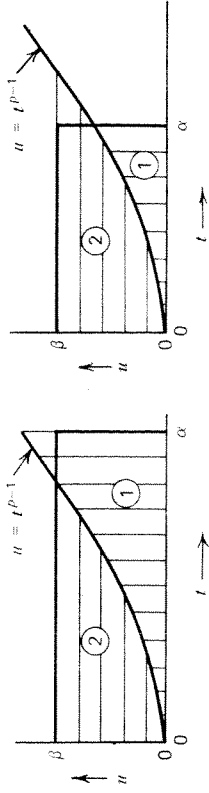


Fig. 5. Inequality (6), where ① corresponds to the first integral in (6) and ② to the second

(b) Let (ξ_j) and (η_j) be such that

$$(7) \quad \sum |\xi_j|^p = 1, \quad \sum |\eta_j|^q = 1.$$

Setting $\alpha = |\xi_j|$ and $\beta = |\eta_j|$, we have from (6) the inequality

$$|\xi_j \eta_j| \cong \frac{1}{p} |\xi_j|^p + \frac{1}{q} |\eta_j|^q.$$

If we sum over j and use (7) and (4), we obtain

$$(8) \quad \sum |\xi_j \eta_j| \cong \frac{1}{p} + \frac{1}{q} = 1.$$

We now take any nonzero $x = (\xi_j) \in l^p$ and $y = (\eta_j) \in l^q$ and set

$$(9) \quad \xi_j = \frac{\xi_j}{\left(\sum |\xi_k|^p\right)^{1/p}}, \quad \eta_j = \frac{\eta_j}{\left(\sum |\eta_m|^q\right)^{1/q}}.$$

Then (7) is satisfied, so that we may apply (8). Substituting (9) into (8) and multiplying the resulting inequality by the product of the denominators in (9), we arrive at the **Hölder inequality for sums**

$$(10) \quad \sum_{j=1}^{\infty} |\xi_j \eta_j| \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} \left(\sum_{m=1}^{\infty} |\eta_m|^q \right)^{1/q}$$

where $p > 1$ and $1/p + 1/q = 1$. This inequality was given by O. Hölder (1889).

If $p = 2$, then $q = 2$ and (10) yields the **Cauchy-Schwarz inequality for sums**

$$(11) \quad \sum_{j=1}^{\infty} \xi_j \eta_j \leq \sqrt{\sum_{k=1}^{\infty} |\xi_k|^2} \sqrt{\sum_{m=1}^{\infty} |\eta_m|^2}.$$

It is too early to say much about this case $p = q = 2$ in which p equals its conjugate q , but we want to make at least the brief remark that this case will play a particular role in some of our later chapters and lead to a space (a Hilbert space) which is "nicer" than spaces with $p \neq 2$.

(c) We now prove the **Minkowski inequality for sums**

$$(12) \quad \left(\sum_{j=1}^{\infty} |\xi_j + \eta_j|^p \right)^{1/p} \leq \left(\sum_{k=1}^{\infty} |\xi_k|^p \right)^{1/p} + \left(\sum_{m=1}^{\infty} |\eta_m|^p \right)^{1/p}$$

where $x = (\xi_j) \in l^p$ and $y = (\eta_j) \in l^p$, and $p \geq 1$. For finite sums this inequality was given by H. Minkowski (1896).

For $p = 1$ the inequality follows readily from the triangle inequality for numbers. Let $p > 1$. To simplify the formulas we shall write $\xi_j + \eta_j = \omega_j$. The triangle inequality for numbers gives

$$\begin{aligned} |\omega_j|^p &= |\xi_j + \eta_j| |\omega_j|^{p-1} \\ &\leq (|\xi_j| + |\eta_j|) |\omega_j|^{p-1}. \end{aligned}$$

Summing over j from 1 to any fixed n , we obtain

$$(13) \quad \sum |\omega_j|^p \leq \sum |\xi_j| |\omega_j|^{p-1} + \sum |\eta_j| |\omega_j|^{p-1}.$$

To the first sum on the right we apply the Hölder inequality, finding

$$\sum |\xi_j| |\omega_j|^{p-1} \leq \left[\sum |\xi_k|^p \right]^{1/p} \left[\sum (\omega_m |^{p-1})^q \right]^{1/q}.$$

1.2 Further Examples of Metric Spaces

On the right we simply have

$$(p-1)q = p$$

because $pq = p + q$; see (5). Treating the last sum in (13) in a similar fashion, we obtain

$$\sum |\eta_j| |\omega_j|^{p-1} \leq \left[\sum |\eta_k|^p \right]^{1/p} \left[\sum |\omega_m|^p \right]^{1/q}.$$

Together,

$$\sum |\omega_j|^p \leq \left\{ \left[\sum |\xi_k|^p \right]^{1/p} + \left[\sum |\eta_k|^p \right]^{1/p} \right\} \left(\sum |\omega_m|^p \right)^{1/q}.$$

Dividing by the last factor on the right and noting that $1 - 1/q = 1/p$, we obtain (12) with n instead of ∞ . We now let $n \rightarrow \infty$. On the right this yields two series which converge because $x, y \in l^p$. Hence the series on the left also converges, and (12) is proved.

(d) From (12) it follows that for x and y in l^p the series in (2) converges. (12) also yields the triangle inequality. In fact, taking any $x, y, z \in l^p$, writing $z = (\zeta_j)$ and using the triangle inequality for numbers and then (12), we obtain

$$\begin{aligned} d(x, y) &= \left(\sum |\xi_j - \eta_j|^p \right)^{1/p} \\ &\leq \left(\sum |\xi_j - \zeta_j| + |\zeta_j - \eta_j| \right)^{1/p} \\ &\leq \left(\sum |\xi_j - \zeta_j|^p \right)^{1/p} + \left(\sum |\zeta_j - \eta_j|^p \right)^{1/p} \\ &= d(x, z) + d(z, y). \end{aligned}$$

This completes the proof that l^p is a metric space. ■

The inequalities (10) to (12) obtained in this proof are of general importance as indispensable tools in various theoretical and practical problems, and we shall apply them a number of times in our further work.